Section 15.8, Problem 26

Use Lagrange multipliers to prove that the triangle with maximum area that has a given perimeter p is equilateral. We use Heron's formula for the area

$$A = \sqrt{s(s-x)(s-y)(s-z)} \tag{1}$$

where $s = p/2$ is half of the perimeter and x, y, z are the lengths of the sides.

First we formulate the constraint. By definition, the perimeter is the sum of the side lengths,

$$x + y + z = p = 2s \tag{2}$$

In the context of this problem, where we want to maximize area with a fixed perimeter, p and s are constant. Thus our constraint function is

$$g(x, y, z) = x + y + z \tag{3}$$

and its gradient is

$$\nabla g = \langle 1, 1, 1 \rangle \tag{4}$$

The function to maximize is the area A. Thus

$$f(x, y, z) = A = \sqrt{s(s-x)(s-y)(s-z)} \tag{5}$$

Taking the gradient of f:

$$\nabla f = \left\langle \frac{s(-1)(s-y)(s-z)}{2\sqrt{s(s-x)(s-y)(s-z)}}, \frac{s(s-x)(-1)(s-z)}{2\sqrt{s(s-x)(s-y)(s-z)}}, \frac{s(s-x)(s-y)(-1)}{2\sqrt{s(s-x)(s-y)(s-z)}} \right\rangle \tag{6}$$

$$= \frac{-s}{2\sqrt{s(s-x)(s-y)(s-z)}} \Big\langle (s-y)(s-z), (s-x)(s-z), (s-x)(s-y) \Big\rangle \tag{7}$$

By the method of Lagrange multipliers, we have

$$\nabla f = \lambda \nabla g = \lambda \langle 1, 1, 1 \rangle = \langle \lambda, \lambda, \lambda \rangle \tag{8}$$

Regardless of the value of λ, this equation implies that all the components of ∇f are equal to each other. Since each component of ∇f involves a factor of $-s/(2\sqrt{s(s-x)(s-y)(s-z)})$, and this factor is never zero (it is essentially the perimeter divided by the area, neither of which can be zero), we can cancel it from the equations to obtain the system

$$(s-y)(s-z) = (s-x)(s-z) \tag{9}$$

$$(s-x)(s-z) = (s-x)(s-y) \tag{10}$$

$$(s-x)(s-y) = (s-y)(s-z) \tag{11}$$

$$x + y + z = 2s \tag{12}$$

Each of the expressions $(s-x)$, $(s-y)$, and $(s-z)$ cannot be zero. For if one is zero, say $(s-x) = 0$, then $x = s = p/2$. Then $y + z = p/2$ as well, and we have $x = y + z$. But for a proper triangle, $x < y + z$. (The triangle inequality: each side is less than the sum of the other two). Thus we may cancel such an expression whenever it appears on both sides of an equation, obtaining

$$(s - y) = (s - x) \tag{13}$$

$$(s - z) = (s - y) \tag{14}$$

$$(s - x) = (s - z) \tag{15}$$

$$x + y + z = 2s \tag{16}$$

The first three equations imply $x = y = z$, so the triangle must be equilateral.