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A Boundary Meshless Method for Neumann Problem

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Abstract

Boundary integral equations (BIE) are reformulations of boundary value problems for partial differential equations. There is a plethora of research on numerical methods for all types of these equations such as solving by discretization which includes numerical integration. In this paper, the Neumann problem is reformulated to a BIE, and then moving least squares as a meshless method is described for solving this integral equation. Error analysis of this method is discussed and then its application and accuracy are illustrated by some case studies.

Keywords: Laplace's equation; Neumann problem; Boundary integral equation; Meshless method; Moving least squares method

Introduction

Laplace's equation is a second order partial differential equation as

$$\Delta u = 0 \ \ in \ \ D,$$

where Δ is the Laplace operator and u is a scalar function of two or three variables. The general theory of solution to Laplace's equation is known as potential theory. The solutions of Laplace's equation are all harmonic functions which are important in many fields of science, notably the fields of electromagnetism, astronomy and fluid dynamics. The Dirichlet problem for Laplace's equation consists of finding a solution u on some domain D such that u on the boundary of D satisfies

$$u = f \ , \ on \ \ S = \partial D,$$

where f is a given function. The Neumann boundary conditions for Laplace's equation specify not the function u itself on the boundary of D, but its normal derivative

$$\frac{\partial u}{\partial n} = f \ , \ on \ \ S = \partial D.$$

Physically, this corresponds to the construction of a potential for a vector field whose effect is known at the boundary of D alone. An important question is whether a solution to a Dirichlet or Neumann problem exists, and if it does, whether it is unique or not. This question is of great importance in both mathematics and physics. Obviously, if we could not guarantee the existence of a solution, the effort of finding it can be in vain. In addition, even if a solution exists, it may not be unique, and then we could not tie the solution to the unique physical state that we are modeling. The question of uniqueness is easier to answer than that of existence.

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For the Dirichlet problem, if a solution exists, it is unique; and for the Neumann problem, it is unique to within an arbitrary constant. The existence theorem, however, is more difficult to prove (see [1]).

There are various numerical methods to solve these problems. One of them is the finite difference method (FDM) for the solution of differential equations, especially, for boundary value problems.

Another numerical technique for finding approximate solutions of partial differential equation is the finite element method (FEM). The solution approach is based either on eliminating the differential equation completely or rendering the partial differential equation into an approximating system of ordering differential equation, which are then numerically integrated using standard techniques such as Euler's method, Runge-Kutta, etc. In solving partial differential equations, the primary challenge is to create an equation that approximates the equation under study, and is also numerically stable. This means that errors in the input and intermediate calculations do not accumulate and, therefore, do not cause the resulting output to be meaningless. In so doing, there are several ways, all of which have both advantages and disadvantages. The Finite Element Method is a good choice for solving partial differential equations over complicated domains (like cars and oil pipelines), when the domain changes (as during a solid state reaction with a moving boundary), when the desired precision varies over the entire domain, or when the solution lacks smoothness. For instance, in a frontal crash simulation it is possible to increase prediction accuracy in important areas like the front of the car and reduce it in its rear. The differences between FEM and FDM are:

- The most attractive feature of the FEM is its ability to handle complicated geometries (and boundaries) with relative ease. While FDM in its basic form is restricted to handle rectangular shapes and simple alterations thereof, the handling of geometries in FEM is theoretically straightforward.
- The most attractive feature of finite differences is that it can be very easy to implement.
- There are several ways one could consider the FDM a special case of the FEM approach. One might choose basis functions as either piecewise constant functions or Dirac delta functions. In both approaches, the approximations are defined on the entire domain, but need not be continuous. Alternatively, one might define the function on a discrete domain, with the result that the continuous differential operator no longer makes sense, however this approach is not FEM.
- There are reasons to consider the mathematical foundation of the finite element approximation more sound, for instance, because the quality of the approximation between grid points is poor in FDM.
- The quality of a FEM approximation is often higher than in the corresponding FDM approach, but this extremely depend on problem and several examples to the contrary can be provided.

The finite volume method (FVM) is a method for representing and evaluating partial differential equations in the form of algebraic equations (see [2]). Similar to the finite difference method or finite element method, values are calculated at discrete places on a meshed geometry. "Finite volume" refers to the small volume surrounding each node point on a mesh. In the finite volume method, volume integrals in a partial differential equation that contain a divergence term are converted to surface integrals. These integrals are then evaluated as fluxes at the surfaces of each finite volume. Because the flux entering a given volume is identical to that leaving the adjacent volume, these methods are conservative. Another advantage of the finite volume method is that it is easily formulated to allow for unstructured meshes.

Since 1970 there has been a significant increase in the popularity of using boundary integral equations to solve Laplace's equation and many other elliptic equations. A given boundary value problem for an elliptic partial differential equation over an open region D can often be reformulated as an equivalent integral equation over the boundary of D. Such a reformulation is called a boundary integral equation, and has the advantage of diminishing the number of space dimension by one and of the capability to handle problems involving infinite domains. This leads to a reduction in the number of algebraic equations generated for the solutions. There exist some numerical methods to solve the obtained boundary integral equation such as the Wavelet-Galerkin method introduced in [3]. In this paper we describe reformulation of a boundary value problem (Neumann problem) to a boundary integral equation. Then we introduce a meshless method based on moving least squares to solve the obtained integral equation. The moving least squares (MLS) as an approximating method has been introduced by Shepard [4], and Lancaster and Salkauskas [5]. Since the numerical approximations of MLS are based on a cluster of scattered nodes instead of an interpolation on elements, there have many meshless methods which are based on the MLS method for the numerical solution of differential equations in recent years. Error analysis and convergence of this method are discussed and its accuracy and applications are illustrated by some numerical examples.

Boundary Integral Method

In this section, we employ boundary integral equation to solve the Neumann problem which in turn is an important kind of boundary value problems. For this purpose, we use the notation which has been used in [6]. Let D be a bounded open simply connected region in the plane, and let its boundary S be a simple closed curve with a parameterization

$$\mathbf{r}(t) = (\xi(t), \eta(t)),\ 0 \le t \le L, \tag{1}$$

with $\mathbf{r} \in C^2[0, L]$ and $|\mathbf{r}'(t)| \neq 0$ for $0 \le t \le L$. We assume the parameterization traverses S in a counter-clockwise direction and then introduce the interior unit normal $\mathbf{n}(t)$ that is orthogonal to the curve S at $\mathbf{r}(t)$:

$$\mathbf{n}(t) = \frac{(-\eta'(t), \xi'(t))}{\sqrt{\xi'(t)^2 + \eta'(t)^2}}.$$

The Neumann problem is as follows: Find $u \in C^1(\bar{D}) \cap C^2(D)$ that satisfies

$$\Delta u(P) = 0,\ P \in D,$$

$$\frac{\partial u(P)}{\partial \mathbf{n}_P} = f(P),\ P \in S,$$

with $f \in C(S)$ a given boundary function. The following theorem in [6] guarantees the uniqueness of the solution.

Theorem 1. Let the function $f \in C(S)$, and for the parameterization of (1), assume $\mathbf{r} \in C^2[0, L]$. Then the Neumann problem has a unique solution, up to the addition of an arbitrary constant, provided

$$\int_S f(Q) dS = 0. \tag{2}$$

A very important tool for studying elliptic partial differential equations is the divergence theorem or Gauss's theorem.

Theorem 2. Assume $\mathbf{F}: \bar{D} \to R^2$ with each component of **F** contained in $C^1(\bar{D})$. Then

$$\int_D \nabla . \mathbf{F}(Q) dD = -\int_S \mathbf{F}(Q) . \mathbf{n}(Q) dS. \tag{3}$$

Using the divergence theorem, one can obtain Green's identities and Green's representation formula. Assuming $u \in C^1(\bar{D})$ and $v \in C^2(\bar{D})$, one can prove Green's first identity by letting $\mathbf{F} = u \nabla v$ in (3)

$$\int_D u \Delta v dD + \int_D \nabla u . \nabla v dD = -\int_S u \frac{\partial v}{\partial \mathbf{n}} dS. \tag{4}$$

Next, assume $u, v \in C^2(\bar{D})$. Interchanging the roles of *u* and *v* in (4) and then subtracting the two identities, one may obtain Green's second identity

$$\int_D [u \Delta v - v \Delta u] dD = \int_S [v \frac{\partial u}{\partial \mathbf{n}} - u \frac{\partial v}{\partial \mathbf{n}}] dS. \tag{5}$$

The identity (4) can be used to prove (i) if the Neumann problem has a solution, then it is unique up to the addition of an arbitrary constant; and (ii) if the Neumann problem is to have a solution, then the condition (2) is necessary.

Another tool that we need is fundamental solution for Laplace's equation in two variables. The fundamental solution of a partial differential equation is a weighting function that is used to boundary integral reformulation of the equation. For this purpose, we have to introduce Dirac delta function. Suppose $\{w_n(x)\}_{n=1}^{\infty}$ be a sequence of unit force distributions which are defined as below

$$w_n(x) = \begin{cases} \frac{n}{2} & |x| < \frac{1}{n} \\ 0 & |x| > \frac{1}{n} \end{cases},$$

which shown in Figure 1.

Then we define the Dirac delta function as the limit of the sequence

$$\delta(x) = \lim_{n \to \infty} w_n(x).$$

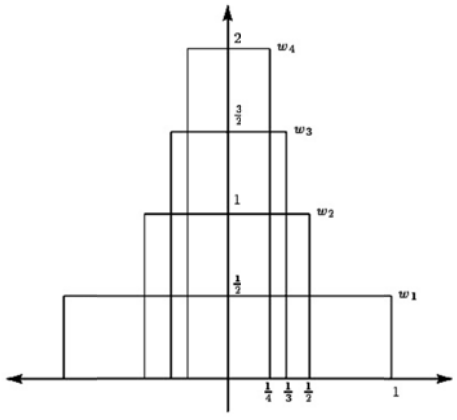

Figure 1. Illustrations of unit force distributions.

The Dirac delta function is not a function in the usual sense, because we have $\delta(x)=0$ for $x\neq 0$ and $\delta(0)=\infty$, it is, therefore, more correctly referred to as the Dirac delta distribution. It has properties which are mentioned below

a) We have $\int_{-\infty}^{\infty}\delta(x)dx=1$ since $\int_{-\infty}^{\infty}w_n(x)dx=1$ for all $n\geq 1$.

b) For any continuous function $h(x)$: $\int_{-\infty}^{\infty}\delta(x)h(x)dx=h(0).$

c) In general, for any constant ξ: $\int_{-\infty}^{\infty}\delta(\xi-x)h(x)dx=h(\xi).$

d) The Dirac delta function is the slope of the Heaviside step function $H(\xi-x)=\begin{cases}0 & if \quad \xi<x\\ 1 & if \quad \xi>x\end{cases}$ (where ξ is a constant): $\delta(\xi-x)=H'(\xi-x)$.

Laplace's equation, as well as most of the common equations, has a well-known fundamental solution. The fundamental solution is a solution of the following equation for an arbitrary point $A\in D$

$$\Delta w+\delta(A-Q)=0,$$

It means w is a solution of $\Delta w=0$ which has a singularity at the point *A*. This is not difficult to find this solution, and one can see [7] for a description of the manner in which w can be determined. In 2-D Laplace's equation the fundamental solution is $\mathrm{w(Q)}=-\frac{1}{2\pi}\log|\mathrm{A}-\mathrm{Q}|$.

Assume $u\in C^2(\bar{D})$, where u(Q) be a solution of Laplace's equation, and let $\mathrm{v(Q)}=\log|\mathrm{A}-\mathrm{Q}|$, with $A\in D$. Since $w(Q)=-\frac{1}{2\pi}v(Q)$ and w(Q) is the fundamental solution, then

$$-\frac{1}{2\pi}\Delta v+\delta(A-Q)=0,$$

or

$$\Delta v=2\pi\delta(A-Q).$$

Also we have $\Delta u=0$, because we supposed that u(Q) be a solution of Laplace's equation. Now by replacing them into the Green's second identity (5) we will have the following relation

$$2\pi u(A)=\int_S[\frac{\partial u(Q)}{\partial \mathbf{n}(Q)}\log|A-Q| -u(Q)\frac{\partial}{\partial \mathbf{n}(Q)}[\log|A-Q|]]dS_Q,\quad A\in D, \tag{6}$$

where the property (c) of Dirac delta function is used in the left side. Assume $P\in S$, and *A* tends to *P*, then we obtain the following limits where one can find a proof of them in [8] or other text on Laplace's equation

$$\lim_{A\to P}\int_S\frac{\partial u(Q)}{\partial \mathbf{n}(Q)}\log|A-Q|dS_Q=\int_S\frac{\partial u(Q)}{\partial \mathbf{n}(Q)}\log|P-Q|dS_Q,$$

and

$$\lim_{A\to P}\int_S u(Q)\frac{\partial}{\partial \mathbf{n}(Q)}[\log|A-Q|]dS_Q=-\pi u(P)+\int_S u(Q)\frac{\partial}{\partial \mathbf{n}(Q)}[\log|P-Q|]dS_Q.$$

By taking limit for both side of the equation (6) and using the above relations we have

$$u(P)=\frac{1}{\pi}\int_S[\frac{\partial u(Q)}{\partial \mathbf{n}(Q)}\log|P-Q| -u(Q)\frac{\partial}{\partial \mathbf{n}(Q)}[\log|P-Q|]]dS_Q,\quad P\in S, \tag{7}$$

which gives a relationship between the values of u and its normal derivative on *S*. By using the boundary condition of the Neumann problem, $\partial u/\partial \mathbf{n}=f$ on S, we rewrite (7) as follows

$$u(P)+\frac{1}{\pi}\int_S u(Q)\frac{\partial}{\partial \mathbf{n}(Q)}[\log|P-Q|]dS_Q=\frac{1}{\pi}\int_S f(Q)\log|P-Q|dS_Q,\quad P\in S. \tag{8}$$

The equation (8) is an integral equation of the second kind which is solvable if and only if the boundary function f satisfies the condition (2). Unfortunately, it is not uniquely solvable. The simplest way to deal with the lack of uniqueness in solving (8) is to introduce an additional condition such as

$$u(P^*)=0,$$

for some fixed point $P^*\in S$. Let $\mathrm{u_1(Q)}$ and $\mathrm{u_2(Q)}$ be two disjoint solutions of (8) which both satisfy in the

above condition. By choosing $P = P^*$ and $u \equiv u_1$ or u_2, the equation (8) implies

$$\int_S u_1(Q)\frac{\partial}{\partial \mathbf{n}(Q)}[\log|P^* - Q|]dS_Q = \int_S f(Q)\log|P^* - Q|\,dS_Q,$$

$$\int_S u_2(Q)\frac{\partial}{\partial \mathbf{n}(Q)}[\log|P^* - Q|]dS_Q = \int_S f(Q)\log|P^* - Q|\,dS_Q,$$

which in turn imply $u_1(Q)=u_2(Q)$. Thus the equation (8) has a unique solution. When the representation $\mathbf{r}(t) = (\xi(t), \eta(t))$ of (1) for S is applied to integral equation (8), this equation becomes

$$-\pi u(t) + \int_0^L K(t,s)u(s)ds = g(t),\ 0 \le t \le L, \qquad (9)$$

with

$$K(t,s) = \frac{\eta'(s)[\xi(t)-\xi(s)] - \xi'(s)[\eta(t)-\eta(s)]}{[\xi(t)-\xi(s)]^2 + [\eta(t)-\eta(s)]^2}, s \ne t,$$

$$K(t,t) = \frac{\eta'(t)\xi''(t) - \xi'(t)\eta''(t)}{2[\xi'(t)^2 + \eta'(t)^2]},$$

and the right hand side

$$g(t) = \int_0^L f(\mathbf{r}(s))\sqrt{\xi'(t)^2 + \eta'(t)^2}\log|\mathbf{r}(t) - \mathbf{r}(s)|\,ds.$$

The solution of this Fredholm integral equation of the second kind is the solution of the Neumann problem on the boundary of *D*. Let $u_i \in C^1(\overline{D_i})$ be a solution of Laplace's equation in D_i (interior of *D*), then u_i satisfies the following relation (see [6])

$$u_i(A) = \int_S u(Q)\frac{\partial}{\partial \mathbf{n}(Q)}[\log|A - Q|]dS_Q,\ A \in D_i,$$

with using (1) for *S* it takes the form

$$u_i(x,y) = \int_0^L M(x,y,s)u(s)ds, (x,y) \in D_i, \qquad (10)$$

$$M(x,y,s) = \frac{\eta'(s)[\xi(s)-x] + \xi'(s)[\eta(s)-y]}{[\xi(s)-x]^2 + [\eta(s)-y]^2}.$$

Fredholm integral equation (9) can be solved with any methods which have been described in books on integral equations. In this paper, we solve it by a meshless method which is based on the moving least square (MLS) method (see [9]). This method does not require any background interpolation or approximation cells and does not depend on the geometry of domain. It is, therefore, suitable for problems with any kind of domains.

Moving Least Squares Method

For a better sense of moving least squares (MLS), we introduce least squares (LS) and weighted least square (WLS) at first. Given n points where located at positions $\mathbf{x}_i \in \mathbf{R}^d$ for $i = 1,...,n$. We wish to obtain a globally defined function $u^h(\mathbf{x})$ that approximates the given scalar values u_i at points $\mathbf{x}_i$ in the least-squares sense with the error functional $J_{LS} = \sum_i \| u^h(\mathbf{x}_i) - u_i \|^2$. Thus, we pose the following minimization problem

$$\min_{u^h \in \Pi_m^d} \sum_i \| u^h(\mathbf{x}_i) - u_i \|^2, \qquad (11)$$

where u^h is taken from Π_m^d, the space of polynomials of total degree *m* in *d* spatial dimensions, and it can be written as

$$u^h(\mathbf{x}) = \mathbf{p}(\mathbf{x})^T \mathbf{a} = \mathbf{p}(\mathbf{x}).\mathbf{a}, \qquad (12)$$

where $\mathbf{p}(\mathbf{x}) = [p_1(\mathbf{x}),..., p_k(\mathbf{x})]^T$ is the polynomial basis vector and $\mathbf{a} = [a_1,...,a_k]^T$ is the vector of unknown coefficients, which we want to minimize in (11). We can minimize (11) by setting the partial derivatives of the error functional J_{LS} to zero, i.e. $\nabla J_{LS} = 0$, where $\nabla = [\partial/\partial a_1,..., \partial/\partial a_k]^T$, which is a necessary condition for a minimization. Then we obtain a linear system of equations

$$\sum_i \mathbf{p}(\mathbf{x}_i)\mathbf{p}(\mathbf{x}_i)^T \mathbf{a} = \sum_i \mathbf{p}(\mathbf{x}_i)u_i,$$

which is solved as

$$\mathbf{a} = [\sum_i \mathbf{p}(\mathbf{x}_i)\mathbf{p}(\mathbf{x}_i)^T]^{-1} \sum_i \mathbf{p}(\mathbf{x}_i)u_i. \qquad (13)$$

If the square matrix $A_{LS} = \sum_i \mathbf{p}(\mathbf{x}_i)\mathbf{p}(\mathbf{x}_i)^T$ is nonsingular, replacing Eq. (13) into Eq. (12) provides the fit function $u^h(\mathbf{x})$.

In the weighted least squares formulation, for a fixed point $\bar{\mathbf{x}} \in \mathbf{R}^d$, we minimize

$$\min_{u^h \in \Pi_m^d} \sum_i w(\| \bar{\mathbf{x}} - \mathbf{x}_i \|)\| u^h(\mathbf{x}_i) - u_i \|^2. \qquad (14)$$

The unknown coefficients we want to obtain from the solution to (14) are weighted by distance to $\bar{\mathbf{x}}$ and therefore a function of $\bar{\mathbf{x}}$. Thus, the local, weighted

least squares approximation in $\bar{\mathbf{x}}$ is written as

$$u^h_{\bar{\mathbf{x}}}(\mathbf{x}) = \mathbf{p}(\mathbf{x})^T \mathbf{a}(\bar{\mathbf{x}}) = \mathbf{p}(\mathbf{x}).\mathbf{a}(\bar{\mathbf{x}}). \tag{15}$$

By taking partial derivatives of the error functional J_{WLS} with respect to the unknown coefficients $\mathbf{a}(\bar{\mathbf{x}})$ we obtain

$$\mathbf{a}(\bar{\mathbf{x}}) = [\sum_i w(\| \bar{\mathbf{x}} - \mathbf{x}_i \|)\mathbf{p}(\mathbf{x}_i)\mathbf{p}(\mathbf{x}_i)^T]^{-1} .\sum_i w(\| \bar{\mathbf{x}} - \mathbf{x}_i \|)\mathbf{p}(\mathbf{x}_i)u_i . \tag{16}$$

Obviously, the only difference between Eqs. (13) and (16) is the weighting terms. Note that whereas the coefficients $\mathbf{a}$ in Eq. (13) are global, the coefficients $\mathbf{a}(\bar{\mathbf{x}})$ are local and need to be recomputed for every $\bar{\mathbf{x}}$. If the square matrix $A_{WLS} = \sum_i w(\| \bar{\mathbf{x}} - \mathbf{x}_i \|)\mathbf{p}(\mathbf{x}_i)\mathbf{p}(\mathbf{x}_i)^T$ is nonsingular, replacing Eq. (16) into Eq. (15) provides the fit function $u^h_{\bar{\mathbf{x}}}(\mathbf{x})$.

The MLS method was proposed by Lancaster and Salkauskas [5] for smoothing and interpolating data. The idea is to start with a weighted least squares formulation for an arbitrary fixed point in $\mathbf{R}^d$, and then move this point over the entire parameter domain, where a weighted least squares fit is computed and evaluated for each point individually. It can be shown that the global function $u^h(\mathbf{x})$, obtained from a set of local functions,

$$u^h(\mathbf{x}) = u^h_{\mathbf{x}}(\mathbf{x}),$$

$$\min_{u^h_{\mathbf{x}} \in \Pi^d_m} \sum_i w(\| \mathbf{x} - \mathbf{x}_i \|) \| u^h_{\mathbf{x}}(\mathbf{x}_i) - u_i \|^2,$$

is continuously differentiable if and only if the weighting function is continuously differentiable (see Levins work [10,11]).

Now we extend the MLS method for BIE which is obtained in the previous section. For this purpose, we follow [9]. Given data values $\mathbf{u} = \{u_j\}_{j=1}^N$ at nodes $\mathbf{x}_j$, the MLS method produces a function $u^h \in C^s(\mathbf{R}^d)$ that approximates data $\mathbf{u}$ in a weighted square sense. Let Π^d_q be the space of polynomials of degree q, $q \ll N$ and $q \ll s$, and let $\{p_0, p_1, ..., p_m\}$ be any basis of Π^d_q. The MLS approximation $u^h(\mathbf{x})$ of $u(\mathbf{x})$ for all $\mathbf{x} \in \Omega$, can be defined as

$$u^h(\mathbf{x}) = \mathbf{P}^T \mathbf{a}(\mathbf{x}), \quad \forall \mathbf{x} \in \bar{\Omega}, \tag{17}$$

where $\mathbf{P}^T(\mathbf{x}) = [p_0, p_1, ..., p_m]$ and $\mathbf{a}(\mathbf{x})$ is a vector with components $a_j(\mathbf{x})$, $j = 0,1,...,m$, which are functions of the space coordinates $\mathbf{x}$. In this paper we suppose that the basis of Π^d_q is a complete monomial basis of order m .For example, for a 1-D problem, the linear basis is $\{1, x\}$, and the quadratic basis is $\{1, x, x^2\}$. For a 2-D problem, the linear basis is $\{1, x, y\}$, and the quadratic basis is $\{1, x, y, x^2, xy, y^2\}$ where $\mathbf{x} = (x, y)$.

The coefficient vector $\mathbf{a}(\mathbf{x})$ is determined by minimizing a weight discrete square norm, which is defined as (u is a scalar function)

$$J(\mathbf{x}) = \sum_{j=1}^n w_j(\mathbf{x})(P^T(\mathbf{x}_j)\mathbf{a}(\mathbf{x}) - u_j)^2 = [P.\mathbf{a}(\mathbf{x}) - \mathbf{u}]^T W .[P.\mathbf{a}(\mathbf{x}) - \mathbf{u}], \tag{18}$$

where $w_j(\mathbf{x})$ is the weight function associated with the node j and n is the number of nodes in $\bar{\Omega}$ for which the weight function $w_j(\mathbf{x}) > 0$ and u_j are the fictitious nodal values, but not the nodal values of the unknown trial function $u^h(\mathbf{x})$ i.e. $u^h(\mathbf{x}_j) \neq u_j$. The matrices P and W are defined as

$$P = \begin{bmatrix} \mathbf{p}^T(\mathbf{x}_1) \\ \mathbf{p}^T(\mathbf{x}_2) \\ \vdots \\ \mathbf{p}^T(\mathbf{x}_n) \end{bmatrix}_{n \times (m+1)}, W = \begin{bmatrix} w_1(\mathbf{x}) & \cdots & 0 \\ \cdots & \ddots & \cdots \\ 0 & \cdots & w_n(\mathbf{x}) \end{bmatrix}_{n \times n}.$$

The stationary point of J, in Eq. (18), with respect to $\mathbf{a}(\mathbf{x})$ leads to the following linear relation between $\mathbf{a}(\mathbf{x})$ and $\mathbf{u}$

$$A(\mathbf{x})\mathbf{a}(\mathbf{x}) = B(\mathbf{x})\mathbf{u}, \tag{19}$$

where the matrices $A(\mathbf{x})$ and $B(\mathbf{x})$ are defined by

$$A(\mathbf{x}) = P^T W P = B(\mathbf{x})P = \sum_{j=1}^n w_j(x)\mathbf{p}(\mathbf{x}_j)\mathbf{p}^T(\mathbf{x}_j),$$

$$B(\mathbf{x}) = P^T W = [w_1(\mathbf{x})\mathbf{p}(\mathbf{x}_1), w_2(\mathbf{x})\mathbf{p}(\mathbf{x}_2), ..., w_n(\mathbf{x})\mathbf{p}(\mathbf{x}_n)].$$

The matrix A is often called the moment matrix, it is of size $(m+1)\times(m+1)$. Computing $\mathbf{a}(\mathbf{x})$, using Eq. (19) and substituting it into Eq. (17), gives

$$u^h(\mathbf{x}) = \Phi^T(\mathbf{x}).\mathbf{u} = \sum_{j=1}^n \phi_j(\mathbf{x})u_j, \mathbf{x} \in \bar{\Omega}, \tag{20}$$

where

$$\Phi^T(\mathbf{x}) = \mathbf{p}^T(\mathbf{x})A^{-1}(\mathbf{x})B(\mathbf{x}),$$

or

$$\phi_j(\mathbf{x}) = \sum_{k=0}^{m} p_k(\mathbf{x})[A^{-1}(\mathbf{x})B(\mathbf{x})]_{kj}. \quad (21)$$

$\phi_j(\mathbf{x})$'s are called the shape functions of the MLS approximation, corresponding to nodal points $\mathbf{x}_j$.

Remark 3. In Eq. (21), to avoid calculating A^{-1}, we can get $C = A^{-1}B$ and then solve the systems $AC = B$ and since for $\phi_j(\mathbf{x})$, only components of *j*-th column of *C* are needed, it is enough to solve only linear system $AC_j = B_j$ which C_j and B_j are *j*-th column of *C* and *B*, respectively. This system can be solved by iterative methods.

If $w_j(\mathbf{x}) \in C^r(\Omega)$ and $p_k(\mathbf{x}) \in C^s(\Omega)$, $j = 1,\ldots,n$, $k = 0,1,\ldots,m$ then $\phi_j(\mathbf{x}) \in C^{\min(r,s)}(\Omega)$. The Gaussian weight function is applied in the present work as

$$w_j(x) = \begin{cases} \dfrac{exp[-(d_j/\alpha)^2] - exp[-(h_j/\alpha)^2]}{1 - exp[-(h_j/\alpha)^2]}, & 0 \le d_j \le h_j, \\ 0, & d_j > h_j, \end{cases}$$

where $d_j = \| \mathbf{x} - \mathbf{x}_j \|$ (the Euclidean distance between $\mathbf{x}$ and $\mathbf{x}_j$), α is a constant which controls the shape of the weight function w_j, and h_j is the size of the support domain.

Integral Equation and MLS

In this section, we follow [9] to solve an integral equation by MLS method. Then we use it for BIE which has been obtained in section 2. Consider Fredholm integral equation

$$\lambda u(x) + \int_a^b \kappa(x,\xi)u(\xi)d\xi = g(x), x \in [a,b], \quad (22)$$

where *u* is unknown function, λ is a real parameter, *a* and *b* are finite real numbers and $\kappa(x,\xi)$ is called kernel function. Assume that u, λ, κ and g have necessary conditions for solvability of the equation and analyzing the method.

To apply the method, at first *N* nodal points $\{x_i\}$ are selected on interval $[a,b]$ where $a \le x_1 < x_2 < \ldots < x_N \le b$. The distribution of nodes could be selected regularly or randomly. Then we can use u^h from Eq. (20) instead of u. So Eq. (22) becomes

$$\lambda u^h(x) + \int_a^b \kappa(x,\xi)u^h(\xi)d\xi = g(x), x \in [a,b], \quad (23)$$

or equivalently

$$\sum_{j=1}^{N}[\lambda\phi_j(x) + \int_a^b \kappa(x,\xi)\phi_j(\xi)d\xi]u_j = g(x), x \in [a,b],$$

Assume that the above Eq. holds at x_i

$$\sum_{j=1}^{N}[\lambda\phi_j(x_i) + \int_a^b \kappa(x_i,\xi)\phi_j(\xi)d\xi]u_j = g(x_i), i=1,2,\ldots N. \quad (24)$$

Using a m_1-point quadrature formula with the coefficients $\{\xi_k\}$ and weights $\{\omega_k\}$ in interval $[a,b]$ to numerically solve the integration in (24), yielding

$$\sum_{j=1}^{N}[\lambda\phi_j(x_i) + \sum_{k=1}^{m_1}\kappa(x_i,\xi_k)\phi_j(\xi_k)\omega_k]\hat{u}_j = g(x_i), i=1,2,\ldots N. \quad (25)$$

where $\hat{u}_j$ are the approximate values of u_j when we use a quadrature rule instead of the exact integral. Now if we set F as an *N* by *N* matrix which is defined by

$$F_{i,j} = \lambda\phi_j(x_i) + \sum_{k=1}^{m_1}\kappa(x_i,\xi_k)\phi_j(\xi)d\xi_k\omega_k,$$

and vectors

$$\hat{\mathbf{u}} = [\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_N]^T,$$

and

$$\mathbf{g} = [g_1, g_2, \ldots, g_N]^T,$$

then we have the following linear system of equations

$$F\hat{\mathbf{u}} = \mathbf{g}. \quad (26)$$

By solving Eq. (26) with a proper numerical method such as Gauss elimination method or LU factorization, we find values of $\hat{u}_j$. Then the value of u(x) at any point $x \in [a,b]$ can be approximated by Eq. (20) as

$$u(x) \approx u_N(x) = \sum_{j=1}^{n}\phi_j(x)\hat{u}_j, x \in [a,b].$$

We use this method to solve BIE (9), and then we calculate the values of the solution of the Neumann problem at any point of the domain by (10).

Error Analysis

The error analysis of this method is based on what has been described in [9]. We write Eq. (22) in operator form as

$$(\lambda - \mathbb{F})u = g, \tag{27}$$

where

$$\mathbb{F}u = \int_a^b \kappa(x,\xi)u(\xi)d\xi.$$

Similarly Eq. (23) can be written as

$$(\lambda - \mathbb{F})u^h = g, \tag{28}$$

and assume that $\mathbb{F}$ is a compact operator (for more details about the compact integral operators see chapter 1 of [6]). From [9] we have the following lemma and theorems.

Theorem 4. There exist constants C_r, $r = 1,2$, such that for each $u \in C^{r,1}(\bar{\Omega})$

$$\| D^\mu u - D^\mu u^h \|_{L^\infty(\Omega)} \le C_r R^{r+1-|\mu|} |u|_{r,1}, 0 \le |\mu| \le r.$$

where $R$is an upper bound of the mesh-size.

Lemma 5. If (27) be uniquely solvable and $\| u - u^h \| \to 0$ then (28) is uniquely solvable.

The approximation scheme for numerical integration can be written as

$$(\lambda - \mathbb{F}_M)u_N = g.$$

Now the following theorem shows the convergence analysis of the method in L^∞ norm.

Theorem 6. Let $u \in C^{r,1}(\bar{\Omega})(r = 1,2)$ where Ω be a closed, bounded set in $\mathbf{R}^d$ and let $\kappa(x,\xi)$ be continuous for $x, \xi \in \Omega$. Assume the quadrature scheme is convergent for all continuous functions on Ω. Further, assume that the integral equation (27) is uniquely solvable for given $g \in C(\Omega)$ with $\lambda \ne 0$. Moreover take a suitable approximation u^h of u. Then for all sufficiently large values of M, the approximate inverses $(\lambda - \mathbb{F}_M)^{-1}$ exist and are uniformly bounded,

$$\| (\lambda - \mathbb{F}_M)^{-1} \| \le \frac{1 + \| (\lambda - \mathbb{F})^{-1} \| . \| \mathbb{F}_M \|}{|\lambda| - \| (\lambda - \mathbb{F}_M)^{-1} \| . \| (\mathbb{F} - \mathbb{F}_M)\mathbb{F}_M \|} \le Q,$$

with a suitable constant $Q < \infty$. For the equations $(\lambda - \mathbb{F})u = g$ and $(\lambda - \mathbb{F}_M)u_N = g$, we have

$$\| u - u_N \|_{L^\infty(\Omega)} \le C_r R^{r+1} |u|_{r,1} (1 + Q \| F - F_M \|_{L^\infty(\Omega)}) + Q \| F - F_M \|_{L^\infty(\Omega)} \| u \|_{L^\infty(\Omega)},$$

where R and C_r are introduced in Theorem 4.

Let $u_i^{(N)}(x,y)$ be the solution of (10) by using the approximation solution u_N

$$u_i^{(N)}(x,y) = \int_0^L M(x,y,s)u_N(s)ds, \tag{29}$$

and let $u_i^{(N,m)}$ denote the result of approximation $u_i^{(N)}(x,y)$ using a quadrature rule

$$u_i^{(N)}(x,y) = \sum_{j=1}^m w_j M(x,y,x_j)u_N(x_j).$$

Such as [6], for a convex region

$$\max_{(x,y)\in D_i} |u_i(x,y) - u_i^N(x,y)| \le 2\pi \| u - u_N \|_{L^\infty(\Omega)}.$$

The total error is given by

$$u_i(x,y) - u_i^{(N,m)}(x,y) = [u_i(x,y) - u_i^{(N)}(x,y)] + [u_i^{(N)}(x,y) - u_i^{(N,m)}(x,y)].$$

Results and Discussion

In this section, efficiency and accuracy of the method are illustrated by some numerical examples. The routines are written in Fortran 90.

Example 1. This example is solved in [6] as a Dirichlet problem and we solve it as a Neumann problem with our proposed method. Let the boundary S be the ellipse

$$r(t) = (a\cos t, b\sin t), 0 \le t \le 2\pi.$$

In this case, the kernel K of (9) can be reduced to

$$K(t,s) = \kappa(\frac{s+t}{2}),$$

$$\kappa(\theta) = \frac{-ab}{2[a^2 sin^2(\theta) + b^2 cos^2(\theta)]},$$

and the integral equation (9) becomes

$$-\pi u(t) + \int_0^{2\pi} \kappa(\frac{s+t}{2})u(s)ds = g(t), 0 \le t \le 2\pi.$$

Table 1, indicates the results of solving this equation with

$$f(x,y) = e^{x}\cos y, (x,y) \in S.$$

The exact solution u is not known explicitly; but we obtain a highly accurate solution by using a large value of N, and then this solution is used to calculate the errors shown in the table. Results are given for $(a,b) = (1,5)$. We put $\delta = 1/(N-1)$ then for the linear case $h_j = 2\times\delta$ and for the quadratic case $h_j = 2.5\times\delta$ to ensure the regularity of moment matrix A in MLS approximation. Besides, $\alpha = 0.6\times\delta$ in both cases. Because of increasing the condition number of A, the error increases in quadratic case as Table 1 shows (see [12]). To evaluate $u_i^{(N)}(x,y)$ from Eq. (29), we use the trapezoidal rule and as shown in Table 1, we choose m = N. The error along the line

$$\mathbf{c}(q) = q(a\cos\frac{\pi}{4}, 0 \le q < 1,$$

is shown in Table 2.

Example 2. Consider the boundary S be the circle

$$r(t) = (\cos(2\pi t), \sin(2\pi t)), 0 \le t \le 1,$$

and boundary condition with

$$f(t) = \cos(2\pi t).$$

The unique exact solution of this problem is

$$u(t) = cos(2\pi t).$$

Errors of the approximation solution are given in Table 3, and $u_i - u_i^{(N,m)}$ along the line

$$\mathbf{c}(q) = q(a\cos\frac{\pi}{4}, b\sin\frac{\pi}{4}),\ 0 \le q < 1,$$

is shown in Table 4.

According to these numerical experiments, the obtained errors and the advantages mentioned in the previous sections guarantee the efficiency and trustworthy of the proposed method.

In this paper, we used the property of boundary integral equation to reduce the 2-D Neumann problem to an integral equation in one dimensional space. In addition, we utilize MLS as a meshless method for solving arisen integral equation. Since this method does not need to take a mesh in domain and only requires some scattering nodes, it is very useful for problems in high dimensions. One can easily extend this method to three dimensional Neumann problem and Dirichlet problem.

Table 1. Maximum errors for different values of N in example 1

N	Linear (r=1)	Quadratic (r=2)
	$\|\mathbf{e}\|_\infty$	$\|\mathbf{e}\|_\infty$
5	2.945E-3	2.184E-5
7	1.383E-3	2.484E-4
9	6.852E-4	1.215E-5
17	9.926E-5	7.326E-4
33	1.011E-7	3.629E-4

Table 2. Error $u_i(\mathbf{c}(q)) - u_i^{(N,m)}(\mathbf{c}(q))$ for some q in example 1

q	m=7	m=9	m=17	m=33
0.0	3.623E-1	2.071E-2	2.071E-3	3.671E-4
0.2	3.031E-1	2.165E-2	8.701E-4	3.235E-5
0.4	2.573E-2	8.763E-3	3.763E-4	2.238E-5
0.6	1.150E-2	5.132E-4	1.670E-4	4.060E-6
0.8	1.335E-3	6.411E-4	2.495E-5	3.101E-6
0.9	3.592E-2	5.739E-3	3.145E-3	2.056E-5

Table 3. Maximum errors for different values of N in example 2

N	Linear (r=1)	Quadratic (r=2)
	$\|\mathbf{e}\|_\infty$	$\|\mathbf{e}\|_\infty$
5	4.320E-4	1.787E-4
7	1.534E-4	2.318E-5
9	7.481E-5	3.881E-6
17	1.249E-6	4.676E-6
33	3.985E-8	5.019E-6

Table 4. Error $u_i(\mathbf{c}(q)) - u_i^{(N,m)}(\mathbf{c}(q))$ for some q in example 2

q	m=7	m=9	m=17	m=33
0.0	2.670E-1	2.124E-2	1.038E-3	4.104E-4
0.2	2.964E-2	2.114E-3	8.806E-3	1.223E-5
0.4	1.421E-2	9.190E-3	5.919E-4	5.551E-5
0.6	8.365E-3	4.849E-4	2.449E-5	3.669E-6
0.8	5.227E-3	5.903E-5	1.229E-6	2.873E-7
0.9	3.572E-2	7.771E-3	2.939E-4	3.059E-5

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