9

The Mathematics of Finance

In this chapter, we will discuss the mathematics of finance—the rules that govern investing and borrowing money.

Careers and Mathematics

Actuary Actuaries use their broad knowledge of statistics, finance, and business to design insurance policies, pension plans, and other financial strategies, and ensure that these plans are maintained on a sound financial basis. They assemble and analyze data to estimate the probability and likely cost of an event such as death, sickness, injury, disability, or loss of property.

Most actuaries are employed in the insurance industry, specializing in either life and health insurance or property and casualty insurance. They produce probability tables or use modeling techniques that determine the likelihood that a potential event will generate a claim. From these, they estimate the amount a company can expect to pay in claims. Actuaries ensure that the premiums charged for such insurance will enable the company to cover claims and other expenses.

Actuaries held about 18,000 jobs in 2006.

Education Actuaries need a strong background in mathematics and general business. Actuaries usually earn an undergraduate degree in mathematics, statistics, or actuarial science, or a business-related field such as finance, economics or business. Actuaries must pass a series of examinations to gain full professional status.

Job Outlook Employment of actuaries is expected to increase by about 24 percent through 2016. Median annual earnings of actuaries were $82,800 in 2006.

For a sample application, see Example 3 in Section 9.3. For more information, see www.bls.gov.oco.ocos.041.htm.

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9.1 Interest

Objectives

1. Compute Simple Interest
2. Compute Compound Interest
3. Borrow Money Using Bank Notes
4. Compute Effective Rate of Interest
5. Compute Present Value

Wages, rent, and interest are three common ways to earn money:

- A wage refers to money received for letting someone use your labor.
- Rent refers to money received for letting someone use your property, especially real estate.
- Interest refers to money received for letting someone use your money.

Few people become wealthy by receiving wages. Unless you receive a large hourly rate of pay, there will not be enough left after daily living expenses to amass true wealth.

You will have a better chance of becoming wealthy by supplementing wages with rent. For example, if you borrow money to buy an apartment building, the rent received from tenants can pay off the loan, and eventually you will own the building without spending your own money.

Perhaps the easiest way to build wealth is to use money to earn interest. If you can earn a good rate of interest, compounded continuously, and keep the investment for a long time, it is amazing how large an investment can grow. In fact, it is said that *compound interest* is the eighth wonder of the world.

In this first section, we will discuss this important money-making tool: *interest.*

When money is borrowed, the lender expects to be paid back the amount of the loan plus an additional charge for the use of the money. This additional charge is called **interest.** When money is deposited in a bank, the bank pays the depositor for the use of the money. The money the deposit earns is also called *interest.*

Interest can be computed in two ways: either as *simple interest* or as *compound interest.*

1. Compute Simple Interest

Simple interest is computed by finding the product of the **principal** (the amount of money on deposit), the **rate of interest** (usually written as a decimal), and the **time** (usually expressed in years).

$$\text{Interest} = \text{principal} \cdot \text{rate} \cdot \text{time}$$

This word equation suggests the following formula.

Simple Interest The simple interest I earned on a principal P in an account paying an annual interest rate r for a length of time t is given by the formula

$$I = Prt$$

EXAMPLE 1 Find the simple interest earned on a deposit of \$5,750 that is left on deposit for $3\frac{1}{2}$ years and earns an annual interest rate of $4\frac{1}{2}\%$.

Solution We write $3\frac{1}{2}$ and $4\frac{1}{2}\%$ as decimals and substitute the given values in the formula for simple interest.

$I = Prt$	**This is the formula for simple interest.**
$I = 5{,}750 \cdot 0.045 \cdot 3.5$	**Substitute 5,750 for P, 0.045 for r, and 3.5 for t.**
$I = 905.625$	**Perform the multiplications.**

In $3\frac{1}{2}$ years, the account will earn \$905.63 in simple interest.

Self Check 1 Find the simple interest earned on a deposit of \$12,275 that is left on deposit for $5\frac{1}{4}$ years and earns an annual interest rate of $3\frac{3}{4}\%$.

EXAMPLE 2 Three years after investing \$15,000, a retired couple received a check for \$3,375 in simple interest. Find the annual interest rate their money earned during that time.

Solution The couple invested \$15,000 (the principal) for 3 years (the time) and earned \$3,375 (the simple interest). We must find the annual interest rate r. To do so, we substitute the given numbers into the simple interest formula and solve for r.

$I = Prt$	
$3{,}375 = 15{,}000 \cdot r \cdot 3$	**Substitute 3,375 for I, 15,000 for P, and 3 for t.**
$3{,}375 = 45{,}000r$	**Multiply.**
$\frac{3{,}375}{45{,}000} = \frac{45{,}000}{45{,}000}r$	**Divide both sides by 45,000.**
$0.075 = r$	**Perform the divisions.**
$r = 7.5\%$	**Write 0.075 as a percent.**

The couple received an annual rate of 7.5% for the 3-year period.

Self Check 2 Find the length of time it will take for the interest to grow to \$9,000.

2. Compute Compound Interest

When interest is left in an account and also earns interest, we say that the account earns **compound interest.**

EXAMPLE 3 A woman deposits \$10,000 in a savings account paying 6% interest, compounded annually. Find the balance in her account after each of the first three years.

Solution At the end of the first year, the interest earned is 6% of the \$10,000, or

$$0.06(\$10{,}000) = \$600$$

This interest is added to the \$10,000 to get a new balance. After one year, this balance will be \$10,600.

The second year's earned interest is 6% of \$10,600, or

$$0.06(\$10{,}600) = \$636$$

This interest is added to \$10,600, giving a second-year balance of \$11,236.

The interest earned during the third year is 6% of \$11,236, or

$$0.06(\$11{,}236) = \$674.16$$

This interest is added to \$11,236 to give the woman a balance of \$11,910.16, after three years.

Self Check 3 Find the balance in the woman's account after two more years.

We can generalize the method used in Example 3 to find a formula for compound interest calculations. Suppose that the original deposit in the account is A_0 dollars, that interest is paid at an **annual rate** r, and that the **accumulated amount** or the **future value** in the account at the end of the first year is A_1. Then the interest earned that year is A_0r, and

The amount after one year	equals	the original deposit	plus	the interest earned on the original deposit.

$$A_1 = A_0 + A_0r$$
$$= A_0(1 + r) \quad \text{Factor out the common factor, } A_0.$$

The amount, A_1, at the end of the first year is the balance in the account at the beginning of the second year. So, the amount at the end of the second year, A_2, is

The amount after two years	equals	the amount after one year	plus	the interest earned on the amount after one year.

$$A_2 = A_1 + A_1r$$
$$= A_1(1 + r) \quad \text{Factor out the common factor, } A_1.$$
$$= A_0(1 + r)(1 + r) \quad \text{Substitute } A_0(1 + r) \text{ for } A_1.$$
$$= A_0(1 + r)^2 \quad \text{Simplify.}$$

By the end of the third year, the amount will be

$$A_3 = A_0(1 + r)^3$$

The pattern continues with the following result.

Compound Interest (Annual Compounding) A single deposit A_0, earning compound interest for n years at an annual rate r, will grow to a **future value** A_n according to the formula

$$A_n = A_0(1 + r)^n$$

EXAMPLE 4 For their newborn child, parents deposit \$10,000 in a college account that pays 8% interest, compounded annually. How much will be in the account on the child's 17th birthday?

Solution We substitute $A_0 = 10{,}000$, $r = 0.08$, and $n = 17$ into the compound interest formula to find the future value A_{17}.

$$A_n = A_0(1 + r)^n$$
$$A_{17} = 10{,}000(1 + 0.08)^{17}$$
$$= 10{,}000(1.08)^{17}$$
$$\approx 37{,}000.18054801 \quad \text{Use a calculator.}$$

To the nearest cent, \$37,000.18 will be available on the child's 17th birthday.

Self Check 4

If the parents leave the money on deposit for two more years, what amount will be available?

Interest compounded once each year is **compounded annually.** Many financial institutions compound interest more often. For example, instead of paying an annual rate of 8% once a year, a bank might pay 4% twice each year, or 2% four times each year. The annual rate, 8%, is also called the **nominal rate,** and the time between interest calculations is called the **conversion period.** If there are k periods each year, interest is paid at the **periodic rate** given by the following formula.

Periodic Rate

$$\text{Periodic rate} = \frac{\text{annual rate}}{\text{number of periods per year}}$$

This formula is often written as

$$i = \frac{r}{k}$$

where i is the periodic interest rate, r is the annual rate, and k is the number of times interest is paid each year.

If interest is calculated k times each year, in n years there will be kn conversions. Each conversion is at the periodic rate i. This leads to another form of the compound interest formula.

Compound Interest Formula

An amount A_0, earning interest compounded k times a year for n years at an annual rate r, will grow to the future value A_n according to the formula

$$A_n = A_0(1 + i)^{kn}$$

where $i = \frac{r}{k}$ is the periodic interest rate.

Interest paid twice each year is called **semiannual** compounding, four times each year **quarterly** compounding, twelve times each year **monthly** compounding, and 360 or 365 times each year **daily** compounding.

EXAMPLE 5

If the parents of Example 4 invested that \$10,000 in an account paying 8%, compounded quarterly, how much more money would they have after 17 years?

Solution

We first calculate the periodic rate, i.

$$i = \frac{r}{k}$$

$$i = \frac{0.08}{4} \quad \text{Substitute } r = 0.08 \text{ and } k = 4.$$

$$i = 0.02$$

We then substitute $A_0 = 10{,}000$, $i = 0.02$, $k = 4$, and $n = 17$ into the compound interest formula.

$$A_n = A_0(1 + i)^{kn}$$
$$A_{17} = 10{,}000(1 + 0.02)^{4 \cdot 17}$$
$$= 10{,}000(1.02)^{68}$$
$$\approx 38{,}442.50502546 \qquad \text{Use a calculator.}$$

To the nearest cent, \$38,442.51 will be available, an increase of \$1,442.33 over annual compounding.

Self Check 5

a. What would \$10,000 become in 17 years if compounded monthly at a nominal rate of 8%?

b. How does this compare with quarterly compounding?

Accent on Technology

Growth of Money

We can use a graphing calculator to find the time it would take a \$10,000 investment to triple, assuming an 8% annual rate, compounded quarterly.

In n years, \$10,000 earning 8% interest, compounded quarterly, will become the future value

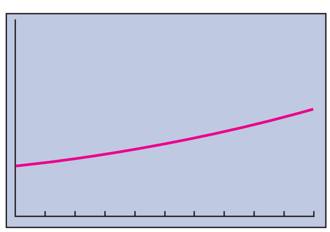
$$10{,}000(1.02)^{4n}$$

To watch this value grow, we enter the function

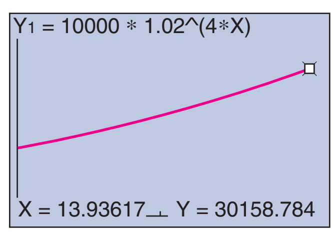
Y$_1$ = 10000*1.02^(4*X)

in a graphing calculator, and set the window to $0 \leq X \leq 10$ (for 10 years) and $0 \leq Y \leq 40000$ (for the dollar amount).

The graph appears in Figure 9-1(a). To find the time it would take for the investment to triple, we use TRACE to move to the point with a Y-value close to 30,000. The X-value in Figure 9-1(b) shows that the investment would triple in about 13.9 years.



(a)



(b)

Figure 9-1

3. Borrow Money Using Bank Notes

When a customer borrows money from a bank, the bank is making an investment in that person. The amount of the loan is the bank's deposit, and the bank expects to be repaid with interest in a single **balloon payment** at a later date. These loans, called **notes,** are based on a 360-day year, and they are usually written for 30 days, 90 days, or 180 days. We use the formula for compound interest to calculate the terms of the loan.

EXAMPLE 6 A student needs \$4,000 for tuition. If his bank writes a 9%, 180-day note, with interest compounded daily, what will he owe at the end of 180 days?

Solution In granting the loan, the bank invests \$4,000. The amount to be repaid is the expected future value

$$A_n = A_0(1 + i)^{kn}$$

where A_0 is \$4,000, the frequency of compounding k is 360, the periodic rate i is $\frac{0.09}{360} = 0.00025$, and the term n is 0.5 (180 days is one-half of 360 days). To determine what the student will owe, we substitute these numbers into the compound interest formula and solve for A_n.

$$\begin{aligned} A_n &= A_0(1 + i)^{kn} \\ A_{0.5} &= 4{,}000(1 + 0.00025)^{360 \cdot 0.5} \\ &= 4{,}000(1.00025)^{180} \\ &\approx 4{,}184.087907996 && \text{Use a calculator.} \\ &\approx 4{,}184.09 && \text{Round to the nearest cent.} \end{aligned}$$

The student must repay \$4,184.09.

Self Check 6 A woman borrows \$7,500 for 90 days at 12%. If interest is compounded daily, how much will she owe at the end of 90 days?

4. Compute Effective Rate of Interest

The true performance of an investment depends on both the frequency of compounding and the annual rate. To help investors compare different savings plans, financial institutions are required by law to provide the **effective rate**—the rate that, if compounded annually, would provide the same yield as a plan that is compounded more frequently.

To derive a formula for an effective rate, we assume that A_0 dollars are invested for n years at an annual rate r, compounded k times per year. That same investment of A_0 dollars, compounded annually at the effective rate R, would produce the same accumulated value. Since these amounts are to be equal, we have the equation

Accumulated amount at effective rate R, compounded annually	equals	accumulated amount at annual rate r, compounded k times per year.

$$A_0(1 + R)^n = A_0(1 + i)^{kn} \qquad \text{i is the periodic rate, } i = \tfrac{r}{k}.$$

We can solve this equation for R.

$$\begin{aligned} A_0(1 + R)^n &= A_0(1 + i)^{kn} \\ (1 + R)^n &= (1 + i)^{kn} && \text{Divide both sides by } A_0. \\ [(1 + R)^n]^{1/n} &= [(1 + i)^{kn}]^{1/n} && \text{Raise both sides to the } 1/n \text{ power.} \\ 1 + R &= (1 + i)^k && \text{Multiply the exponents.} \\ R &= (1 + i)^k - 1 && \text{Subtract 1 from both sides.} \end{aligned}$$

This result establishes the following formula.

Effective Rate of Interest

The **effective rate of interest** R for an account paying a nominal rate r, compounded k times per year, is

$$R = (1 + i)^k - 1$$

where i is the periodic rate, $i = \frac{r}{k}$.

EXAMPLE 7 A bank offers the savings plans shown in the table. Calculate the effective interest rates for each investment.

	a. Money market fund	b. Certificate of deposit
Annual rate	6.5%	7%
Compounding	quarterly	monthly
Effective rate		

Solution

a. For the money market fund, $r = 0.065$ and $k = 4$, so $i = \frac{r}{k} = \frac{0.065}{4} = 0.01625$. To find the effective rate, we substitute $k = 4$ and $i = 0.01625$ in the formula for effective rate.

$$\begin{aligned} R &= (1 + \mathbf{i})^{\mathbf{k}} - 1 \\ R &= (1 + \mathbf{0.01625})^{\mathbf{4}} - 1 \\ &\approx 0.0666016088 && \text{Use a calculator.} \\ &\approx 0.0666 && \text{Round to the nearest ten thousandth.} \end{aligned}$$

As a percent, the effective rate is 6.66%, or approximately $6\frac{2}{3}\%$.

b. For the certificate of deposit, $r = 0.07$ and $k = 12$, so

$$i = \frac{0.07}{12} \approx 0.00583333$$

and

$$\begin{aligned} R &= (1 + 0.00583333)^{12} - 1 \\ &\approx 0.0722900809 && \text{Use a calculator.} \\ &\approx 0.0723 && \text{Round to the nearest ten thousandth.} \end{aligned}$$

As a percent, the effective rate is 7.23%.

Self Check 7 A passbook savings account offers daily compounding (365 days per year) at an annual rate of 6%. Find the effective rate to the nearest hundredth.

5. Compute Present Value

For an initial deposit A_0, the compound interest formula gives the future value A_n of the account after n years. This is the situation suggested by Figure 9-2, where we know the beginning amount and need to find its future value.

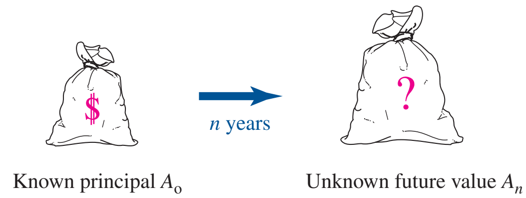


Figure 9-2

Often, the situation is reversed: We need to make a deposit now that will become a specific amount several years from now—perhaps enough to buy a car or pay tuition. As Figure 9-3 suggests, we need to know what single deposit *now* will accomplish that goal: What **present value** A_0 will yield a *specific* future value A_n?

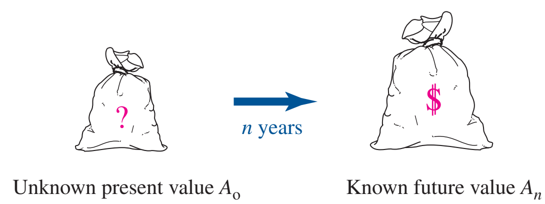


Figure 9-3

To derive the formula for present value, we solve the compound interest formula for A_0.

$$A_n = A_0(1 + i)^{kn} \quad \text{The original deposit is the present value, } A_0.$$

$$\frac{A_n}{(1 + i)^{kn}} = \frac{A_0(1 + i)^{kn}}{(1 + i)^{kn}} \quad \text{Divide both sides by } (1 + i)^{kn}.$$

$$\frac{A_n}{(1 + i)^{kn}} = A_0 \quad \text{Divide on both sides: } \frac{(1 + i)^{kn}}{(1 + i)^{kn}} = 1.$$

$$A_n(1 + i)^{-kn} = A_0 \quad \text{Use the definition of negative exponent: } \frac{1}{a^x} = a^{-x}.$$

This result establishes the following formula.

Present Value

The **present value** A_0 is the amount that must be deposited now to provide a **future value** A_n after n years is given by the formula

$$A_0 = A_n(1 + i)^{-kn}$$

where interest is compounded k times per year at an annual rate r $\left(i \text{ is the periodic rate, } \frac{r}{k}\right)$.

EXAMPLE 8

When a medical student graduates in 8 years, she will need \$25,700 to buy furniture for her medical office. What amount must she deposit now (at 8%, compounded twice per year) to meet this future obligation?

Solution

Use the annual rate ($r = 0.08$) and the frequency of compounding ($k = 2$) to find the periodic rate:

$$i = \frac{r}{k} = \frac{0.08}{2} = 0.04$$

In the present value formula, we substitute

the number of years, $n = 8$,
the periodic rate, $i = 0.04$,
the frequency of compounding, $k = 2$, and
the future value in 8 years, $A_8 = 25{,}700$.

$$A_0 = A_n(1 + i)^{-kn}$$

$$A_0 = 25{,}700(1 + 0.04)^{-2 \cdot 8} \quad A_n = A_8 = 25{,}700$$

$$= 25{,}700(1.04)^{-16} \quad \text{Use a calculator.}$$

$$\approx 13{,}721.44 \quad \text{Round to the nearest cent.}$$

She must deposit \$13,721.44 now to have \$25,700 in 8 years.

Self Check 8 If the student decides to take two extra years to complete medical school, her obligation will be \$27,000. What present value will meet her goal?

Self Check Answers **1.** \$2,416.64 **2.** 8 yr **3.** \$13,382.26 **4.** \$43,157.01 **5. a.** \$38,786.48 **b.** \$343.97 more than with quarterly compounding **6.** \$7,728.37 **7.** 6.18% **8.** \$12,322.45

9.1 Exercises

Vocabulary and Concepts *Fill in the blanks.*

1. A bank pays ______ for the privilege of using your money.
2. If interest is left on deposit to earn more interest, the account earns ________ interest.
3. Interest compounded once each year is called ______ compounding.
4. The initial deposit is called the _______ or the ______ value.
5. After a specific time, the principal grows to a _____ value.
6. Interest is calculated as a ______ of the amount on deposit.
7. Future value = principal + ______ earned
8. The annual rate also is called the _______ rate.
9. __________ $= \dfrac{\text{annual rate}}{\text{number of periods per year}}$
10. The time between interest calculations is the ________ period.
11. In the future value formula $A_n = A_0(1 + i)^{kn}$,
    A_0 is the _______,
    i is the __________,
    k is the ___________________, and
    n is the ____________.
12. In the present value formula $A_0 = A_n(1 + i)^{-kn}$,
    A_n is the _________,
    i is the __________,
    k is the ___________________, and
    n is the ____________.
13. To help consumers compare savings plans, banks advertise the _______ rate of interest.
14. If after one year, \$100 grows to \$110, the effective rate is ___%.

Practice

15. Find the simple interest earned in an account where \$4,500 is on deposit for 4 years at $3\frac{1}{4}\%$ annual interest.
16. Find the simple interest earned in an account where \$12,400 is on deposit for $8\frac{1}{4}$ years at $4\frac{1}{2}\%$ annual interest.
17. Find the principal necessary to earn \$814 in simple interest if the money is to be left on deposit for 4 years and earns $5\frac{1}{2}\%$ annual interest.
18. Find the time necessary for a deposit of \$11,500 to earn \$3,450 in simple interest if the money is to earn $3\frac{3}{4}\%$ annual interest.

19. Find the annual rate necessary for a deposit of \$50,000 to earn \$7,500 in simple interest if the money is to be left on deposit for $2\frac{1}{2}$ years.

20. Find the time necessary for a deposit of \$5,000 to double in an account paying $6\frac{1}{4}\%$ simple interest.

Assume that* \$1,200 *is deposited in an account in which interest is compounded annually at a rate of* 8%. *Find the accumulated amount after the given number of years.

21. 1 year **22.** 3 years
23. 5 years **24.** 20 years

Assume that* \$1,200 *is deposited in an account in which interest is compounded annually at the given rate. Find the accumulated amount after* 10 *years.

25. 3% **26.** 5%
27. 9% **28.** 12%

Assume that* \$1,200 *is deposited in an account in which interest is compounded at the given frequency, at an annual rate of* 6%. *Find the accumulated amount after* 15 *years.

29. $k = 2$ **30.** $k = 4$
31. $k = 12$ **32.** $k = 365$

Find the effective interest rate with the given annual rate r and compounding frequency k.

33. $r = 6\%, k = 4$ **34.** $r = 8\%, k = 12$
35. $r = 9\frac{1}{2}\%, k = 2$ **36.** $r = 10\%, k = 360$

Find the present value of* \$20,000 *due in* 6 *years, at the given annual rate and compounding frequency.

37. 6%, semiannually **38.** 8%, quarterly
39. 9%, monthly **40.** 7%, daily (360 days/year)

Applications

41. Small business To start a mobile dog-grooming service, a woman borrowed \$2,500. If the loan was for 2 years and the amount of interest was \$175, what simple interest rate was she charged?

42. Banking Three years after opening an account that paid 6.45% simple interest, a depositor withdrew the \$3,483 in interest earned. How much money was left in the account?

43. Saving for college At the birth of their child, the Fieldsons deposited \$7,000 in an account paying 6% interest, compounded quarterly. How much will be available when the child turns 18?

44. Planning a celebration When the Fernandez family made reservations at the end of 2008 for the December 2014 New Year's celebration in Paris, they placed \$5,700 into an account paying 8% interest, compounded monthly. What amount will be available at the time of the celebration?

45. Planning for retirement When Jim retires in 12 years, he expects to live lavishly on the money in a retirement account that is earning $7\frac{1}{2}\%$ interest, compounded semiannually. If the account now contains \$147,500, how much will be available at retirement?

46. Pension fund management The managers of a pension fund invested \$3 million in government bonds paying 8.73% annual interest, compounded semiannually. After 8 years, what will the investment be worth?

47. Real estate investing Property values in the suburbs have been appreciating about 11% annually. If this trend continues, what will a \$137,000 home be worth in four years? Give the result to the nearest dollar.

48. Real estate investing Property in suburbs closer to the city is appreciating about 8.5% annually. If this trend continues, what will a \$47,000 one-acre lot be worth in five years? Give the result to the nearest dollar.

49. Gas consumption The gas utilities expect natural gas consumption to increase at 7.2% per year for the next decade. Monthly consumption for one county is currently 4.3 million cubic feet. What monthly demand for gas is expected in ten years?

50. Comparing banks Bank One offers a passbook account with 4.35% annual rate, compounded quarterly. Bank Two offers a money market account at 4.3%, compounded monthly. Which account provides the better growth? (*Hint:* Find the effective rates.)

51. Comparing accounts A savings and loan offers the two accounts shown in the table. Find the effective rates.

	Annual rate	Compounding	Effective rate
NOW account	7.2%	quarterly	
Money market	6.9%	monthly	

52. Comparing accounts A credit union offers the two accounts shown in the table. Find the effective rates.

	Annual rate	Compounding	Effective rate
Certificate of deposit	6.2%	semiannually	
Passbook	5.25%	quarterly	

53. Car repair Craig borrows $1,230 for unexpected car repair costs. His bank writes a 90-day note at 12%, with interest compounded daily. What will Craig owe?

54. Fly now, pay later For a 7-day Hawaii vacation, Beth borrowed $2,570 for 9 months at an annual rate of 11.4%, compounded monthly. What did she owe?

55. Buying a computer A man estimates that the computer he plans to buy in 18 months will cost $4,200. To meet this goal, how much should he deposit in an account paying 5.75%, compounded monthly?

56. Buying a copier An accounting firm plans to deposit enough money now in an account paying 7.6% interest, compounded quarterly, to finance the purchase of a $2,780 copier in 18 months. What should be the amount of that deposit?

Discovery and Writing

57. Adding to an investment To prepare for his retirement in 14 years, Jay deposited $12,000 in an account paying 7.5% annual interest, compounded monthly. Ten years later, he deposited another $12,000. How much will be available at retirement?

58. Changing rates Ten years ago, a man invested $1,100 in a 5-year certificate of deposit paying 10%, compounded monthly. When the CD matured, he invested the proceeds in another 5-year CD paying 8%, compounded semiannually. How much is available now?

59. The power of time "A young person's most powerful money-making scheme," said an investment advisor, "is *time.*" Write a paragraph explaining what the advisor meant.

60. Watching money grow $10,000 is invested at 10%, compounded annually. Use a graphing calculator to find how long it will take for the accumulated value to exceed $1 million.

61. Explain why the compound interest formula on page 727 is equivalent to the one on page 385.

62. Explain why the present value formula on page 731 is equivalent to the compound interest formula on page 727.

Review ***Simplify each expression. Assume that all variables represent positive numbers.***

63. $\dfrac{x^2 - 2x - 15}{2x^2 - 9x - 5}$

64. $3x(x^2 - 5) - (x^3 - 2x)$

65. $\dfrac{(3 - x)(x + 3)}{-x^2 + 9}$

66. $-\sqrt{x^2 - 6x + 9} \quad (x \geq 3)$

9.2 Annuities and Future Value

Objectives

1. Find the Future Value of an Annuity
2. Work with Sinking Funds

Now that we understand how interest works, we will discuss how to use interest to build wealth to pay for retirement, a vacation, or some other special event. Two instruments for doing this are annuities and sinking funds.

1. Find the Future Value of an Annuity

Financial plans that involve a series of payments are called **annuities.** Monthly mortgage payments, for example, are part of an annuity, as are regular contributions to a retirement plan.

Annuity A plan involving payments made at regular intervals is called an **annuity.**

Future Value The **future value** of an annuity is the sum of all the payments and the interest those payments earn.

Term The time over which the payments are made is called the **term** of the annuity.

Ordinary Annuity In an **ordinary annuity,** the payments are made at the *end* of each time interval.

In this book, we will consider only ordinary annuities with equal periodic payments made for a fixed term.

To understand how an annuity works, assume that a savings account pays 12% annual interest, compounded monthly. Its periodic rate is $\frac{12\%}{12}$, or 1%. Also assume that each month for the next year, $100 will be deposited in that account.

To determine the future value of this annuity, we think of each monthly payment as a one-time initial contribution to a compound-interest savings account. The future value of the annuity is the sum of the accumulated values of 12 individual accounts. As Figure 9-4 suggests, the first deposit earns interest for 11 months, the second for 10 months, and so on. Because the last deposit is made at the end of the year, it earns no interest. A few minute's work with a calculator will show that the total value of the account is $1,268.25.

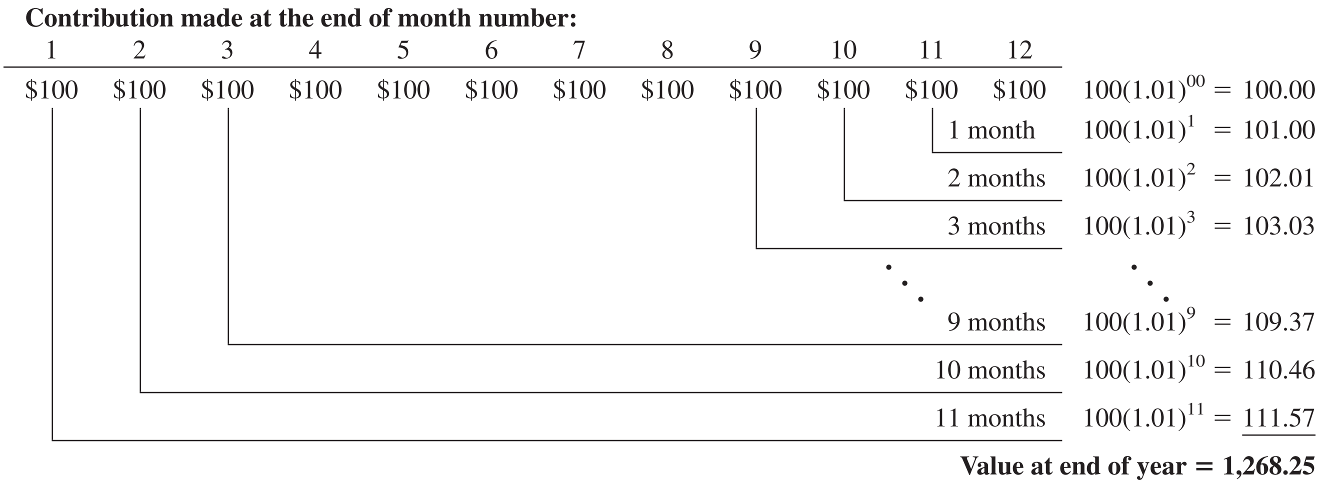


Figure 9-4

We will now generalize this example to derive a formula for the future value of an annuity. Consider an annuity with regular deposits of P dollars and with interest compounded k times per year for n years, at an annual rate r (periodic rate $i = \frac{r}{k}$). During n years, there will be kn periods and kn deposits.

The last deposit, made at the end of the last period, earns no interest. The next-to-last deposit earns interest for one period, and so on. The first deposit, made at the end of the first period, earns compound interest for $kn - 1$ periods.

Future value of the annuity = future value of the last deposit + future value of the next-to-last deposit + ⋯ + future value of the first deposit.

$$A_n = P + P(1 + i)^1 + P(1 + i)^2 + P(1 + i)^3 + \cdots + P(1 + i)^{kn-1}$$

This is a geometric sequence with first term P and a common ratio of $(1 + i)$. Its sum is given by

$$A_n = \frac{P[(1 + i)^{kn} - 1]}{i}$$

Recall that the sum of the terms of the geometric sequence $S_n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}$ is $S_n = \frac{a(r^n - 1)}{r - 1}$.

We summarize this result.

Future Value of an Annuity

The future value A_n of an ordinary annuity with deposits of P dollars made regularly k times each year for n years, with interest compounded k times per year at an annual rate r, is

$$A_n = \frac{P[(1 + i)^{kn} - 1]}{i}$$

where i is the periodic rate, $i = \frac{r}{k}$.

EXAMPLE 1 Verify the future value of the annuity outlined in Figure 9-4.

Solution From Figure 9-4, we find

the term, in years:	$n = 1$
the frequency of compounding:	$k = 12$

the annual rate: $r = 0.12$
the regular deposit: $P = 100$

We then calculate the periodic interest rate: $i = \frac{r}{k} = \frac{0.12}{12} = 0.01$, and substitute these numbers in the formula for the future value of an annuity.

$$A_n = \frac{P[(1+i)^{kn} - 1]}{i}$$

$$A_1 = \frac{100[(1+0.01)^{12\cdot 1} - 1]}{0.01}$$

$$= \frac{100[(1.01)^{12} - 1]}{0.01}$$

$$\approx 1{,}268.25030132 \quad \text{Use a calculator.}$$

$$\approx 1{,}268.25 \quad \text{Round to the nearest cent.}$$

The annuity of Figure 9-4 will provide \$1,268.25 by the end of the year.

Self Check 1 Under the payroll savings plan, a student contributes \$50 a month to an ordinary annuity paying $7\frac{1}{2}\%$ annual interest, compounded monthly. How much will he have in 5 years?

Accent on Technology

Growth of Money

The value of an annuity grows rapidly. To watch the value of the annuity of Example 1 grow, we graph the function

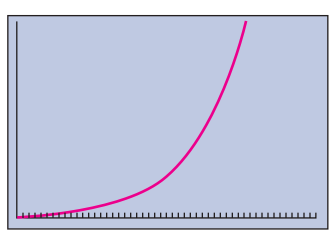
$$A_n = \frac{100[(1.01)^{12n} - 1]}{0.01}$$

$$= 10{,}000(1.01^{12n} - 1) \quad \frac{100}{0.01} = 10{,}000$$

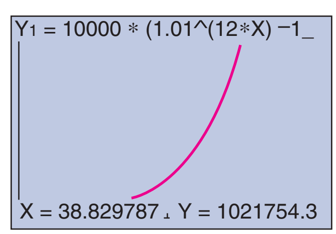
We enter the function

$$Y_1 = 10000*(1.01^{\wedge}(12*X) - 1)$$

on a graphing calculator, set the window values to be $0 \le X \le 50$ and $0 \le Y \le 1{,}000{,}000$, and graph it, as in Figure 9-5(a). Note how the amount increases more rapidly as the years go by. Using TRACE, as in Figure 9-5(b), we see that over \$1 million accumulates in just 39 years.



(a)



(b)

Figure 9-5

2. Work with Sinking Funds

If we know the amount of each deposit, we can calculate the future value of an annuity using the formula on page 736. This situation is often reversed: What regular deposits, made over time will provide a specific future amount? An annuity created to produce a fixed future value is called a **sinking fund.** To determine the required periodic payment P, we solve the future value formula for P.

$$A_n = \frac{P[(1+i)^{kn}-1]}{i}$$

$$A_n \frac{i}{(1+i)^{kn}-1} = \frac{P[(1+i)^{kn}-1]}{i} \cdot \frac{i}{(1+i)^{kn}-1}$$ **To isolate P, multiply both sides by $\frac{i}{(1+i)^{kn}-1}$.**

$$\frac{A_n i}{(1+i)^{kn}-1} = P$$ **Simplify.**

This result establishes the following formula.

Sinking Fund Payment

For an annuity to provide a future value A_n, regular payments P are made k times per year for n years, with interest compounded k times per year at an annual rate n. The payment P is given by

$$P = \frac{A_n i}{(1+i)^{kn}-1}$$

where i is the periodic rate, $i = \frac{r}{k}$.

EXAMPLE 2

An accounting firm will need $17,000 in 5 years to replace its computer system. What periodic deposits to a sinking fund paying quarterly interest at a 9% annual rate will achieve that goal?

Solution

The sinking fund will have the following characteristics:

future value:	$A_n = \$17{,}000$
annual rate:	$r = 0.09$
term:	$n = 5$
number of periods per year:	$k = 4$
periodic rate:	$i = \frac{r}{k} = \frac{0.09}{4} = 0.0225$

We substitute the given values in the formula to find the sinking fund payment.

$$P = \frac{A_n i}{(1+i)^{kn}-1}$$

$$= \frac{(17{,}000)(0.0225)}{(1+0.0225)^{4\cdot 5}-1}$$

$$\approx 682.415203056$$ **Use a calculator.**

$$\approx 682.42$$ **Round to the nearest cent.**

Quarterly payments of $682.42 will accumulate to $17,000 in 5 years.

Self Check 2

What quarterly deposits to the above account are required to raise the $50,000 startup cost of a branch office in 7 years?

Self Check Answers

1. $3,626.36 **2.** $1,301.26

9.2 Exercises

Vocabulary and Concepts *Fill in the blanks.*

1. Plans involving payments made at regular intervals are called ________.
2. In an ordinary annuity, payments are made at the ____ of each period.
3. The future value of an annuity is the sum of all the ________ and ______.
4. The time over which the payments are made is the _____ of the annuity.
5. In the future value formula, $A_n = \dfrac{P[(1+i)^{kn} - 1]}{i}$

   P is the _____________,
   i is the __________,
   k is the ____________________, and
   n is the _____________.
6. An annuity created to fund a specific future obligation is a ______ fund.

Practice

Assume that* $100 *is deposited at the end of each year in an account in which interest is compounded annually at a rate of* 6%. *Find the accumulated amount after the given number of years.

7. 10 years
8. 5 years
9. 3 years
10. 20 years

Assume that* $100 *is deposited at the end of each year into an account in which interest is compounded annually at the given rate. Find the accumulated amount after* 10 *years.

11. 4%
12. 7%
13. 9.5%
14. 8.5%

Assume that* $100 *is deposited at the end of each period in an account in which interest is compounded at the given frequency, at an annual rate of* 8%. *Find the accumulated amount after* 15 *years.

15. $k = 2$
16. $k = 4$
17. $k = 12$
18. $k = 1$

Find the amount of each regular payment to provide* $20,000 *in* 10 *years, at the given annual rate and compounding frequency.

19. 4%, annually
20. 6%, quarterly
21. 9%, semiannually
22. 8%, monthly

Applications

23. **Saving for a vacation** For next year's vacation, the Phelps family is saving $200 each month in an account paying 6% annual interest, compounded monthly. How much will be available a year from now?
24. **Planning for retirement** Hank's regular $1,300 quarterly contributions to his retirement account have earned 6.5% annual interest, compounded quarterly, since he started 21 years ago. How much is in his account now?
25. **Pension fund management** The managers of a company's pension fund invest the monthly employee contributions of $135,000 into a government fund paying 8.7%, compounded monthly. To what value will the fund grow in 20 years?
26. **Saving for college** A mother has been saving regularly for her daughter's college—$25 each month for 11 years. The money has been earning $7\frac{1}{2}$% annual interest, compounded monthly. How much is now in the account?
27. **Buying office machines** A company's new corporate headquarters will be completed in $2\frac{1}{2}$ years. At that time, $750,000 will be needed for office equipment. How much should be invested monthly to fund that expense? Assume 9.75% interest, compounded monthly.
28. **Retirement lifestyle** A woman would like to receive a $500,000 lump-sum distribution from her retirement account when she retires in 25 years. She begins making monthly contributions now to an annuity paying 8.5%, compounded monthly. Find the amount of that monthly contribution.
29. **Comparing accounts** Which account will require the lower annual contributions to fund a $10,000 obligation in 20 years? (*Hint:* Compare the yearly total contributions.)

Bank A	5.5%; annually
Bank B	5.35%; monthly

30. Avoiding a balloon payment The last payment of a home mortgage is a balloon payment of \$47,000, which the owner is scheduled to pay in 12 years. How much *extra* should he start including in each monthly payment to eliminate the balloon payment? His mortgage is at 10.2%, compounded monthly.

Discovery and Writing

31. Retirement strategy Jim will retire in 30 years. He will invest \$100 each month for 15 years and then let the accumulated value continue to grow for the next 15 years. How much will be available at retirement? Assume 8%, compounded monthly.

32. Retirement strategy (See Exercise 31.) Jim's brother Jack also will retire in 30 years. He plans on doing nothing during the first 15 years, then contributing twice as much—\$200 monthly—to "catch up." How much will be available at retirement? Assume 8%, compounded monthly.

33. Changing plans A woman needs \$13,500 in 10 years. She would like to make regular annual contributions for the first 5 years and then let the amount grow at compound interest for the next 5 years. What should her contributions be? Assume 9%, compounded annually.

34. Talking financial sense How would you explain to a friend who has just been hired for her first job that now is the time to start thinking about retirement?

Review *Solve each equation.*

35. $\dfrac{2(5x - 12)}{x} = 8$

36. $\dfrac{2(5x - 12)}{x} = x$

37. $\sqrt{2x + 3} = 3$

38. $\sqrt{2x + 3} = x$

9.3 Present Value of an Annuity; Amortization

Objectives

1. Compute Present Value of an Annuity
2. Amortization

Suppose you are lucky enough to come into a great sum of money. Perhaps you will receive an inheritance, sell a business, or win the lottery. Would you spend the money wisely, or would you waste it as have so many lottery winners? In this section, we will discuss how to invest a portion of this money to guarantee that you will receive regular payments in the future.

1. Compute Present Value of an Annuity

Instead of using an annuity to create a future value A_n, we might ask, "What *single* deposit made *now* would create that same future value?" The one deposit that gives the same final result as an annuity is called the **present value** of that annuity.

To find a formula for the present value of an annuity, we combine two previous formulas. A series of regular payments of P dollars for n years will grow to a future value A_n given by

$$(1) \qquad A_n = \frac{P[(1 + i)^{kn} - 1]}{i}$$

From a formula in Section 9.1, the present value of a future asset is given by

(2) $A_0 = A_n(1 + i)^{-kn}$

We can find the present value of a series of future payments by substituting the right side of Equation 1 into Equation 2.

$A_0 = A_n(1 + i)^{-kn}$ — This is Equation 2.

$A_0 = \dfrac{P[(1 + i)^{kn} - 1]}{i}(1 + i)^{-kn}$ — Substitute $\dfrac{P[(1 + i)^{kn} - 1]}{i}$ for A_n in Equation 2.

$= \dfrac{P[(1 + i)^{kn}(1 + i)^{-kn} - 1(1 + i)^{-kn}]}{i}$ — Use the distributive property.

$= \dfrac{P[1 - (1 + i)^{-kn}]}{i}$ — Simplify: $x^m x^{-m} = x^0 = 1$.

This establishes the following formula.

Present Value of an Annuity

The present value A_0 of an annuity with payments of P dollars made k times per year for n years, with interest compounded k times per year at an annual rate r, is

$$A_0 = \frac{P[1 - (1 + i)^{-kn}]}{i}$$

where i is the periodic rate, $i = \dfrac{r}{k}$.

EXAMPLE 1

To buy a boat in 2 years, the Higgins family plans to save $200 a month in an account that pays 12% interest, compounded monthly.

a. Find the total amount of the payments.

b. Find the value of the account in 2 years.

c. Find the single deposit in that account that would give the same future value.

Solution

a. At $200 per month for 24 months, the total amount contributed is

$200(24) = $4,800

b. To find the value after 2 years, we use the formula for future value of an annuity found on page 736:

$$A_n = \frac{P[(1 + i)^{kn} - 1]}{i}$$

$$A_2 = \frac{200[1.01^{12 \cdot 2} - 1]}{0.01}$$

$\approx 5{,}394.692971$ — Use a calculator.

$\approx 5{,}394.69$ — Round to the nearest cent.

c. To find the present value of the annuity, we substitute:

the term, in years:	$n = 2$
the frequency of compounding:	$k = 12$
the annual rate:	$r = 0.12$

the payment: $P = 200$

the periodic interest rate: $i = \frac{r}{k} = \frac{0.12}{12} = 0.01$

in the present value formula.

$$A_0 = \frac{P[1 - (1 + i)^{-kn}]}{i}$$

$$A_0 = \frac{200[1 - (1 + 0.01)^{-12 \cdot 2}]}{0.01}$$

$$= \frac{200[1 - (1.01)^{-24}]}{0.01} \quad \text{Simplify.}$$

$$\approx 4{,}248.677451 \quad \text{Use a calculator.}$$

$$\approx 4{,}248.68 \quad \text{Round to the nearest cent.}$$

The present value of the annuity is \$4,248.68. That one deposit now will provide the same final amount, \$5,394.69, as the annuity.

Self Check 1

For his retirement in 30 years, a man plans to make monthly contributions of \$25 to an ordinary annuity paying $8\frac{1}{2}\%$ annually, compounded monthly.

a. Find the total amount of his contributions.

b. Find the single deposit now that will provide the same retirement benefit.

State lottery winnings are usually paid as a 20-year annuity. That is to the state's advantage, because it can fund the annuity with a single amount that is much smaller than the total prize.

EXAMPLE 2

Britta won the lottery. She will receive \$75,000 per month for the next 20 years—a total of \$18 million. What single deposit should the lottery commission make now to fund Britta's annuity? Assume 8.4% annual interest, compounded monthly.

Solution

The lottery commission finds the present value of the annuity, with

the payment: $P = 75{,}000$

the annual rate: $r = 0.084$

the frequency of compounding: $k = 12$

the periodic rate: $i = \frac{r}{k} = \frac{0.084}{12} = 0.007$

the term, in years: $n = 20$

These values are used in the formula for the present value of an annuity.

$$A_0 = \frac{P[1 - (1 + i)^{-kn}]}{i}$$

$$A_0 = \frac{75{,}000[1 - (1.007)^{-12 \cdot 20}]}{0.007}$$

$$\approx 8{,}705{,}700.365 \quad \text{Use a calculator.}$$

$$\approx 8{,}705{,}700.37 \quad \text{Round to the nearest cent.}$$

To fund the \$18 million prize, the commission must deposit \$8,705,700.37.

Self Check 2 The lottery pays a total prize of $120,000 in monthly installments, as a 10-year annuity. Assuming 8.4% interest, compounded monthly, what current deposit is needed to fund the annuity?

EXAMPLE 3 As a settlement in an automobile injury lawsuit, Robyn will receive $30,000 each year for the next 25 years, for a total of $750,000. The insurance company is offering a one-payment settlement of $300,000, now. Should she accept? Assume that the money can be invested at 9% annual interest.

Solution Robyn should calculate the present value of an annuity with:

the payment:	$P = 30{,}000$
the annual rate:	$r = 0.09$
the frequency of compounding:	$k = 1$ (annual)
the periodic rate:	$i = \frac{r}{k} = \frac{0.09}{1} = 0.09$
the term, in years:	$n = 25$

She should use these values in the formula for the present value of an annuity.

$$A_0 = \frac{P[1 - (1 + i)^{-kn}]}{i}$$

$$A_0 = \frac{30{,}000[1 - (1.09)^{-1 \cdot 25}]}{0.09}$$

$$\approx 294{,}677.3881 \quad \text{Use a calculator.}$$

$$\approx 294{,}677.39 \quad \text{Round to the nearest cent.}$$

Since the annuity is worth $294,677.39 and the company is offering $300,000, Robyn should accept the $300,000.

Self Check 3 If Robyn could invest the settlement at 8% interest, should she still accept the lump-sum offer?

When a worker is employed, regular contributions are usually made to a retirement fund. After retirement, those funds are given back, either as an annuity or as a lump-sum distribution.

EXAMPLE 4 Carlos wants to fund an annuity to supplement his retirement income. How much should he deposit now to generate retirement income of $1,000 a month for the next 20 years? Assume that he can get $9\frac{3}{4}$% interest, compounded monthly.

Solution Carlos must calculate the present value of a future stream of income, with:

the payment:	$P = 1{,}000$
the annual rate:	$r = 0.0975$
the frequency of compounding:	$k = 12$
the periodic rate:	$i = \frac{r}{k} = \frac{0.0975}{12} = 0.008125$
the term, in years:	$n = 20$

He should use these values in the formula for the present value of an annuity.

$$A_0 = \frac{P[1 - (1 + i)^{-kn}]}{i}$$

$$A_0 = \frac{1{,}000[1 - (1.008125)^{-12 \cdot 20}]}{0.008125}$$

$$\approx 105{,}428$$

If \$105,428 is deposited now, Carlos will receive \$1,000 per month in retirement income for 20 years.

Self Check 4 If Carlos can invest at $8\frac{3}{4}\%$, what deposit is needed now?

2. Amortization

Before a bank will lend money, you must sign a **promissory note** indicating that you will pay the money back. We discussed one-payment notes in Section 9.1. Most loans, however, are repaid in installments instead of all at once. Spreading the repayment over several equal payments is called **amortization.**

When a such a loan is made, the bank is buying an annuity from the borrower, and the bank pays the borrower a certain amount and expects regular payments in return. To calculate the amount of these regular installment payments, we solve the present value formula for P to get

$$A_0 = \frac{P[1 - (1 + i)^{-kn}]}{i}$$

$$A_0 i = P[1 - (1 + i)^{-kn}] \quad \text{Multiply both sides by } i.$$

$$P = \frac{A_0 i}{1 - (1 + i)^{-kn}} \quad \text{Divide both sides by } 1 - (1 + i)^{-kn}.$$

In this context, the present value A_0 is the amount of the loan.

Installment Payments The periodic payment P required to repay an amount A_0 is given by

$$P = \frac{A_0 i}{1 - (1 + i)^{-kn}}$$

where

r is the annual rate,

k is the frequency of compounding (usually monthly),

i is the periodic rate, $i = \frac{r}{k}$, and

n is the term of the loan.

EXAMPLE 5 The Almondi family takes a 15-year mortgage of \$200,000 for their new home, at 10.8%, compounded monthly.

a. Find their monthly payments.

b. Find the total of their payments over the full term.

Solution **a.** The mortgage has the following characteristics.

the amount:	$A_0 = 200{,}000$
the annual rate:	$r = 0.108$

the frequency of compounding: $k = 12$

the periodic rate: $i = \frac{r}{k} = \frac{0.108}{12} = 0.009$

the term, in years: $n = 15$

We substitute these values into the formula for installment payments to get

$$P = \frac{A_0 i}{1 - (1 + i)^{-kn}}$$

$$P = \frac{(200{,}000)(0.009)}{1 - (1.009)^{-12 \cdot 15}}$$

$$= \frac{1{,}800}{1 - 0.1993379912}$$

$$= 2{,}248.13964$$

Each monthly mortgage payment will be \$2,248.14.

b. There are $12 \cdot 15 = 180$ payments of \$2,248.14 each, for a total of \$404,665—more than twice the amount borrowed!

Self Check 5 Instead of a 15-year mortgage, the Almondis considered a 30-year mortgage. Answer the previous two questions again.

Everyday Connections

Mortgage Rates

LIBOR is an abbreviation for "London Interbank Offered Rate," and is the interest rate offered by a specific group of London banks for U.S. dollar deposits of a stated maturity. LIBOR is used as a base index for setting rates of some adjustable rate financial instruments, including Adjustable Rate Mortgages (ARMs) and other loans.

One week in 2000, the average rate on a one-year adjustable mortgage surged to 6.51 percent, the highest since January 2001, from 5.84 percent the prior week. The rate also surpassed the cost of a 30-year fixed loan for the first time.

Suppose a prospective homeowner obtains a mortgage loan with the following terms:

- Mortgage amount \$324,000
- Mortgage term 25 years
- Annual interest rate (fixed) 5.64%

Calculate the annual monthly mortgage payment (principal and interest).

Source: http://immobilienblasen.blogspot.com/2007_08_01_archive.html

Self Check Answers **1. a.** \$9,000 **b.** \$3,251.34 **2.** about \$81,000 **3.** No; the annuity is now worth more than \$320,243. **4.** \$113,159 **5. a.** \$1,874.48 **b.** \$674,814

9.3 Exercises

Vocabulary and Concepts ***Fill in the blanks.***

1. The current worth of a future stream of income is the __________ of an annuity.
2. The amount required now to produce a future stream of income is the __________ of an annuity.
3. A loan is called a ____________ because you promise to repay it.
4. Often, loan repayment is spread out over several _________.
5. Spreading repayment of a loan over several *equal* payments is called ________ the loan.
6. An amortized loan is also called a _______.

Practice ***Find the present value of an annuity with the given terms.***

7. Annual payments of $3,500 at 5.25%, compounded annually for 25 years
8. Semiannual payments of $375 at a 4.92% annual rate, compounded semiannually for 10 years

Find the periodic payment required to repay a loan with the given terms.

9. $25,000 repaid over 15 years, with monthly payments at a 12% annual rate
10. $1,750 repaid in 18 monthly installments, at an annual rate of 19%
11. **Funding retirement** Instead of making quarterly contributions of $700 to a retirement fund for the next 15 years, a man would rather make only one contribution, now. How much should that be? Assume $6\frac{1}{4}\%$ annual interest, compounded quarterly.
12. **Funding a lottery** To fund Jamie's lottery winnings of $15,000 per month for the next 20 years, the lottery commission needs to make a single deposit now. Assuming 9.2% compounded monthly, what should the deposit be?
13. **Money up front** Instead of receiving an annuity of $12,000 each year for the next 15 years, a young woman would like a one-time payment, now. Assuming she could invest the proceeds at $8\frac{1}{2}\%$, what would be a fair amount?
14. **Funding retirement** What single amount deposited now into an account paying $7\frac{2}{3}\%$ annual interest, compounded quarterly, would fund an annuity paying $5,000 quarterly for the next 25 years?
15. **Buying a car** The Jepsens are buying a $21,700 car and financing it over the next 4 years. They secure an 8.4% loan. What will their monthly payments be?
16. **Total cost of buying a car** What will be the total amount the Jepsens will pay over the life of the loan? (See Exercise 15.)
17. **Choosing a mortgage** One lender offers two mortgages—a 15-year mortgage at 12%, and a 20-year mortgage at 11%. For each, find the monthly payment to repay $130,000.
18. **Total cost of a mortgage** For each of the mortgages in Exercise 17, find the total of the monthly payments.

Discovery and Writing

19. **Getting an early start** As Jorge starts working now at the age of 20, he decides to make regular contributions to a savings account. He wants to accumulate enough by age 55 to fund an annuity of $5,000 per month until age 80. What should his monthly contributions be? Assume that both accounts pay 8.75%, compounded monthly.
20. **Comparing annuities** Which of these 20-year plans is best, and why? All are at 8% annually.
    a. $1,000 each year for 10 years, and then let the accumulated amount grow for 10 years
    b. $500 each year for 20 years
    c. Do nothing for 10 years, and then contribute $2,000 each year for 10 years
    d. One payment of $8,000 now, and let it grow
21. **Changing the payment** A woman contributed $500 per quarter for the first 10 years of an annuity, but changed to quarterly payments of $1,500 for the last 10 years. Assuming $7\frac{1}{4}\%$ annual interest compounded quarterly, what is her accumulated value?
22. **Changing the rate** A woman contributed $150 per month for 10 years to an account that paid 5% for the first 5 years, but 6.5% for the last 5 years. How much has she saved?

Review ***Simplify each expression. Assume that all variables represent positive numbers.***

23. $\frac{6\sqrt{30}}{3\sqrt{5}}$

24. $\frac{6}{\sqrt{7} - 2}$

25. $3\sqrt{5x} + 5\sqrt{20x}$

26. $\sqrt{\frac{x^3y^5}{x^5y^6}}$

CHAPTER REVIEW

9.1 Interest

Definitions and Concepts	Examples
If funds in a savings account earn **simple interest** at an **annual rate** r, the amount deposited is the **principal** P, and the length of **time** is t, the amount of **interest** I earned is given by the formula $I = Prt$	Find the simple interest on a deposit of \$8,000 that is left on deposit for 15 years at an annual rate of 4.5%. $I = Prt$ $I = 8{,}000 \cdot 0.045 \cdot 15$ $= 5{,}400$ The interest earned is \$5,400 and the account will contain \$13,400.
Compound interest, annual compounding: A single deposit A_0 earning compound interest for n years at an annual rate r, will grow to a future value A_n according to the formula $A_n = A_0(1 + r)^n$	Find the amount in an account where \$8,000 is left on deposit for 15 years at an annual rate of 4.5%, compounded annually. $A_n = A_0(1 + r)^n$ $A_{15} = 8{,}000(1 + 0.045)^{15}$ $= 15{,}482.25954$ The amount will be \$15,482.26. This is \$2,082.26 more than when the money was deposited at simple interest.
Compound interest formulas: An amount A_0, earning interest compounded k times a year for n years at an annual rate r, will grow to a future value A_n according to the formula $A_n = A_0(1 + i)^{kn}$ where $i = \frac{r}{k}$ is the periodic interest rate.	Find the amount in an account in which \$8,000 is left on deposit for 15 years at an annual rate of 4.5%, compounded monthly. The periodic interest rate is $i = \frac{r}{k} = \frac{0.045}{12} = 0.00375$. $A_n = A_0(1 + i)^{kn}$ $A_n = 8{,}000(1 + 0.00375)^{12 \cdot 15}$ $= 15{,}692.44007$ The amount will be \$15,692.44, \$210.18 more than annual compounding.

The **effective rate** R is used to compare different savings plans. $R = (1 + i)^k - 1$	Find the effective rate in the example above. $R = (1 + i)^k - 1$ $= (1 + 0.00375)^{12} - 1$ $= (1.00375)^{12} - 1$ $= 0.045939825$ The effective rate is about 4.6%.
The **present value** A_0 is the single deposit *now* that will yield a specific future value, A_n. $A_0 = A_n(1 + i)^{-kn}$ where interest is compounded k times a year at an annual rate r. $\left(i \text{ is the periodic rate } \frac{r}{k}.\right)$	Find the amount that must be deposited now to grow to be \$15,692.44 in 15 years in an account earning 4.5%, compounded monthly. As shown above, the periodic interest rate is 0.00375 $A_0 = A_n(1 + i)^{-nk}$ $A_0 = 15{,}692.44(1 + 0.00375)^{-12\cdot 15}$ $= 7{,}999.999967$ The present value is \$8,000. In the example above, we saw that \$8,000 grew to be \$15,692.44. So it is expected that the present value of \$15,692.44 is \$8,000.

Exercises

1. \$2,000 is deposited in an account that earns 9% simple interest. Find the value of the account in 5 years.
2. \$2,000 is deposited in an account in which interest is compounded annually at 9%. Find the value in 5 years.
3. Brian borrows \$2,350 for medical bills. The bank writes a 60-day note at 14%, with interest compounded daily. What will Brian owe?
4. \$2,000 earns interest, compounded quarterly, at an annual rate of 7.6% for 16 years. Find the future value.
5. BigBank advertises a savings account at a 6.3% rate, compounded quarterly. BestBank offers 6.21%, compounded daily. Calculate each effective rate and choose the better account.
6. What amount deposited now in an account paying 5.75% interest, compounded semiannually, will yield \$7,900 in 6 years?

9.2 Annuities and Future Value

Definitions and Concepts	Examples
An **annuity** is a series of payments P made at regular intervals. Its **future value** A_n is the sum of all the payments and the interest those payments earn. The time over which the payments are made is called the **term** of the annuity. In an **ordinary annuity,** the payments are made at the *end* of each time interval. The future value A_n of an ordinary annuity with deposits of P dollars made regularly k times each year for n years, with interest compounded k times per year at an annual rate r is $A_n = \dfrac{P[(1 + i)^{kn} - 1]}{i}$ where i is the periodic rate, $i = \frac{r}{k}$.	Under a company savings plan, a worker contributes \$100 a month to an ordinary annuity paying 8%, compounded monthly. How much will the annuity be worth in 50 years? The periodic interest rate is $i = \frac{r}{k} = \frac{0.08}{12} \approx 0.0067$. So, $A_n = \dfrac{P[(1 + i)^{kn} - 1]}{i}$ $A_{50} = \dfrac{100[(1 + 0.0067)^{12\cdot 50} - 1]}{0.0067}$ $= 805{,}362.6379$ The future value is \$805,362.64.

An annuity with the purpose of funding a future obligation is a **sinking fund.** To yield a specific future value A_n, regular deposits of P dollars are made k times per year for n years, with interest compounded k times per year at an annual rate r. The payment P is

$$P = \frac{A_n i}{(1+i)^{kn} - 1}$$

where i is the periodic rate, $i = \frac{r}{k}$.

What periodic deposit to a sinking fund paying 12% interest, compounded monthly, will amount to $50,000 in 30 years?

The periodic interest rate is $i = \frac{0.12}{12} = 0.01$.

So,

$$\begin{aligned} P &= \frac{A_n i}{(1+i)^{kn} - 1} \\ &= \frac{50{,}000(0.01)}{(1+0.01)^{12\cdot 30} - 1} \\ &= 14.30629846 \\ &\approx 14.31 \end{aligned}$$

Exercises

7. $500 is deposited at the end of each year into an annuity in which interest is compounded annually at 5%. Find the accumulated amount after 13 years.

8. $150 is deposited monthly into an account that pays 8% annual interest, compounded monthly. Find the future value after 20 years.

9. The owners of a small dry cleaning shop will need $40,700 to open a second shop in 7 years. What monthly payments to a sinking fund earning 7.5% interest, compounded monthly, will meet that obligation?

9.3 Present Value of an Annuity; Amortization

Definitions and Concepts

The **present value** of an annuity is the current value of a future stream of income.

The present value A_0 of an annuity with payments of P dollars made k times per year for n years, with interest compounded k times per year at an annual rate r, is

$$A_0 = \frac{P[1-(1+i)^{-kn}]}{i}$$

where i is the periodic rate, $i = \frac{r}{k}$.

Examples

To buy a car in 2 years, a man intends to pay $300 each month into an account that pays 6% annual interest, compounded monthly. Find the present value of the annuity.

The periodic interest rate is $i = \frac{r}{k} = \frac{0.06}{12} = 0.005$.

So,

$$\begin{aligned} A_0 &= \frac{P[1-(1+i)^{-kn}]}{i} \\ &= \frac{300[1-(1+0.005)^{-12\cdot 2}]}{0.005} \\ &= 6{,}768.859867 \end{aligned}$$

The present value of the annuity is $6,769.

Loans are often paid off in **installments.** If equal installments are paid over a fixed time, the payments are **amortized.**

The periodic payment P required to repay an amount A is given by

$$P = \frac{A_0 i}{1 - (1 + i)^{-kn}}$$

where

r is the annual rate,

k is the frequency of compounding,

i is the periodic rate, $i = \frac{r}{k}$, and

n is the term of the loan.

A family buys a new house for \$250,000. To do so, they get a 30-year mortgage at 6% interest, compounded monthly. Find their monthly payment.

The periodic interest rate is $i = \frac{r}{k} = \frac{0.06}{12} = 0.005$.

So,

$$\begin{aligned} P &= \frac{A_0 i}{1 - (1 + i)^{-kn}} \\ &= \frac{250{,}000(0.005)}{1 - (1 + 0.005)^{-12 \cdot 30}} \\ &= 1{,}498.876313 \end{aligned}$$

The monthly payment for principal and interest will be \$1,498.88. Taxes and insurance will be extra.

Exercises

10. An annuity pays \$250 semiannually for 20 years. At a semiannually compounded rate of 6.5%, what is the present value?

11. The lottery must fund a 20-year annuity of \$50,000 per year. At 9.6%, compounded annually, what must be invested now?

12. What are the monthly payments for a \$150,500, 15-year, 10.75% mortgage? What is the total amount paid?

13. Answer the previous question, but for a 30-year mortgage.

CHAPTER TEST

Fill in the blanks.

1. When interest is left on deposit to earn more interest, the account earns ________ interest.

2. The annual rate of interest divided by the number of periods is called the _______ interest rate.

3. To compare different savings plans, compare the _______ rates of interest.

4. The nominal rate of interest is also called the ______ rate.

5. Plans involving regular periodic payments are called ________.

6. An annuity to fund a specific future obligation is a __________.

7. The current value of a series of future payments is the ___________ of an annuity.

8. Repaying a loan over several regular, equal installments is called _________ the loan.

9. \$1,300 is deposited in a new account that earns 5% simple interest. What will the account be worth in 10 years?

10. \$1,300 is deposited in a new account that earns 5% interest, compounded annually. What will the account be worth in 10 years?

11. \$1,300 is deposited in an account that earns 5% annual interest, compounded monthly. What will it be worth in 10 years?

12. What is the effective rate of the savings plan in Problem 11?

13. What single deposit now will yield \$5,000 in 10 years? Assume 7% annual interest, compounded quarterly.

14. Each month for 5 years, a student made \$700 payments to an account paying 7.3% annual interest, compounded monthly. Find the accumulated amount.

15. What monthly payment to a sinking fund will raise \$8,000 in 5 years? Assume 6.5% annual interest, compounded monthly.

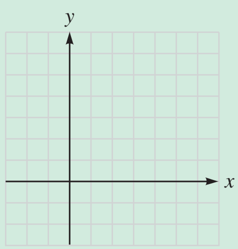
16. Find the present value of an annuity that pays \$1,000 each month for 15 years. Assume 6.8% annual interest, compounded monthly.

17. What are the monthly payments for a 15-year, \$90,000 mortgage at 8.95%?

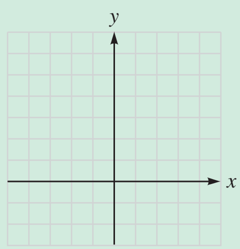
CUMULATIVE REVIEW EXERCISES

Solve each system by graphing.

1. $\begin{cases} 2x + y = 8 \\ x - 2y = -1 \end{cases}$



2. $\begin{cases} 3x = -y + 2 \\ y + x - 4 = -2x \end{cases}$



Solve each system.

3. $\begin{cases} 5x = 3y + 12 \\ 2x - 3y = 3 \end{cases}$

4. $\begin{cases} 2x + y - z = 7 \\ x - y + z = 2 \\ x + y - 3z = 2 \end{cases}$

Solve each system using matrices.

5. $\begin{cases} 2x + y - z = 0 \\ x - y + z = 3 \\ x + y - 3z = -5 \end{cases}$

6. $\begin{cases} 2x - 2y + 3z + t = 2 \\ x + y + z + t = 5 \\ -x + 2y - 3z + 2t = 2 \\ x + y + 2z - t = 4 \end{cases}$

Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix}$, ***and*** $C = \begin{bmatrix} 2 & 0 & -1 \\ -1 & 2 & 2 \end{bmatrix}$. ***Find each matrix.***

7. $A + B$

8. $B - A$

9. AC

10. $B^2 + 2A$

Find the inverse of each matrix, if possible.

11. $\begin{bmatrix} 2 & 6 \\ 2 & 4 \end{bmatrix}$

12. $\begin{bmatrix} 1 & -1 & 1 \\ 1 & 4 & 0 \\ 2 & 4 & 1 \end{bmatrix}$

Evaluate each determinant.

13. $\begin{vmatrix} -3 & 5 \\ 4 & 7 \end{vmatrix}$

14. $\begin{vmatrix} 2 & -3 & 2 \\ 0 & 1 & -1 \\ 1 & -2 & 1 \end{vmatrix}$

Set up the determinants to find x and y in the system $\begin{cases} 4x + 3y = 11 \\ -2x + 5y = 24 \end{cases}$. ***Do not evaluate the determinants.***

15. $x =$

16. $y =$

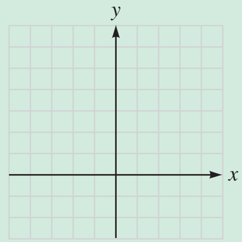
Decompose each fraction into partial fractions.

17. $\dfrac{-x + 1}{(x + 1)(x + 2)}$

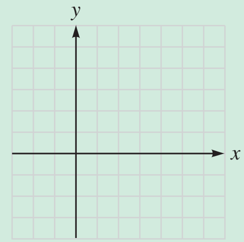
18. $\dfrac{x - 4}{(2x - 5)^2}$

Find each solution by graphing.

19. $y \leq 2x + 6$



20. $\begin{cases} 2x + 3y \geq 6 \\ 2x - 3y \leq 6 \end{cases}$



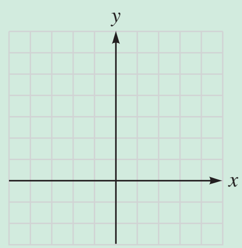
Write the equation of each circle with the given center O and radius r.

21. $O(0, 0)$; $r = 4$

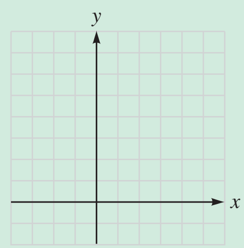
22. $O(2, -3)$; $r = 11$

Complete the square on x and/or y and graph each equation.

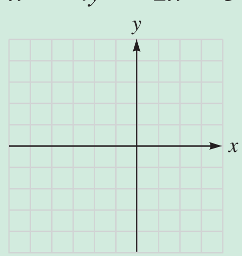
23. $x^2 + y^2 - 4y = 12$



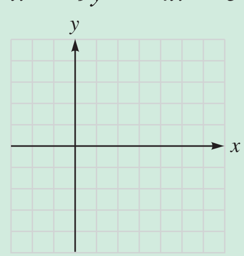
24. $x^2 - 2y - 2x = -7$



25. $x^2 + 4y^2 + 2x = 3$



26. $x^2 - 9y^2 - 4x = 5$



Write the equation of each ellipse.

27. Center (0, 0); horizontal major axis of 12; minor axis of 8

28. Center (2, 3); $a = 5$; $c = 2$, major axis vertical

Write the equation of each hyperbola.

29. Center (0, 0); focus (3, 0); vertex (2, 0)

30. Center (2, 4); area of fundamental rectangle is 36 square units; $a = b$; transverse axis parallel to y-axis

Find the required term of the expansion of $(x + 2y)^8$.

31. 2nd term

32. 6th term

Find each sum.

33. $\sum_{k=1}^{5} 2$

34. $\sum_{k=2}^{6} (3x + 1)$

Find the sum of the first six terms of each sequence.

35. $-2, 1, 4, \ldots$

36. $\frac{1}{9}, \frac{1}{3}, 1, \ldots$

Find each value.

37. $P(8, 4)$

38. $P(24, 0)$

39. $C(12, 10)$

40. $P(4, 4) \cdot C(6, 6)$

41. In how many ways can 6 men and 4 women be placed in a line if the women line up first?

42. In how many ways can a committee of 4 people be selected from a group of 12 people?

Find each probability.

43. Rolling 11 on one roll of two dice

44. Being dealt an all-red 5-card poker hand from a standard deck

45. If the probability that a person is married is 0.6 and the probability that a married person has children is 0.8, find the probability that a randomly chosen person is married with children.

46. Prove the formula by induction:

$$4 + 7 + 10 + \cdots + (3n + 1) = \frac{n(3n + 5)}{2}$$

47. What single deposit made now in an account that pays $8\frac{1}{2}\%$ interest, compounded annually, will grow to \$10,000 in 12 years?

48. A bank offers a \$110,000, 20-year mortgage at 8.75%. Find the monthly payment.