Determinants

Introduction

Determinants are mathematical objects which have applications in engineering mathematics. For example, they can be used in the solution of simultaneous equations, and to evaluate vector products. This leaflet will show you how to calculate the value of a determinant.

1. Evaluating a determinant

The symbol $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ represents the expression $ad - bc$ and is called a **determinant**.

For example

$$\begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix} \qquad \text{means} \qquad 3 \times 4 \quad - \quad 2 \times 1 = 12 - 2 = 10$$

Because $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ has two rows and two columns we describe it as a '2 by 2' or second-order determinant. Its value is given by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

If we are given values for a, b, c and d we can use this to calculate the value of the determinant. Note that, once we have worked it out, a determinant is a single number.

Exercises

Evaluate the following determinants:

a) $\begin{vmatrix} 3 & 4 \\ 6 & 5 \end{vmatrix}$, b) $\begin{vmatrix} 2 & -2 \\ 1 & 4 \end{vmatrix}$, c) $\begin{vmatrix} 8 & 5 \\ -2 & 4 \end{vmatrix}$, d) $\begin{vmatrix} 6 & 10 \\ -3 & -5 \end{vmatrix}$, e) $\begin{vmatrix} x & 5 \\ y & 2 \end{vmatrix}$.

Answers

a) $15 - 24 = -9$, b) $8 - (-2) = 10$, c) $32 - (-10) = 42$, d) $-30 - (-30) = 0$, e) $2x - 5y$.

2. Third-order determinants

A third-order, or '3 by 3' determinant can be written

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

One way in which it can be evaluated is to use second-order determinants as follows:

$$a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

Note in particular the way that the signs alternate between $+$ and $-$.

For example

$$\begin{aligned}\begin{vmatrix} 1 & 2 & 1 \\ -1 & 3 & 4 \\ 5 & 1 & 2 \end{vmatrix} &= 1\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} - 2\begin{vmatrix} -1 & 4 \\ 5 & 2 \end{vmatrix} + 1\begin{vmatrix} -1 & 3 \\ 5 & 1 \end{vmatrix} \\ &= 1(2) - 2(-22) + 1(-16) \\ &= 2 + 44 - 16 \\ &= 30\end{aligned}$$

Exercises.

1. Evaluate each of the following determinants.

a) $\begin{vmatrix} 2 & 4 & 1 \\ 1 & 0 & 4 \\ 5 & -1 & 3 \end{vmatrix}$ b) $\begin{vmatrix} 0 & -3 & 2 \\ -9 & 4 & 1 \\ 6 & 0 & 2 \end{vmatrix}$ c) $\begin{vmatrix} 7 & -2 & 3 \\ -1 & -4 & -4 \\ 6 & -2 & 12 \end{vmatrix}$ d) $\begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix}$

2. Evaluate each of the following determinants.

a) $\begin{vmatrix} 9 & 12 & 1 \\ 1 & 4 & 1 \\ 1 & 5 & 3 \end{vmatrix}$, b) $\begin{vmatrix} 3 & 12 & 1 \\ -3 & 4 & 1 \\ 4 & 5 & 3 \end{vmatrix}$, c) $\begin{vmatrix} 3 & 9 & 1 \\ -3 & 1 & 1 \\ 4 & 1 & 3 \end{vmatrix}$, d) $\begin{vmatrix} 3 & 9 & 12 \\ -3 & 1 & 4 \\ 4 & 1 & 5 \end{vmatrix}$

Answers.

1. a) 75, b) -120, c) -290, d) abc. 2. a) 40, b) 146, c) 116, d) 198.

3. Fourth-order determinants

These are evaluated using third-order determinants. Once again note the alternating plus and minus sign.

Example

$$\begin{aligned}\begin{vmatrix} 5 & 2 & 6 & 3 \\ 3 & 9 & 12 & 1 \\ -3 & 1 & 4 & 1 \\ 4 & 1 & 5 & 3 \end{vmatrix} &= 5\begin{vmatrix} 9 & 12 & 1 \\ 1 & 4 & 1 \\ 1 & 5 & 3 \end{vmatrix} - 2\begin{vmatrix} 3 & 12 & 1 \\ -3 & 4 & 1 \\ 4 & 5 & 3 \end{vmatrix} + 6\begin{vmatrix} 3 & 9 & 1 \\ -3 & 1 & 1 \\ 4 & 1 & 3 \end{vmatrix} - 3\begin{vmatrix} 3 & 9 & 12 \\ -3 & 1 & 4 \\ 4 & 1 & 5 \end{vmatrix} \\ &= 5(40) - 2(146) + 6(116) - 3(198) \\ &= 200 - 292 + 696 - 594 \\ &= 10\end{aligned}$$

Determinants can be used in the solution of simultaneous equations using Cramer's Rule - see the leaflet *5.2 Cramer's Rule*.