DESCRIPTION OF LECTURES

In this lecture series we give an overview of some topics in algebra and topology where posets have played an important role; and along the way we survey and exemplify some of the techniques that have emerged. We illustrate how posets arise and are put to good use in connection with 1) matroids and oriented matroids, 2) subspace arrangements, 3) Coxeter groups, 4) certain commutative rings, 5) cell complexes and convex polytopes. Many open problems will be mentioned.

The general plan for the series is as follows.

Lectures 1–2. ***Posets and order complexes***

We review basic definitions and facts from the combinatorics of posets and lattices, such as Möbius function, etc. Then we define the order complex, which attaches a topological space (and hence homology groups, etc) and a commutative ring (the Stanley-Reisner ring) to every poset. Shellable and Cohen-Macaulay posets are defined and exemplified, and the idea of "algebras with straightening laws" is sketched. Cell complexes and CW posets are mentioned. Some interesting classes of posets, such as face lattices of convex polytopes, lattices of subgroups, and posets of words, are discussed.

Lecture 3–4. ***Graphs and matroids***

We begin by explaining how monotone graph properties can be viewed as simplicial complexes, and mention cases where questions about such complexes have come up in research in algebra and topology. Several of these questions (e.g. concerning matchings and k-connectivity) have been answered using poset tools and techniques. We then go on to explain some key tools, such as fiber theorems and Morse matchings, illustrating their use on various graph complexes. The action of the symmetric group on a graph complex induces representations on homology, and this leads to another set of tools for the analysis of such complexes. Among the applications we mention Vassiliev stratification of spaces of knots and resolution of certain determinantal rings.

Matroids and geometric lattices are introduced and discussed in terms of the relevant simplicial complexes and the representations as hyperplane arrangements. The case of real hyperplane arrangements and the dual zonotope is of particular importance. We then move on to oriented matroids and explain the topological representation theorem, which relies on poset-theoretic constructions. The final topic is random walks on the regions of an oriented matroid.

Lectures 5–6. *Subspace arrangements*

We define the intersection lattice of a subspace arrangement and discuss properties of the arrangement that can be read or constructed from this lattice. One such result is the theorem of Ziegler and Zivaljevic, which gives the homotopy type of the singularity link at the origin. We sketch how the Z-Z theorem is obtained via a poset fiber argument and an appropriate cell complex. Such complexes are constructed via the zonotope of a hyperplane embedding. We illustrate this general and useful cell complex construction with some additional examples, e.g. for complex hyperplane arrangements.

The Z-Z theorem implies the Goresky-MacPherson theorem, which gives a formula for the cohomology of the complement of a real subspace arrangement in terms of the order complex of its intersection lattice. This allows for useful computations in special cases. For instance, the solution to some decision tree complexity problems is given as an application.

Subspace arrangements over finite fields are examples of singular varieties, so the conclusions of the Weil conjectures don't all apply. But still a lot can be said. We review some of the technical tools, such as zeta function and ℓ-adic cohomology. We recall some well-known combinatorial q-identities that illustrate the deep connections between q-counting and complex Betti numbers. The ℓ-adic Goresky-MacPherson formula for subspace arrangements over a finite field is presented, showing that the summands known from the original Goresky-MacPherson formula are here eigenspaces of the Frobenius map. Discussion: Are there complexity-theoretic implications also here?

Time permitting, we briefly discuss the vanishing ideals of subspace arrangements (over general fields). What is sought is a good combinatorial construction of generators for such ideals. This is known in some interesting cases motivated by graph theory, and there are partial results related to the poset-theoretic "blocker" construction.

Lectures 7–8. *Bruhat order*

We begin with a 5-10 minute introduction to finite reflection groups and (more generally) Coxeter groups. Then we introduce the Bruhat order on such a group and give a glimpse of its geometric and algebraic origins. What can be said about the structure of intervals in Bruhat order? The basic fact is sphericity, which is proved via shellability. Short intervals in the finite groups have been classified. For longer intervals there is some information on the rank-generating function, particularly for intervals beginning at the identity element. We survey results of this type and the methods used to obtain them (combinatorial and cohomological tools).

We end with a discussion of some remarkable recent discoveries, initiated by work in algebraic geometry. One such discovery is that much of the combinatorial structure of Bruhat order is inherited by the subposet of involutions (group elements of order two). Another that the poset of intervals of Bruhat order appears as the face poset of certain cell decompositions.

Lecture 9–10. ***Counting faces***

We now turn to f-vector theory, that is, the study of rank-generating functions for order ideals in face lattices of polytopes (and other CW posets). We review the fundamental theorems of Kruskal-Katona and Macaulay for complexes of sets and of multisets. Then we discuss the circle of ideas around the still open g-conjecture for simplicial spheres and its solution in the polytope case some 25 years ago. As a consequence of the g-theorem we derive a generalized Upper Bound Theorem for simplicial polytopes.

The method of algebraic shifting of simplicial complexes was introduced some 20 years ago by Kalai. It is a wonderful tool for extracting information about a complex by simplifying its structure without destroying too much of its essential properties. This method has in the last 10 years attracted the interest of some algebraists, who view it in terms of "generic initial ideals" within Gröbner basis theory.

As an application of algebraic shifting we derive a complete set of relations characterizing f-vectors and Betti numbers coming from a simplicial complex (thus extending the Euler-Poincaré formula). More generally, we present a theorem that contains this characterization, as well as the Kruskal-Katona theorem and the characterization of f-vectors of Cohen-Macaulay complexes, as special cases.

We end with a brief introduction to Scarf complexes determined by real matrices. Some open problems and some results for f-vectors of Scarf complexes are discussed. In the case of integer matrices there are connections with minimal free resolutions of lattice ideals.