Complex Analysis

Lecture Notes for MATH 4023 (Spring 2017)

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Preface

「虛則實之，實則虛之。」

《孫子兵法》

This is the lecture note written for the course MATH 4023 - Complex Analysis, taught by the author at the Hong Kong University of Science and Technology (HKUST) in Spring 2017.

The purpose of this lecture note and the course is to introduce both theory and applications of complex-valued functions of one variable. It begins with basic notions of complex differentiability (i.e. holomorphic) functions. The central part of the course is the Cauchy's integral formula, which is a fundamental theorem leading many important and exciting results in the later half of the course. The last chapter of the course explains the statement of the Riemann Hypothesis, a famous unsolved problem that worths US$1,000,000 in Pure Mathematics.

The prerequisites of the course include Multivariable Calculus (in which students should be familiar with line integrals), and Analysis I (basic $\varepsilon - \delta$ languages are needed). It is also recommended that students have taken Analysis II (MATH 3033/3043) before taking this course, as some toolkits such as Weierstass's M-test, Lebesgue Dominated Convergence Theorem, Fubini's Theorem, etc. will appear frequently in the later half of the course. Students without MATH 3033/3043 are recommended to first go through some examples in these topics of MATH 3033/3043. A list of theorems in MATH 3033/3043 which are essential for this course can be found in the appendix of this lecture note.

The author welcomes any students and/or readers to point out typographical errors and mistakes of this lecture note.

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20 January, 2017
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Chapter 1

Preliminaries

1.1. Complex Numbers

1.1.1. Basic Arithmetics. From middle/high school, we learned that the quadratic equation $x^2 + 1 = 0$ does not have any real root because $x^2 + 1 > 0$ for any $x \in \mathbb{R}$. Complex numbers are introduced to make it possible for the equation $x^2 + 1 = 0$ to have roots. We denote:

$$i = \sqrt{-1} \quad \text{so that} \quad i^2 = -1.$$

While complex numbers make their appearance for purely algebraic purposes, their uses branch out to many scientific fields beyond Mathematics, including Quantum Mechanics, String Theory, Electrical Engineering, Fluid Mechanics, etc.

Definition 1.1 (Complex Numbers). A *complex number* z is a number of the form:

$$z = a + bi$$

where a and b are real numbers, and $i = \sqrt{-1}$. We call:

- a is the *real part* of z and is denoted by $a =: \mathrm{Re}(z)$; and
- b is the *imaginary part* of z and is denoted by $b =: \mathrm{Im}(z)$.

The set of all complex numbers is denoted by $\mathbb{C}$. Precisely, we have:

$$\mathbb{C} := \{a + bi : a, b \in \mathbb{R}\}.$$

Remark 1.2. Note that a real number is also considered as a complex number, since $a = a + 0i$. In other words, we have $\mathbb{R} \subset \mathbb{C}$.

A complex number $z = x + yi$ can be geometrically represented by the point (x, y) in $\mathbb{R}^2$ (see Figure 1.1). The x-axis is now called the *real axis* as it represents numbers of the form $a + 0i$. Likewise, the y-axis is called the *imaginary axis*, which represents numbers of the form $0 + bi$.

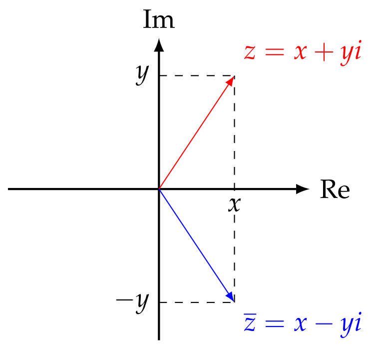


Figure 1.1. geometry of complex numbers

Given two complex numbers $z_1 = a + bi$ and $z_2 = c + di$, the arithmetics between them are defined by:

$$\begin{aligned} z_1 + z_2 &= (a+c) + (b+d)i \\ z_1 - z_2 &= (a-c) + (b-d)i \\ z_1 z_2 &= (a+bi)(c+di) \\ &= (ac-bd) + (ad+bc)i \\ \frac{z_1}{z_2} &= \frac{a+bi}{c+di} \cdot \frac{c-di}{c-di} && \text{(where } z_2 \neq 0\text{)} \\ &= \frac{(ac+bd)}{c^2+d^2} + \frac{(bc-ad)i}{c^2+d^2} \end{aligned}$$

1.1.2. Conjugate and Modulus. Two important operations on complex numbers are taking *conjugates* and *modulus*:

Definition 1.3 (Conjugate and Modulus)**.** Given $z = a + bi \in \mathbb{C}$, we denote and define:

- $\overline{z} := a - bi$ as the *conjugate* of z; and
- $|z| := \sqrt{a^2 + b^2}$ as the *modulus* of z.

Remark 1.4. It is important to note that complex numbers are *un-ordered*. It does not make sense to say $z_1 < z_2$ or $z_1 > z_2$. However, since $|z|$ is a real number, it makes sense to make comparison of $|z_1|$ and $|z_2|$.

Remark 1.5. Geometrically, $\overline{z}$ is obtained by reflecting z across the Re-axis (see Figure 1.1), and $|z|$ is the magnitude of the position vector representing z.

Listed below are some very useful properties of complex numbers. Given any $z, z_1, z_2 \in \mathbb{C}$, we have:

$$\begin{array}{ccc} z\overline{z} = |z|^2 & \overline{\overline{z}} = z & |\overline{z}| = |z| \\ \mathrm{Re}(z) = \dfrac{z+\overline{z}}{2} & \mathrm{Im}(z) = \dfrac{z-\overline{z}}{2i} & \\ \overline{z_1 \pm z_2} = \overline{z_1} \pm \overline{z_2} & \overline{z_1 z_2} = \overline{z_1}\,\overline{z_2} & \overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}} \end{array}$$

The proofs are all straight-forward and hence omitted. Simply let $z = x + yi$ and verify LHS and RHS are equal in each property. Let's look at some examples on how to make good use of these properties:

Example 1.1. Show that for any $z_1, z_2 \in \mathbb{C}$, we have:

$$|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\text{Re}(z_1\overline{z_2}).$$

Solution

The key step is to use the property that $|z|^2 = z\overline{z}$ for any $z \in \mathbb{C}$.

$$\begin{aligned}
|z_1 + z_2|^2 &= (z_1 + z_2)\overline{(z_1 + z_2)} = (z_1 + z_2)(\overline{z_1} + \overline{z_2}) \\
&= z_1\overline{z_1} + z_1\overline{z_2} + \overline{z_1}z_2 + z_2\overline{z_2} \\
&= |z_1|^2 + z_1\overline{z_2} + \overline{z_1\overline{z_2}} + |z_2|^2 \\
&= |z_1|^2 + |z_2|^2 + 2\text{Re}(z_1\overline{z_2})
\end{aligned}$$

Example 1.2. Let $\alpha, \beta \in \mathbb{C}\backslash\{0\}$. Show that $\alpha\overline{\beta} \in \mathbb{R}$ if and only if $\frac{\beta}{\alpha} \in \mathbb{R}$.

Solution

($\Longrightarrow$) Suppose $\alpha\overline{\beta} \in \mathbb{R}$, then we have $\overline{\alpha\overline{\beta}} = \alpha\overline{\beta}$, and so $\overline{\alpha}\beta = \alpha\overline{\beta}$. Since $\alpha, \beta \neq 0$, by rearrangement we get:

$$\frac{\beta}{\alpha} = \frac{\overline{\beta}}{\overline{\alpha}} = \overline{\left(\frac{\beta}{\alpha}\right)}$$

Therefore, $\frac{\beta}{\alpha}$ is equal to its conjugate. It concludes that $\frac{\beta}{\alpha} \in \mathbb{R}$.

($\Longleftarrow$) Conversely, let $\frac{\beta}{\alpha} = \lambda \in \mathbb{R}$. Then: $\alpha\overline{\beta} = \alpha\overline{\lambda\alpha} = \lambda\alpha\overline{\alpha} = \lambda\,|\alpha|^2 \in \mathbb{R}$.

It is important to note that in general $|z_1 + z_2| \neq |z_1| + |z_2|$. However, we do have:

Proposition 1.6 (Triangle Inequality). *Let $z_1, z_2 \in \mathbb{C}$, we have:*

$$|z_1 + z_2| \leq |z_1| + |z_2|\,.$$

Proof. From Example 1.1, we have:

$$|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\text{Re}(z_1\overline{z_2}).$$

Let $z_1\overline{z_2} = u + vi$, where $u, v \in \mathbb{R}$. Then, we have:

$$2\text{Re}(z_1\overline{z_2}) = 2u \leq 2\sqrt{u^2 + v^2} = 2\,|z_1\overline{z_2}| = 2\,|z_1|\,|\overline{z_2}| = 2\,|z_1|\,|z_2|\,.$$

Finally, we get:

$$|z_1 + z_2|^2 \leq |z_1|^2 + |z_2|^2 + 2\,|z_1|\,|z_2| = (|z_1| + |z_2|)^2$$

and it completes the proof by taking square root on both sides. $\square$

Exercise 1.1. Let $z_1, z_2 \in \mathbb{C}$, show that:

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2).$$

Exercise 1.2. Let $\alpha, \beta \in \mathbb{C}$. Suppose $\alpha\overline{z} + \beta z \in \mathbb{R}$ for *any* $z \in \mathbb{C}$. Show that $\alpha = \overline{\beta}$.

Exercise 1.3. Let $z_1, z_2 \in \mathbb{C}$. Show that $||z_1| - |z_2|| \leq |z_1 - z_2|$.

Exercise 1.4. Let p be the polynomial $p(z) = c_0 + c_1 z + \cdots + c_d z^d$ where $d \geq 1$ and $\{c_0, c_1, c_2, \ldots, c_d\}$ is a (monotone) decreasing sequence of positive real numbers. Prove that the polynomial equation $p(z) = 0$ does not have any roots with modulus (strictly) less than 1.

1.1.3. Polar Form. There are two common types of coordinates in $\mathbb{R}^2$, namely rectangular and polar. Apart from the standard (rectangular) form $x + yi$ for representing a complex number, we can also represent a complex number by a polar form. The conversion rule between rectangular and polar coordaintes is given by:

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$

Therefore, a complex number $z = x + yi$ can be written as:

$$z = (r\cos\theta) + i(r\sin\theta) = r(\cos\theta + i\sin\theta).$$

The form $z = r(\cos\theta + i\sin\theta)$ is commonly called the *polar form* of z.

Note that $|\cos\theta + i\sin\theta| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$. When $z = r(\cos\theta + i\sin\theta)$, it is easy to see that $r = |z|$. However, the value of θ is not unique as both sin and cos are periodic functions of period 2π. We define the principal argument of a complex number to be the angle θ with a specified range described below:

Definition 1.7 (Principal Argument)**.** Given a complex number z, the *principal argument* of z, denoted by $\operatorname{Arg}(z)$, is defined to be the angle $\theta_0 \in (-\pi, \pi]$ such that:

$$z = |z|\,(\cos\theta_0 + i\sin\theta_0).$$

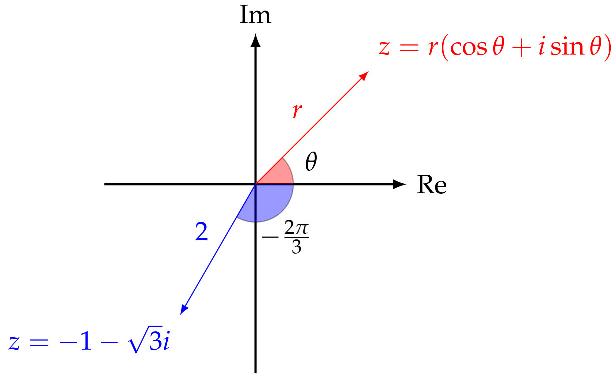
For example, $-1 - \sqrt{3}i$ has modulus 2 and so the r-coordinate is 2:

$$-1 - \sqrt{3}i = 2\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right).$$

To find the θ-coordinate, we solve $\cos\theta = -\frac{1}{2}$ and $\sin\theta = -\frac{\sqrt{3}}{2}$. From standard trigonometry, we get $\theta = \frac{4\pi}{3} + 2k\pi$ for any integer k. The only θ that falls into the range $(-\pi, \pi]$ is $-\frac{2\pi}{3} = \frac{4\pi}{3} - 2\pi$. Therefore, we have:

$$-1 - \sqrt{3}i = 2\left(\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right)$$

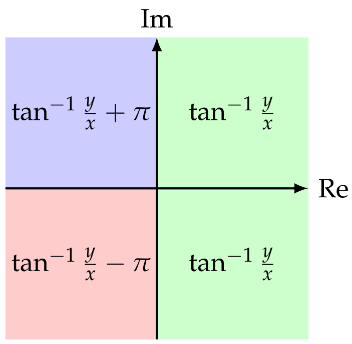
and $\operatorname{Arg}(-1 - \sqrt{3}i) = -\frac{2\pi}{3}$.



In general, $\operatorname{Arg}(x+yi)$ can be found using $\tan^{-1}\frac{y}{x}$ since if $x = r\cos\theta$ and $y = r\sin\theta$, then $\tan\theta = \frac{y}{x}$. However, it is important to note that $\operatorname{Arg}(x+yi)$ is NOT simply equal to $\tan^{-1}\frac{y}{x}$ because by definition of the inverse tangent, $\tan^{-1}\frac{y}{x}$ takes the value in $(-\frac{\pi}{2}, \frac{\pi}{2})$ only. Precisely, we have (when $x \neq 0$):

$$\operatorname{Arg}(x+yi) = \begin{cases} \tan^{-1}\frac{y}{x} & \text{if } (x,y) \text{ is in 1st and 4th quadrants;} \\ \tan^{-1}\frac{y}{x} + \pi & \text{if } (x,y) \text{ is in 2nd quadrant;} \\ \tan^{-1}\frac{y}{x} - \pi & \text{if } (x,y) \text{ is in 3rd quadrant;} \end{cases}$$

Furthermore, $\operatorname{Arg}(0+yi) = \frac{\pi}{2}$ when $y > 0$; and $\operatorname{Arg}(0+yi) = -\frac{\pi}{2}$ when $y < 0$. Note that $\operatorname{Arg}(0+0i)$ is undefined.



Exercise 1.5. Express the following complex numbers in polar form, and find their principal arguments Arg:

(a) $1+2i$

(b) $1-2i$

(c) $\cos(-\pi) + i\sin(-\pi)$

(d) $-i$

Exercise 1.6. Given $|z| = 1$, show that:

(a) $\operatorname{Re}\left(\frac{1+z}{1-z}\right) = 0$

(b) $\left|\frac{z-\omega}{1-\overline{\omega}z}\right| = 1$ for any $\omega \in \mathbb{C}$ such that $\overline{\omega}z \neq 1$.

Exercise 1.7. Given $z, \omega \in \mathbb{C}$ such that $|z+\omega| = |z-\omega|$, show that:

(a) $iz\overline{\omega} \in \mathbb{R}$

(b) $\text{Arg}(z) - \text{Arg}(\omega) = \dfrac{\pi}{2}$ or $\dfrac{3\pi}{2}$

Exercise 1.8. Show that the real-valued function $f : \mathbb{R}^2 \backslash \{(x,0) : x \leq 0\}$ defined by $f(x,y) := \text{Arg}(x+yi)$ is continuous.

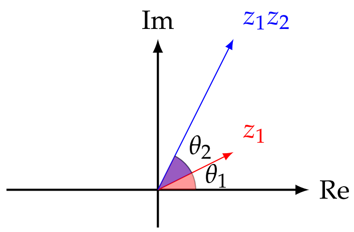
1.1.4. De Moivre's Theorem. By expressing complex numbers using polar form, one can see that multiplications and divisions between two complex numbers are *rotations* in the complex plane $\mathbb{C}$. It thanks to the fact that:

$$\begin{aligned}(\cos\theta + i\sin\theta)&(\cos\phi + i\sin\phi)\\ &= (\cos\theta\,\cos\phi - \sin\theta\,\sin\phi) + i(\cos\theta\,\sin\phi + \sin\theta\,\cos\phi)\\ &= \cos(\theta+\phi) + i\sin(\theta+\phi)\end{aligned} \tag{1.1}$$

Using (1.1), we can see that given $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then we have:

$$\begin{aligned}z_1z_2 &= r_1r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\\ &= r_1r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))\end{aligned}$$

Therefore, z_1z_2 is obtained by rotating z_1 by $\text{Arg}(z_2)$, and lengthen (or shorten) z_1 by a factor of $|z_2|$. See the figure below:



An important consequence of (1.1) is the following celebrated theorem:

Theorem 1.8 (De Moivre's Theorem). *For any $\theta \in \mathbb{R}$ and $n \in \mathbb{Z}$, we have:*

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta). \tag{1.2}$$

Proof. We prove by induction for positive n's. Clearly (1.2) is true when $n = 1$. Assume that (1.2) is true when $n = k$ for some positive integer k. Then, for $n = k+1$, we have:

$$\begin{aligned}(\cos\theta + i\sin\theta)^{k+1} &= (\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta)\\ &= (\cos(k\theta) + i\sin(k\theta))(\cos\theta + i\sin\theta) && \text{(induction assumption)}\\ &= \cos(k\theta+\theta) + i\sin(k\theta+\theta) && \text{(from (1.1))}\\ &= \cos((k+1)\theta) + i\sin((k+1)\theta)\end{aligned}$$

Hence (1.2) is true when $n = k+1$. By induction, (1.2) is true for all positive integer n.

When $n = 0$, (1.2) also holds because $(\cos\theta + i\sin\theta)^0 = 1$.

Finally we consider negative integers n. When $n < 0$, let $m = -n$ then m is a positive integer. From above, (1.2) holds for this m:

$$\begin{aligned}
(\cos\theta + i\sin\theta)^m &= \cos(m\theta) + i\sin(m\theta) \\
(\cos\theta + i\sin\theta)^{-n} &= \cos(-n\theta) + i\sin(-n\theta) \\
\frac{1}{(\cos\theta + i\sin\theta)^n} &= \cos(n\theta) - i\sin(n\theta) \\
(\cos\theta + i\sin\theta)^n &= \frac{1}{\cos(n\theta) - i\sin(n\theta)} \\
&= \frac{1}{\cos(n\theta) - i\sin(n\theta)} \cdot \frac{\cos(n\theta) + i\sin(n\theta)}{\cos(n\theta) + i\sin(n\theta)} \\
&= \frac{\cos(n\theta) + i\sin(n\theta)}{\cos^2(n\theta) + \sin^2(n\theta)} = \cos(n\theta) + i\sin(n\theta).
\end{aligned}$$

This proves (1.2) holds for negative integers n, and hence completing the proof of the theorem. □

De Moivre's Theorem can be used to derive some trigonometric identities. For example, consider $(\cos\theta + i\sin\theta)^3$. On one hand, De Moivre's Theorem shows that:

$$(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$$

and on the other hand, by expanding $(\cos\theta + i\sin\theta)^3$ we get:

$$\begin{aligned}
(\cos\theta + i\sin\theta)^3 &= \cos^3\theta + 3(\cos^2\theta)(i\sin\theta) + 3\cos\theta(i\sin\theta)^2 + (i\sin\theta)^3 \\
\cos 3\theta + i\sin 3\theta &= (\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta)
\end{aligned}$$

By equating the real and imaginary parts, we get:

$$\begin{aligned}
\cos 3\theta &= \cos^3\theta - 3\cos\theta\sin^2\theta = \cos^3\theta - 3\cos\theta\,(1-\cos^2\theta) = 4\cos^3\theta - 3\cos\theta \\
\sin 3\theta &= 3\cos^2\theta\sin\theta - \sin^3\theta = 3(1-\sin^2\theta)\sin\theta - \sin^3\theta = 3\sin\theta - 4\sin^3\theta
\end{aligned}$$

Exercise 1.9. Use De Moivre's Theorem to show that:

$$\cos n\theta = \sum_{k=0}^{[\frac{n}{2}]} \sum_{r=0}^{k} C_{2k}^n C_r^k (-1)^{k+r} \cos^{n-2k+2r}\theta$$

for any $n \in \mathbb{N}$. Here $[\frac{n}{2}]$ denotes the integer part of $\frac{n}{2}$.

1.1.5. Roots of Complex Numbers. In the real number system, the root equation $x^n = a$ where $a \neq 0$ and $n \in \mathbb{N}$, has at most two solutions. When n is odd (no matter whether a is positive or negative), the only real solution is $x = \sqrt[n]{a}$. When n is even and $a > 0$, there are two real solutions $x = \sqrt[n]{a}$ or $-\sqrt[n]{a}$. The equation has no solution when n is even and $a < 0$.

However, in the complex number system, the root equation $z^n = a$, where $a \in \mathbb{C}\backslash\{0\}$ and $n \in \mathbb{N}$, always has n solutions! Let's first look at the simplest equation $z^n = 1$:

Certainly, 1 is a solution to the equation. Furthermore, using De Moivre's Theorem, we get:

$$\left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)^n = \cos\left(\frac{2\pi}{n}\cdot n\right) + i\sin\left(\frac{2\pi}{n}\cdot n\right) = \cos(2\pi) + i\sin(2\pi) = 1.$$

Clearly, this shows the complex number $\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$ satisfies the equation $z^n = 1$. In fact, any number which can be expressed in form of $\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$, where k is an integer, is a solution to the root equation $z^n = 1$:

$$\left(\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}\right)^n = \cos(2k\pi) + i\sin(2k\pi) = 1.$$

Note that the set of roots $\left\{\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n} : k \in \mathbb{Z}\right\}$ consists of n distinct elements only (instead of infinitely many), since

$$\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n} = \cos\frac{2m\pi}{n} + i\sin\frac{2m\pi}{n}$$

if and only if $k - m$ is a multiple of n. In other words, when $k = n$, the root $\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$ is the same as the one with $k = 0$. Likewise, the root when $k = n + 1$ gives the same root as the one with $k = 1$, etc. Overall, the set of n-th roots of 1 is essentially given by the finite set:

$$\left\{\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n} : k \in \{0, 1, 2, \ldots, n-1\}\right\}$$

and these n numbers are called the *n-th root of* 1. In terms of notations, we write:

$$1^{\frac{1}{n}} = \left\{\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n} : k \in \{0, 1, 2, \ldots, n-1\}\right\}.$$

It is important to note that unlike the real number system, the n-th root of 1 is no longer a single number. In contrast, $1^{\frac{1}{n}}$ represents a *set* of roots for the equation $z^n = 1$.

Due to this distinctive difference from the real number system, from now on we will use $\sqrt[n]{a}$ to denote the n-th root of a in the *real* number system; while we will use $a^{\frac{1}{n}}$ to denote the n-th root of a in the *complex* number system, which will be discussed in the next paragraph.

Now consider the general root equation $z^n = a$ where $a \neq 0$. Suppose a can be expressed in polar form as:

$$a = |a|\left(\cos\theta + i\sin\theta\right)$$

Then, one can show that:

$$\underbrace{\sqrt[n]{|a|}}_{\text{real } n\text{-th root}} \left(\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right), \qquad k \in \mathbb{Z}$$

are solutions to the root equation $z^n = a$, since:

$$\begin{aligned}
&\left[\sqrt[n]{|a|}\left(\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right)\right]^n \\
&= \left(\sqrt[n]{|a|}\right)^n \left(\cos\left(\frac{\theta + 2k\pi}{n}\cdot n\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\cdot n\right)\right) \\
&= |a|\left(\cos(\theta + 2k\pi) + i\sin(\theta + 2k\pi)\right) \\
&= |a|\left(\cos\theta + i\sin\theta\right) \\
&= a
\end{aligned}$$

Again, two numbers $\cos\left(\frac{\theta+2k\pi}{n}\right)+i\sin\left(\frac{\theta+2k\pi}{n}\right)$ and $\cos\left(\frac{\theta+2m\pi}{n}\right)+i\sin\left(\frac{\theta+2m\pi}{n}\right)$ are equal if and only if $k-m$ is a multiple of n. Therefore, we conclude that the following n complex numbers:

$$\sqrt[n]{|a|}\left(\cos\left(\frac{\theta+2k\pi}{n}\right)+i\sin\left(\frac{\theta+2k\pi}{n}\right)\right)$$

are all the solutions to the root equation $z^n = a$. Similar to the case of roots of 1, we write the n-th root of a as:

Definition 1.9 (Roots of a Complex Number)**.** Given any $a \in \mathbb{C}\backslash\{0\}$ and $n \in \mathbb{N}$, the *n-th roots* of a is a set given by:

$$a^{\frac{1}{n}} = \left\{\sqrt[n]{|a|}\left(\cos\left(\frac{\operatorname{Arg}(a)+2k\pi}{n}\right)+i\sin\left(\frac{\operatorname{Arg}(a)+2k\pi}{n}\right)\right) : k \in \{0,1,\ldots,n-1\}\right\}$$

Example 1.3. Find $i^{\frac{1}{3}}$ and $(1-\sqrt{3}i)^{\frac{1}{2}}$.

Solution

First express i into polar form $i = \cos\frac{\pi}{2}+i\sin\frac{\pi}{2}$. Hence by Definition 1.9, we have:

$$\begin{aligned}
i^{\frac{1}{3}} &= \left\{\cos\frac{\frac{\pi}{2}+2k\pi}{3}+i\sin\frac{\frac{\pi}{2}+2k\pi}{3} : k=0,1,2\right\} \\
&= \left\{\underbrace{\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}}_{k=0},\ \underbrace{\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}}_{k=1},\ \underbrace{\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}}_{k=2}\right\} \\
&= \left\{\frac{\sqrt{3}+i}{2},\ \frac{\sqrt{3}-i}{2},\ -i\right\}
\end{aligned}$$

Similarly, to find $\left\{(1-\sqrt{3}i)^{\frac{1}{2}}\right\}$, we first express:

$$1-\sqrt{3}i = 2\left(\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right)$$

Hence, by Definition 1.9, we have:

$$\begin{aligned}
(1-\sqrt{3}i)^{\frac{1}{2}} &= \left\{\sqrt{2}\left(\cos\left(\frac{-\frac{\pi}{3}+2k\pi}{2}\right)+i\sin\left(\frac{-\frac{\pi}{3}+2k\pi}{2}\right)\right) : k=0,1\right\} \\
&= \left\{\sqrt{2}\left(\frac{\sqrt{3}-i}{2}\right),\ \sqrt{2}\left(\frac{-\sqrt{3}+i}{2}\right)\right\} \\
&= \left\{\frac{\sqrt{3}-i}{\sqrt{2}},\ \frac{-\sqrt{3}+i}{\sqrt{2}}\right\}
\end{aligned}$$

Exercise 1.10. First, show that the roots of $z^4+1=0$ are:

$$\left\{\frac{1+i}{\sqrt{2}},\ \frac{1-i}{\sqrt{2}},\ \frac{-1+i}{\sqrt{2}},\ \frac{-1-i}{\sqrt{2}}\right\}.$$

Then, use this result to factorize z^4+1 into the product of two quadratic polynomials with real coefficients.

Exercise 1.11. By considering the roots of the equation $z^n - 1 = 0$ (where $n > 2$ is an integer), show that $z^n - 1$ can be factorized into a product of linear and quadratic polynomials with real coefficients:

$$z^n - 1 = \begin{cases} (z-1)(z+1)\prod_{r=1}^{k-1}\left(z^2 - 2z\cos\frac{2\pi r}{n} + 1\right) & \text{if } n = 2k \\ (z-1)\prod_{r=1}^{k-1}\left(z^2 - 2z\cos\frac{2\pi r}{n} + 1\right) & \text{if } n = 2k-1 \end{cases}$$

Next we discuss a useful observation about the n-th root of 1. Let

$$\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$$

where n is an integer with $n \geq 1$, then one can show the following identity holds:

$$1 + \omega + \omega^2 + \ldots + \omega^{n-1} = 0.$$

$$\begin{aligned}
&(1-\omega)(1+\omega+\omega^2+\ldots+\omega^{n-1}) \\
&= (1+\omega+\omega^2+\ldots+\omega^{n-1}) - \omega(1+\omega+\omega^2+\ldots+\omega^{n-1}) \\
&= (1+\omega+\omega^2+\ldots+\omega^{n-1}) - (\omega+\omega^2+\ldots+\omega^{n-1}+\omega^n) \\
&= 1 - \omega^n \\
&= 1 - \left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)^n \\
&= 1 - (\cos(2\pi) + i\sin(2\pi)) = 1 - 1 = 0.
\end{aligned}$$

Since $\omega \neq 1$ as $n \geq 1$, we conclude that:

$$1 + \omega + \omega^2 + \ldots + \omega^{n-1} = 0.$$

Using this result, one can derive some trigonometric identities. Express ω in terms of its real and imaginary parts:

$$1 + \underbrace{\left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)}_{\omega} + \left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)^2 + \ldots + \left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)^{n-1} = 0$$

$$\begin{aligned}
1 + \left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right) + \left(\cos\frac{4\pi}{n} + i\sin\frac{4\pi}{n}\right) + \ldots \\
+ \left(\cos\frac{2(n-1)\pi}{n} + i\sin\frac{2(n-1)\pi}{n}\right) = 0
\end{aligned}$$

By equating the real and imaginary parts, we obtain two trigonometric identities:

$$\cos\frac{2\pi}{n} + \cos\frac{4\pi}{n} + \ldots + \cos\frac{2(n-1)\pi}{n} = -1$$

$$\sin\frac{2\pi}{n} + \sin\frac{4\pi}{n} + \ldots + \sin\frac{2(n-1)\pi}{n} = 0$$

Exercise 1.12. Show that for any $z \neq 1$, we have

$$1 + z + z^2 + \ldots + z^n = \frac{1 - z^{n+1}}{1 - z},$$

and use it to show:

$$1 + \cos\theta + \cos 2\theta + \ldots + \cos n\theta = \frac{1}{2} + \frac{\sin\left(\frac{(2n+1)\theta}{2}\right)}{2\sin\frac{\theta}{2}}$$

for any $\theta \in (0, 2\pi)$.

Exercise 1.13. Let $n \geq 2$ be an integer.

(a) Solve the equation $(z+1)^n - 1 = 0$.

(b) Hence, show that $\sin\dfrac{\pi}{n} \cdot \sin\dfrac{2\pi}{n} \cdots \sin\dfrac{(n-1)\pi}{n} = \dfrac{n}{2^{n-1}}$.

(c) Consider a circle of radius 1, and let $P_1, P_2, \ldots, P_n$ be the vertices of a regular n-sided polygon inscribed in the circle. Denote the distance between any pair of points P and Q by $\overline{PQ}$. Using (b), show that:

$$\prod_{k=2}^{n} \overline{P_1 P_k} = n.$$

Exercise 1.14. Let $P_k(x_k, y_k)$, where $k = 1, 2, 3$, be three distinct points in $\mathbb{C}$ and let $z_k := x_k + y_k i$ be the complex number representing P_k. Denote $\omega = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$. Show that $\triangle P_1 P_2 P_3$ is equilateral if and only if

$$z_1 + \omega z_2 + \omega^2 z_3 = 0.$$

Using this, show that it is impossible for $\triangle P_1 P_2 P_3$ being equilateral if $x_k, y_k \in \mathbb{Q}$ for all $k = 1, 2, 3$.

1.2. Sequences and Series

1.2.1. Sequences of Complex Numbers. In this section, we will extend the notion of sequences and series to complex numbers. As we shall see, many results and convergence tests which hold for real numbers will carry over to complex numbers. Let's begin with the definition of convergence of complex sequences:

> **Definition 1.10** (Limit of Sequences)**.** Let $\{z_n\}_{n=1}^{\infty}$ be a sequence of complex numbers. We say z_n *converges to* w *as* $n \to \infty$ if for any $\varepsilon > 0$, there exists an integer $N > 0$ such that whenever $n \geq N$, we have $|z_n - w| < \varepsilon$.

Remark 1.11. We may abbreviate "z_n converges to w as $n \to \infty$" by simply saying:

$$\lim_{n\to\infty} z_n = w.$$

Remark 1.12. It is easy to see that $\lim\limits_{n\to\infty} z_n = w$ is equivalent to $\lim\limits_{n\to\infty} |z_n - w| = 0$.

Remark 1.13. The definition of convergence of complex sequences is almost the same as the that of real sequences. The only difference is now $|\cdot|$ represents the modulus while for real sequence it represents the absolute value. Therefore, many computational rules about limits carry over to complex sequences. For instance, if $\lim\limits_{n\to\infty} z_n = L$ and $\lim\limits_{n\to\infty} w_n = M$, then we have

$$\begin{aligned} \lim_{n\to\infty} (z_n \pm w_n) &= L \pm M \\ \lim_{n\to\infty} (z_n w_n) &= LM \\ \lim_{n\to\infty} \frac{z_n}{w_n} &= \frac{L}{M} && \text{(whenever } M \neq 0\text{)} \end{aligned}$$

Example 1.4. Consider the sequence $z_n = z^n$ where $z \in \mathbb{C}$ is a fixed complex number. Show from the definition of limits that:

- if $|z| < 1$, then z_n converges to 0 as $n \to \infty$;
- if $z = 1$, then z_n converges to 1 as $n \to \infty$;

Solution

First consider the case $|z| < 1$: if $z = 0$, then $z_n = 0$ for any n and the desired result clearly holds. From now on we assume $z \neq 0$. For any $\varepsilon > 0$, we pick a positive integer $N > \frac{\log \varepsilon}{\log |z|}$. Whenever, $n \geq N$, we have:

$$|z_n - 0| = |z^n| = |z|^n \leq |z|^N.$$

Here we have used the fact that $|z| < 1$ and $n \geq N$. By our choice of N, we have:

$$\begin{aligned} |z|^N &< |z|^{\frac{\log \varepsilon}{\log |z|}} \\ &= |z|^{\log_{|z|} \varepsilon} = \varepsilon. \end{aligned}$$

This shows $\lim\limits_{n\to 0} z_n = 0$ in case of $|z| < 1$. The case of $z = 1$ is trivial.

When $|z| \geq 1$ and $z \neq 1$, the sequence $z_n = z^n$ can be shown to diverge using the squeezing principle (see Exercise 1.16). It can also be proved using the following useful fact:

Proposition 1.14. *A sequence $\{z_n\} \in \mathbb{C}$ converges to w as a complex sequence if and only if $\{\mathrm{Re}(z_n)\}$ converges to $\mathrm{Re}(w)$ and $\{\mathrm{Im}(z_n)\}$ converges to $\mathrm{Im}(w)$ as real sequences.*

Proof. $(\Longrightarrow)$-part follows from the inequalities:

$$|\mathrm{Re}(z_n) - \mathrm{Re}(w)| \leq |z_n - w| \quad \text{and} \quad |\mathrm{Im}(z_n) - \mathrm{Im}(w)| \leq |z_n - w|$$

and the squeezing principle.

$(\Longleftarrow)$-part follows from the fact that:

$$|z_n - w| = \sqrt{|\mathrm{Re}(z_n) - \mathrm{Re}(w)|^2 + |\mathrm{Im}(z_n) - \mathrm{Im}(w)|^2}$$

□

Now given a complex number z expressed in polar form as $z = r(\cos\theta + i\sin\theta)$, and suppose $|z| \geq 1$ (i.e. $r \geq 1$) and $z \neq 1$. Consider again the sequence $z_n = z^n$. By De Moivre's Theorem, we have:

$$z_n = r^n(\cos n\theta + i\sin n\theta).$$

It is well known in real analysis that when $\theta \neq 2k\pi$ (where $k \in \mathbb{Z}$), at least one of the real sequences $\{\cos n\theta\}$ and $\{\sin n\theta\}$ diverges as $n \to \infty$. Hence, when $r \geq 1$ and $\theta \neq 2k\pi$ ($k \in \mathbb{Z}$), at least one of the real sequences $\{r^n \cos n\theta\}$ and $\{r^n \sin n\theta\}$ diverges. This shows z_n diverges.

Exercise 1.15. Show that if $\lim\limits_{n\to\infty} z_n = L$, then $\lim\limits_{n\to\infty} \overline{z_n} = \overline{L}$ and $\lim\limits_{n\to\infty} |z_n| = |L|$.

Exercise 1.16. Show (without using Proposition 1.14) that if $|z| \geq 1$ and $z \neq 1$, then the sequence $\{z^n\}$ must diverge. [*Hint: First prove the following inequality:*

$$|z - 1| \leq \left|z^{n+1} - w\right| + |z^n - w|$$

for any $z \in \mathbb{C}$ such that $|z| \geq 1$, and any $w \in \mathbb{C}$.]

In Real Analysis, there is a notion of *Cauchy sequences* which describe sequences that are closer and closer to each other. It is *a priori* different from *convergent sequences*, which are sequences that are closer and closer to a certain limit. However, it is well-known that for sequences in $\mathbb{R}$, the Cauchy condition will guarantee convergence. This important fact is known as *completeness of real numbers*.

In Complex Analysis, we have a similar notion of *Cauchy sequences* and *completeness*, to be discussed below.

Definition 1.15 (Cauchy Sequence)**.** A sequence $\{z_n\}_{n=1}^{\infty}$ of complex numbers is said to be a *Cauchy sequence* if and only if for any $\varepsilon > 0$, there exists an integer $N \in \mathbb{N}$ such that whenever $m, n \geq N$, we have $|z_n - z_m| < \varepsilon$.

Theorem 1.16 (Completeness of $\mathbb{C}$)**.** *Every Cauchy sequence of complex numbers converges to a certain complex number. In other words, $\mathbb{C}$ is complete.*

Proof. Let $\{z_n\}$ be a Cauchy sequence of complex numbers. We need to show it converges. Write $z_n = x_n + iy_n$, where $x_n, y_n \in \mathbb{R}$. Since we have:

$$|x_n - x_m| \leq |z_n - z_m|$$
$$|y_n - y_m| \leq |z_n - z_m|$$

and given that $\{z_n\}$ is a Cauchy sequence, the real sequences $\{x_n\}$ and $\{y_n\}$ are also Cauchy sequences. By Completeness of $\mathbb{R}$, both $\{x_n\}$ and $\{y_n\}$ converge to some real numbers x_∞ and y_∞ respectively. By Proposition 1.14, the complex sequence $\{z_n\}$ converges to $x_\infty + iy_\infty$. □

Exercise 1.17. Suppose $\{z_n\}_{n=0}^{\infty}$ is a complex sequence. Suppose there exists a real constant $\alpha \in [0,1)$ such that:

$$|z_{n+1} - z_n| \leq \alpha\, |z_n - z_{n-1}| \quad \text{for any } n \in \mathbb{N}.$$

Show that the complex sequence $\{z_n\}_{n=0}^{\infty}$ converges.

1.2.2. Series of Complex Numbers. An (infinite) series $\sum\limits_{n=1}^{\infty} z_n$ of complex numbers $z_n \in \mathbb{C}$ is the limit (if exists) of the N-th partial sums $\sum\limits_{n=1}^{N} z_n$ as $N \to \infty$. In Real Analysis, we learned that many series convergence tests rely on the fact that $\mathbb{R}$ is complete. Now that we know $\mathbb{C}$ is also complete (Theorem 1.16), we can generalize many (although not all) series convergence tests for $\mathbb{C}$.

Definition 1.17 (Absolute and Conditional Convergences)**.** A series of complex numbers $\sum\limits_{n=1}^{\infty} z_n$ is said to converge *absolutely* if the series $\sum\limits_{n=1}^{\infty} |z_n|$ converges. A series $\sum\limits_{n=1}^{\infty} z_n$ is said to converge *conditionally* if it converges but does not converge absolutely.

Proposition 1.18 (Absolute Convergence Test)**.** *If the series* $\sum\limits_{n=1}^{\infty} |z_n|$ *converges, then the complex series* $\sum\limits_{n=1}^{\infty} z_n$ *also converges.*

Proof. Given that $\sum\limits_{n=1}^{\infty} |z_n|$ converges, its N-th partial sum $\sum\limits_{n=1}^{N} |z_n|$ is a Cauchy sequence. Now consider the sequence of N-th partial sums $\sum\limits_{n=1}^{N} z_n$. We want to show the later is also a Cauchy sequence.

For any $\varepsilon > 0$, there exists an integer $K > 0$ such that whenever $M > N \geq K$, we have

$$\sum_{n=1}^{M} |z_n| - \sum_{n=1}^{N} |z_n| < \varepsilon.$$

It implies:

$$\left| \sum_{n=1}^{M} z_n - \sum_{n=1}^{N} z_n \right| = \left| \sum_{n=N+1}^{M} z_n \right| \leq \sum_{n=N+1}^{M} |z_n| = \sum_{n=1}^{M} |z_n| - \sum_{n=1}^{N} |z_n| < \varepsilon.$$

Therefore, $\sum_{n=1}^{N} z_n$ is also a Cauchy sequence. By completeness of $\mathbb{C}$ (Theorem 1.16), the N-th partial sum $\sum_{n=1}^{N} z_n$ (and hence the infinite series $\sum_{n=1}^{\infty} z_n$) converges. $\square$

Example 1.5. Does the series $\sum_{n=1}^{\infty} \frac{i^n}{n}$ converge absolutely, conditionally, or does not converge? How about the series $\sum_{n=1}^{\infty} \frac{i^n}{n^2}$?

Solution

The series $\sum_{n=1}^{\infty} \left|\frac{i^n}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges by p-test. The N-th partial sum can be decomposed into:

$$\sum_{n=1}^{N} \frac{i^n}{n} = \begin{cases} \left(-\frac{1}{2} + \frac{1}{4} - \ldots + \frac{(-1)^k}{2k}\right) + \left(1 - \frac{1}{3} + \frac{1}{5} + \ldots + \frac{(-1)^{k-1}}{2k-1}\right) i & \text{if } N = 2k \\ \left(-\frac{1}{2} + \frac{1}{4} - \ldots + \frac{(-1)^k}{2k}\right) + \left(1 - \frac{1}{3} + \frac{1}{5} + \ldots + \frac{(-1)^{k+1}}{2k+1}\right) i & \text{if } N = 2k+1 \end{cases}$$

In either case, the real and imaginary parts converge by alternating series test. By Proposition 1.14, the series $\sum_{n=1}^{\infty} \frac{i^n}{n}$ converges, and so it converges conditionally.

Now consider $\sum_{n=1}^{\infty} \frac{i^n}{n^2}$. The series $\sum_{n=1}^{\infty} \left|\frac{i^n}{n^2}\right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p-test. Therefore, the series $\sum_{n=1}^{\infty} \frac{i^n}{n^2}$ converges absolutely.

One good property of an absolute convergent series is that we can *rearrange* the terms as we wish without changing the value of the series. Precisely, given an absolute convergent series $\sum_{n=1}^{\infty} z_n =: L$ and a bijection $\sigma : \mathbb{N} \to \mathbb{N}$, then the rearranged series $\sum_{n=1}^{\infty} z_{\sigma(n)}$ also converges absolutely to the limit L. The proof is the same as in the real case (hence omitted here).

Recall from Real Analysis that the ratio test and root test follow from the absolute convergence test and completeness of $\mathbb{R}$. Now we learned that both hold on $\mathbb{C}$, hence the ratio test and root test can be extended to complex series:

Proposition 1.19 (Ratio Test). *Consider the complex series* $\sum_{n=1}^{\infty} z_n$*:*

- *If* $\lim_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| < 1$, *then* $\sum_{n=1}^{\infty} z_n$ *converges absolutely.*
- *If* $\lim_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| > 1$, *then* $\sum_{n=1}^{\infty} z_n$ *diverges.*
- *If* $\lim_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| = 1$, *then no conclusion can be drawn.*

Proposition 1.20 (Root Test). *Consider the complex series* $\sum_{n=1}^{\infty} z_n$*:*

- *If* $\lim_{n\to\infty} \sqrt[n]{|z_n|} < 1$*, then* $\sum_{n=1}^{\infty} z_n$ *converges absolutely.*
- *If* $\lim_{n\to\infty} \sqrt[n]{|z_n|} > 1$*, then* $\sum_{n=1}^{\infty} z_n$ *diverges.*
- *If* $\lim_{n\to\infty} \sqrt[n]{|z_n|} = 1$*, then no conclusion can be drawn.*

Remark 1.21. The proofs of the ratio and root tests are the same as in the real case. We omit their proofs but we encourage readers to write down their proofs as an exercise.

Example 1.6. Show that for any $z \in \mathbb{C}$, the complex series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges absolutely.

Solution

We use the ratio test. Consider:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{z^{n+1}/(n+1)!}{z^n/n!}\right| &= \lim_{n\to\infty}\left|\frac{z^{n+1}}{z^n}\,\frac{n!}{(n+1)!}\right| \\ &= \lim_{n\to\infty}\left|\frac{z}{n+1}\right| = \lim_{n\to\infty}\frac{|z|}{n+1} \\ &= 0 < 1 \quad \text{for any } z \in \mathbb{C}.\end{aligned}$$

Hence, the series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges absolutely by ratio test (Proposition 1.19).

Alternatively, we can also use the root test (Proposition 1.20) by showing that:

$$\lim_{n\to\infty}\sqrt[n]{\left|\frac{z^n}{n!}\right|} = \lim_{n\to\infty}\frac{|z|}{\sqrt[n]{n!}} = 0 < 1$$

for any $z \in \mathbb{C}$. Here we have used the fact that $\lim_{n\to\infty}\sqrt[n]{n!} = \infty$.

Example 1.7. Determine all complex numbers z such that the series $\sum_{n=0}^{\infty} nz^n$ converges.

Solution

Consider the limit $\lim_{n\to\infty}\left|\frac{(n+1)z^{n+1}}{nz^n}\right| = \lim_{n\to\infty}\frac{(n+1)}{n}|z| = |z|$. Therefore, by ratio test (Proposition 1.19), the series converges absolutely when $|z| < 1$; and diverges when $|z| > 1$.

When $|z| = 1$, the ratio test fails to conclude anything. In this case, we let $z = \cos\theta + i\sin\theta$ where $\theta \in \mathbb{R}$. Then, the series is given by $\sum_{n=0}^{\infty}(n\cos n\theta + in\sin n\theta)$, and the real and imaginary parts are:

$$\text{Re}\left(\sum_{n=0}^{\infty} nz^n\right) = \sum_{n=0}^{\infty} n\cos(n\theta) \quad \text{and} \quad \text{Im}\left(\sum_{n=0}^{\infty} nz^n\right) = \sum_{n=0}^{\infty} n\sin(n\theta).$$

By Proposition 1.14, if the complex series converges, then both their real and imaginary parts converge, and in particular we have:

$$\lim_{n\to\infty} n\cos(n\theta) = 0 \quad \text{and} \quad \lim_{n\to\infty} n\sin(n\theta) = 0.$$

By Squeeze Theorem, it will imply:

$$\lim_{n\to\infty} \cos(n\theta) = \lim_{n\to\infty} \sin(n\theta) = 0.$$

However, it would contradict the fact that $\cos^2(n\theta) + \sin^2(n\theta) = 1$; and so the series $\sum_{n=0}^{\infty} nz^n$ does not converge when $|z| = 1$.

Conclusion: the series $\sum_{n=0}^{\infty} nz^n$ converges if and only if $|z| < 1$.

Exercise 1.18. Determine whether each of the following complex series converges absolutely, conditionally, or diverge:

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(1-3i)^n}{(4+i)^{2n}}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n+n^3 i}$

(c) $\displaystyle\sum_{n=1}^{\infty} (\cos n - i\sin n)$

Exercise 1.19. In each of the following complex series: (i) determine all complex numbers z such that the series converges, (ii) sketch the range of these z's on the complex plane $\mathbb{C}$.

(a) $\displaystyle\sum_{n=1}^{\infty} z^n$

(b) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{z}{z+1}\right)^n$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n z^{5n}}{n!}$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{z^{n!}}{n^2}$

Exercise 1.20. Suppose $z \in \mathbb{C}$.

(a) Assume $|z| \neq 1$ and $z \neq 0$, show that for any $n \in \mathbb{N}$, we have:
$$\left|\frac{z^n}{1+z^{2n}}\right| \leq \frac{1}{\left||z|^n - |z|^{-n}\right|}$$

(b) Using (a), or otherwise, find all $z \in \mathbb{C}$ such that the sequence $\left\{\frac{z^n}{1+z^{2n}}\right\}_{n=1}^{\infty}$ converges.

(c) Find all $z \in \mathbb{C}$ such that the series $\sum_{n=1}^{\infty} \frac{z^n}{1+z^{2n}}$ converges.

1.2.3. Euler's Identity. The series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ considered in Example 1.6 is an important one – it *defines* the complex exponential function. When $z = x$ is a real number, the value of the series is given by e^x. Given that the series converges for any $z \in \mathbb{C}$, we define e^z to be the limit of this series:

Definition 1.22 (Complex Exponential)**.** Let $z \in \mathbb{C}$, the exponential e^z, or equivalently $\exp(z)$, of z is defined by:
$$e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Remark 1.23. Please do NOT ask *why* $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$, as it is by the *definition*. A more appropriate question is *what* motivates such a definition. One motivation is that by such a definition, many nice properties about e^x in the real case can be extended to e^z in the complex case. These properties may include $e^{z+w} = e^z e^w$, $e^z \neq 0$, etc. We will look into them soon.

Here is the famous Euler's identity that relates complex exponentials with the polar form of a complex number:

Theorem 1.24 (Euler's Identity)**.** *For any $\theta \in \mathbb{R}$, we have:*
$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{1.3}$$

Proof. The key idea is to split the defining series into real and imaginary parts.
$$\begin{aligned}
e^{i\theta} &= \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = \lim_{N\to\infty} \sum_{n=0}^{2N} \frac{i^n\theta^n}{n!} \\
&= \lim_{N\to\infty} \left(\sum_{k=0}^{N} \frac{i^{2k}\theta^{2k}}{(2k)!} + \sum_{k=0}^{N-1} \frac{i^{2k+1}\theta^{2k+1}}{(2k+1)!} \right) && \text{[by rearrangement]} \\
&= \underbrace{\lim_{N\to\infty} \sum_{k=0}^{N} \frac{(-1)^k\theta^{2k}}{(2k)!}}_{=\cos\theta} + i \underbrace{\left(\lim_{N\to\infty} \sum_{k=0}^{N-1} \frac{(-1)^k\theta^{2k+1}}{(2k+1)!} \right)}_{=\sin\theta} && \text{[using } i^{2k} = (i^2)^k = (-1)^k\text{]} \\
&= \cos\theta + i\sin\theta
\end{aligned}$$
□

Remark 1.25. From (1.3), it is evident that we have:

$$e^{i\pi} + 1 = 0$$

which is a single identity involving 5 most important constants in mathematics, namely 1, 0, e, π and i.

Remark 1.26. From the Euler's identity, we can now write down the polar form of a complex number in a simpler way: if $z = r(\cos\theta + i\sin\theta)$, then we can write:

$$z = re^{i\theta}.$$

In particular, any $z \in \mathbb{C}$ such that $|z| = 1$ can be expressed as $z = e^{i\theta}$ for some $\theta \in \mathbb{R}$.

We are going to show that the complex exponential has the property that $e^z e^w = e^{z+w}$ just like the real case. *Informally*, we express both e^z and e^w into two infinite series. After multiplying the two series, we express the double sum diagonally:

$$\begin{aligned}
e^z e^w &= \left(\sum_{n=0}^{\infty} \frac{z^n}{n!}\right)\left(\sum_{m=0}^{\infty} \frac{w^m}{m!}\right) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{z^n w^m}{n!m!} \\
&= \sum_{k=0}^{\infty}\sum_{m+n=k} \frac{z^n w^m}{n!m!} = \sum_{k=0}^{\infty}\sum_{n=0}^{k} \frac{z^n w^{k-n}}{n!(k-n)!} && [\text{since } m = k-n] \\
&= \sum_{k=0}^{\infty}\sum_{n=0}^{k} \frac{C_n^k z^n w^{k-n}}{k!} && [\text{since } C_n^k = \frac{k!}{(n-k)!n!}] \\
&= \sum_{k=0}^{\infty} \frac{(z+w)^k}{k!} = e^{z+w} && [\text{Binomial Theorem}]
\end{aligned}$$

Although this "proof" above seems convincing and neat, there is a little step we need to justify, namely why we can rearrange the *infinite* double sum $\sum_n \sum_m$ in a diagonal way: $\sum_k \sum_{m+n=k}$? We have seen in Real Analysis that even switching $\sum_n$ and $\sum_m$ may sometimes result in a different sum. Below we give a rigorous (and more refined) proof of this fact:

Proposition 1.27. *For any $z, w \in \mathbb{C}$, we have $e^z e^w = e^{z+w}$.*

Proof. Consider the N-th partial sums $\sum_{n=0}^{N} \frac{z^n}{n!}$ and $\sum_{m=0}^{N} \frac{w^m}{m!}$, then:

$$\begin{aligned}
\left(\sum_{n=0}^{N} \frac{z^n}{n!}\right)\left(\sum_{m=0}^{N} \frac{w^m}{m!}\right) &= \underbrace{\sum_{n=0}^{N}\sum_{m=0}^{N} \frac{z^n w^m}{n!m!}}_{\text{Region I in Fig. 1.2}} \\
&= \underbrace{\sum_{k=0}^{2N}\sum_{m+n=k} \frac{z^n w^m}{n!m!}}_{\text{Region I+II+III in Fig. 1.2}} - \underbrace{\sum_{m=0}^{N}\sum_{n=N+1}^{2N-m+1} \frac{z^n w^m}{n!m!}}_{\text{Region II in Fig. 1.2}} - \underbrace{\sum_{n=0}^{N}\sum_{m=N+1}^{2N-n+1} \frac{z^n w^m}{n!m!}}_{\text{Region III in Fig. 1.2}}
\end{aligned}$$

Here we break down the finite double sum $\sum_n^N \sum_m^N$ into three triangular sums. See Figure 1.2 for illustration. For the sum corresponding to the large triangle (Region I+II+III in Figure 1.2), we can rewrite it as:

$$\sum_{k=0}^{2N}\sum_{m+n=k} \frac{z^n w^m}{n!m!} = \sum_{k=0}^{2N}\sum_{n=0}^{k} \frac{z^n w^{k-n}}{n!(k-n)!} = \sum_{k=0}^{2N}\sum_{n=0}^{k} \frac{C_n^k z^n w^{k-n}}{k!} = \sum_{k=0}^{2N} \frac{(z+w)^k}{k!} \to e^{z+w}$$

as $N \to \infty$.

For Region II in Figure 1.2, we can show that it converges to 0 as $N \to \infty$:

$$\begin{aligned}
\left|\sum_{m=0}^{N}\sum_{n=N+1}^{2N-m+1}\frac{z^n w^m}{n!m!}\right| &\leq \sum_{m=0}^{N}\sum_{n=N+1}^{2N-m+1}\left|\frac{z^n w^m}{n!m!}\right| \\
&\leq \sum_{m=0}^{N}\sum_{n=N+1}^{\infty}\frac{|z|^n}{n!}\frac{|w|^m}{m!} \\
&= \left(\sum_{m=0}^{N}\frac{|w|^m}{m!}\right)\left(\sum_{n=N+1}^{\infty}\frac{|z|^n}{n!}\right) \\
&\leq e^{|w|}\left(\sum_{n=N+1}^{\infty}\frac{|z|^n}{n!}\right) \to 0
\end{aligned}$$

as $N \to \infty$ since the infinite sum $\sum_{n=0}^{\infty}\frac{|z|^n}{n!}$ converges (to $e^{|z|}$). The sum corresponding to Region III in Figure 1.2 can be shown to converge to 0 by switching m and n, and z and w in the above argument.

Overall, we have shown:

$$\begin{aligned}
&\lim_{N\to\infty}\left(\sum_{n=0}^{N}\frac{z^n}{n!}\right)\left(\sum_{m=0}^{N}\frac{w^m}{m!}\right) \\
&= \lim_{N\to\infty}\sum_{k=0}^{2N}\sum_{m+n=k}\frac{z^n w^m}{n!m!} - \lim_{N\to\infty}\sum_{m=0}^{N}\sum_{n=N+1}^{2N-m+1}\frac{z^n w^m}{n!m!} - \lim_{N\to\infty}\sum_{n=0}^{N}\sum_{m=N+1}^{2N-n+1}\frac{z^n w^m}{n!m!} \\
&= e^{z+w} - 0 - 0,
\end{aligned}$$

which implies $e^z e^w = e^{z+w}$ as desired. □

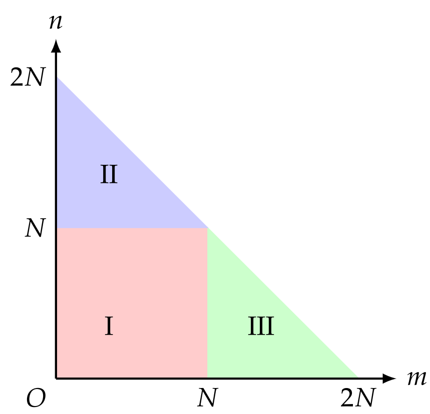


Figure 1.2

Exercise 1.21. Given two series $\sum_{n=0}^{\infty} z_n$ and $\sum_{n=0}^{\infty} w_n$ which converge absolutely to A and B respectively, show that the series below converges *absolutely* to AB:

$$\sum_{k=0}^{\infty}\left(\sum_{n=0}^{k} z_n w_{k-n}\right)$$

Exercise 1.22. Suppose $\{a_n\}_{n=1}^{\infty}$ is a (monotonically) decreasing sequence of *real* numbers such that $\lim_{n\to\infty} a_n = 0$, and $\{z_n\}_{n=1}^{\infty}$ be a sequence of complex numbers with the property that there is a constant $C > 0$ such that $\left|\sum_{n=1}^{N} z_n\right| \leq C$ for any N. Show that the series $\sum_{n=1}^{\infty} a_n z_n$ converges. [*Hint: First prove the following summation-by-parts formula*

$$\sum_{n=1}^{N} a_n z_n = \sum_{n=1}^{N} a_{N+1} z_n + \sum_{n=1}^{N} \sum_{k=1}^{n} (a_n - a_{n+1}) z_k,$$

and make good use of the given conditions.]

Furthermore, use the above result to prove the alternating series test in Real Analysis.

Exercise 1.23. Using the result from Exercise 1.22, show that the series $\sum_{n=1}^{\infty} \frac{z^n}{n}$ converges for any z such that $|z| = 1$ and $z \neq 1$.

Using the multiplicative property $e^z e^w = e^{z+w}$, one can show the following properties about the complex exponential function. We leave the proofs for readers.

Remark 1.28. For any $z = x + yi \in \mathbb{C}$ where $x, y \in \mathbb{R}$, we have:

- $(e^z)^n = e^{nz}$ for any integer n.
- $e^z = e^x e^{iy} = e^x(\cos y + i \sin y)$, and hence $|e^z| = e^x$.
- $e^z \neq 0$.

The complex exponential a^z with other real base $a > 0$ is defined via the natural exponential e^z. Recall that $a = e^{\ln a}$, and we define:

$$a^z := e^{(\ln a)z}.$$

Using this definition, some properties of e^z extend to complex exponentials a^z with an arbitrary real base $a > 0$. Proofs are again left for readers.

Remark 1.29. For any real $a, b > 0$ and $z, w \in \mathbb{C}$ we have:

- $(a^z)^n = a^{nz}$ for any integer n.
- $a^z a^w = a^{z+w}$
- $|a^z| = a^{\mathrm{Re}(a)}$
- $a^z \neq 0$
- $(ab)^z = a^z b^z$

Remark 1.30. For any positive integer n, the rational number $\frac{1}{n}$ is no doubt also a complex number. Therefore, now $e^{\frac{1}{n}}$ could mean two different things, namely the value of the series $\sum_{k=0}^{\infty} \frac{\left(\frac{1}{n}\right)^k}{k!}$, or the n-th roots of e. It is a confusing ambiguity but fortunately we seldom deal with both of them in the same context. One way to avoid such a confusion is to represent the n-th roots of e by $e^{\frac{1}{n}}$, and use $\exp(\frac{1}{n})$ to represent the value of the aforesaid series.

1.2.4. Riemann ζ Function: the first encounter. The Riemann zeta function, denoted by $\zeta(z)$, is of central importance in Complex Analysis and Number Theory. It is an infinite series defined as:

$$\zeta(z) := \sum_{n=1}^{\infty} \frac{1}{n^z}$$

for $\mathrm{Re}(z) > 1$. This complex series motivates the discussions of the famous *Riemann Hypothesis*, which is a conjecture purposed by Riemann in 1859 and remains unsolved as of today (January 20, 2017). The statement of the Riemann Hypothesis will be explained after we learn about analytic continuation of holomorphic functions. The Riemann zeta function has deep connections with Number Theory, in particular on the study of distribution of prime numbers. It is used to show the renowned *Prime Number Theorem*, which asserts that:

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\ln x} = 1$$

where $\pi(x)$ is the number of positive prime numbers less than or equal to x.

The deep connection between $\zeta(z)$ and prime numbers is beyond the scope of this course. Meanwhile, we would like to point out that this series converges absolutely when $\mathrm{Re}(z) > 1$ by the (real) p-test. The main reason is as follows. Write $z = x + yi$ where $x, y \in \mathbb{R}$, then we have:

$$\left|\frac{1}{n^z}\right| = \left|\frac{1}{e^{z\log n}}\right| = \frac{1}{\left|e^{x\log n}e^{iy\log n}\right|} = \frac{1}{n^x} = \frac{1}{n^{\mathrm{Re}(z)}}.$$

By (real) p-test, the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\mathrm{Re}(z)}}$ converges if and only if $\mathrm{Re}(z) > 1$. Therefore, by the (complex) absolute convergence test, the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^z}$ converges absolutely when $\mathrm{Re}(z) > 1$.

1.3. Point-Set Topology of $\mathbb{C}$

In this section, we will introduce several terminologies and topological concepts about subsets of $\mathbb{C}$. These topological concepts will come up from time to time in the course.

To begin, let's define some standard notations we will use in the rest of the course. Let $z_0 \in \mathbb{C}$ and $r > 0$. From now on, we will denote:

$$\begin{aligned} B_r(z_0) &= \{z \in \mathbb{C} : |z - z_0| < r\} \\ \overline{B_r(z_0)} &= \{z \in \mathbb{C} : |z - z_0| \leq r\} \\ \partial B_r(z_0) &= \{z \in \mathbb{C} : |z - z_0| = r\} \end{aligned}$$

which are respectively the open ball, closed ball and circle with radius r centered at z_0. In the literature of Complex Analysis, it is often that the term *disc* is used instead of *ball*.

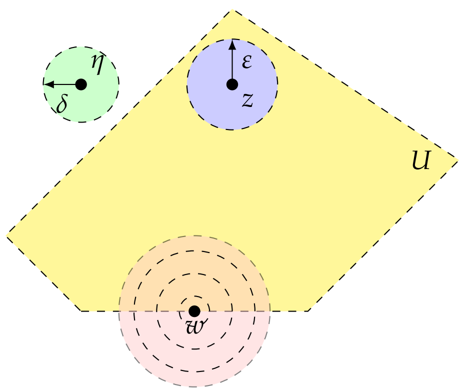
1.3.1. Open and Closed Subsets. Intuitively, an open subset Ω in $\mathbb{C}$ is one that does not have a boundary. However, this "definition" is not rigorous enough since the term "boundary" has not been defined so far. We are going to give a rigorous definition of open and closed subsets here. We first define:

Definition 1.31 (Interior, Boundary and Exterior Points)**.** Consider a set $U \subset \mathbb{C}$. We say that $z \in \mathbb{C}$ is an *interior point* of U if there exists $\varepsilon > 0$ such that $B_\varepsilon(z) \subset U$. We say that $w \in \mathbb{C}$ is a *boundary point* of U if for any $\varepsilon > 0$, both $B_\varepsilon(w) \cap U$ and $B_\varepsilon(w) \cap (\mathbb{C}\backslash U)$ are non-empty. We say $\eta \in \mathbb{C}$ is an *exterior point* of U if there exists $\delta > 0$ such that $B_\delta(\eta) \subset \mathbb{C}\backslash U$.

In the figure below, the yellow set is the subset $U \subset \mathbb{C}$. The point $z \in U$ is an interior ball because by drawing a ball with a small enough radius (i.e. the blue ball), the ball is completely inside U. In *layman* terms, an interior point of U is a point z whose "nearby" points are contained in U.

On the other hand, the point w in the figure below is a boundary point. No matter how small the ball you draw around w, that ball must contain some points in U, and some points not in U. In *layman* terms, a boundary point of U is a point w at which if you look around it, you can see "nearby" some points in U and some point not in U.

The point η in the figure is an exterior point of U. In *layman* terms, it is a point whose "nearby" are outside U.



Remark 1.32. Since $z \in B_\varepsilon(z)$ for any $z \in \mathbb{C}$ and $\varepsilon > 0$, if z is an interior point of U, it is automatically that $z \in U$. In other words, an interior point of a set must belong to that set. However, a boundary point of a set can be contained or not contained in the set. Furthermore, according to the definitions, interior points, boundary points and exterior points are mutually exclusive.

Example 1.8. Find all interior, boundary and exterior points of the set:

$$U = \{z \in \mathbb{C} : \operatorname{Re}(z) > 1\}.$$

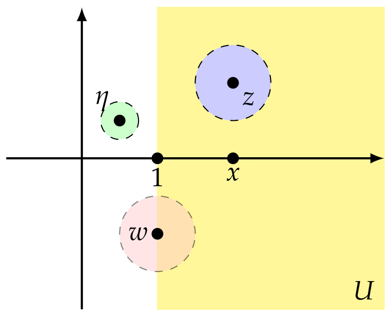
Solution

We claim that the set of interior points is U itself. For any $z \in U$, we have $\operatorname{Re}(z) > 1$. Write $z = x + yi$, then we have $x > 1$. We need to find a small $\varepsilon > 0$ such that $B_\varepsilon(x + yi) \subset U$. According to the figure below, an appropriate choice of ε should be $\varepsilon = \frac{x-1}{2}$. We next verify that it is indeed $B_\varepsilon(z) \subset U$ for this choice of ε.

For any $\alpha \in B_\varepsilon(z)$, we have $|\alpha - z| < \varepsilon = \dfrac{x-1}{2}$. Then, by $\operatorname{Re}(z - \alpha) \leq |z - \alpha|$, we know that:

$$\operatorname{Re}(z - \alpha) < \frac{x-1}{2} \quad \Longrightarrow \quad x - \operatorname{Re}(\alpha) < \frac{x-1}{2}.$$

By rearrangement, we get $\operatorname{Re}(\alpha) > x - \dfrac{x-1}{2} = \dfrac{x+1}{2} > \dfrac{1+1}{2} = 1$, which is equivalently to saying that $\alpha \in U$. This shows $B_\varepsilon(z) \subset U$, and hence z is an interior point.



Next we show that every point w with $\operatorname{Re}(w) = 1$ is a boundary point of U. Given any $\varepsilon > 0$, we consider the ball $B_\varepsilon(w)$. The point $w - \frac{\varepsilon}{2}$ lies in the ball $B_\varepsilon(w)$ and has real part $1 - \frac{\varepsilon}{2}$ and hence is not in U; while the point $w + \frac{\varepsilon}{2}$ is also in the ball $B_\varepsilon(w)$ but has real part $1 + \frac{\varepsilon}{2}$ and so is inside U. Therefore, both $B_\varepsilon(w) \cap U$ and $B_\varepsilon(w) \cap (\mathbb{C}\backslash U)$ are non-empty, it concludes that w is a boundary point of U.

Finally, we claim that any point $\eta \in \mathbb{C}$ with $\operatorname{Re}(\eta) < 1$ is an exterior point of U. To prove this claim, we choose a $\delta = \frac{1-\operatorname{Re}(\eta)}{2}$ and show that $B_\delta(\eta) \subset \mathbb{C}\backslash U$: Given any $\beta \in B_\delta(\eta)$, we have:

$$\operatorname{Re}(\beta - \eta) \leq |\beta - \eta| < \delta = \frac{1 - \operatorname{Re}(\eta)}{2} \quad \Longrightarrow \quad \operatorname{Re}(\beta) < \frac{1 + \operatorname{Re}(\eta)}{2} < 1.$$

Therefore, $\beta \notin U$, and it shows $B_\delta \subset \mathbb{C}\backslash U$. It completes the claim that η is an exterior point of U.

Exercise 1.24. Find all the interior, boundary and exterior points of each set below:

(a) $U_1 = \{z \in \mathbb{C} : \mathrm{Re}(z) \geq 0 \text{ and } \mathrm{Im}(z) < 0\}$.

(b) $U_2 = B_r(z_0)$ where $z_0 \in \mathbb{C}$ is a fixed number and $r > 0$

(c) $U_3 = \overline{B_r(z_0)}$ where $z_0 \in \mathbb{C}$ is a fixed number and $r > 0$.

(d) $U_4 = \partial B_r(z_0)$ where $z_0 \in \mathbb{C}$ is a fixed number and $r > 0$.

(e) $U_5 = \mathbb{C}$.

From now on, given any set $U \subset \mathbb{C}$, we denote and define:

$$\begin{aligned} U^c &:= \mathbb{C} \backslash U = \text{the complement of } U \text{ in } \mathbb{C} \\ U^\circ &:= \text{set of all interior points of } U \\ \partial U &:= \text{set of all boundary points of } U \\ \overline{U} &:= U \cup \partial U = \text{the closure of } U \end{aligned}$$

There is no standard notation for the set of all exterior points though. According to the definition of interior points, we must have $U^\circ \subset U$.

We are now ready to define what are open sets and closed sets. The way we define open sets is very common in many other textbooks, while the way we define closed sets may sound different from some textbooks but it is more intuitive and is nonetheless equivalent to the definition found in other textbooks.

Definition 1.33 (Open and Closed Sets)**.** A set $\Omega \subset \mathbb{C}$ is said to be *open* if every point $z \in \Omega$ is an interior point of Ω (i.e. $\Omega = \Omega^\circ$). A set $E \subset \mathbb{C}$ is said to be *closed* if all boundary points of E belong to E (i.e. $\partial E \subset E$).

Remark 1.34. Note that it is always true that $\Omega^\circ \subset \Omega$.

Let's look at some examples. Consider the set $\Omega = \{z \in \mathbb{C} : \mathrm{Re}(z) > 1\}$:

$$\begin{aligned} \Omega^\circ &= \{z \in \mathbb{C} : \mathrm{Re}(z) > 1\} = \Omega \\ \partial\Omega &= \{z \in \mathbb{C} : \mathrm{Re}(z) = 1\} \not\subset \Omega \end{aligned}$$

Therefore, Ω is an open set, but is not closed.

Let's look at another example: $E = \{z \in \mathbb{C} : \mathrm{Re}(z) \geq 1\}$. By inspection (we left the detail for readers), we can see:

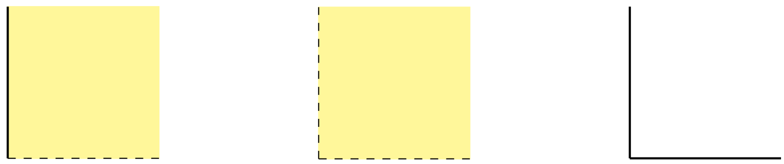
$$\begin{aligned} E^\circ &= \{z \in \mathbb{C} : \mathrm{Re}(z) > 1\} \neq E \\ \partial E &= \{z \in \mathbb{C} : \mathrm{Re}(z) = 1\} \subset E \end{aligned}$$

Therefore, E is not open, but is closed.

There are sets which are not open and not closed! For instance, consider the unit circle $W = \{z \in \mathbb{C} : \mathrm{Re}(z) \geq 0 \text{ and } \mathrm{Im}(z) > 0\}$. We can see from Figure 1.3 that:

$$\begin{aligned} W^\circ &= \{z \in \mathbb{C} : \mathrm{Re}(z) > 0 \text{ and } \mathrm{Im}(z) > 0\} \neq W \\ \partial W &= \{x + 0i \in \mathbb{C} : x \geq 0\} \cup \{0 + yi \in \mathbb{C} : y \geq 0\} \not\subset W. \end{aligned}$$

W is neither open nor closed.



(a) W **(b)** W° **(c)** ∂W

Figure 1.3. The set $W = \{z \in \mathbb{C} : \mathrm{Re}(z) \geq 0 \text{ and } \mathrm{Im}(z) > 0\}$ and its interior and boundary sets

Surprisingly, there are sets which are *both* open and closed (so "open" and "closed" are not exactly opposite, which is a linguistic nightmare)! For subsets of $\mathbb{C}$, there are not many though. They are the empty set $\varnothing$ and the whole $\mathbb{C}$. It is easy to see that $\mathbb{C}^\circ = \mathbb{C}$ and $\partial\mathbb{C} = \varnothing \subset \mathbb{C}$ (the empty-set is a subset of every set). This shows $\mathbb{C}$ is both open and closed.

The argument that shows $\varnothing$ is both open and closed has a bit of *philosophical* favor. We claim that $\varnothing^\circ = \varnothing$. Suppose otherwise, then we must have $\varnothing^\circ \not\subset \varnothing$ (since $\varnothing$ is a subset of every set). This means there exists $z \in \varnothing^\circ$ such that $z \notin \varnothing$. Then, z being an interior point of $\varnothing$ implies there exists $\varepsilon > 0$ such that $B_\varepsilon(z) \subset \varnothing$, which is clearly impossible! This shows $\varnothing^\circ = \varnothing$ and so the empty set is open. To show $\varnothing$ is closed as well, we claim $\partial\varnothing = \varnothing$. Suppose $\partial\varnothing$ is non-empty, then we can pick $w \in \partial\varnothing$, then for any $\delta > 0$, both $B_\delta(w) \cap \varnothing$ and $B_\delta(w) \cap (\mathbb{C}\backslash\varnothing)$ are non-empty. However, the former cannot happen! This concludes $\partial\varnothing = \varnothing$, and so $\varnothing$ is closed as well!

Remark 1.35. There is an interesting YouTube video titled "Hitler learns Topology".

Exercise 1.25. Determine whether each set U_1 to U_5 in Exercise 1.24 is open, closed, neither or both.

Readers who have learned a bit point-set topology may have seen another definition of closed sets, namely a set E is closed if its complement E^c is open. We are going to show that this is equivalent to our definition:

Proposition 1.36. *For any set $E \subset \mathbb{C}$, we have*

$$\partial E \subset E \quad \Longleftrightarrow \quad E^c \text{ is open.}$$

Proof. ($\Longrightarrow$)-part: Suppose $\partial E \subset E$. Consider $z \in E^c$, by the given condition $\partial E \subset E$, we know $z \notin \partial E$. This shows there exists $\varepsilon > 0$ such that at least one of the sets $B_\varepsilon(z) \cap E$ or $B_\varepsilon(z) \cap E^c$ is empty. Since $z \in E^c$, we must have $B_\varepsilon(z) \cap E = \varnothing$, which is equivalent to saying $B_\varepsilon(z) \subset E^c$. This shows E^c is open.

($\Longleftarrow$)-part: Suppose E^c is open. Consider $w \in \partial E$, and we need to show $w \in E$. Suppose not, then $w \in E^c$. By the openness of E^c, there exists $\delta > 0$ such that $B_\delta(w) \subset E^c$. However, it would imply $B_\delta(w) \cap E = \varnothing$, contradicting to the fact that $w \in \partial E$. This shows $w \in E$, completing the proof that $\partial E \subset E$. $\square$

Therefore, from now on we can say a set is closed if and only if its complement is open, which is more convenient sometimes. For instance, this fact can be used to show an important and nice property about a closed set E: if there is a *convergent* sequence in E, then the limit must be inside E.

Proposition 1.37. *Let $E \subset \mathbb{C}$ be a closed set. Suppose $\{z_n\}_{n=1}^\infty$ is a complex sequence such that $z_n \in E$ for any n. If $\lim\limits_{n\to\infty} z_n = w$, then $w \in E$.*

Proof. We prove by contradiction. The key idea is that if $w \notin E$, then one can draw a small ball around w such that the ball is completely outside E. However, then z_n which approaches w must go within the ball, and hence outside E, when n is large (see Figure 1.4).

Here we present the detail: suppose $w \notin E$, then $w \in E^c$. By Proposition 1.36, E^c is open and so there exists $\varepsilon > 0$ such that $B_\varepsilon(w) \subset E^c$. By the fact that $z_n \to w$, there exists $N \in \mathbb{N}$ such that whenever $n \geq N$, we have $|z_n - w| < \varepsilon$. However, it implies:

$$z_n \in B_\varepsilon(w) \subset E^c \quad \Longrightarrow \quad z_n \notin E$$

which is clearly a contradiction. It proves $w \in E$. □

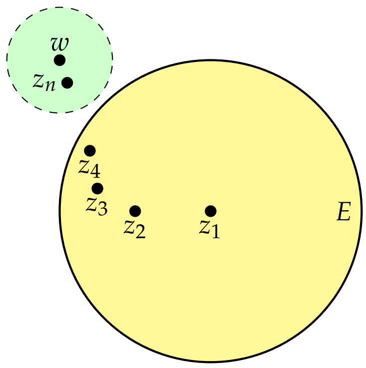


Figure 1.4. If E is closed, $w \notin E$ and $z_n \to w$, then z_n must go outside E for large n.

Below is a list of useful facts about open and closed sets. We will prove some of them and leave the others as exercises for readers.

Proposition 1.38. *Open and closed sets in $\mathbb{C}$ have the following properties:*

- *The union $\bigcup_\alpha U_\alpha$ of any family (finite or infinite) of open sets $\{U_\alpha\}$ in $\mathbb{C}$ is open.*
- *The intersection $\bigcap_{k=1}^{N} U_k$ of a finite family of open sets $U_1, \ldots, U_N$ in $\mathbb{C}$ is open.*
- *The union $\bigcup_{k=1}^{N} E_k$ of a finite family of closed sets $E_1, \ldots, E_N$ in $\mathbb{C}$ is closed.*
- *The intersection $\bigcap_\alpha E_\alpha$ of any family (finite or infinite) of closed sets $\{E_\alpha\}$ in $\mathbb{C}$ is closed.*

Proof. Let's prove the second statement only, that if $U_1, \ldots, U_N$ are open, then their intersection is also open. Let $z \in \bigcap_{k=1}^{N} U_k$, then $z \in U_k$ for any $k = 1, \ldots, N$. For each k, since U_k is open, z is an interior point of U_k and so there exists $\varepsilon_k > 0$ such that $B_{\varepsilon_k}(z) \subset U_k$. Let $\varepsilon = \min\{\varepsilon_1, \ldots, \varepsilon_N\}$, which is positive, then $\varepsilon \leq \varepsilon_k$ for any k, and so we have:

$$B_\varepsilon(z) \subset B_{\varepsilon_k}(z) \subset U_k \text{ for any } k = 1, \ldots, N.$$

Therefore, $B_\varepsilon(z) \subset \bigcap_{k=1}^{N} U_k$. This shows z is an interior point of $\bigcap_{k=1}^{N} U_k$. As a result, $\bigcap_{k=1}^{N} U_k$ is an open set.

We leave the proof of the first statement as an exercise for readers. Once the first two statements are established, the third and fourth statements about closed sets easily follow from Proposition 1.36 and De Morgan's Rule: $\left(\bigcup_k E_k\right)^c = \bigcap_k E_k^c$ and $\left(\bigcap_\alpha E_\alpha\right)^c = \bigcup_\alpha E_\alpha^c$. $\square$

Exercise 1.26. Prove all the other three statements in Proposition 1.38. Give an example of a family of open sets whose intersection is not open. Also give an example of a family of closed sets whose union is not closed.

Exercise 1.27. Given any two sets U and V in $\mathbb{C}$, show that:

(a) $\partial(U \cup V) \subset \partial U \cup \partial V$

(b) $\partial(\partial U) = \partial U$

(c) $\overline{U} := U \cup \partial U$ is always closed.

Here are two more terminologies which we will use sometimes:

- A set Ω in $\mathbb{C}$ is said to be *bounded* if there exists $M > 0$ such that $|z| < M$ for any $z \in \Omega$, i.e. $\Omega \subset B_M(0)$.
- A set Ω in $\mathbb{C}$ is said to be *compact* if it is *closed* and *bounded*.

Exercise 1.28. Use the Bolzano-Weierstrass's Theorem for $\mathbb{R}$ to show the Bolzano-Weierstrass's Theorem for $\mathbb{C}$, which asserts that if $\{z_n\}_{n=1}^\infty$ is a complex sequence in a bounded set Ω, then there exists a convergent subsequence $\{z_{n_k}\}_{k=1}^\infty$ of $\{z_n\}_{n=1}^\infty$.

Exercise 1.29. Supoose $\Omega_1 \supset \Omega_2 \supset \Omega_3 \supset \cdots$ is an infinite sequence of non-empty *compact* sets in $\mathbb{C}$. Show that:

$$\bigcap_{k=1}^{\infty} \Omega_k \neq \varnothing.$$

[*Hint: Pick* $z_k \in \Omega_k$ *for each* k. *What can you say about* $\{z_k\}_{k=1}^\infty$*?*]

1.3.2. Connected Sets. Intuitively, a connected set is one that is in one "piece". However, such a definition is not rigorous as the word "piece" is quite vague. To define connectedness, we first need to understand what it means by a *disconnected set*:

Definition 1.39 (Disconnected Sets). A set $\Omega \subset \mathbb{C}$ is said to be *disconnected* if there exists two disjoint open sets U and V (disjoint means $U \cap V = \varnothing$) such that:

$$\Omega \subset U \cup V, \quad \Omega \cap U \neq \varnothing \quad \text{and} \quad \Omega \cap V \neq \varnothing.$$

Remark 1.40. The condition $\Omega \subset U \cup V$ means that U and V together cover the whole set Ω. The condition $\Omega \cap U$ and $\Omega \cap V$ being non-empty means that Ω has something inside U and something inside V. Since the definition requires U and V are disjoint (i.e. separated in some sense), these sets U and V create a separation for the set Ω, and hence we say Ω is disconnected (see Figure 1.5).

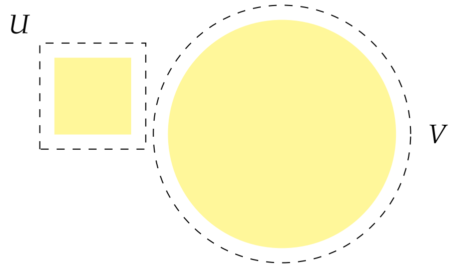
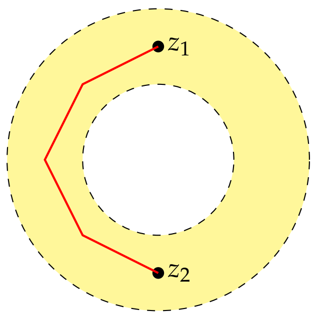


Figure 1.5. Ω is the yellow set. It is disconnected with disjoint open sets U and V that separate Ω.

A set Ω is said to be *connected* if it is *not disconnected*, meaning that whenever there are disjoint open sets U and V covering the set Ω, then at least one of $\Omega \cap U$ or $\Omega \cap V$ must be empty. In practice, it is not straight-forward to verify that a set is connected using the definition, even for simple examples such as an open ball $B_r(z)$, an open rectangle or an annulus $1 < |z| < 2$. However, thanks for a proposition that we will state, one can verify that they are all connected easily. Before we state the proposition, we need to define:

Definition 1.41 (Polygonally Path-Connected Sets)**.** A non-empty set $\Omega \subset \mathbb{C}$ is said to be *polygonally path-connected* if any pair of points in Ω can be joined by a continuous path consisting of finitely many line segments contained inside Ω.

For instance, any convex set is polygonally path-connected since every pair of points can be joined by a single line segment contained inside the set. The annulus $1 < |z| < 2$ is also polygonally path-connected (see the figure below):



The following proposition asserts that for any *open* set Ω, *connectedness* and *polygonal-path-connectedness* are equivalent:

Proposition 1.42. *An open set Ω in $\mathbb{C}$ is connected if and only if it is polygonally path-connected.*

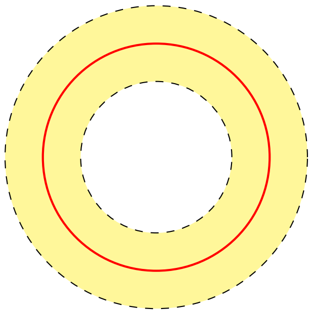
We omit the proof in this lecture note. Interested readers may consult Stein-Shakarchi's book (Exercise 5 in P.25) for an outline of the proof and try to complete the detail as an exercise. Using this proposition, it is easy to see that any convex open sets (and many other non-convex open sets) are connected.

The last notion about sets in $\mathbb{C}$ to be introduced is *simply-connectedness*. Readers should have encountered this concept in Multivariable Calculus (typically in the chapter about conservative vector field).

Definition 1.43 (Simply-Connected Sets)**.** A set Ω is said to be *simply-connected* if Ω is connected and that every closed loop in Ω can continuously contract to a point without leaving Ω.

The concept of simply-connectedness will come up frequently when we talk contour integrals and Cauchy's Integral Formula.

A ball and a rectangle (either open or closed) are simply-connected, while an annulus $1 < |z| < 2$ is not, because the red circle in the figure below cannot shrink to a point unless it steps into the "hole" which is not a part of the annulus.



On $\mathbb{C}$, simply-connected sets have one nice property concerning simple closed curves ("simple closed" means closed without self-intersections). If γ is a simple closed curve contained inside a simply-connected set Ω, then the region enclosed by γ will be a subset of Ω. Some textbooks put this as the definition of simply-connected sets in $\mathbb{C}$.

Exercise 1.30. For each set described below, sketch the region on $\mathbb{C}$, and determine whether it is (i) open, (ii) closed, (iii) bounded, (iv) compact, (v) connected and (vi) simply-connected or not.

(a) $\Omega_1 = \{z \in \mathbb{C} : |z+1| \geq 4\,|z-1|\}$
(b) $\Omega_2 = \{z \in \mathbb{C} : |z+1| < 4\,|z-1|\}$
(c) $\Omega_3 = \{z \in \mathbb{C} : |z| \leq \operatorname{Re}(z) + 1\}$
(d) $\Omega_4 = \{e^z \in \mathbb{C} : 1 \leq \operatorname{Re}(z) \leq 2\}$
(e) $\Omega_5 = \{z \in \mathbb{C} : \left|z^2 - 1\right| \leq 1\}$

Chapter 2

Holomorphic Functions

2.1. Complex-Valued Functions

A real-valued function $f : (a, b) \to \mathbb{R}$ with domain (a, b) maps a real number $x \in (a, b)$ to a unique real number $f(x) \in \mathbb{R}$. To visualize such a function, we can consider its graph $y = f(x)$ in the xy-plane.

In this chapter and in this course, we are mostly concerned about *complex-valued* functions. They are functions $f : \Omega \to \mathbb{C}$ with inputs z inside an open domain $\Omega \subset \mathbb{C}$, and also with complex numbers $f(z)$ as the outputs. By writing the inputs as $x + yi$ and the output as $u + vi$, then a complex-valued function $f : \Omega \to \mathbb{C}$ can be generally expressed as:

$$f(x + yi) = u(x, y) + iv(x, y)$$

where $u(x, y)$ and $v(x, y)$ are real-valued functions. Essentially, both inputs and outputs are two-dimensional, and so the graph of f is *four* dimensional! It is not possible for us to visualize such a graph. In this section, we will learn how to visualize a complex-valued function by various other ways.

2.1.1. Examples of Complex-Valued Functions. From now on, unless otherwise is stated, we will write $z = x + yi$, and $f(z) = u + iv$.

Example 2.1. An easy example of a complex-valued function is $f(z) = z^2$. The domain of this function is $\mathbb{C}$. By writing $z = x + yi$, we can see that:

$$f(z) = (x + yi)^2 = x^2 + 2xyi + (yi)^2 = (x^2 - y^2) + 2xyi$$

Therefore, we denote its real and imaginary parts by:

$$\begin{aligned} u(x, y) &= x^2 - y^2 \\ v(x, y) &= 2xy. \end{aligned}$$

Example 2.2. Consider another function $f(z) = e^z$. By writing $z = x + yi$, we get:

$$f(z) = e^{x+yi} = e^x e^{yi} = e^x(\cos y + i \sin y).$$

Therefore, $u(x, y) = e^x \cos y$ and $v(x, y) = e^x \sin y$.

Exercise 2.1. For each function below, state its domain and find its real and imaginary parts:

(a) $f(z) = \dfrac{1}{\bar{z}}$

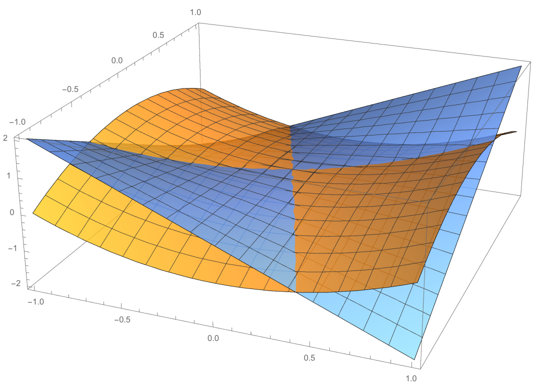
(b) $f(z) = |z|^2$

(c) $f(z) = e^{2z}$

(d) $f(z) = \dfrac{iz+1}{\bar{z}-i}$

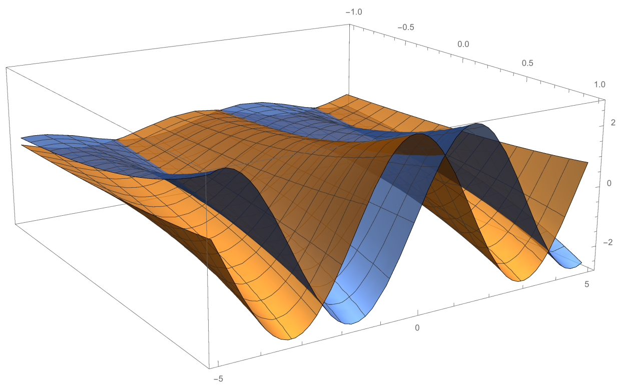
2.1.2. Visualizing Complex-Valued Functions using Graphs. Although one cannot visualize the graph $w = f(z)$ of a complex-valued function, we can *separately* visualize the graphs of the real and imaginary parts of $f(z)$.

Take $f(z) = z^2 = (x^2 - y^2) + 2xyi$ as an example. One can plot two separate graphs $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$ to represent this function:



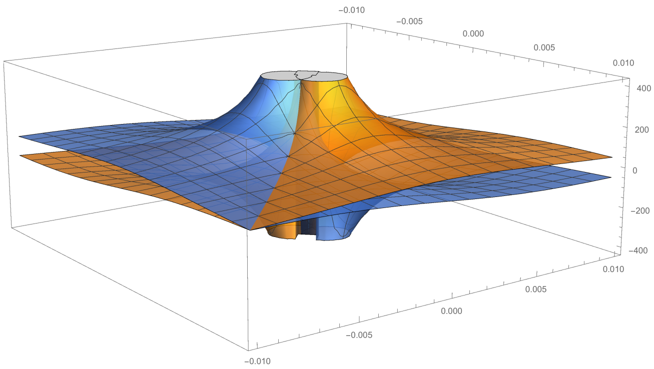
The orange graph represents the real part $u(x, y) = x^2 - y^2$, whereas the blue graph represents the imaginary part $v(x, y) = 2xy$.

Similarly, the exponential function $e^z = e^x \cos y + ie^x \sin y$ can be visualized as two separate graphs



Again the orange graph is the real part, and the blue graph is the imaginary part.

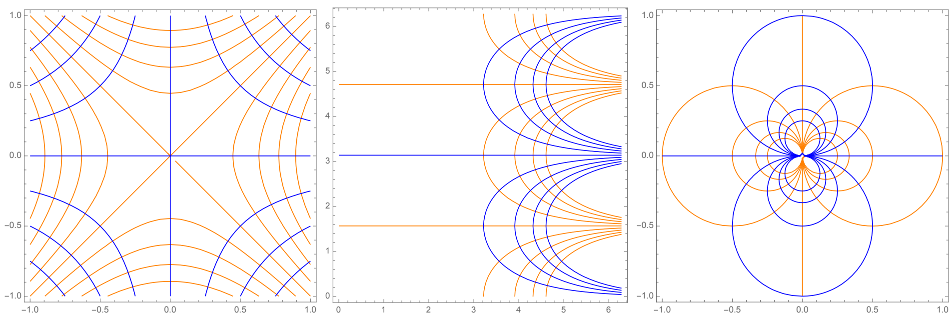
Some functions such as $f(z) = \dfrac{1}{z} = \dfrac{x}{x^2+y^2} - \dfrac{yi}{x^2+y^2}$ are not defined everywhere on $\mathbb{C}$. Let's see how its graphs look:



From the graph, one can see easily that both real and imaginary parts tend to $\pm\infty$ as (x, y) approaches $(0, 0)$.

2.1.3. Visualizing Complex-Valued Functions using Level Curves. Recall that a level set of a real-valued function $u(x, y) : \mathbb{R}^2 \to \mathbb{R}$ is a set in $\mathbb{R}^2$ such that $u(x, y) =$ constant. Different constants will give different curves or points on the plane, and a collection of these level sets is called a level set diagram of the function.

Level set diagrams are another good way to visualize a complex-valued function. Below are level-sets of some complex-valued functions. The orange curves are level curves of the real part $u(x, y)$, and the blue curves are level curves of the imaginary part.



(a) $f(z) = z^2$ **(b)** $f(z) = e^z$ **(c)** $f(z) = 1/z$

One can see the orange and blue level curves intersect each other orthogonally at almost all points. It is in fact not coincident! This orthogonality phenomenon is related to *complex differentiability* as we will see in the next section.

2.1.4. Visualizing Complex-Valued Functions via Mappings. One elegant way to visualize a complex-valued function is by its mapping properties. A function $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ can be regarded as a map that assigns a point z in the domain Ω to a unique point in $f(z)$ in $\mathbb{C}$.

Example 2.3. Take $f(z) = z^2 = (x^2 - y^2) + 2xyi$ as an example. We are going to see how it maps a unit square in the xy-plane to the uv-plane. Consider the straight-line L_1 parametrized by $(x, y) = (t, 0)$ where $t \in [0, 1]$. The image of L_1 under the mapping f is given by:

$$f(t + 0i) = t^2 + 0i$$

which is a straight-line joining $0+0i$ and $1+0i$ in the uv-plane. Similarly, consider the straight-line L_2 parametrized by $(x,y)=(1,t)$, where $t\in[0,1]$. The image of L_2 under the mapping f is:

$$f(1+ti)=(1-t^2)+2ti.$$

Consider the (u,v)-coordinates, we have $u=1-t^2$ and $v=2t$, which simplifies to:

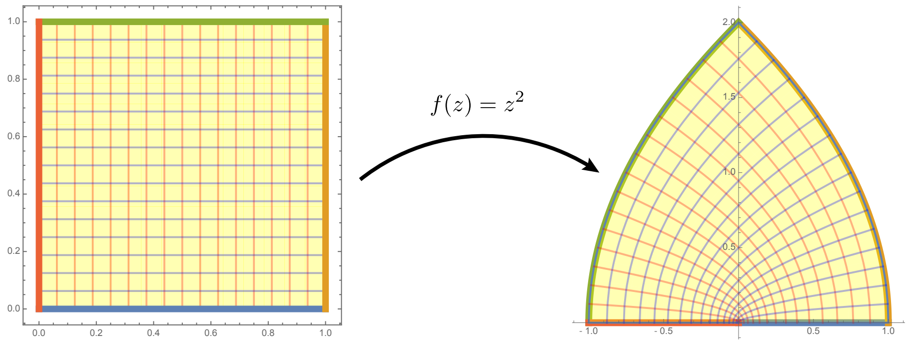
$$u=1-\frac{v^2}{4}$$

which is a parabola in the uv-plane joining $(1,0)$ and $(0,2)$

Likewise, one can figure out that f maps the straight-line L_3 joining $(0,0)$ and $(1,1)$ in the xy-plane to the parabola

$$u=\frac{v^2}{4}-1$$

joining $(0,2)$ and $(-1,0)$. It maps the straight-line L_4 joining $(0,1)$ and $(0,0)$ in the xy-plane to the straight-line joining $(-1,0)$ and $(0,0)$ in the uv-plane.



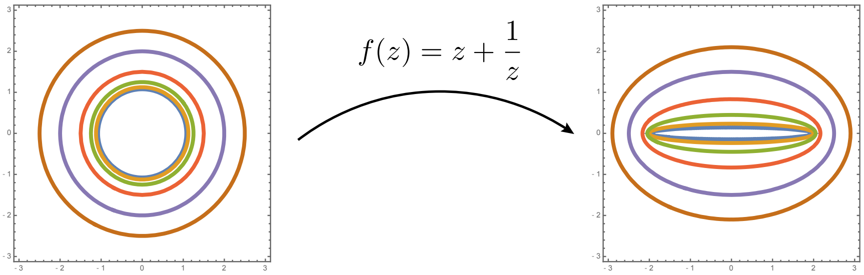
Example 2.4. Consider another example $f(z)=z+\dfrac{1}{z}$. We want to find the image of the circle $|z|=r_0$ under this map. Write $z=r_0e^{i\theta}$ where $0\le\theta\le 2\pi$, then we have:

$$\begin{aligned} f(z) &= r_0e^{i\theta}+\frac{1}{r_0e^{i\theta}} \\ &= r_0e^{i\theta}+\frac{1}{r_0}e^{-i\theta} \\ &= r_0\left(\cos\theta+i\sin\theta\right)+\frac{1}{r_0}\left(\cos\theta-i\sin\theta\right) \\ &= \underbrace{\left(r_0+\frac{1}{r_0}\right)\cos\theta}_{u}+i\underbrace{\left(r_0-\frac{1}{r_0}\right)\sin\theta}_{v}, \end{aligned}$$

which gives the following ellipse in the uv-plane (when $r_0\neq 1$):

$$\frac{u^2}{\left(r_0+\frac{1}{r_0}\right)^2}+\frac{v^2}{\left(r_0-\frac{1}{r_0}\right)^2}=1.$$

As $r_0\to 1$, the image of the circle $|z|=r_0$ degenerates to a straight-line on the real-axis:



Exercise 2.2. Describe and sketch the image of the semi-infinite strip:

$$\Sigma = \{z \in \mathbb{C} : 0 \leq \mathrm{Re}(z) \leq a \text{ and } \mathrm{Im}(z) \geq 0\}$$

under the map $f(z) = z^2$. Here a is a positive real number.

Exercise 2.3. Describe and sketch the image of the square:

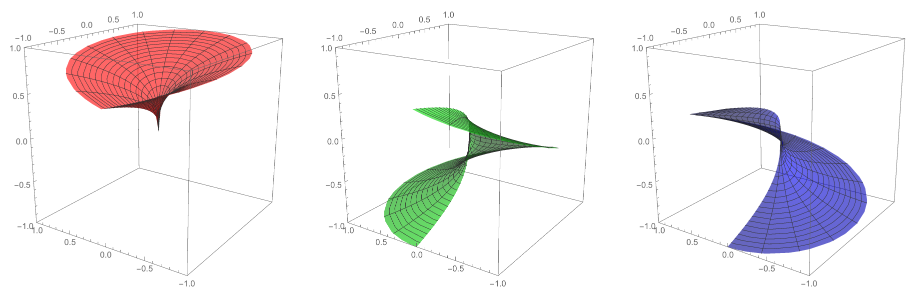
$$\Sigma = \{z \in \mathbb{C} : 0 \leq \mathrm{Re}(z) \leq a \text{ and } 0 \leq \mathrm{Im}(z) \leq b\}$$

under the map $f(z) = e^z$. Here a and b are positive real numbers.

2.1.5. Multi-Valued "Functions". In first year courses, we learned that a function needs to be well-defined, meaning that one input z gives exactly one output $f(z)$. In Complex Analysis, we often come across the term *multi-valued functions*, which are "functions" with more than one outputs for a given input. For example, the cubic root $f(z) = z^{\frac{1}{3}}$ is multi-valued as discussed in Section 1.1. Given any $z = re^{i\theta}$ with $r \neq 0$, there are three possible cubic roots:

$$z^{\frac{1}{3}} = \left(|z|\, e^{i\mathrm{Arg}(z)}\right)^{\frac{1}{3}} = \left\{|z|^{\frac{1}{3}}\, e^{\frac{\mathrm{Arg}(z)}{3}i}, |z|^{\frac{1}{3}}\, e^{\frac{\mathrm{Arg}(z)+2\pi}{3}i}, |z|^{\frac{1}{3}}\, e^{\frac{\mathrm{Arg}(z)+4\pi}{3}i}\right\}.$$

Therefore, the map $z \mapsto z^{\frac{1}{3}}$ is not rigorously a function from $\mathbb{C}$ to $\mathbb{C}$, but is a function with a *set* as the output. To visualize such a function, one can separate the graph into different *branches*. Below are the graphs of the real parts:



(a) $z \mapsto \mathrm{Re}\left(|z|^{\frac{1}{3}}\, e^{\frac{\mathrm{Arg}(z)}{3}i}\right)$ **(b)** $z \mapsto \mathrm{Re}\left(|z|^{\frac{1}{3}}\, e^{\frac{\mathrm{Arg}(z)+2\pi}{3}i}\right)$ **(c)** $z \mapsto \mathrm{Re}\left(|z|^{\frac{1}{3}}\, e^{\frac{\mathrm{Arg}(z)+4\pi}{3}i}\right)$

Each branch is then a well-defined function (each input gives a unique output). If we plot all three branches in a single graph, we obtain a beautiful surface below:

Another example of a multi-valued function is the argument map with domain $\mathbb{C}\backslash\{0\}$. It is denoted and defined as:

$$\arg(z) = \{\mathrm{Arg}(z) + 2k\pi i : k \in \mathbb{Z}\}.$$

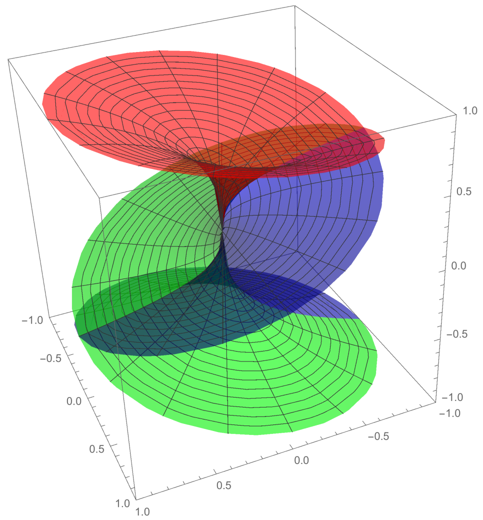
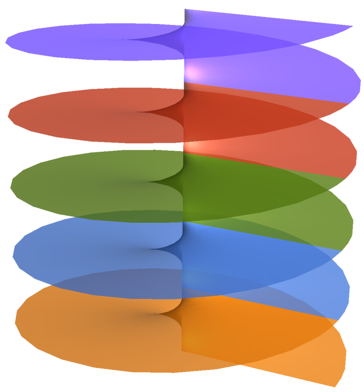


Figure 2.3. Three branches of $z \mapsto \text{Re}\left(z^{\frac{1}{3}}\right)$.

Recall that $\text{Arg}(z)$ is defined to be the unique angle $\theta_0 \in (-\pi, \pi]$ such that $z = |z|\, e^{i\theta_0}$. An element in the set $\arg(z)$ is any angle θ such that $z = |z|\, e^{i\theta}$. For example,

$$\arg(i) = \left\{\ldots, \frac{\pi}{2} - \pi, \frac{\pi}{2}, \frac{\pi}{2} + \pi, \frac{\pi}{2} + 2\pi, \ldots\right\}$$
$$\arg(1+i) = \left\{\ldots, \frac{\pi}{4} - \pi, \frac{\pi}{4}, \frac{\pi}{4} + \pi, \frac{\pi}{4} + 2\pi, \ldots\right\}$$

For each fixed $k \in \mathbb{Z}$, we regard $z \mapsto \text{Arg}(z) + 2k\pi i$ as a branch of the multi-valued function $\arg(z)$. Below is the graph of five of its branches:



The "infinite spiral" is call a *helicoid*, which is a *minimal surface* in Differential Geometry.

2.2. Complex Differentiability

In Real Analysis, we learned about continuity and differentiability of functions $F(x, y) : \mathbb{R}^2 \to \mathbb{R}^2$. The function F is said to be continuous at (x_0, y_0) if for any $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $\sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$, we have:

$$|F(x,y) - F(x_0, y_0)| < \varepsilon.$$

Analogously, a complex-valued function $f : \mathbb{C} \to \mathbb{C}$ is said to be continuous at z_0 if for any $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $|z - z_0| < \delta$, we have:

$$|f(z) - f(z_0)| < \varepsilon.$$

If we match $z = x + yi$ with (x, y), $z_0 = x_0 + y_0 i$ with (x_0, y_0), and $F(x, y)$ with $f(z)$, we can see the notions of real continuity and complex continuity are essentially the same.

However, we will see that *complex differentiability* is very distinguished from *real differentiability* (and this is why we have a separate course on Complex Analysis). In single-variable calculus, the derivative of a function $f : \mathbb{R} \to \mathbb{R}$ is defined to be:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}. \tag{2.1}$$

If such a limit exists, then we say $f'(x)$ is the derivative of f at x, and that f is said to be *(real) differentiable* at x.

Now for a complex-valued function $f : \mathbb{C} \to \mathbb{C}$, we define the derivative $f'(z)$ in an analogous way, except that we replace $h \to 0$ (where $h \in \mathbb{R}$) by $w \to 0$ where $w \in \mathbb{C}$:

$$f'(z) = \lim_{w \to 0} \frac{f(z+w) - f(z)}{w}. \tag{2.2}$$

If such a limit exists, then we say f is *complex differentiable* at z.

Example 2.5. Let $f(z) = z^n$ where n is a positive integer. Find $f'(z)$ from the definition of complex derivative, i.e. (2.2).

Solution

$$\begin{aligned}
f'(z) &= \lim_{w \to 0} \frac{(z+w)^n - z^n}{w} \\
&= \lim_{w \to 0} \frac{\overbrace{z^n + C_1^n z^{n-1} w + C_2^n z^{n-2} w^2 + \ldots + C_{n-1}^n z w^{n-1} + w^n}^{(z+w)^n} - z^n}{w} \\
&= \lim_{w \to 0} \frac{C_1^n z^{n-1} w + C_2^n z^{n-2} w^2 + \ldots + C_{n-1}^n z w^{n-1} + w^n}{w} \\
&= \lim_{w \to 0} \left(C_1^n z^{n-1} + C_2^n z^{n-2} w + \ldots + C_{n-1}^n z w^{n-2} + w^{n-1} \right) \\
&= C_1^n z^{n-1} + 0 + \ldots + 0 \\
&= n z^{n-1}.
\end{aligned}$$

A natural question: How is it the different from single-variable calculus? The key distinction is that (2.2) is a multivariable limit since w is a complex number! By writing $w = h + ki$, $z = x + yi$ and $f(x + yi) = u(x, y) + iv(x, y)$ where u, v, x, y, h, k are real, the limit in (2.2) can be rewritten as:

$$f'(x+yi) = \lim_{(h,k)\to(0,0)} \frac{\overbrace{(u(x+h,y+k)+iv(x+h,y+k))}^{f(z+w)=f((x+yi)+(h+ki))} - \overbrace{(u(x,y)+iv(x,y))}^{f(z)}}{h+ki}.$$

Past experience in MATH 2023/3033/3043 tell us that a multivariable limit exists less likely than a single variable limit, since we require the limit not only exist, but also equal when (h,k) approaches $(0,0)$ in all possible directions in the hk-plane. Therefore, there are many complex-valued functions, which look simple and elementary, are not complex differentiable. For instance, let $f(z) = |z|^2 = x^2 + y^2$, then:

$$\begin{aligned}\lim_{w\to 0} \frac{f(z+w)-f(z)}{w} &= \lim_{(h,k)\to(0,0)} \frac{(x+h)^2+(y+k)^2-(x^2+y^2)}{h+ki} \\ &= \lim_{(h,k)\to(0,0)} \frac{2hx+h^2+2ky+k^2}{h+ki}\end{aligned}$$

Along the path $\{k=0\}$, the limit goes to:

$$\lim_{\substack{h\to 0\\k=0}} \frac{2hx+h^2+2ky+k^2}{h+ki} = \lim_{h\to 0} \frac{2hx+h^2}{h} = 2x.$$

However, along the path $\{h=0\}$, the limit goes to:

$$\lim_{\substack{k\to 0\\h=0}} \frac{2hx+h^2+2ky+k^2}{h+ki} = \lim_{k\to 0} \frac{2ky+k^2}{ki} = -2yi.$$

In general, $2x \neq -2yi$ (unless $x = y = 0$) and so $f'(z)$ does not exist for almost all $x + yi \in \mathbb{C}$. Nonetheless, $f(z) = x^2 + y^2$ is a polynomial of x and y and so it is *real differentiable* everywhere on the xy-plane $\mathbb{R}^2$. From this example, we can see that complex differentiability is much more restrictive than real differentiability!

Exercise 2.4. Show that $f(z) = \operatorname{Re}(z)$ is *nowhere* complex differentiable on $\mathbb{C}$.

Exercise 2.5. Show that $f(z) = 1/z$ is complex differentiable at any $z \in \mathbb{C}\backslash\{0\}$.

Exercise 2.6. Find all possible constants $\alpha, \beta \in \mathbb{C}$ such that $f(z) = \alpha z + \beta\bar{z}$ is complex differentiable at every $z \in \mathbb{C}$.

2.2.1. Cauchy-Riemann Equations. In this section, we will examine a necessary condition for a function being complex differentiable. This necessary condition is called the Cauchy-Riemann equations (or Cauchy-Riemann relations in some textbooks).

Proposition 2.1. *If $f(z) = u(x,y) + iv(x,y)$ is complex differentiable at $z = x + yi$, then all first derivatives of u and v exist, and the following equations hold at (x,y):*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \tag{2.3}$$

This set of equations is called the Cauchy-Riemann equations.

Proof. Given that f is complex differentiable at $z = x + yi$, the following limit exists:

$$\lim_{(h,k)\to(0,0)} \frac{f((x+yi)+(h+ki)) - f(x+yi)}{h+ki} \tag{2.4}$$

and the values of the limit are equal as (h,k) approaches $(0,0)$ from any direction.

In particular, along the path $\{k=0\}$, the above limit equals to:

$$\begin{aligned}
&\lim_{h\to 0}\frac{f(x+yi+h)-f(x+yi)}{h}\\
&=\lim_{h\to 0}\frac{u(x+h,y)+iv(x+h,y)-u(x,y)-iv(x,y)}{h}\\
&=\lim_{h\to 0}\left[\left(\frac{u(x+h,y)-u(x,y)}{h}\right)+i\left(\frac{v(x+h,y)-v(x,y)}{h}\right)\right]\\
&=\frac{\partial u}{\partial x}(x,y)+i\frac{\partial v}{\partial x}(x,y).
\end{aligned}$$

Along the path $\{h=0\}$, the limit (2.4) equals to:

$$\begin{aligned}
&\lim_{k\to 0}\frac{f(x+yi+ki)-f(x+yi)}{ki}\\
&=\lim_{k\to 0}\frac{u(x,y+k)+iv(x,y+k)-u(x,y)-iv(x,y)}{ki}\\
&=\lim_{k\to 0}\left[\left(\frac{u(x,y+k)-u(x,y)}{ki}\right)+i\left(\frac{v(x,y+k)-v(x,y)}{ki}\right)\right]\\
&=\frac{1}{i}\left(\frac{\partial u}{\partial y}(x,y)+i\frac{\partial v}{\partial y}(x,y)\right)\\
&=\frac{\partial v}{\partial y}(x,y)-i\frac{\partial u}{\partial y}(x,y)
\end{aligned}$$

As the limits along these two paths must be equal, we get:

$$\frac{\partial u}{\partial x}(x,y)+i\frac{\partial v}{\partial x}(x,y)=\frac{\partial v}{\partial y}(x,y)-i\frac{\partial u}{\partial y}(x,y)$$

which proves the desired result (2.3). □

It is notable that the converse of Proposition 2.1 is *not* true. If $f(z)=u(x,y)+iv(x,y)$ satisfies the Cauchy-Riemann equations (2.3) at (x,y), it may not be true that f is complex differentiable at $z=x+yi$. Here is one counter-example:

$$f(z)=\sqrt{|\mathrm{Re}(z)\mathrm{Im}(z)|}=\sqrt{|xy|}.$$

Then $u(x,y)=\sqrt{|xy|}$ and $v(x,y)=0$. We claim that $\frac{\partial u}{\partial x}(0,0)=\frac{\partial u}{\partial y}(0,0)=0$:

$$\begin{aligned}
\frac{\partial u}{\partial x}(0,0)&=\lim_{h\to 0}\frac{u(0+h,0)-u(0,0)}{h}\\
&=\lim_{h\to 0}\frac{0-0}{h}=0.
\end{aligned}$$

Similarly, one can also show $\frac{\partial u}{\partial y}(0,0)=0$. It is obvious that $\frac{\partial v}{\partial x}=\frac{\partial v}{\partial y}=0$. Therefore, the Cauchy-Riemann equation (2.3) holds at $(x,y)=(0,0)$.

However, the function $f(z)$ is *not* complex differentiable at $z=0+0i$. It is because when computing the limit (2.4) at $(x,y)=(0,0)$, we get:

$$\lim_{(h,k)\to(0,0)}\frac{f((0+0i)+(h+ki))-f(0+0i)}{h+ki}=\lim_{(h,k)\to(0,0)}\frac{\sqrt{|hk|}}{h+ki}.$$

Along the path $\{h = 0\}$, the limit equals to 0. However, along the path $\{h = k\}$, the limit becomes:

$$\lim_{h\to 0} \frac{\sqrt{h^2}}{h + hi} = \lim_{h\to 0} \frac{|h|}{h(1+i)}$$

which does not exist as it approaches $\frac{1}{1+i}$ as $h \to 0^+$, while it approaches $-\frac{1}{1+i}$ as $h \to 0^-$.

This $f(z)$ is one example that the Cauchy-Riemann equation holds at a point, but the function is not complex differentiable at that point. Fortunately, if we assume *further* that $u(x, y)$ and $v(x, y)$ are (real) differentiable functions, then the Cauchy-Riemann equations imply complex differentiability.

Proposition 2.2. *Let $f : \Omega \to \mathbb{C}$ be a complex-valued function defined on an open set $\Omega \subset \mathbb{C}$ such that $u(x, y) := \mathrm{Re} f(x + yi)$ and $v(x, y) := \mathrm{Im} f(x + yi)$ are both (real) differentiable functions on Ω. If the Cauchy-Riemann equations* (2.3) *hold at $(x_0, y_0) \in \Omega$, then f is complex differentiable at $z_0 = x_0 + y_0 i$.*

Proof. We only sketch the outline and leave the detail for readers (see Exercise 2.7). We define:

$$E_1(x,y) = u(x,y) - u(x_0,y_0) - \left.\frac{\partial u}{\partial x}\right|_{(x_0,y_0)} (x - x_0) - \left.\frac{\partial u}{\partial y}\right|_{(x_0,y_0)} (y - y_0)$$

$$E_2(x,y) = v(x,y) - v(x_0,y_0) - \left.\frac{\partial v}{\partial x}\right|_{(x_0,y_0)} (x - x_0) - \left.\frac{\partial v}{\partial y}\right|_{(x_0,y_0)} (y - y_0).$$

The proof then consists of three major steps:

(1) Since both u and v are differentiable at (x_0, y_0), we have:

$$\lim_{(x,y)\to(x_0,y_0)} \frac{E_1(x,y)}{|z - z_0|} = \lim_{(x,y)\to(x_0,y_0)} \frac{E_2(x,y)}{|z - z_0|} = 0.$$

(2) Derive using the Cauchy-Riemann equations that:

$$\frac{f(z) - f(z_0)}{z - z_0} = \left.\frac{\partial u}{\partial x}\right|_{(x_0,y_0)} + i \left.\frac{\partial v}{\partial x}\right|_{(x_0,y_0)} + \frac{E_1(x,y) + iE_2(x,y)}{z - z_0}$$

(3) Finally, by taking $z \to z_0$ and using the results from step (1), we can deduce:

$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \left.\frac{\partial u}{\partial x}\right|_{(x_0,y_0)} + i \left.\frac{\partial v}{\partial x}\right|_{(x_0,y_0)}.$$

□

Exercise 2.7. Complete the detail of the proof of Proposition 2.2.

Remark 2.3. The condition of (real) differentiability is sometimes difficult to verify. Fortunately, if u and v are C^1 on Ω, i.e. $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ all exist and are continuous on Ω, then from MATH 3033/3043 we learned that u and v are then (real) differentiable on Ω.

In short, combining Propositions 2.1 and 2.2, for any functions $f = u + iv$ such that u and v are (real) differentiable on an open domain Ω, complex differentiability is *equivalent* to the Cauchy-Riemann equations. Many functions we will encounter are (real) differentiable. Therefore, to show such a function is complex differentiable it suffices to verify the Cauchy-Riemann equations. Let's look at some examples.

Example 2.6. Determine all z at which the following functions are complex differentiable.

(a) $f(z) = \overline{z}$

(b) $f(z) = |z|^2$

(c) $f(z) = e^z$

Solution

(a) $f(z) = \overline{z} = x - yi$. Hence $u = x$ and $v = -y$, which are clearly C^1.

$$\frac{\partial u}{\partial x} = 1 \qquad \frac{\partial u}{\partial y} = 0$$
$$\frac{\partial v}{\partial x} = 0 \qquad \frac{\partial v}{\partial x} = -1$$

Cauchy-Riemann equations do not hold at every point, hence f is *not* complex differentiable at any point $z \in \mathbb{C}$.

(b) $f(z) = |z|^2 = x^2 + y^2$. Hence $u = x^2 + y^2$ and $v = 0$, which are clearly C^1.

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial u}{\partial y} = 2y$$
$$\frac{\partial v}{\partial x} = 0 \qquad \frac{\partial v}{\partial y} = 0$$

Cauchy-Riemann equations hold if and only if $(x, y) = (0, 0)$. Therefore, f is complex differentiable at $z = 0$ only.

(c) $f(z) = e^z = e^x \cos y + ie^x \sin y$. Hence $u = e^x \cos y$ and $v = e^x \sin y$, which are C^1 functions

$$\frac{\partial u}{\partial x} = e^x \cos y \qquad \frac{\partial u}{\partial y} = -e^x \sin y$$
$$\frac{\partial v}{\partial x} = e^x \sin y \qquad \frac{\partial v}{\partial y} = e^x \cos y$$

which are all continuous. Clearly the Cauchy-Riemann equations hold at every (x, y), hence by Proposition 2.2, the function $f(z) = e^z$ is complex differentiable at every $z \in \mathbb{C}$.

From now on, we will use a more *professional* term to describe complex differentiable functions:

Definition 2.4 (Holomorphic Functions)**.** Let Ω be an open set of $\mathbb{C}$. A complex-valued function f is said to be *holomorphic on* Ω if f is complex differentiable at every $z \in \Omega$. A function which is holomorphic on $\mathbb{C}$ is said to be an *entire function*.

Remark 2.5. It is a *custom* to say a function is holomorphic *on* an open domain while saying a function is complex differentiable *at* a point. Therefore, whenever we say holomorphic on a set Ω, the set Ω must be open (and non-empty). We never say a function is holomorphic on the real-axis in $\mathbb{C}$, since the real-axis is not an open set. Instead, we should say the function is complex differentiable at every point on the real-axis.

Exercise 2.8. Determine all z's in the complex plane at which the following functions are complex differentiable:

(a) $f(z) = 1/z$

(b) $f(z) = z\,|z|^2$

(c) $f(z) = z^2$

Exercise 2.9. Consider the function:

$$f(z) = \begin{cases} e^{-1/z^4} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$

Show that the Cauchy-Riemann equations hold everywhere on $\mathbb{C}$, but f is not complex differentiable at $z = 0$.

Exercise 2.10. Suppose $f(z)$ is complex differentiable at every $z \in \mathbb{C}$. Prove that *any one* of the following conditions imply that f is constant:

(i) $\mathrm{Re}(f)$ is constant.

(ii) $\mathrm{Im}(f)$ is constant.

(iii) $|f|$ is constant.

Exercise 2.11. Find a function $f : \mathbb{C} \to \mathbb{C}$ which is complex differentiable everywhere in $\mathbb{C}$ such that

$$\mathrm{Re}(f(x + yi)) = y^3 - 2x^2y.$$

Exercise 2.12. Show that the Cauchy-Riemann equations (2.3) for the function $f = u + iv$ is equivalent to:

$$\frac{\partial f}{\partial y} = i\frac{\partial f}{\partial x}.$$

Give a *geometric* interpretation of this result.

Exercise 2.13. Show that any holomorphic function $f = u + iv : \Omega \to \mathbb{C}$ satisfies

$$|\nabla u| = |\nabla v| = \left|\frac{\partial f}{\partial x}\right| = \left|\frac{\partial f}{\partial y}\right| = |f'(z)| = \sqrt{\det \frac{\partial(u, v)}{\partial(x, y)}}$$

on the domain Ω.

Exercise 2.14. Fix a complex number w such that $|w| < 1$. Consider the map $f : B_1(0) \to \mathbb{C}$ defined by:

$$f(z) = \frac{w - z}{1 - \overline{w}z}.$$

Show that:

(a) f is well-defined on $B_1(0)$;

(b) f holomorphic on $B_1(0)$;

(c) The image of f is $B_1(0)$;

(d) f is bijective as a map $f : B_1(0) \to B_1(0)$.

Exercise 2.15. Let $f : \mathbb{C} \to \mathbb{C}$ be the function defined by:

$$f(z) = \begin{cases} z^2 & \text{if } \mathrm{Re}(z) \in \mathbb{Q} \text{ and } \mathrm{Im}(z) \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$$

Show that f is complex differentiable at $z = 0$, but it is not holomorphic on any open set containing 0.

Example 2.7. Find the largest open subset $\Omega \subset \mathbb{C}$ such that the function:

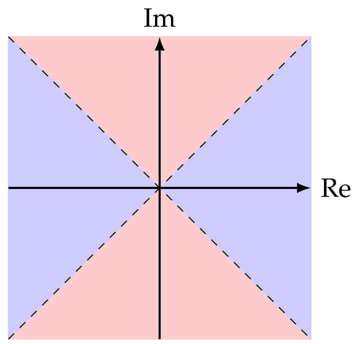
$$f(x + yi) = \left|x^2 - y^2\right| + 2xyi$$

is holomorphic on Ω.

Solution

First we divide the complex plane $\mathbb{C}$ into open regions:

$$U = \{x + yi : x^2 - y^2 > 0\} \text{ and } V = \{x + yi : x^2 - y^2 < 0\}.$$



On U, we have $f(z) = (x^2 - y^2) + 2xyi$ which is clearly C^1 on U. It clearly satisfies the Cauchy-Riemann equations on U (straight-forward computations, left as an exercise). Hence f is holomorphic on U.

On the other hand, we have $f(z) = (y^2 - x^2) + 2xyi$ on V, which does not satisfy the Cauchy-Riemann equations (straight-forward computations). Therefore, f is not complex differentiable at any $z_0 \in V$.

The largest open set Ω on which f is holomorphic is U, since any *open* set which is larger than Ω must contain some point in V.

Exercise 2.16. Find the largest open subset $\Omega \subset \mathbb{C}$ such that the function:

$$f(x + yi) = |x| + |y|\, i$$

is holomorphic on Ω.

2.2.2. Geometric Interpretation of Cauchy-Riemann Equations. Recall from MATH 2023 that $\nabla u(a, b)$, where $u(x, y)$ is a real-valued functions of (x, y), is a normal vector to the level curve of u at (a, b) whenever $\nabla u(a, b) \neq 0$.

Now consider a complex-valued function $f = u + iv$. If f is holomorphic on its domain Ω, then the Cauchy-Riemann equations, i.e. $u_x = v_y$ and $u_y = -v_x$ hold on Ω. As result, we get easily see that:

$$\nabla u \cdot \nabla v = \left(\frac{\partial u}{\partial x}\mathbf{i} + \frac{\partial u}{\partial y}\mathbf{j}\right) \cdot \left(\frac{\partial v}{\partial x}\mathbf{i} + \frac{\partial v}{\partial y}\mathbf{j}\right) = \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} = 0.$$

Therefore, ∇u and ∇v are perpendicular to each other provided that they are both non-zero. Geometrically speaking, it means that the level sets of u and v are perpendicular!

There are more we can say about ∇u and ∇v. Define the matrix:

$$\mathsf{J} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Given any vector $\mathbf{x} \in \mathbb{R}^2$, the product $\mathsf{J}(\mathbf{x})$ is the vector obtained by rotating $\mathbf{x}$ counter-clockwisely by $\frac{\pi}{2}$. The Cauchy-Riemann equations can be rewritten as:

$$\begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} -u_y \\ u_x \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix}.$$

As a result, we have $\nabla v = \mathsf{J}(\nabla u)$. It means that for a holomorphic function f, the vector ∇v can be obtained by rotating ∇u counter-clockwisely by $\frac{\pi}{2}$.

2.2.3. Conformal Mappings. Another important geometric significance of holomorphic functions is that it preserves angles. Let $\gamma_1(t) = x_1(t) + iy_1(t)$ and $\gamma_2(t) = x_2(t) + iy_2(t)$ be two C^1 curves in the complex plane, and assume that they intersect at $t = 0$, i.e. $\gamma_1(0) = \gamma_2(0) =: z_0$, then the angle between the curves at the point z_0 is measured by the angle between the tangent vectors $\gamma_1'(0)$ and $\gamma_2'(0)$.

Now consider a map $f : \mathbb{C} \to \mathbb{C}$. The images of the curves γ_1 and γ_2 under the map f are the curves $f \circ \gamma_1(t)$ and $f \circ \gamma_2(t)$, and hence that angle between them at the point $f(z_0)$ is measured by the angle between vectors $(f \circ \gamma_1)'(0)$ and $(f \circ \gamma_2)'(0)$. We will show that if f is complex differentiable at z_0 and that $f'(z_0) \neq 0$, then the angle between $\gamma_1'(0)$ and $\gamma_2'(0)$, is the same as that between $(f \circ \gamma_1)'(0)$ and $(f \circ \gamma_2)'(0)$.

First, we leave the following elementary fact as an exercise:

Exercise 2.17. Let $z_1, z_2, w_1, w_2 \in \mathbb{C}$ such that $z_2 \neq 0$ and $w_2 \neq 0$. Suppose we have $\dfrac{z_1}{z_2} = \dfrac{w_1}{w_2}$. Show that the angle between z_1 and z_2, is the same as that between w_1 and w_2.

Proposition 2.6. *Let $\gamma_k(t) = x_k(t) + iy_k(t)$ where $k = 1, 2$ be two C^1 curves in $\mathbb{C}$ which intersect at $t = 0$ at the point $z_0 \in \mathbb{C}$. Suppose $f : B_\varepsilon(z_0) \to \mathbb{C}$ is complex differentiable at z_0 and $f'(z_0) \neq 0$ then we have:*

$$\frac{(f \circ \gamma_1)'(0)}{(f \circ \gamma_2)'(0)} = \frac{\gamma_1'(0)}{\gamma_2'(0)}.$$

As a result, the angle between γ_1 and γ_2 at z_0 is preserved under f. See Figure 2.4.

Proof. Let $f = u + iv$ (as usual). Recall from Exercise 2.12 that the Cauchy-Riemann equation is equivalent to

$$\frac{\partial f}{\partial y} = i\frac{\partial f}{\partial x}.$$

Using this and the chain rule, we have:

$$\begin{aligned} \frac{d}{dt}(f \circ \gamma_1) &= \frac{\partial f}{\partial x}\frac{dx_1}{dt} + \frac{\partial f}{\partial y}\frac{dy_1}{dt} \\ (f \circ \gamma_1)'(0) &= \left.\frac{\partial f}{\partial x}\right|_{z_0} x_1'(0) + i \left.\frac{\partial f}{\partial x}\right|_{z_0} y_1'(0) \\ &= \left.\frac{\partial f}{\partial x}\right|_{z_0} \gamma_1'(0). \end{aligned}$$

Likewise, we have:

$$(f \circ \gamma_2)'(0) = \left.\frac{\partial f}{\partial x}\right|_{z_0} \gamma_2'(0)$$

By Exercise 2.13, $|f'(z_0)| = \left|\frac{\partial f}{\partial x}(z_0)\right|$. Given $f'(z_0) \neq 0$, we also have $\frac{\partial f}{\partial x}(z_0) \neq 0$ and hence:

$$\frac{(f \circ \gamma_1)'(0)}{(f \circ \gamma_2)'(0)} = \frac{f_x(z_0)\,\gamma_1'(0)}{f_x(z_0)\,\gamma_2'(0)} = \frac{\gamma_1'(0)}{\gamma_2'(0)}$$

as desired. □

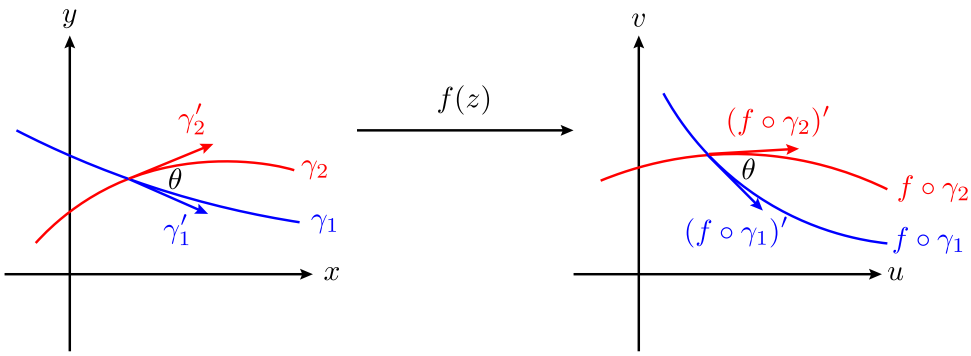


Figure 2.4. A holomorphic map preserves angles

2.3. Logarithmic and Trigonometric Functions

In the real world, the exponential function $x \mapsto e^x$ is injective, and hence it makes perfect sense to define its inverse function, which is known as the logarithm $x \mapsto \log x$.

However, in the complex world, the exponential function is no longer injective. In particular, $e^{z+2k\pi i} = e^z$ for any integer k. Therefore, $z \mapsto e^z$ an infinity-to-one map. Therefore, just like the n-th root function $z \mapsto z^{1/n}$ and the argument function arg, the complex logarithm is *multi-valued*.

Another type of functions that is conceptually different from the real world is trigonometric functions such as sin, cos and tan. In the real world, $\sin\theta$ and $\cos\theta$ are defined in a geometric way using a right-angled triangle of angle θ. However, if z is a complex number, it doesn't make any sense to say a triangle with a "complex" angle z, and so the complex trigonometric functions such as $\sin z$, $\cos z$ and $\tan z$ are defined in a different way.

In this section, we will study the logarithmic and trigonometric functions in detail.

2.3.1. Logarithmic Functions.

As discussed before, the complex exponential function $f(z) = e^z$ is not injective, since for any integer k, we have:

$$e^{z+2k\pi i} = e^z e^{2k\pi i} = e^z(\cos 2k\pi + i\sin 2k\pi) = e^z(1+0i) = e^z.$$

As such, the complex logarithm is multi-valued and so for each $z \neq 0$, $\log z$ is a set rather than a single number.

Definition 2.7 (Complex Logarithm)**.** Given any $z \in \mathbb{C}$ and $z \neq 0$, we define the *complex logarithm* to be the set:

$$\log z := \{w \in \mathbb{C} : e^w = z\}.$$

Example 2.8. If w is a complex number such that $e^w = 1$, by writing $w = u + vi$, then we have:

$$1 = e^{u+iv} = e^u e^{iv} = e^u(\cos v + i\sin v) = e^u\cos v + ie^u\sin v.$$

The only possible solutions for (u, v) are $u = 0$ and $v = 2k\pi$ where $k \in \mathbb{Z}$. Therefore, we get:

$$\log 1 = \{w \in \mathbb{C} : e^w = 1\} = \{2k\pi i : k \in \mathbb{Z}\} = \{\cdots, -4\pi i, -2\pi i, 0, 2\pi i, 4\pi i, \cdots\}.$$

Let's derive a general formula for $\log z$. Given any $z \neq 0$, we first express it in polar form $z = |z|\, e^{i\operatorname{Arg}(z)}$. Then, $w = u + iv$ satisfies $e^w = z$ if and only if $e^{u+iv} = |z|\, e^{i\operatorname{Arg}(z)}$:

$$e^u e^{iv} = |z|\, e^{i\operatorname{Arg}(z)} \iff u = \ln|z| \text{ and } v = \operatorname{Arg}(z) + 2k\pi i \text{ where } k \in \mathbb{Z}.$$

As a result, we have $w = \ln|z| + i(\operatorname{Arg}(z) + 2k\pi)$ for any $k \in \mathbb{Z}$, and using set notations:

$$\log z = \{\ln|z| + i(\operatorname{Arg}(z) + 2k\pi) : k \in \mathbb{Z}\}.$$

Remark 2.8. To avoid confusion, from now on we will denote ln as the real, single-valued logarithm, and use log for the complex multi-valued logarithm.

Recall that $\arg(z) = \{\operatorname{Arg}(z) + 2k\pi : k \in \mathbb{Z}\}$, we can also write:

$$\log z = \ln|z| + i\arg(z)$$

for any $z \neq 0$. For example:

$$\begin{aligned}
\log(1) &= \ln|1| + i\arg(1) = \{0 + i(\underbrace{\mathrm{Arg}(1)}_{0} + 2k\pi) : k \in \mathbb{Z}\} = \{2k\pi i : k \in \mathbb{Z}\} \\
\log(2i) &= \ln|2i| + i\arg(2) = \{\ln 2 + i(\underbrace{\mathrm{Arg}(2i)}_{\frac{\pi}{2}} + 2k\pi) : k \in \mathbb{Z}\} \\
&= \left\{\ln 2 + \left(\frac{\pi}{2} + 2k\pi\right) i : k \in \mathbb{Z}\right\} \\
\log(-1) &= \ln|-1| + i\arg(-1) = \{0 + i(\underbrace{\mathrm{Arg}(-1)}_{\pi} + 2k\pi) : k \in \mathbb{Z}\} \\
&= \{(2k+1)\pi i : k \in \mathbb{Z}\}
\end{aligned}$$

Remark 2.9. Recall that for real logarithms, $\ln(a)$ is undefined if $a < 0$. However, for complex logarithms, $\log(a)$ is defined even for $a < 0$ since e^z can be a negative real number. On the other hand, both $\log 0$ and $\ln 0$ are undefined, since $e^z \neq 0$ for any $z \in \mathbb{C}$.

Recall that for the argument function $\arg(z)$, we define the *principal argument* $\mathrm{Arg}(z)$ as the unique angle $\theta \in (-\pi, \pi]$ such that $z = |z|\, e^{i\theta}$. Similarly, we define the *principal logarithm* to be:

Definition 2.10 (Principal Logarithm)**.** For any $z \neq 0$, we define its *principal logarithm* to be:

$$\mathrm{Log}(z) := \ln|z| + i\mathrm{Arg}(z).$$

For example, we have:

$$\begin{aligned}
\mathrm{Log}(1) &= 1 \\
\mathrm{Log}(2i) &= \ln 2 + \frac{\pi}{2} i \\
\mathrm{Log}(-1) &= \pi i
\end{aligned}$$

Exercise 2.18. Find $\log(z)$ and $\mathrm{Log}(z)$ for each of the following z's:

(a) $z = -2$

(b) $z = 1 + \sqrt{3}i$

(c) $z = -i$

Exercise 2.19. Give an example of z_1 and z_2 such that

$$\mathrm{Log}(z_1 z_2) \neq \mathrm{Log}(z_1) + \mathrm{Log}(z_2).$$

Exercise 2.20. For any two subsets S and T of $\mathbb{C}$, we define the sum of the two sets to be:

$$S + T := \{s + t : s \in S \text{ and } t \in T\}.$$

Show that for any $z_1, z_2 \neq 0$, we have:

$$\begin{aligned}
\arg(z_1 z_2) &= \arg(z_1) + \arg(z_2) \\
\log(z_1 z_2) &= \log(z_1) + \log(z_2).
\end{aligned}$$

Exercise 2.21. Let $z \neq 0$, and $n \in \mathbb{N}$.

(a) Show that:

$$\log(z^{1/n}) = \left\{ \frac{1}{n} \ln|z| + \frac{\operatorname{Arg}(z) + 2(pn+k)\pi}{n} i : p \in \mathbb{Z}, k \in \{0, 1, \ldots, n-1\} \right\}.$$

(b) Hence, show that $\log(z^{1/n}) = \dfrac{1}{n} \log z$.

2.3.2. Complex Derivatives of Logarithms. Recall that if $f(z)$ is holomorphic, then from the proof of Proposition 2.1, we have $f'(z) = \dfrac{\partial f}{\partial x}$. We will use this fact to compute the complex derivatives of both e^z and $\log(z)$.

For the exponential function $f(z) = e^z$, we have already shown in Example 2.6 that it is holomorphic on $\mathbb{C}$. Its complex derivative is given by:

$$\begin{aligned} \frac{d}{dz} e^z &= \frac{\partial f}{\partial x} \\ &= \frac{\partial}{\partial x}(e^x \cos y + i e^x \sin y) \\ &= e^x \cos y + i e^x \sin y \\ &= e^z. \end{aligned} \tag{2.5}$$

We next compute $\dfrac{d}{dz} \log(z)$. Note that $\log(z)$ is multi-valued. Differentiating $\log(z)$ basically means differentiating each branch of $\log(z)$ individually. We first verify that every branch of $\log(z)$ is holomorphic on $\mathbb{C} \backslash \{x + 0i : x \leq 0\}$:

$$\log(z) = \{\ln|z| + i(\operatorname{Arg}(z) + 2k\pi) : k \in \mathbb{Z}\}$$

Let $u(x, y) = \ln|z| = \frac{1}{2} \ln(x^2 + y^2)$ and $v(x, y) = \operatorname{Arg}(z) + 2k\pi$. We leave it as an exercise for readers to verify that on $\mathbb{C} \backslash (-\infty, 0]$:

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{x}{x^2 + y^2} & \frac{\partial u}{\partial y} &= \frac{y}{x^2 + y^2} \\ \frac{\partial v}{\partial x} &= -\frac{y}{x^2 + y^2} & \frac{\partial v}{\partial y} &= \frac{x}{x^2 + y^2}. \end{aligned}$$

Clearly, they are all continuous and satisfy the Cauchy-Riemann equation. By Proposition 2.2, each branch $\ln|z| + i(\operatorname{Arg}(z) + 2k\pi)$ is holomorphic on $\mathbb{C} \backslash \{x + 0i : x \leq 0\}$, and so:

$$\frac{d}{dz}(\ln|z| + i(\operatorname{Arg}(z) + 2k\pi)) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{x - yi}{x^2 + y^2} = \frac{\bar{z}}{z\bar{z}} = \frac{1}{z}.$$

Therefore, every branch of $\log(z)$ has the same complex derivative, so we may simply write:

$$\frac{d}{dz} \log(z) = \frac{1}{z}. \tag{2.6}$$

2.3.3. Complex Powers. For any $z, w \in \mathbb{C}$ where $z \neq 0$, we define *z to the power of w* as follows:

$$z^w := e^{w \log z}. \tag{2.7}$$

For example, we have

$$i^i = e^{i \log(i)} = e^{i(\ln|i| + i \arg(i))} = \{e^{i(i(\frac{\pi}{2} + 2k\pi))} : k \in \mathbb{Z}\} = \{e^{-\frac{\pi}{2} - 2k\pi} : k \in \mathbb{Z}\}.$$

Maybe to your surprise, i^i is real-valued!

Using (2.7), one can recover the n-th root formula stated in Definition 1.9. For any complex number $a \neq 0$, according to (2.7), we have:

$$\begin{aligned}
a^{\frac{1}{n}} &= e^{\frac{1}{n}\log a} \\
&= \left\{ e^{\frac{1}{n}(\ln|a| + (\operatorname{Arg}(a)+2k\pi)i)} : k \in \mathbb{Z} \right\} \\
&= \left\{ e^{\frac{1}{n}\ln|a|} e^{\frac{\operatorname{Arg}(a)+2k\pi}{n} i} : k \in \mathbb{Z} \right\} \\
&= \left\{ \sqrt[n]{|a|} \left(\cos \frac{\operatorname{Arg}(a)+2k\pi}{n} + i \sin \frac{\operatorname{Arg}(a)+2k\pi}{n} \right) : k \in \mathbb{Z} \right\} \\
&= \left\{ \sqrt[n]{|a|} \left(\cos \frac{\operatorname{Arg}(a)+2k\pi}{n} + i \sin \frac{\operatorname{Arg}(a)+2k\pi}{n} \right) : k = 0, 1, 2, \ldots, n-1 \right\}
\end{aligned}$$

Using the chain rule, one can derive the differentiation formula for complex powers. Regard w as a fixed complex numbers, we can derive:

$$\begin{aligned}
\frac{d}{dz} z^w &= \frac{d}{dz} e^{w \log z} = e^{w \log z} \frac{d}{dz} w \log z \\
&= z^w \cdot \frac{w}{z} = w z^{w-1}
\end{aligned} \tag{2.8}$$

$$\begin{aligned}
\frac{d}{dz} w^z &= \frac{d}{dz} e^{z \log w} = e^{z \log w} \frac{d}{dz} z \log w \\
&= w^z \log w
\end{aligned} \tag{2.9}$$

Exercise 2.22. Find the following complex powers:

(a) $(1+i)^{1-i}$

(b) 2^{1-i}

(c) 3^{3i}

Exercise 2.23. Compute the complex derivative of each function below:

(a) $f(z) = (1-z)^{1/2}$

(b) $f(z) = (1-z^2)^{1/3}$

Exercise 2.24. Fix a non-zero complex number w. On what domains are the following functions holomorphic?

(a) $f(z) = w^z$

(b) $g(z) = z^w$

For complex logarithms, we use $\log z$ and $\operatorname{Log} z$ to distinguish the multi-valued logarithm, or the principal branch of logarithm. Similarly, for a complex power z^w, which is defined via logarithms, we can also define its principal branch by $e^{w\operatorname{Log}(z)}$. For instance, the principal branch of i^i is $e^{-\frac{\pi}{2}}$. Unlike logarithms and arguments, we do not introduce a new symbol to denote the principal branch of z^w. With abuse of notations, we may sometimes simply write, for instance $i^i = e^{-\frac{\pi}{2}}$, to mean $e^{-\frac{\pi}{2}}$ is the principal value of i^i. It may cause ambiguity, but such an ambiguity will cause less nuisance if we state clearly in the context whether z^w means a multi-valued power or the principal branch.

2.3.4. Trigonometric Functions. From Euler's formula (1.3), we saw that:

$$e^{ix} = \cos x + i \sin x \quad \text{and} \quad e^{-ix} = \cos x - i \sin x$$

for any $x \in \mathbb{R}$. Hence by writing $\sin x$ and $\cos x$ in terms of the exponentials, we get:

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \qquad \cos x = \frac{e^{ix} + e^{-ix}}{2}.$$

When z is a complex number, it does not make sense to define $\sin z$ and $\cos z$ by regarding z as an "angle". Thanks for the above identities, we *define* the complex sin and cos functions as:

Definition 2.11 (Complex Sine and Cosine)**.** For any $z \in \mathbb{C}$, we define:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}.$$

Using the chain rule, we get:

$$\frac{d}{dz} e^{iz} = ie^{iz} \qquad \frac{d}{dz} e^{-iz} = -ie^{-iz},$$

and so it is easy to verify that both $\sin z$ and $\cos z$ are entire functions and their derivatives are:

$$\frac{d}{dz} \sin z = \cos z \qquad \frac{d}{dz} \cos z = -\sin z. \tag{2.10}$$

The other trigonometric functions are defined similarly as in the real case:

$$\tan z = \frac{\sin z}{\cos z} \qquad \sec z = \frac{1}{\cos z}$$
$$\cot z = \frac{\cos z}{\sin z} \qquad \csc z = \frac{1}{\sin z}$$

Using the product and quotient rules, one can easily derive that:

$$\frac{d}{dz} \tan z = \sec^2 z \qquad \frac{d}{dz} \sec z = \sec z \tan z \tag{2.11}$$

$$\frac{d}{dz} \cot z = -\csc^2 z \qquad \frac{d}{dz} \csc z = -\csc z \cot z \tag{2.12}$$

Exercise 2.25. Prove (2.10), (2.11) and (2.12).

Exercise 2.26. Prove that for any $z, z_1, z_2 \in \mathbb{C}$:

(a) $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$
(b) $\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$
(c) $\sin^2 z + \cos^2 z = 1$
(d) $1 + \tan^2 z = \sec^2 z$
(e) $1 + \cot^2 z = \csc^2 z$
(f) $\sin \bar{z} = \overline{\sin z}$
(g) $\cos \bar{z} = \overline{\cos z}$
(h) $\sin(-z) = -\sin z$
(i) $\cos(-z) = \cos z$

2.3.5. Inverse Trigonometric Functions. It is well-known that trigonometric functions are not injective even in the real case. The real inverse sine $\sin^{-1} x$ is *defined* to be the number $\theta \in (-\pi/2, \pi/2]$ such that $x = \sin\theta$, making $\sin^{-1}$ a well-defined, single-valued function of x.

In the complex world, we accept multi-valued functions. Therefore, given any $z \in \mathbb{C}$, we regard $\sin^{-1} z$ to be:

$$\sin^{-1} z := \{w \in \mathbb{C} : \sin w = z\}.$$

It is possible to rewrite $\sin^{-1} z$ using complex logarithms. Suppose $\sin w = z$, let's try to solve for w in terms of z. By definition of $\sin z$, we have:

$$\frac{e^{iw} - e^{-iw}}{2i} = z.$$

By rearrangement, we get:

$$\begin{aligned}
e^{iw} - e^{-iw} - 2iz &= 0 \\
e^{iw}\left(e^{iw} - e^{-iw} - 2iz\right) &= 0 \\
\left(e^{iw}\right)^2 - 2iz\left(e^{iw}\right) - 1 &= 0
\end{aligned}$$

which can be regarded as a quadratic equation of e^{iw}. Solving it, we get:

$$e^{iw} = \frac{2iz + \left((-2iz)^2 - 4(1)(-1)\right)^{1/2}}{2} = iz + \left(1 - z^2\right)^{1/2}.$$

Note that $\left(1 - z^2\right)^{1/2}$ is multi-valued and it is not necessary to write $iz \pm \left(1 - z^2\right)^{1/2}$. From this, we get:

$$iw = \log\left(iz + \left(1 - z^2\right)^{1/2}\right) \quad \Longrightarrow \quad w = -i\log\left(iz + \left(1 - z^2\right)^{1/2}\right).$$

As a result, we have:

$$\sin^{-1} z = -i\log\left(iz + \left(1 - z^2\right)^{1/2}\right).$$

For example,

$$\begin{aligned}
\sin^{-1}(1) &= -i\log(i) \\
\sin^{-1}(i) &= -i\log(-1 + 2^{1/2}) = -i\log\left(-1 \pm \sqrt{2}\right)
\end{aligned}$$

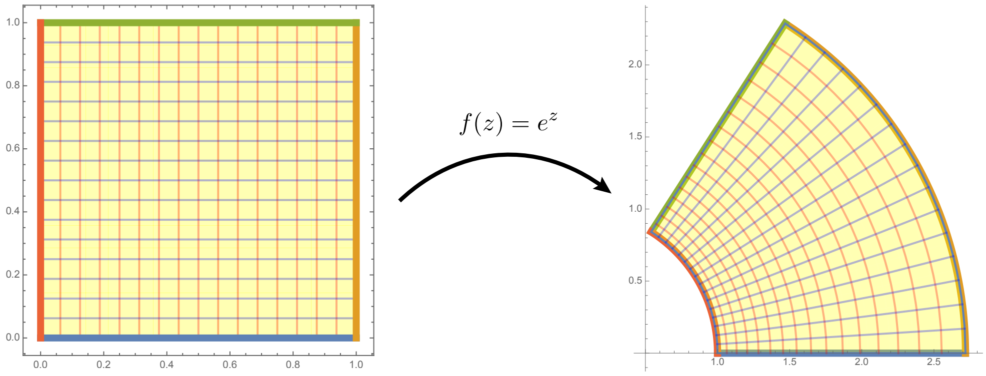
Exercise 2.27. Show that:

$$\cos^{-1} z = -i\log\left(z + i\left(1 - z^2\right)^{1/2}\right) \qquad \tan^{-1} z = \frac{i}{2}\log\frac{i+z}{i-z}$$

Exercise 2.28. Derive the following differentiation rules:

$$\begin{aligned}
\frac{d}{dz}\sin^{-1} z &= \frac{1}{(1-z^2)^{1/2}} \\
\frac{d}{dz}\cos^{-1} z &= -\frac{1}{(1-z^2)^{1/2}} \\
\frac{d}{dz}\tan^{-1} z &= \frac{1}{1+z^2}
\end{aligned}$$

2.3.6. Mapping Properties. Recall that given any $z = x + yi$, we have $e^z = e^x e^{iy} = e^x \cos y + ie^x \sin y$. Therefore, along the straight-path $z = x_0 + yi$ (where x_0 is fixed and y varies), the image under the map $z \mapsto e^z$ is given by $(u, v) = e^{x_0}(\cos y + i \sin y)$, which is a circle with radius e^{x_0} centered at the origin. In other words, the complex exponential maps vertical lines in the xy-plane to concentric circles in the uv-plane. On the other hand, along the horizontal path $z = x + y_0 i$, the image is given by $(u, v) = e^x(\cos y_0 + i \sin y_0)$, which is a half-line from the origin (but not passing through it).



Exercise 2.29. Show that the complex sine function $f(z) = \sin z$ maps horizontal lines in the xy-plane to ellipses in the uv-plane, and maps vertical lines in the xy-plane to hyperbolas in the uv-plane.

Chapter 3

Contour Integrals

We start discussing complex integrations in this chapter. Given a function $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ and a C^1 curve γ in the domain of f, the *contour integral* of f over γ is denoted by:

$$\int_\gamma f(z)\, dz.$$

We will learn how they are defined and how they can be computed soon. In the first glance, it appears quite similar to line integrals in Multivariable Calculus. However, when combining with properties of holomorphic functions, there are many beautiful and amazing results concerning complex contour integrals which did not appear in line integrals. One notable result is Cauchy's integral formula, an elegant theorem which leads to many important results in Complex Analysis and beyond.

3.1. Complex Integrations

3.1.1. Contour Integrals.

Consider a C^1 curve γ in $\mathbb{C}$ parametrized by:

$$z(t) = x(t) + iy(t), \quad t \in [a, b].$$

The differential dz is regarded as:

$$dz = \frac{dz}{dt}\, dt = \left(x'(t) + iy'(t)\right)\, dt.$$

For example, if γ is the unit circle centered at the origin, then it is parametrized by:

$$z(t) = \cos t + i \sin t = e^{it}, \quad t \in [0, 2\pi].$$

Hence, we have $dz = \dfrac{d(e^{it})}{dt}\, dt = ie^{it}\, dt$.

Definition 3.1 (Contour Integrals). Let $f : \Omega \to \mathbb{C}$ be a continuous function on the open domain $\Omega \subset \mathbb{C}$, and γ be a C^1 curve in Ω. Suppose γ is parametrized by

$$z(t) = x(t) + iy(t), \quad t \in [a, b],$$

then the *contour integral of f over γ* is denoted and defined by:

$$\int_\gamma f(z)\, dz := \int_a^b f(z(t))\, \underbrace{z'(t)\, dt}_{dz}.$$

Remark 3.2. If γ is a piecewise C^1 curve, meaning that it can be decomposed into $\gamma = \gamma_1 + \ldots + \gamma_k$ where each of $\gamma_1, \ldots, \gamma_k$ is C^1, and that the whole curve γ is continuous, then we define:

$$\int_\gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \ldots + \int_{\gamma_k} f(z)\,dz.$$

Furthermore, if γ is closed, we usually denote the contour integral by:

$$\oint_\gamma f(z)\,dz.$$

Example 3.1. Compute the line integral $\oint_\gamma f(z)\,dz$ for each of the functions below. Here γ is the circle with radius 2 centered at the origin.

(a) $f(z) = z^2$

(b) $f(z) = \dfrac{1}{z}$

(c) $f(z) = \overline{z}$

Solution

γ can be parametrized by:

$$z(t) = 2e^{it}, t \in [0, 2\pi].$$

Therefore $dz = 2ie^{it}\,dt$.

(a)

$$\begin{aligned}\oint_\gamma z^2\,dz &= \int_0^{2\pi} \underbrace{\left(2e^{it}\right)^2}_{z^2} \cdot \underbrace{2ie^{it}\,dt}_{dz} = \int_0^{2\pi} 8ie^{3it}\,dt \\ &= 8i\left[\frac{1}{3i}e^{3it}\right]_{t=0}^{t=2\pi} \\ &= \frac{8}{3}\left(e^{6\pi i} - e^0\right) = \frac{8}{3}(1-1) = 0.\end{aligned}$$

(b)

$$\oint_\gamma \frac{1}{z}\,dz = \int_0^{2\pi} \frac{1}{2e^{it}} \cdot 2ie^{it}\,dt = \int_{t=0}^{t=2\pi} i\,dt = 2\pi i.$$

(c)

$$\oint_\gamma \overline{z}\,dz = \int_0^{2\pi} 2e^{-it} \cdot 2e^{it}\,dt = \int_0^{2\pi} 4\,dt = 8\pi i.$$

Remark 3.3. In part (a) of the above example, we have used the fact that $\frac{d}{dt}\left(\frac{1}{3i}e^{3it}\right) = e^{3it}$, and also Fundamental Theorem of Calculus. In general, just like in the real case, if $F(t)$ is a differentiable function of t on $[a, b]$ such that $F'(t) = \varphi(t)$ on $[a, b]$, then we have

$$\int_{t=a}^{t=b} \varphi(t)\,dt = F(b) - F(a).$$

However, we sometimes need to be more careful when applying this. Try to find out what's wrong with the calculation below:

$$\begin{aligned}\oint_{|z|=1} \frac{1}{1-2z}\,dz &= \int_0^{2\pi} \frac{1}{1-2e^{it}} ie^{it}\,dt = \left[-\frac{1}{2i}\mathrm{Log}(1-2e^{it})\right]_0^{2\pi} \\ &= -\frac{1}{2i}\left(\mathrm{Log}(-1) - \mathrm{Log}(-1)\right) = 0???\end{aligned}$$

Example 3.2. Consider the line segment L from a point z_1 to a point z_2 in $\mathbb{C}$. Compute the following contour integral (in terms of z_1 and z_2):

$$\int_L e^z\,dz.$$

Solution

First we parametrize L:

$$z(t) = (1-t)z_1 + tz_2, \quad t \in [0,1].$$

Then, we have $dz = (z_2 - z_1)\,dt$, and so:

$$\begin{aligned}\int_L e^z\,dz &= \int_0^1 e^{z_1+t(z_2-z_1)} \cdot (z_2 - z_1)\,dt \\ &= \left[\frac{1}{z_2-z_1} e^{z_1+t(z_2-z_1)} \cdot (z_2-z_1)\right]_0^1 \\ &= e^{z_2} - e^{z_1}.\end{aligned}$$

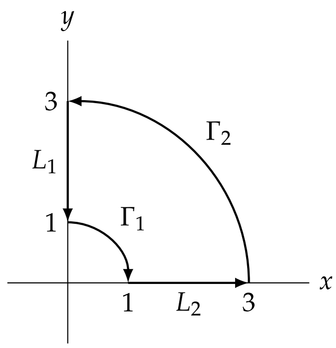


Figure 3.1. the path in Example 3.1

Exercise 3.1. Compute the contour integrals

$$\oint_\gamma \frac{1}{z^2}\,dz, \quad \oint_\gamma \bar{z}\,dz \quad \text{and} \quad \oint_\gamma |z|\,dz$$

where $\gamma = \Gamma_1 + L_2 + \Gamma_2 + L_1$ is the curve in Figure 3.1.

3.1.2. Primitive Functions. In Calculus I, we learned that if $F'(x) = f(x)$ on $x \in [a,b]$, then:

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

This is the celebrated Fundamental Theorem of Calculus. In Complex Analysis, we have an analogous result:

Theorem 3.4. *Let $f : \Omega \to \mathbb{C}$ be a continuous function defined on an open domain $\Omega \subset \mathbb{C}$, and γ be a piecewise C^1 curve in Ω with starting point z_1 and ending point z_2. If $F : \Omega \to \mathbb{C}$ is a (single-valued) holomorphic function on Ω such that $F'(z) = f(z)$ for every $z \in \Omega$, then we have:*

$$\int_\gamma f(z)\,dz = F(z_2) - F(z_1).$$

Proof. First assume that γ is C^1. Suppose the path γ can be parametrized by:

$$z(t) = x(t) + iy(t), \quad t \in [a, b].$$

Then, we have $dz = z'(t)\,dt$, and hence:

$$\begin{aligned}
\int_\gamma f(z)\,dz &= \int_a^b f(z(t)) \cdot z'(t)\,dt \\
&= \int_a^b \underbrace{F'(z(t))}_{f(z(t))} \cdot z'(t)\,dt \\
&= \int_a^b \frac{d}{dt} F(z(t))\,dt && \text{(chain rule)} \\
&= F(z(b)) - F(z(a)) \\
&= F(z_2) - F(z_1).
\end{aligned}$$

If γ is only piecewise C^1, we can decompose $\gamma = \gamma_1 + \cdots + \gamma_k$ so that each γ_i is C^1. Then, one can argue as above for each γ_i, and finally obtain the desired result by adding a telescope sum. □

Remark 3.5. If such an $F(z)$ in Theorem 3.4 exists, then we call $F(z)$ a *primitive function* of $f(z)$.

The above theorem is particularly useful when the anti-derivative of f is easy to find. For example, if γ is any continuous piecewise C^1 path from z_1 to z_2, we can find easily that:

$$\int_\gamma z^2\,dz = \left[\frac{z^3}{3}\right]_{z_1}^{z_2} = \frac{z_2^3 - z_1^3}{3}$$

$$\int_\gamma e^z\,dz = [e^z]_{z_1}^{z_2} = e^{z_2} - e^{z_1}.$$

In particular, if C is a closed path, then we have:

$$\oint_C z^2\,dz = 0 \quad \text{and} \quad \oint_C e^z\,dz = 0.$$

Exercise 3.2. Let γ_1 be the path which starts from $(0, 0)$, first to $(1, 1)$, then to $(0, 2)$. Let γ_2 be the path which starts from $(0, 0)$, then straight to $(0, 2)$. Verify the following by direct computations:

$$\int_{\gamma_1} \cos\frac{\pi z}{2}\,dz = \int_{\gamma_2} \cos\frac{\pi z}{2}\,dz.$$

Then, verify that Theorem 3.4 gives the same result.

However, it is important to note that Theorem 3.4 requires the curve γ to be inside Ω (on which $F'(z) = f(z)$ holds). Let's consider the function $f(z) = \frac{1}{z}$. Although we usually simply write $\frac{d}{dz}\text{Log}(z) = \frac{1}{z}$, it is only true for $z \in \mathbb{C}\backslash\{x + 0i : x \leq 0\}$ since $\text{Log}(z)$ is not continuous on the negative x-axis.

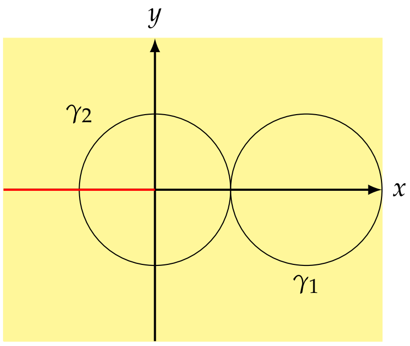
Therefore, we can only apply Theorem 3.4 when the curve γ lies inside $\Omega := \mathbb{C}\backslash\{x + 0i : x \leq 0\}$. For instance, we still have

$$\oint_{\gamma_1} \frac{1}{z} = 0$$

where γ_1 is the unit circle centered at $2 + 0i$ with radius 1. This closed curve γ_1 is contained inside Ω.

However, it is incorrect to claim $\oint_{\gamma_2} \frac{1}{z} = 0$ where γ_2 is the unit circle centered at the origin. The reason is that this closed curve passes through the negative x-axis (hence not contained inside Ω). In fact we can directly verify that:

$$\oint_{\gamma_2} \frac{1}{z} = 2\pi i.$$



Fortunately, we can still apply Theorem 3.4 on $f(z) = \frac{1}{z^2}$ when the integration curve γ does not pass through the origin. The reason is that $F(z) = -\frac{1}{z}$ is a primitive function for f such that $F'(z) = f(z)$ holds on $\mathbb{C}\backslash\{0\}$. Therefore, we have:

$$\oint_{\gamma} \frac{1}{z^2} = 0$$

for any closed curve γ not passing through the origin. Also, for a path L in $\mathbb{C}\backslash\{0\}$ connecting z_1 to z_2, we have:

$$\int_L \frac{1}{z^2}\, dz = \left[-\frac{1}{z}\right]_{z_1}^{z_2} = \frac{1}{z_1} - \frac{1}{z_2}.$$

Exercise 3.3. Consider the path γ parametrized by:

$$z(t) = \cos^{3033} t + i \sin^{2033} t, \quad \text{where } t \in [0, \pi].$$

Find the contour integrals $\displaystyle\int_\gamma \frac{1}{z^{1014}}\, dz$ and $\displaystyle\int_\gamma (1 + iz)^{1013}\, dz$.

Exercise 3.4. Evaluate the integral $\displaystyle\int_\gamma |z|\, dz$ where γ is each of the following:

(a) a line segment joining $-i$ to i.

(b) a counter-clockwise semi-circular path joining $-i$ to i

Does it exist an entire function $F : \mathbb{C} \to \mathbb{C}$ such that $F'(z) = |z|$ for any $z \in \mathbb{C}$? Why or why not?

Exercise 3.5. First verify that on an appropriate domain, we have:

$$\frac{d}{dz} i\left(\operatorname{Log}(i+z) - \operatorname{Log}(i-z)\right) = \frac{1}{1+z^2}.$$

Using this, show that:

$$\oint_{|z|=r} \frac{1}{1+z^2}\, dz = 0 \text{ when } r < 1.$$

In your solution, explain clearly where the condition $r < 1$ is needed.

3.1.3. Integral Estimates.

Estimation of a contour integral is an important technique in Complex Analysis. It will appear in many parts of the course. If we know an upper bound for $|f(z)|$ on the curve γ, and the upper bound for the length of γ, then we are able to bound the contour integral $\int_\gamma f(z)\, dz$ without calculating it.

Lemma 3.6. *Let $f : \Omega \to \mathbb{C}$ be defined on an open domain Ω. Suppse γ is a curve in Ω such that:*

- *$|f(z)| \leq M$ for any $z \in \gamma$, and*
- *the arc-length of γ is bounded above by L.*

Then, we have:

$$\left| \int_\gamma f(z)\, dz \right| \leq ML.$$

Proof. There is a nice trick in the proof that readers are recommended to learn. Let

$$I = \int_\gamma f(z)\, dz.$$

Express I in polar form: $I = |I|\, e^{i\theta}$, then we have $e^{-i\theta} I = |I|$ which is real! Suppose γ is parametrized by $z(t) = x(t) + iy(t)$ where $a \leq t \leq b$, then:

$$\begin{aligned}
e^{-i\theta} I = e^{-i\theta} \int_\gamma f(z)\, dz &= \int_\gamma e^{-i\theta} f(z)\, dz \\
&= \int_a^b \left[\operatorname{Re}\left(e^{-i\theta} f(z)\right) + i \operatorname{Im}\left(e^{-i\theta} f(z)\right) \right] (x'(t) + iy'(t))\, dt \\
&= \int_a^b \left[\operatorname{Re}\left(e^{-i\theta} f(z)\right) x'(t) - \operatorname{Im}\left(e^{-i\theta} f(z)\right) y'(t) \right]\, dt.
\end{aligned}$$

The last equality above follows from the fact that $e^{-i\theta} I$ is real.

Then, we use Cauchy-Schwarz's inequality to bound the integrand:

$$\begin{aligned}
&\left| \operatorname{Re}\left(e^{-i\theta} f(z)\right) x'(t) - \operatorname{Im}\left(e^{-i\theta} f(z)\right) y'(t) \right| \\
&\leq \sqrt{\left(\operatorname{Re}\left(e^{-i\theta} f(z)\right)\right)^2 + \left(\operatorname{Im}\left(e^{-i\theta} f(z)\right)\right)^2} \sqrt{(x'(t))^2 + (y'(t))^2} \\
&= \left| e^{-i\theta} f(z) \right| |z'(t)| = |f(z)|\, |z'(t)| \leq M\, |z'(t)|.
\end{aligned}$$

Finally, we get:

$$\left| e^{-i\theta} I \right| \leq \int_a^b M\, |z'(t)| = ML,$$

and hence $|I| \leq ML$, completing the proof. □

Remark 3.7. If we estimate the integral $\left|\int_\gamma f(z)\,dz\right|$ in a more direct way by writing $f = u + iv$ and then consider the following:

$$\begin{aligned}\left|\int_\gamma f(z)\,dz\right| &= \left|\int_\gamma (u+iv)(dx+idy)\right| = \left|\int_a^b (ux' - vy') + i(vx' + uy')\,dt\right| \\ &= \sqrt{\left(\int_a^b (ux' - vy')\,dt\right)^2 + \left(\int_a^b (vx' + uy')\,dt\right)^2}.\end{aligned}$$

then after applying Cauchy-Schwarz's inequality to each integral, the best we can achieve is

$$\left|\int_\gamma f(z)\,dz\right| \leq \sqrt{2}ML,$$

which is weaker than the result in Lemma 3.6.

Example 3.3. Find an upper bound for the contour integral:

$$\left|\oint_{|z|=1} e^{\frac{1}{z}}\,dz\right|.$$

Solution

For any $z \in \mathbb{C}$ such that $|z| = 1$, we have:

$$\begin{aligned}e^{\frac{1}{z}} &= e^{\frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}} = e^{x-iy} = e^x e^{-iy}, \\ \left|e^{\frac{1}{z}}\right| &= e^x \leq e^1 = e.\end{aligned}$$

Here we have used the fact that $-1 \leq x \leq 1$ along the curve $|z| = 1$.

Therefore, by Lemma 3.6, we have:

$$\left|\oint_{|z|=1} e^{\frac{1}{z}}\,dz\right| \leq \underbrace{2\pi}_{L}\,\underbrace{e}_{M}.$$

Example 3.4. Show that:

$$\lim_{R\to+\infty} \oint_{|z|=R} \frac{1}{(z-1)^2}\,dz = 0.$$

Solution

We are interested in the limit when $R \to +\infty$, so we can assume $R > 1$ so that the contour circle $|z| = R$ does not pass through 1 (at which the integrand is undefined).

On the contour $|z| = R$, we have $|z-1| \geq R - 1$ (draw a diagram to convince yourself on that), so we have:

$$\left|\frac{1}{(z-1)^2}\right| = \frac{1}{|z-1|^2} \leq \underbrace{\frac{1}{(R-1)^2}}_{M} \quad \text{on } |z| = R.$$

The length of the contour $|z| = R$ is $2\pi R$. Hence, by Lemma 3.6, we get

$$\left|\oint_{|z|=R} \frac{1}{(z-1)^2}\,dz\right| \leq 2\pi R \cdot \frac{1}{(R-1)^2}.$$

From elementary calculus, we have $\lim_{R\to+\infty} \frac{2\pi R}{(R-1)^2} = 0$, and the desired result follows from the squeeze theorem.

Exercise 3.6. Let $f : \mathbb{C} \to \mathbb{C}$ be a continuous function, and consider a fixed point $\alpha \in \mathbb{C}$. Show that:

$$\left| \oint_{|z|=R} \frac{f(z)}{z-\alpha}\, dz \right| \leq \frac{2\pi R}{R-|\alpha|} \max_{|z|=R} |f(z)| \qquad \text{when } R > |\alpha| .$$

Exercise 3.7. Suppose $f : \mathbb{C} \to \mathbb{C}$ is a continuous function such that:

$$\lim_{R\to+\infty} \sup_{|z|\geq R} \frac{|f(z)|}{R} = 0.$$

Show that:

$$\lim_{R\to+\infty} \oint_{|z|=R} \frac{f(z)}{z^2}\, dz = 0.$$

Exercise 3.8. Let $f : \mathbb{C} \to \mathbb{R}$ be a continuous *real-valued* function such that $|f(z)| \leq 1$ for any $z \in \mathbb{C}$. Show that:

$$\left| \oint_{|z|=1} f(z)\, dz \right| \leq 4.$$

[Hint: Define $I = \oint_{|z|=1} f(z)\, dz$, then write $I = |I|\, e^{i\theta}$.]

3.2. Cauchy-Goursat's Theorem

In this section, we will prove a very fundamental theorem in Complex Analysis, the Cauchy-Goursat's Theorem, which asserts that if $f : \Omega \to \mathbb{C}$ is a holomorphic function on a simply-connected domain Ω, then the contour integral $\oint_\gamma f(z)\,dz$ must be zero for any closed curve γ in Ω. The statement of the theorem sounds simple, but the proof is quite delicate. We will discuss the proof of this theorem in detail.

Cauchy-Goursat's Theorem is fundamental because it is used to prove the Cauchy's integral formula, which provides a very elegant way for computing contour integral of the form $\oint_\gamma \frac{f(z)}{z-\alpha}\,dz$ and leading many exciting results. We will see later in the course that the Cauchy integral formula is the *heart* of complex analysis.

Theorem 3.8 (Cauchy-Goursat's Theorem). *Let $\Omega \subset \mathbb{C}$ be a simply-connected open domain, γ be any closed piecewise C^1 curve in Ω, and $f : \Omega \to \mathbb{C}$ be any holomorphic function defined on Ω, then we have:*

$$\oint_\gamma f(z)\,dz = 0.$$

Using Cauchy-Goursat's Theorem, we can immediately conclude that all the integrals below over any closed curve $\gamma \in \mathbb{C}$ are zero, without performing any calculation:

$$\oint_\gamma e^z\,dz, \quad \oint_\gamma \sin z\,dz, \quad \oint_\gamma z^2\,dz, \quad \text{etc.}$$

Both conditions of Ω being simply-connected and f being holomorphic on Ω are essential. If Ω is not simply-connected, say $\Omega = \mathbb{C}\backslash\{0\}$, Cauchy-Goursat's Theorem does not hold. Here is a quick counter-example:

$$\oint_{|z|=1} \frac{1}{z}\,dz = 2\pi i \neq 0.$$

Moreover, the holomorphic condition on f is also necessary, and here is a counter-example:

$$\oint_{|z|=1} \bar{z}\,dz = 2\pi i \neq 0.$$

We will prove this theorem soon. The proof consists of several steps:

Step 1: First prove a special case when the contour γ is a *triangle* (while Ω is any simply-connected open domain);

Step 2: Then prove a special case when Ω is *convex* (while γ is any closed piecewise C^1 contour).

Step 3: Use results from previous steps to deduce the general case: Ω is any simply-connected open domain, and γ is any closed piecewise C^1 contour.

3.2.1. Step 1: Cauchy-Goursat's Theorem for Triangle Contours. Let's begin by assuming that T is a triangle contour in Ω. We bisect each side of the triangle T to create four smaller triangles $T_1^{(1)}$, $T_2^{(1)}$, $T_3^{(1)}$ and $T_4^{(1)}$ as shown in the Figure 3.2.

By cancellations of common sides, we have:

$$\oint_T f(z)\,dz = \sum_{j=1}^{4} \oint_{T_j^{(1)}} f(z)\,dz.$$

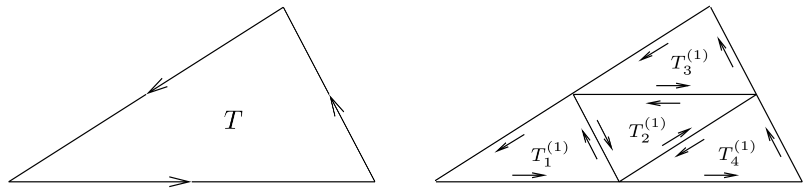


Figure 3.2. Divide the contour T into 4 triangles

Triangle inequality then shows:

$$\left|\oint_T f(z)\,dz\right| \leq \sum_{j=1}^{4}\left|\oint_{T_j^{(1)}} f(z)\,dz\right|.$$

Let $T^{(1)}$ be the triangle among all $T_j^{(1)}$'s (where $j = 1,2,3,4$) with the largest value of $\left|\oint_{T_j^{(1)}} f(z)\,dz\right|$, then one has:

$$\left|\oint_T f(z)\,dz\right| \leq 4\left|\oint_{T^{(1)}} f(z)\,dz\right|$$

Repeat the above procedure on $T^{(1)}$: sub-divide $T^{(1)}$ into four congruent triangles $T_j^{(2)}$ (where $j = 1,2,3,4$), and pick the one with the largest value of $\left|\oint_{T_j^{(2)}} f(z)\,dz\right|$ and label it as $T^{(2)}$. Then, one has:

$$\left|\oint_{T^{(1)}} f(z)\,dz\right| \leq 4\left|\oint_{T^{(2)}} f(z)\,dz\right| \implies \left|\oint_T f(z)\,dz\right| \leq 4^2\left|\oint_{T^{(2)}} f(z)\,dz\right|.$$

Continuing this process, we obtain a sequence of triangles:

$$T^{(0)}, T^{(1)}, T^{(2)}, T^{(3)}, \ldots$$

(where we denote $T^{(0)} := T$) such that

$$\left|\oint_{T^{(0)}} f(z)\,dz\right| \leq 4^n\left|\oint_{T^{(n)}} f(z)\,dz\right| \quad \text{for any } n \geq 0. \tag{3.1}$$

Denote $\Delta^{(j)}$ to be the closed triangular region enclosed by $T^{(j)}$. Then, we have:

$$\Delta^{(0)} \supset \Delta^{(1)} \supset \Delta^{(2)} \supset \ldots$$

By Exercise 1.29, there is at least one point z_0 contained inside all of $\Delta^{(n)}$.

Our goal is to bound the RHS term $4^n\left|\oint_{T^{(n)}} f(z)\,dz\right|$ of (3.1), so as to show that $\left|\oint_{T^{(0)}} f(z)\,dz\right|$ is arbitrarily small, concluding that it must be zero. To achieve our goal, we recall that f is holomorphic on Ω, and in particular, it is complex differentiable at z_0 (which is a point in all of $\Delta^{(n)}$'s). By considering the derivative $f'(z_0)$, and by rearrangement:

$$f'(z_0) = \lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0} \implies \lim_{z\to z_0}\frac{f(z)-f(z_0)-f'(z_0)(z-z_0)}{z-z_0} = 0.$$

For simplicity, denote the numerator by $E(z) := f(z) - f(z_0) - f'(z_0)(z - z_0)$, then we have:

$$\lim_{z \to z_0} \frac{E(z)}{z - z_0} = 0. \tag{3.2}$$

Since the function $f(z_0) + f'(z_0)(z - z_0)$ has a primitive function $zf(z_0) + \frac{f'(z_0)}{2}(z - z_0)^2$ (note that z_0 is a fixed point), we have

$$\oint_{T^{(n)}} E(z)\,dz = \oint_{T^{(n)}} \left[f(z) - f(z_0) - f'(z_0)(z - z_0)\right]\,dz = \oint_{T^{(n)}} f(z)\,dz.$$

Therefore, to bound the RHS of (3.1), we can consider the integral of $E(z)$ instead, which is very small according to (3.2).

Now, given any $\varepsilon > 0$, by (3.2), there exists $\delta > 0$ such that whenever $z \in B_\delta(z_0)$, we have $\left|\dfrac{E(z)}{z - z_0}\right| < \varepsilon$. Recall that $\{\Delta^{(n)}\}_{n=0}^\infty$ is a strictly decreasing sequence of triangles "converging" to the point z_0. Hence, for sufficiently large n, $\Delta^{(n)}$ must lie inside the ball $B_\delta(z_0)$, and so $|E(z)| < \varepsilon\,|z - z_0|$ for any $z \in \Delta^{(n)} \subset B_\delta(z_0)$.

Recall that $|z - z_0|$ is the distance between z and z_0, both of which are in $\Delta^{(n)}$. By elementary geometry, the distance between any two points in a triangle must be bounded by the perimeter of the triangle. Hence, we have for any $z \in \Delta^{(n)}$,

$$|E(z)| < \varepsilon\,|z - z_0| \le \varepsilon L_n = \frac{\varepsilon L_0}{2^n} \tag{3.3}$$

where L_n denotes the perimeter of the triangle $T^{(n)}$.

Using (3.3), we can apply Lemma 3.6 to show:

$$\left|\oint_{T^{(n)}} E(z)\,dz\right| \le \frac{\varepsilon L_0}{2^n} \cdot L_n = \frac{\varepsilon L_0^2}{4^n}.$$

Finally, by considering (3.1), we have proved:

$$\left|\oint_T f(z)\,dz\right| \le 4^n \left|\oint_{T^{(n)}} f(z)\,dz\right| = 4^n \left|\oint_{T^{(n)}} E(z)\,dz\right| \le 4^n \cdot \frac{\varepsilon L_0^2}{4^n} = \varepsilon L_0^2.$$

Since $\varepsilon > 0$ is arbitrary, by letting $\varepsilon \to 0^+$, we get:

$$\oint_T f(z)\,dz = 0,$$

completing Step 1.

Exercise 3.9. Using the result proved so far, show that Cauchy-Goursat's Theorem holds for any closed *polygon* γ.

Exercise 3.10. Show that if $\triangle ABC$ is contained inside a simply-connected open set Ω on which f is holomorphic, then we have:

$$\int_{L(A,C)} f(z)\,dz = \int_{L(A,B)} f(z)\,dz + \int_{L(B,C)} f(z)\,dz.$$

Here $L(A, B)$, for instance, is the straight path from A to B.

Exercise 3.11. Which part in the proof of Step 1 will break down if f is not holomorphic? Also, why will the proof break down if Ω is not simply-connected?

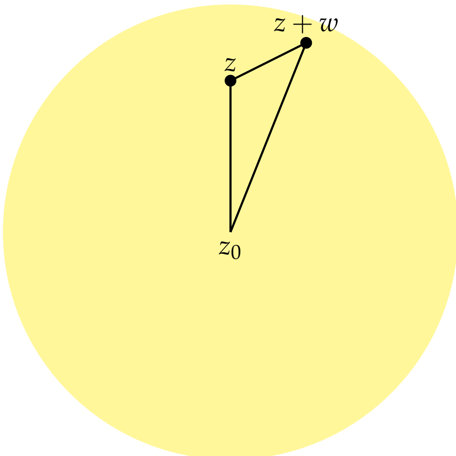
3.2.2. Step 2: Cauchy-Goursat's Theorem for Convex Domains. Now we are given any closed piecewise C^1 curve γ (not necessarily a triangle) in an open convex domain Ω. We want to show that if $f : \Omega \to \mathbb{C}$ is holomorphic, then

$$\oint_\gamma f(z)\,dz = 0.$$

We show that by finding a primitive function $F : \Omega \to \mathbb{C}$ such that $F'(z) = f(z)$ on Ω, then this step is proved using Theorem 3.4. To define such a function F, we first fix a point $z_0 \in \Omega$, and denote $L(z_0, z)$ to be the straight path from z_0 to z. Note that by convexity of Ω, such a path must be contained in Ω. Next, we define:

$$F(z) := \int_{L(z_0,z)} f(\xi)\,d\xi.$$

We claim that $F'(z) = f(z)$ by showing that the quotient $\dfrac{F(z+w) - F(z)}{w}$ tends to $f(z)$ as $w \to 0$.



From Step 1 (note that z_0, z and $z+w$ form a triangle), we know that:

$$\begin{aligned}\frac{F(z+w) - F(z)}{w} &= \frac{1}{w}\left(\int_{L(z_0,z+w)} f(\xi)\,d\xi - \int_{L(z_0,z)} f(\xi)\,d\xi\right)\\ &= \frac{1}{w}\int_{L(z,z+w)} f(\xi)\,d\xi.\end{aligned}$$

By observing that $\int_{L(z,z+w)} f(z)\,d\xi = [f(z)\,\xi]_{\xi=z}^{\xi=z+w} = w\,f(z)$, we have:

$$\begin{aligned}\frac{F(z+w) - F(z)}{w} &= \frac{1}{w}\int_{L(z,z+w)} f(\xi)\,d\xi = \frac{1}{w}\int_{L(z,z+w)} (f(\xi) - f(z)) + f(z)\,d\xi\\ &= \frac{1}{w}\int_{L(z,z+w)} (f(\xi) - f(z))\,d\xi + f(z).\end{aligned} \tag{3.4}$$

The next task will be to show that $\dfrac{1}{w}\displaystyle\int_{L(z,z+w)} (f(\xi) - f(z))\,d\xi$ tends to 0 as $w \to 0$. For any $\varepsilon > 0$, by the continuity of f, there exists $\delta > 0$ such that whenever $\xi \in B_\delta(z)$, we have $|f(\xi) - f(z)| < \varepsilon$. In particular, if $|w| < \delta$, then the path $L(z, z+w) \subset B_\delta(z)$, and so for any $\xi \in L(z, z+w)$, we have:

$$|f(\xi) - f(z)| < \varepsilon.$$

Applying Lemma 3.6 on the integral $\int_{L(z,z+w)} (f(\xi) - f(z))\,d\xi$, we have:

$$\left|\int_{L(z,z+w)} (f(\xi) - f(z))\,d\xi\right| \leq \varepsilon \cdot \underbrace{|w|}_{\text{length of contour}},$$

which implies $\left|\frac{1}{w}\int_{L(z,z+w)} (f(\xi) - f(z))\,d\xi\right| \leq \varepsilon$ (whenever $0 < |w| < \delta$), or equivalently,

$$\lim_{w\to 0} \frac{1}{w}\int_{L(z,z+w)} (f(\xi) - f(z))\,d\xi = 0.$$

Finally, from (3.4), we conclude:

$$\lim_{w\to 0} \frac{F(z+w) - F(z)}{w} = f(z) \quad \Longrightarrow \quad F'(z) = f(z).$$

This shows $f(z)$ has a primitive function on Ω, and hence

$$\oint_\gamma f(z)\,dz = 0$$

for any closed curve γ in Ω, completing Step 2.

Remark 3.9. It is worthwhile to note that the whole argument in Step 2 remains valid as long as f is continuous on Ω, and that

$$\oint_T f(z)\,dz = 0$$

for any triangle T in the domain Ω. These two conditions are enough to prove, using the same argument, that $F'(z) = f(z)$ on Ω, even if we don't assume f is holomorphic. This observation will be important in the proof of Morera's Theorem in later section.

Exercise 3.12. Discuss: In the above proof, we require Ω to be convex so that $L(z_0, z)$ is contained in Ω. Now suppose Ω is not convex, but is polygonally path-connected, and we define F as:

$$F(z) = \int_{\gamma(z_0,z)} f(\xi)\,d\xi$$

where $\gamma(z_0, z)$ is *any* polygonal path from z_0 to z. Can we still claim that $F'(z) = f(z)$ with the same proof? If not, where does the proof break down?

3.2.3. Step 3: Completion of the Proof.

We have by far proved that Cauchy-Goursat's Theorem holds when at least one of the conditions holds:

(i) γ is a closed polygon; or

(ii) Ω is convex.

Now we deduce the general case based on these special cases.

Given any simply-connected domain Ω and any closed piecewise C^1 curve $\gamma \subset \Omega$, and a holomorphic function $f : \Omega \to \mathbb{C}$, the key idea to show $\oint_\gamma f(z)\,dz = 0$ is to break the region enclosed by γ into small rectangles $\{R_j\}_{j=1}^N$ and "partial rectangles" $\{\gamma_k\}_{k=1}^M$ (see Figure 3.3). By breaking the region into small enough of these rectangles and partial rectangles, we may assume that these partial rectangles are contained inside an convex subset of Ω. This is intuitively true, but the proof involves some deep knowledge on analysis and topology beyond the scope of this course.

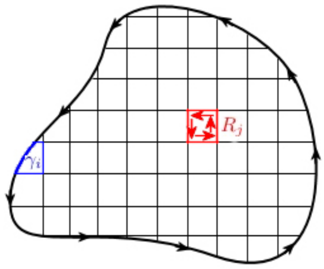


Figure 3.3

For each rectangle R_j and partial rectangle γ_k, results from Steps 1 and 2 show

$$\oint_{R_j} f(z)\,dz = \oint_{\gamma_k} f(z)\,dz = 0.$$

Note that by cancellation of common sides, we can see:

$$\oint_{\gamma} f(z)\,dz = \sum_j \oint_{R_j} f(z)\,dz + \sum_k \oint_{\gamma_k} f(z)\,dz = 0.$$

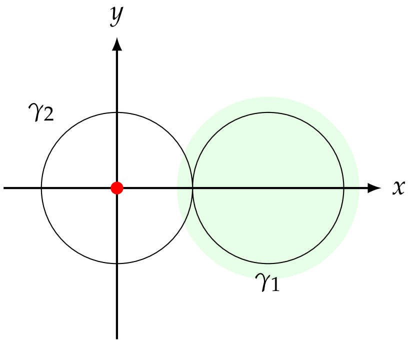
It completes the proof of Cauchy-Goursat's Theorem.

Exercise 3.13. Consider a holomorphic $f = u + iv : \Omega \to \mathbb{C}$ on a simply-connected domain Ω, and a closed piecewise C^1 curve γ in Ω. Now, we *further assume* that f is C^1, i.e. $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ are all continuous on C^1, show that then Cauchy-Goursat's Theorem can be *easily* proved using Green's Theorem.

3.3. Cauchy's Integral Formula I

Cauchy-Goursat's Theorem requires that the function f involved is defined and holomorphic in the region enclosed by the closed curve γ. When the integrand has some "singularities" such as $f(z) = \frac{1}{z}$, Cauchy-Goursat's Theorem may not hold.

Consider the closed curves γ_1 and γ_2 shown below:



For γ_1, there is no issue to apply Cauchy-Goursat's Theorem by taking Ω to be the green region, and it shows

$$\oint_{\gamma_1} \frac{1}{z}\, dz = 0$$

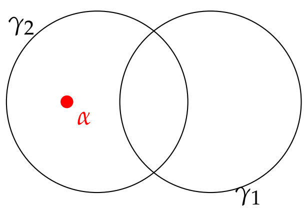
since $\frac{1}{z}$ is holomorphic on the green region. However, we cannot do the same for γ_2. Any simply-connected region containing γ_2 must contain 0 at which $\frac{1}{z}$ is undefined. In this section, we will introduce Cauchy's integral formula to deal with contour integrals of the form $\displaystyle\oint_\gamma \frac{f(z)}{z-\alpha}\, dz$.

Theorem 3.10 (Cauchy's Integral Formula)**.** *Let $f : \Omega \to \mathbb{C}$ be a holomorphic function defined on a simply-connected domain Ω, and γ be a simple closed curve in Ω. Then, we have:*

$$\frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z-\alpha}\, dz = \begin{cases} f(\alpha) & \text{if } \gamma \text{ encloses } \alpha \\ 0 & \text{if } \gamma \text{ does not enclose } \alpha \end{cases}$$

For instance, given an entire function $f : \mathbb{C} \to \mathbb{C}$, a point α, and two closed curves γ_1 and γ_2 below. Cauchy's Integral Formula asserts that:

$$\oint_{\gamma_1} \frac{f(z)}{z-\alpha}\, dz = 0 \quad \text{whereas} \quad \oint_{\gamma_2} \frac{f(z)}{z-\alpha}\, dz = 2\pi i f(\alpha).$$



It is a very powerful theorem as it tells us that the evaluation of some contour integrals can be done by just substituting a point into the numerator function. Let's first see some examples, and then we will prove the theorem.

3.3.1. Elementary Examples. We first illustrate the use of Cauchy's integral formula by a toy example:

$$\oint_\gamma \frac{1}{z}\,dz = \oint_\gamma \frac{1}{z-0}\,dz = \begin{cases} 2\pi i \cdot 1 = 2\pi i & \text{if } \gamma \text{ encloses } 0 \\ 0 & \text{if } \gamma \text{ does not enclose } 0 \end{cases}$$

Here we take $f(z) = 1$ which is an entire function on $\mathbb{C}$.

Example 3.5. Evaluate the following contour integrals:

(a) $\displaystyle\oint_{|z|=2} \frac{z}{(z+3i)(z-i)}\,dz$

(b) $\displaystyle\oint_{|z|=4} \frac{z}{(z+3i)(z-i)}\,dz$

(c) $\displaystyle\oint_{|z|=2} \frac{e^z}{z^2+1}\,dz$

Solution

(a) The integrand has two singularities: $z = -3i$ and $z = i$. First observe that the curve $|z| = 2$ enclose i only, and hence near the simply-connected region $|z| \leq 2$, the function $f(z) := \frac{z}{z+3i}$ is holomorphic. Apply Cauchy's integral formula with this f, we get:

$$\begin{aligned}\oint_{|z|=2} \frac{z}{(z+3i)(z-i)}\,dz &= \oint_{|z|=2} \frac{\frac{z}{z+3i}}{z-i}\,dz = 2\pi i \cdot \left.\frac{z}{z+3i}\right|_{z=i} \\ &= 2\pi i \cdot \frac{i}{i+3i} = \frac{\pi i}{2}.\end{aligned}$$

(b) Note that the curve $|z| = 4$ enclose both singularities $-3i$ and i of the integrand. We cannot apply Cauchy's integral formula by writing the integrand as either:

$$\frac{\frac{z}{z+3i}}{z-i} \quad \text{or} \quad \frac{\frac{z}{z-i}}{z+3i}.$$

The way out is to do partial fractions for the denominator. Let A and B be complex numbers such that:

$$\frac{1}{(z+3i)(z-i)} = \frac{A}{z+3i} + \frac{B}{z-i}.$$

We need to solve for A and B:

$$\begin{aligned}\frac{1}{(z+3i)(z-i)} &= \frac{A(z-i)+B(z+3i)}{(z+3i)(z-i)} \\ 1 &= (A+B)z + (-Ai+3Bi)\end{aligned}$$

Equating coefficients, we need $A + B = 0$ and $(-Ai + 3Bi) = 1$. Solving these equations, we get $A = \frac{1}{4}i$ and $B = -\frac{1}{4}i$, and hence:

$$\frac{1}{(z+3i)(z-i)} = \frac{\frac{1}{4}i}{z+3i} - \frac{\frac{1}{4}i}{z-i}.$$

Now applying Cauchy's integral formula:

$$\begin{aligned}\oint_{|z|=4} \frac{z}{(z+3i)(z-i)}\,dz &= \oint_{|z|=4} \frac{\frac{1}{4}zi}{z+3i} - \frac{\frac{1}{4}zi}{z-i}\,dz \\ &= 2\pi i\left(\left[\frac{1}{4}zi\right]_{z=-3i} - \left[\frac{1}{4}zi\right]_{z=i}\right) \\ &= 2\pi i\left(\frac{1}{4}\cdot(-3i)i - \frac{1}{4}i^2\right) = 2\pi i.\end{aligned}$$

(c) The integrand has z^2+1 as the denominator. Be careful that it can be zero in the complex world and so $\frac{e^z}{z^2+1}$ is NOT holomorphic everywhere. By partial fractions, we get:

$$\frac{1}{z^2+1} = \frac{1}{(z-i)(z+i)} = \frac{1}{2i}\left(\frac{1}{z-i} - \frac{1}{z+i}\right).$$

Hence, Cauchy's integral formula shows:

$$\begin{aligned}\oint_{|z|=2} \frac{e^z}{z^2+1}\,dz &= \frac{1}{2i}\oint_{|z|=2}\left(\frac{e^z}{z-i} - \frac{e^z}{z+i}\right)dz \\ &= \frac{1}{2i}\cdot 2\pi i\cdot\left(e^i - e^{-i}\right) \\ &= \pi\left((\cos 1 + i\sin 1) - (\cos 1 - i\sin 1)\right) \\ &= 2\pi i \sin 1.\end{aligned}$$

Exercise 3.14. Use Cauchy's integral formula to evaluate the following contour integrals:

(a) $\displaystyle\oint_{|z|=2} \frac{1}{z^2+i}\,dz$

(b) $\displaystyle\oint_{|z-e^{\pi i/4}|=1} \frac{1}{z^2+i}\,dz$

(c) $\displaystyle\oint_{|z|=2} \frac{1}{z^3-1}\,dz$

Try to do the problems in a rather *tedious* way using partial fractions. We will provide another approach soon.

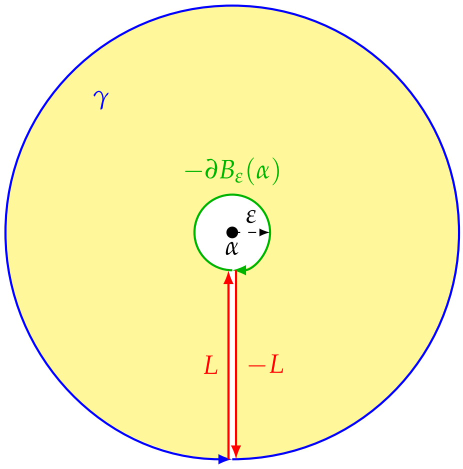
Exercise 3.15. Let $f : \Omega \to \mathbb{C}$ be a holomorphic function defined on a domain Ω containing $B_r(\alpha)$. Prove the following Mean-Value Identity:

$$f(\alpha) = \frac{1}{2\pi}\int_0^{2\pi} f(\alpha + re^{i\theta})\,d\theta.$$

3.3.2. Proof of Cauchy's Integral Formula. The proof of Cauchy's integral formula is a reminiscence of the proof of generalized (i.e. with holes) Green's Theorem in Multivariable Calculus. Fix $\alpha \in \mathbb{C}$ and consider a simple closed curve γ enclosing α. We want to find out the value of the integral:

$$\oint_\gamma \frac{f(z)}{z-\alpha}\,dz.$$

We drill a circular hole near α in the region enclosed by γ, so that the following "key-hole" contour Γ_ε is produced.



The contour $\Gamma_\varepsilon = \gamma + L - \partial B_\varepsilon(\alpha) - L$ encloses a simply-connected region on which $\dfrac{f(z)}{z-\alpha}$ is holomorphic (since $z \neq \alpha$ in this key-hole region). Therefore, we have:

$$\begin{aligned} 0 = \oint_{\Gamma_\varepsilon} \frac{f(z)}{z-\alpha}\,dz &= \oint_{\gamma} \frac{f(z)}{z-\alpha}\,dz + \oint_{L} \frac{f(z)}{z-\alpha} \underbrace{- \oint_{|z-\alpha|=\varepsilon} \frac{f(z)}{z-\alpha}\,dz}_{\text{orientation!}} - \oint_{L} \frac{f(z)}{z-\alpha} \\ &= \oint_{\gamma} \frac{f(z)}{z-\alpha}\,dz - \oint_{|z-\alpha|=\varepsilon} \frac{f(z)}{z-\alpha}\,dz. \end{aligned}$$

Therefore, we have $\displaystyle\oint_{\gamma} \frac{f(z)}{z-\alpha}\,dz = \oint_{|z-\alpha|=\varepsilon} \frac{f(z)}{z-\alpha}\,dz$ for any sufficiently small $\varepsilon > 0$. To prove the desired result, we try to figure out the contour integral over the circle $|z-\alpha| = \varepsilon$. The key trick is to write $f(z) = f(z) - f(\alpha) + f(\alpha)$, so that:

$$\begin{aligned} \oint_{|z-\alpha|=\varepsilon} \frac{f(z)}{z-\alpha}\,dz &= \oint_{|z-\alpha|=\varepsilon} \left(\frac{f(z)-f(\alpha)}{z-\alpha} + \frac{f(\alpha)}{z-\alpha} \right) dz \\ &= \oint_{|z-\alpha|=\varepsilon} \frac{f(z)-f(\alpha)}{z-\alpha}\,dz + f(\alpha) \oint_{|z-\alpha|=\varepsilon} \frac{1}{z-\alpha}\,dz \end{aligned} \tag{3.5}$$

The second integral can be computed directly by parametrizing the circle: $z = \alpha + \varepsilon e^{it}$, where $t \in [0, 2\pi]$:

$$\begin{aligned} \oint_{|z-\alpha|=\varepsilon} \frac{1}{z-\alpha}\,dz &= \int_0^{2\pi} \frac{1}{\varepsilon e^{it}} \cdot \varepsilon i e^{it}\,dt \\ &= \int_0^{2\pi} i\,dt = 2\pi i. \end{aligned}$$

For the first term, we claim that it tends to 0 as $\varepsilon \to 0^+$: since f is complex differentiable at $z = \alpha$, and so

$$\lim_{z\to\alpha} \frac{f(z)-f(\alpha)}{z-\alpha} = f'(\alpha).$$

By definition of limit, there exists $\delta > 0$ such that whenever $z \in B_\delta(\alpha)$ we have:

$$\left| \frac{f(z)-f(\alpha)}{z-\alpha} - f'(\alpha) \right| < 1,$$

and hence

$$\left| \frac{f(z)-f(\alpha)}{z-\alpha} \right| < 1 + |f'(\alpha)| =: M.$$

As a result, when $\varepsilon < \delta$, the contour $|z-\alpha| = \varepsilon$ lies completely inside the ball $B_\delta(\alpha)$, then by Lemma 3.6, we have:

$$\left| \oint_{|z-\alpha|=\varepsilon} \frac{f(z)-f(\alpha)}{z-\alpha} \right| \leq M \cdot 2\pi\varepsilon \to 0 \text{ as } \varepsilon \to 0^+.$$

Finally, from (3.5), we have:

$$\lim_{\varepsilon\to 0^+} \oint_{|z-\alpha|=\varepsilon} \frac{f(z)}{z-\alpha}\, dz = 2\pi i f(\alpha).$$

Recall that $\oint_\gamma \frac{f(z)}{z-\alpha}\, dz = \oint_{|z-\alpha|=\varepsilon} \frac{f(z)}{z-\alpha}\, dz$ for any sufficiently small $\varepsilon > 0$, so we have:

$$\oint_\gamma \frac{f(z)}{z-\alpha}\, dz = \lim_{\varepsilon\to 0^+} \oint_{|z-\alpha|=\varepsilon} \frac{f(z)}{z-\alpha}\, dz = 2\pi i f(\alpha),$$

completing the proof of Cauchy's integral formula.

3.3.3. Cauchy's Integral Formula with Multiple Holes. We have seen how to apply Cauchy's integral formula on fractions such as $\frac{1}{z^2+1}$ which is not defined on $z = i$ and $z = -i$. If a simple closed contour γ encloses both singularities, then we performed partial fractions so that the fraction becomes $\frac{1}{2i}\left(\frac{1}{z-i} - \frac{1}{z+i}\right)$.

Sometimes, partial fractions can be time-consuming especially when there are many singularities. However, using the hole-drilling technique demonstrated in the proof of Cauchy's integral formula, we can break down the contour integral into a sum of several contour integrals, each of which is over a contour that encloses only *one* singularity. Let's look at some examples.

Example 3.6. Evaluate the contour integral:

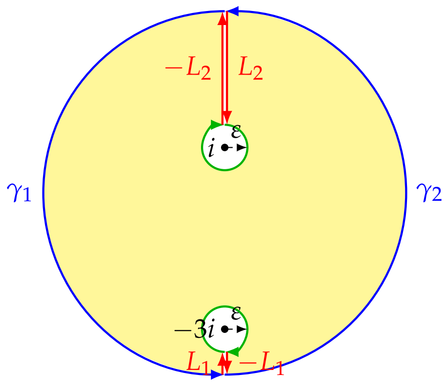
$$\oint_{|z|=4} \frac{z}{(z+3i)(z-i)}\, dz$$

without using partial fractions.

Solution

The two singularities are $z = -3i$ and $z = i$, both are contained inside the contour $|z| = 4$. Draw two little circles with small radii ε around each singularity and consider the key-hole contour:

$$\Gamma = \gamma_1 + L_1 - \partial B_\varepsilon(-3i) - L_1 + \gamma_2 + L_2 - \partial B_\varepsilon(i) - L_2$$



Then, the key hole contour Γ encloses a simply-connected region not containing any singularity of the integrand. Therefore, Cauchy-Goursat's Theorem asserts that

$$\oint_{\Gamma} \frac{z}{(z+3i)(z-i)}\,dz = 0.$$

On the other hand, by cancellation of the common sides, we have:

$$\oint_{\Gamma} = \int_{\gamma_1} + \int_{\gamma_2} - \oint_{|z+3i|=\varepsilon} - \oint_{|z-i|=\varepsilon} = \oint_{|z|=4} - \oint_{|z+3i|=\varepsilon} - \oint_{|z-i|=\varepsilon}.$$

Therefore,

$$\begin{aligned} 0 &= \oint_{\Gamma} \frac{z}{(z+3i)(z-i)}\,dz \\ &= \oint_{|z|=4} \frac{z}{(z+3i)(z-i)}\,dz - \oint_{|z+3i|=\varepsilon} \frac{z}{(z+3i)(z-i)}\,dz \\ &\quad - \oint_{|z-i|=\varepsilon} \frac{z}{(z+3i)(z-i)}\,dz. \end{aligned}$$

Therefore, we can break the required integral into the sum of two integrals:

$$\oint_{|z|=4} \frac{z}{(z+3i)(z-i)}\,dz = \oint_{|z+3i|=\varepsilon} \frac{z}{(z+3i)(z-i)}\,dz + \oint_{|z-i|=\varepsilon} \frac{z}{(z+3i)(z-i)}\,dz$$

Since ε is very small, the function $\dfrac{z}{z-i}$ is holomorphic on $|z+2i| < \varepsilon$, and so Cauchy's integral formula asserts that:

$$\oint_{|z+3i|=\varepsilon} \frac{z}{(z+3i)(z-i)}\,dz = \oint_{|z+3i|=\varepsilon} \frac{\frac{z}{z-i}}{z-(-3i)}\,dz = 2\pi i \cdot \frac{-3i}{-3i-i} = \frac{3\pi i}{2}.$$

For the second integral, we have:

$$\oint_{|z-i|=\varepsilon} \frac{z}{(z+3i)(z-i)}\,dz = \oint_{|z-i|=\varepsilon} \frac{\frac{z}{z+3i}}{z-i}\,dz = 2\pi i \cdot \frac{i}{i+3i} = \frac{\pi i}{2}$$

Adding up the results, we get:

$$\oint_{|z|=4} \frac{z}{(z+3i)(z-i)}\,dz = \frac{3\pi i}{2} + \frac{\pi i}{2} = 2\pi i.$$

Example 3.7. Evaluate the contour integral:

$$\oint_{|z|=2} \frac{1}{z^3-1}\,dz$$

without using partial fractions.

Solution

First factorize the integrand:

$$\frac{1}{z^3-1} = \frac{1}{(z-1)(z-\omega)(z-\omega^2)}$$

where $\omega := e^{\frac{2\pi i}{3}}$ is the cubic root of unity. There are three singularities, namely 1, ω and ω^2, all are enclosed by the given contour $|z| = 2$. By mimicking the hole-drilling argument, one can arrive at:

$$\begin{aligned}
&\oint_{|z|=2} \frac{1}{(z-1)(z-\omega)(z-\omega^2)}\,dz \\
&= \oint_{|z-1|=\varepsilon} \frac{1}{(z-1)(z-\omega)(z-\omega^2)}\,dz + \oint_{|z-\omega|=\varepsilon} \frac{1}{(z-1)(z-\omega)(z-\omega^2)}\,dz \\
&\quad + \oint_{|z-\omega^2|=\varepsilon} \frac{1}{(z-1)(z-\omega)(z-\omega^2)}\,dz \\
&= \oint_{|z-1|=\varepsilon} \frac{\frac{1}{(z-\omega)(z-\omega^2)}}{z-1}\,dz + \oint_{|z-\omega|=\varepsilon} \frac{\frac{1}{(z-1)(z-\omega^2)}}{z-\omega}\,dz + \oint_{|z-\omega^2|=\varepsilon} \frac{\frac{1}{(z-1)(z-\omega)}}{z-\omega^2}\,dz \\
&= 2\pi i \left[\frac{1}{(1-\omega)(1-\omega^2)} + \frac{1}{(\omega-1)(\omega-\omega^2)} + \frac{1}{(\omega^2-1)(\omega^2-\omega)}\right].
\end{aligned}$$

We leave it as an exercise to show that the final answer is 0. [Hint: use the fact that $1+\omega+\omega^2 = 0$]

Exercise 3.16. Evaluate the following contour integrals:

(a) $\displaystyle\oint_{|z|=24601} \frac{1}{z^3+1}\,dz$

(b) $\displaystyle\oint_{|z|=2} \frac{1}{(z^2+1)(z^2+9)}\,dz$

(c) $\displaystyle\oint_{|z-1|=1} \frac{e^z}{z^4+1}\,dz$

(d) $\displaystyle\oint_{|z|=4} \frac{z}{1-e^z}\,dz$

Exercise 3.17. Let n be a positive integer, and $\omega := e^{2\pi i/n}$ denote the n-th root of unity. Express the contour integral:

$$\oint_{|z|=2} \frac{1}{z^n-1}\,dz$$

in terms of ω.

Exercise 3.18. Given any real constant $a \in \mathbb{R}$, by considering the contour integral $\displaystyle\oint_{|z|=1} \frac{e^{az}}{z}\,dz$, prove the following integration formula:

$$\int_0^{\pi} e^{a\cos\theta}\cos(a\sin\theta)\,d\theta = \pi.$$

3.4. Cauchy's Integral Formula II

Recall that Cauchy's integral formula asserts that if $f : \Omega \to \mathbb{C}$ is holomorphic on a simply-connected domain Ω and γ is a closed curve in Ω, then we have:

$$f(\alpha) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z - \alpha}\, dz$$

if γ encloses α.

If the integrand is of the form $\dfrac{f(z)}{(z-\alpha)(z-\beta)}$ whenever $\alpha \neq \beta$, we can still use Cauchy's integral formula in a modified way: either by partial fractions, or by a hole-drilling argument illustrated in the previous section.

However, if the integrand is of the form $\dfrac{f(z)}{(z-\alpha)^2}$, then both partial fractions and the hole-drilling argument do not work well (think about why). Indeed, the contour integral $\displaystyle\oint_\gamma \frac{f(z)}{(z-\alpha)^2}\, dz$ is related to $f'(\alpha)$, and this fact has many deep and surprising consequences as we will see later. These include the celebrated Liouville's Theorem (which implies Fundamental Theorem of Algebra).

Our goal is to prove and discuss the following higher-order Cauchy's integral formula:

Theorem 3.11 (Higher-Order Cauchy's Integral Formula)**.** *Let $f : \Omega \to \mathbb{C}$ be a holomorphic function defined on a simply-connected domain Ω, and α be any point in Ω. Then, for any simple closed curve γ enclosing α, the n-th derivative of f at α is equal to:*

$$f^{(n)}(\alpha) = \frac{n!}{2\pi i} \oint_\gamma \frac{f(z)}{(z-\alpha)^{n+1}}\, dz, \quad n = 0, 1, 2, \ldots$$

Corollary 3.12. *If f is holomorphic on an open domain Ω, then f is complex differentiable for infinitely many times on Ω, i.e. $f^{(n)}$ exists on Ω for any $n \geq 0$.*

The corollary is a very remarkable and surprising result. In Real Analysis, there are many functions which are differentiable for one time but not the second time or further. However, this theorem and the corollary assert that once f is *complex* differentiable on a simply-connected domain (say an open ball), then it is infinitely differentiable on that domain!

3.4.1. Elementary Examples.

Again, we will first see some examples of using the higher-order Cauchy's integral formula, then we will give a proof for it. As a quick example:

$$\oint_{|z|=1} \frac{1}{z^2}\, dz.$$

One way of evaluating it is to argue that its primitive function is $-\frac{1}{z}$, which is well defined and holomorphic near the contour $|z| = 1$. Then by Proposition 3.4, the contour integral is 0.

Let's see how to obtain the same result using Theorem 3.11 (with $n = 1$, and $f(z) \equiv 1$):

$$\frac{1}{2\pi i} \oint_{|z|=1} \frac{1}{z^{1+1}}\, dz = \left.\frac{d}{dz}\right|_{z=0} 1 = 0.$$

Example 3.8. Evaluate the contour integral using higher-order Cauchy's integral formula:

$$\oint_{|z|=1} \frac{e^{2z}}{z^3}\, dz.$$

Solution

In practice, it may be helpful to write the higher-order Cauchy's integral formula as:

$$\oint_{\gamma} \frac{f(z)}{(z-\alpha)^{n+1}}\, dz = \frac{2\pi i}{n!} f^{(n)}(\alpha).$$

Let $f(z) = e^{2z}$ which is entire, then $f'(z) = 2e^{2z}$ and $f''(z) = 4e^{2z}$. By Theorem 3.11 (with $n = 2$), we get:

$$\begin{aligned}\oint_{\gamma} \frac{e^{2z}}{z^3}\, dz &= \oint_{\gamma} \frac{e^{2z}}{(z-0)^{2+1}}\, dz \\ &= \frac{2\pi i}{2!} f''(0) = \frac{2\pi i}{2} \cdot 4 = 4\pi i.\end{aligned}$$

Example 3.9. Evaluate the contour integral:

$$\oint_{|z|=3} \frac{1}{(z+i)^2(z-2i)^3}\, dz.$$

Solution

The contour $|z| = 3$ encloses two singularities of the integrand, namely $-i$ and $2i$. By the hole-drilling technique, we can pick a small $\varepsilon > 0$ such that:

$$\oint_{|z|=3} \frac{1}{(z+i)^2(z-2i)^3}\, dz = \left(\oint_{|z+i|=\varepsilon} + \oint_{|z-2i|=\varepsilon} \right) \frac{1}{(z+i)^2(z-2i)^3}\, dz.$$

Then we calculate each integral on the RHS individually:

$$\begin{aligned}\oint_{|z+i|=\varepsilon} \frac{1}{(z+i)^2(z-2i)^3}\, dz &= \oint_{|z+i|=\varepsilon} \frac{\frac{1}{(z-2i)^3}}{(z+i)^{1+1}}\, dz \\ &= \frac{2\pi i}{1!} \frac{d}{dz}\bigg|_{z=-i} \frac{1}{(z-2i)^3} = -\frac{2\pi i}{3^3} \\ \oint_{|z-2i|=\varepsilon} \frac{1}{(z+i)^2(z-2i)^3}\, dz &= \oint_{|z-2i|=\varepsilon} \frac{\frac{1}{(z+i)^2}}{(z-2i)^{2+1}}\, dz \\ &= \frac{2\pi i}{2!} \frac{d^2}{dz^2}\bigg|_{z=2i} \frac{1}{(z+i)^2} \\ &= \pi i \cdot \left[\frac{6}{(z+i)^4} \right]_{z=2i} \\ &= \frac{2\pi i}{3^3}\end{aligned}$$

Therefore,

$$\oint_{|z|=3} \frac{1}{(z+i)^2(z-2i)^3}\, dz = -\frac{2\pi i}{3^3} + \frac{2\pi i}{3^3} = 0.$$

Exercise 3.19. Evaluate the following contour integrals:

(a) $\displaystyle\oint_{|z|=2} \frac{\sin z}{(z-\pi)^2}\,dz$

(b) $\displaystyle\oint_{|z|=3} \frac{ze^{tz}}{(z+1)^3}\,dz$ where $t > 0$ is real.

(c) $\displaystyle\oint_{|z|=1} \left(2+z+\frac{1}{z}\right)\frac{f(z)}{z}\,dz$, where f is entire and $f(0) = 1$.

Exercise 3.20. Evaluate the contour integral (where n is a positive integer):

$$\oint_{|z|=1} \left(z+\frac{1}{z}\right)^{2n} \frac{1}{z}\,dz.$$

Hence, show that:

$$\int_0^{2\pi} \cos^{2n}\theta\,d\theta = 2\pi\frac{(2n-1)!!}{(2n)!!}.$$

Exercise 3.21. Let $f : \Omega \to \mathbb{C}$ be a holomorphic function defined on a simply-connected domain Ω. Suppose $B_R(z_0) \subset \Omega$, show that:

$$\left|f^{(n)}(z_0)\right| \leq \frac{n!}{R^n} \sup_{|z-z_0|=R} |f(z)|$$

for any integer $n \geq 0$.

3.4.2. Proof of Higher Order Cauchy's Integral Formula. Now we discuss the proof of Theorem 3.11. From the (zeroth order) Cauchy's integral formula, we know:

$$f(\alpha) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z-\alpha}\,dz,$$

where α is a point on the domain Ω, and γ is a simple closed curve in Ω enclosing α.

Note that if $w \in \mathbb{C}$ is very small, $\alpha + w$ will still be enclosed by γ, and so we have:

$$f(\alpha+w) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{(z-\alpha-w)}\,dz.$$

Our first goal is to show Theorem 3.11 holds for $f'(\alpha)$, i.e. $n = 1$. Recall that:

$$f'(\alpha) = \lim_{w\to 0}\frac{f(\alpha+w)-f(\alpha)}{w}.$$

We will use the zeroth order Cauchy's integral formula to evaluate such a limit:

$$\begin{aligned} f'(\alpha) &= \frac{1}{2\pi i}\lim_{w\to 0}\frac{1}{w}\left(\oint_\gamma \frac{f(z)}{z-\alpha-w}\,dz - \oint_\gamma \frac{f(z)}{z-\alpha}\right)dz \\ &= \frac{1}{2\pi i}\lim_{w\to 0}\oint_\gamma f(z)\cdot\frac{1}{w}\left(\frac{1}{z-\alpha-w}-\frac{1}{z-\alpha}\right)dz. \end{aligned} \tag{3.6}$$

By straight-forward computation, we get:

$$\frac{1}{w}\left(\frac{1}{z-\alpha-w}-\frac{1}{z-\alpha}\right) = \frac{1}{(z-\alpha-w)(z-\alpha)}.$$

The integrand of (3.6) becomes $\dfrac{f(z)}{(z-\alpha-w)(z-\alpha)}$, which is bounded as $z \in \gamma$ is away from α and $\alpha + w$ when w is small, and that the holomorphic function f is bounded on γ by Extreme-Value Theorem. The length of γ is also bounded. Using Lebesgue

Dominated Covergence Theorem (commonly called LDCT in short), we can switch the limit and the integral sign of (3.6), and get:

$$\begin{aligned} f'(\alpha) &= \frac{1}{2\pi i} \oint_\gamma \lim_{w\to 0} f(z) \cdot \frac{1}{w} \left(\frac{1}{z-\alpha-w} - \frac{1}{z-\alpha} \right) dz \\ &= \frac{1}{2\pi i} \oint_\gamma \lim_{w\to 0} \frac{f(z)}{(z-\alpha-w)(z-\alpha)}\, dz \\ &= \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{(z-\alpha)^2}\, dz, \end{aligned}$$

proving Theorem 3.11 when $n = 1$.

The second and higher order cases of Theorem 3.11 can be proved by induction. Assume the theorem holds for some integer n:

$$f^{(n)}(\alpha) = \frac{n!}{2\pi i} \oint_\gamma \frac{f(z)}{(z-\alpha)^{n+1}}\, dz$$

for any α enclosed by γ. When w is very small, $\alpha + w$ is also enclosed by γ, hence it is also true that:

$$f^{(n)}(\alpha + w) = \frac{n!}{2\pi i} \oint_\gamma \frac{f(z)}{(z-\alpha-w)^{n+1}}\, dz.$$

Our next goal is to determine $f^{(n+1)}(\alpha)$ from the definition:

$$f^{(n+1)}(\alpha) = \lim_{w\to 0} \frac{f^{(n)}(\alpha+w) - f^{(n)}(\alpha)}{w}.$$

We leave it as an exercise:

Exercise 3.22. Follow the outline listed below, and complete the inductive proof of Theorem 3.11:

(a) Show that:

$$\begin{aligned} &\frac{1}{w}\left(\frac{1}{(z-\alpha-w)^{n+1}} - \frac{1}{(z-\alpha)^{n+1}} \right) \\ &= \frac{1}{(z-\alpha-w)(z-\alpha)} \sum_{j=0}^{n} \frac{1}{(z-\alpha-w)^j (z-\alpha)^{n-j}} \end{aligned}$$

(b) Using the induction assumption and LDCT, show that

$$\begin{aligned} &f^{(n+1)}(\alpha) \\ &= \frac{n!}{2\pi i} \oint_\gamma \lim_{w\to 0} \frac{f(z)}{(z-\alpha-w)(z-\alpha)} \sum_{j=0}^{n} \frac{1}{(z-\alpha-w)^j (z-\alpha)^{n-j}}\, dz. \end{aligned}$$

(c) Finally, complete the proof.

3.4.3. Liouville's Theorem. We now discuss an important consequence (Liouville's Theorem) of the higher order Cauchy's integral formula. Using this theorem, one can give a very short and elegant proof that every non-constant complex polynomial must have at least one root!

Theorem 3.13 (Liouville's Theorem)**.** *Any bounded entire function must be constant.*

Proof. The proof is a consequence of 1st-order Cauchy's integral formula. Suppose $f : \mathbb{C} \to \mathbb{C}$ is a entire function and that there exists $M > 0$ such that $|f(z)| \le M$ for any $z \in \mathbb{C}$.

Take an arbitrary $\alpha \in \mathbb{C}$, and consider the contour $|z-\alpha| = R$. By Theorem 3.11 with $n = 1$, we know:

$$f'(\alpha) = \frac{1}{2\pi i} \oint_{|z-\alpha|=R} \frac{f(z)}{(z-\alpha)^2}\, dz.$$

Then on the contour, we have:

$$\left|\frac{f(z)}{(z-\alpha)^2}\right| \leq \frac{M}{R^2},$$

and by Lemma 3.6, we can estimate that:

$$\left|\oint_{|z-\alpha|=R} \frac{f(z)}{(z-\alpha)^2}\, dz\right| \leq 2\pi R \cdot \frac{M}{R^2} = \frac{2\pi M}{R}.$$

Therefore, we have for any $\alpha \in \mathbb{C}$ and $R > 0$:

$$\left|f'(\alpha)\right| = \left|\frac{1}{2\pi i} \oint_{|z-\alpha|=R} \frac{f(z)}{(z-\alpha)^2}\, dz\right| \leq \frac{1}{2\pi} \cdot \frac{2\pi M}{R} \to 0 \text{ as } R \to +\infty.$$

This shows $f' \equiv 0$, and hence f is a constant function. □

Exercise 3.23. Why is it necessary that f is entire in the proof of Liouville's Theorem? Which step will it break down if f is holomorphic only on a proper subset of $\mathbb{C}$?

Exercise 3.24. Prove the following general version of Liouville's Theorem: Suppose $f : \mathbb{C} \to \mathbb{C}$ is an entire function, and there exists $M > 0$ and a nonnegative integer k such that:

$$|f(z)| \leq M\,|z|^k \text{ for any } z \in \mathbb{C}.$$

Show that f is a polynomial of degree at most k.

Exercise 3.25. Suppose $f : \mathbb{C} \to \mathbb{C}$ is an entire function satisfying:

$$\lim_{R\to+\infty} \sup_{|z|\geq R} \frac{|f(z)|}{R} = 0.$$

Show that f is a constant function.

Liouville's Theorem is a "luxury" for holomorphic functions. There are many non-constant bounded functions $f : \mathbb{R} \to \mathbb{R}$ that are (real) differentiable everywhere, while Liouville's Theorem says there is no non-constant bounded functions $f : \mathbb{C} \to \mathbb{C}$ which are complex differentiable everywhere.

The theorem has many surprising consequences. One of them is:

Corollary 3.14 (Fundamental Theorem of Algebra)**.** *Every non-constant complex polynomial $p(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$ must have at least one complex root.*

Proof. We prove by contradiction. If $p(z)$ has no root, then $\frac{1}{p(z)}$ is an entire function. Note that $|p(z)| \to \infty$ as $|z| \to \infty$, we have: $\frac{1}{p(z)} \to 0$ as $|z| \to \infty$. In particular, $\frac{1}{p(z)}$ is bounded. By Liouville's Theorem, $\frac{1}{p(z)}$ is constant, which is a contradiction. □

Remark 3.15. There are many proofs of Fundamental Theorem of Algebra, at least one in almost all important fields in mathematics. There is one in Topology using the concept of homotopy. There is even one geometric proof using Gauss-Bonnet's Theorem in Differential Geometry! Ironically, despite the name of the theorem, a purely algebraic proof has not yet been found. The most purest algebraic proof uses Galois Theory, but that proof is based on the fact that every real number has a real cubic root (which has to be justified using Intermediate-Value Theorem in Real Analysis).

Exercise 3.26. In the proof of Fundamental Theorem of Algebra (Corollary 3.14), we used the fact that $|p(z)| \to \infty$ as $|z| \to \infty$. Although this fact is intuitively clear since the dominant term $a_n z^n$ of p becomes very large when $|z| \to \infty$, try to prove this fact in a more rigorous way. Hint: try to show that if $p(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0$, then

$$|p(z)| \geq |z|^{n-1} \left(|a_n z| - |a_{n-1}| - \ldots - |a_0|\right)$$

whenever $|z| > 1$.

Exercise 3.27. Using Liouville's Theorem, show that if the image of an entire function $f : \mathbb{C} \to \mathbb{C}$ is disjoint from an open ball $B_\delta(z_0)$, then f is a constant function.

The above exercise gives a very powerful way for showing certain entire function must be constant. For example, if $f : \mathbb{C} \to \mathbb{C}$ is entire and maps $\mathbb{C}$ onto the upper-half plane in $\mathbb{C}$, then the image of f is disjoint from many open balls such as $B_{1/2}(-i)$. Hence it must be a constant.

3.5. Morera's Theorem

Before we stated Morera's Theorem, let's recall the proof of Cauchy-Goursat's Theorem. Using the holomorphic condition on f, Step 1 shows that $\oint_T f(z)\,dz = 0$ for any triangle contour T in the domain. Using this fact, Step 2 shows $F(z) := \int_{L(z_0,z)} f(\xi)\,d\xi$, where $L(z_0, z)$ is the straight path from a fixed point z_0 to z, is a primitive function for f, i.e. $F'(z) = f(z)$ on the convex domain Ω.

It is a nice observation that the proof in Step 2 requires only two facts about f, namely:

(1) f is continuous on Ω; and

(2) $\oint_T f(z)\,dz = 0$ for any triangle T in Ω.

Under these two conditions, the entire argument in Step 2 is still valid even if we don't assume that f is holomorphic on Ω. Step 2 shows $F'(z) = f(z)$ on Ω, hence proving $\oint_\gamma f(z)\,dz = 0$ for any closed curve γ in Ω.

The result that $F'(z) = f(z)$ on Ω has another implication: since the primitive function F is holomorphic on Ω (and its derivative is f), the higher order Cauchy's integral formula (Theorem 3.11) and Corollary 3.12 tell us that F is complex differentiable on Ω for infinitely many times. Certainly, it shows $f = F'$ is also complex differentiable on Ω for infinitely many times too. In particular, f is holomorphic on Ω.

To summarize, the preceding discussion proves the following remarkable result:

Theorem 3.16 (Morera's Theorem)**.** *If $f : \Omega \to \mathbb{C}$ is a continuous function on an open domain Ω, and*

$$\oint_T f(z)\,dz = 0$$

for any triangle contour T in Ω, then f is holomorphic on Ω.

Remark 3.17. Although convexity of the domain is needed in Step 2 of the proof of Cauchy-Goursat's Theorem, we do not need to assume Ω is convex when using Morera's Theorem. It is because complex differentiability is a *local* property. One can first restrict f on an open ball $B_\varepsilon(z_0)$ which is convex, then prove f is holomorphic on $B_\varepsilon(z_0)$. Simply repeat the same argument on all other open balls in the domain. It will show f is holomorphic on the whole Ω.

In practice, it seems more difficult to verify $\oint_T f(z)\,dz = 0$ for any triangle T than to show f is holomorphic directly. Nonetheless, Morera's Theorem can come in handy if we want to show holomorphicity of a function which is not quite explicit. In the last chapter, we may encounter functions defined in an integral form, such as the Gamma's function:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\,dt.$$

It is almost impossible to find an explicit, integral-free expression. Nonetheless, it is possible to show it is a holomorphic function using Morera's Theorem. The key idea is to show that $\oint_T \Gamma(z)\,dz = 0$ for any triangle T in the domain under consideration.

Example 3.10. Define $f : \Omega := \{z : \text{Re}(z) < 0\} \to \mathbb{C}$ by:

$$f(z) = \int_0^\infty \frac{e^{zt}}{t+1}\,dt.$$

Show that $f(z)$ is holomorphic on Ω.

Solution

First we show that f is defined on Ω: for any $z \in \Omega$ and $t \in [0, \infty)$, we have:

$$\left|\frac{e^{zt}}{t+1}\right| \leq \left|e^{zt}\right| \leq e^{xt}$$

(as usual, we denote $z = x + yi$). Note that:

$$\int_0^\infty e^{xt}\,dt = \left[\frac{1}{x}e^{xt}\right]_0^\infty = -\frac{1}{x} < \infty.$$

Hence, $\displaystyle\int_0^\infty \frac{e^{zt}}{t+1}\,dt$ is integrable.

It is quite difficult to find an explicit formula for $f(z)$, let alone its derivative. To show it is holomorphic, we are going to use Morera's Theorem: given any triangle T in Ω, we want to show $\displaystyle\int_T f(z)\,dz = 0$.

$$\begin{aligned}
\int_T f(z)\,dz &= \int_T \int_0^\infty \frac{e^{zt}}{t+1}\,dt\,dz \\
&= \int_0^\infty \int_T \frac{e^{zt}}{t+1}\,dz\,dt && \text{(Fubini's Theorem)} \\
&= \int_0^\infty 0\,dt && \left(\text{since } \frac{e^{zt}}{t+1} \text{ is holomorphic}\right) \\
&= 0
\end{aligned}$$

To justify the legitimacy of using Fubini's Theorem, we require the integral $\displaystyle\int_T \int_0^\infty \left|\frac{e^{zt}}{t+1}\right| dt\,|dz|$ to be finite. To verify this, we consider $\displaystyle\int_0^\infty \left|\frac{e^{zt}}{t+1}\right| dt \leq -\frac{1}{x}$, so that $\displaystyle\int_T \int_0^\infty \left|\frac{e^{zt}}{t+1}\right| dt\,|dz| \leq \int_T -\frac{1}{x}\,|dz|$, which is finite since x is away from 0 when z is on any triangle $T \subset \Omega$.

Hence by Morera's Theorem, f is holomorphic on Ω.

Exercise 3.28. Define $f : \mathbb{C}\backslash[0,1] \to \mathbb{C}$ by:

$$f(z) = \int_0^1 \frac{\sqrt{t}}{t-z}\,dt.$$

Show that f is holomorphic on its domain.

Exercise 3.29. Suppose $\{f_n\}_{n=1}^\infty$ is a sequence of holomorphic functions on an open domain Ω, and that f_n converges uniformly to f on Ω. Show that the limit function f is also holomorphic on Ω.

Exercise 3.30. Recall that the Riemann's zeta function $\zeta : \Omega \to \mathbb{C}$ is defined on $\Omega := \{z : \operatorname{Re}(z) > 1\}$ and by:

$$\zeta(z) := \sum_{n=1}^{\infty} \frac{1}{n^z} = \sum_{n=1}^{\infty} \frac{1}{e^{z \ln n}}.$$

(a) Show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^z}$ converges uniformly on $\Omega_\varepsilon := \{z : \operatorname{Re}(z) > 1 + \varepsilon\}$ for any $\varepsilon > 0$.

(b) Show that ζ is holomorphic on Ω.

Chapter 4

Taylor and Laurent Series

4.1. Taylor Series

4.1.1. Taylor Series for Holomorphic Functions. In Real Analysis, the Taylor series of a given function $f : \mathbb{R} \to \mathbb{R}$ is given by:

$$f(x_0) + f'(x_0)\,(x - x_0) + \frac{f''(x_0)}{2!}\,(x - x_0)^2 + \frac{f'''(x_0)}{3!}\,(x - x_0)^3 + \ldots$$

We have examined some convergence issues and applications of Taylor series in MATH 2033/2043. We also learned that even if the function f is infinitely differentiable everywhere on $\mathbb{R}$, its Taylor series may not converge to that function. In contrast, there is no such an issue in Complex Analysis: as long as the function $f : \mathbb{C} \to \mathbb{C}$ is holomorphic on an open ball $B_\delta(z_0)$, we can show the Taylor series of f:

$$f(z_0) + f'(z_0)\,(z - z_0) + \frac{f''(z_0)}{2!}\,(z - z_0)^2 + \frac{f'''(z_0)}{3!}\,(z - z_0)^3 + \ldots$$

converges pointwise to $f(z)$ on $B_\delta(z_0)$, and uniformly on any smaller ball. As we shall see, it thanks to Cauchy's integral formula. Moreover, the proof of Taylor Theorem in Complex Analysis is also much easier than that in Real Analysis, again thanks to Cauchy's integral formula.

In this chapter, it is more convenient to re-label the variables in the Cauchy's integral formula:

$$f^{(n)}(\alpha) = \frac{n!}{2\pi i} \oint_\gamma \frac{f(z)}{(z-\alpha)^{n+1}}\,dz \quad \longrightarrow \quad f^{(n)}(z) = \frac{n!}{2\pi i} \oint_\gamma \frac{f(\xi)}{(\xi - z)^{n+1}}\,d\xi.$$

For the *re-labelled* Cauchy's integral formula, we require the point z to be enclosed by the simple closed curve γ.

Theorem 4.1 (Taylor Theorem for Holomorphic Functions). *Given a complex-valued function f which is holomorphic on an open ball $B_R(z_0)$, the series:*

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}\,(z - z_0)^n$$

converges (pointwise) to $f(z)$ for any $z \in B_R(z_0)$.

Proof. Given any $z \in B_R(z_0)$, we let $\varepsilon > 0$ be small enough so that $|z - z_0| < R - \varepsilon$. For simplicity, denote $R' = R - \varepsilon$.

By Cauchy's integral formula, for any $z \in B_{R'}(z_0)$, we have:

$$f(z) = \frac{1}{2\pi i} \oint_{|\xi - z_0| = R'} \frac{f(\xi)}{\xi - z}\, d\xi.$$

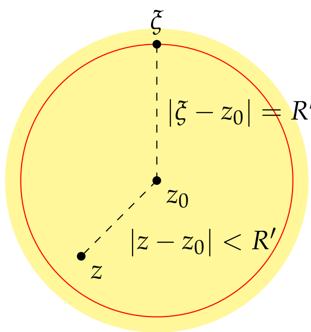
Then, the contour $|z - z_0| = R'$ lies inside the open ball $B_R(z_0)$. The key trick to prove the Taylor Theorem is rewriting $\dfrac{1}{\xi - z}$ as a geometric series. Recall that:

$$\frac{1}{1-w} = 1 + w + w^2 + \dots \quad \text{whenever } |w| < 1.$$

We first rewrite $\dfrac{1}{\xi - z}$ into this form:

$$\begin{aligned} \frac{1}{\xi - z} &= \frac{1}{(\xi - z_0) - (z - z_0)} = \frac{1}{\xi - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{\xi - z_0}} \\ &= \frac{1}{\xi - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\xi - z_0} \right)^n \end{aligned}$$

Here we have used the fact that $\left| \frac{z - z_0}{\xi - z_0} \right| < 1$. See the diagram below. The yellow ball is $B_R(z_0)$, and the red circle is $|\xi - z_0| = R'$.



Then, whenever $z \in B_{R'}(z_0)$, the function $f(z)$ can be expressed as:

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \oint_{|\xi - z_0| = R'} f(\xi) \cdot \frac{1}{\xi - z}\, d\xi \\ &= \frac{1}{2\pi i} \oint_{|\xi - z_0| = R'} \frac{f(\xi)}{\xi - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\xi - z_0} \right)^n d\xi \\ &= \frac{1}{2\pi i} \oint_{|\xi - z_0| = R'} \sum_{n=0}^{\infty} \frac{f(\xi)\,(z - z_0)^n}{(\xi - z_0)^{n+1}}\, d\xi \end{aligned} \tag{4.1}$$

Next we want to see whether we can switch the integral sign $\oint_{|\xi - z_0| = R'}$ and the infinite summation $\sum_{n=0}^{\infty}$. For this we need to show uniform convergence of the series below.

$$\sum_{n=0}^{\infty} \frac{f(\xi)\,(z - z_0)^n}{(\xi - z_0)^{n+1}}.$$

We use Weiestrass's M-test: for any ξ on the circle $\{|\xi - z_0| = R'\}$, we have:

$$\begin{aligned}\left|\frac{f(\xi)\,(z-z_0)^n}{(\xi-z_0)^{n+1}}\right| &\leq \left|\frac{(z-z_0)^n}{(\xi-z_0)^{n+1}}\right| \underbrace{\sup_{|\xi-z_0|=R'} |f(\xi)|}_{=:C_{R'}} \\ &= \frac{C_{R'}}{R'}\left(\frac{|z-z_0|}{R'}\right)^n\end{aligned}$$

Since $|z - z_0| < R'$, the series

$$\sum_{n=0}^{\infty} \frac{C_{R'}}{R'}\left(\frac{|z-z_0|}{R'}\right)^n$$

converges. Note that the above series does not depend on ξ (the integration variable). Hence by Weiestrass's M-test, the series $\displaystyle\sum_{n=0}^{\infty} \frac{f(\xi)\,(z-z_0)^n}{(\xi-z_0)^{n+1}}$ converges uniformly on the circle $\{|\xi - z_0| = R'\}$, thus allowing the switch between the integral sign and the summation sign in (4.1):

$$\begin{aligned}f(z) &= \frac{1}{2\pi i}\sum_{n=0}^{\infty} \oint_{|\xi-z_0|=R'} \frac{f(\xi)\,(z-z_0)^n}{(\xi-z_0)^{n+1}}\,d\xi \\ &= \sum_{n=0}^{\infty} \frac{1}{2\pi i}\left(\oint_{|\xi-z_0|=R'} \frac{f(\xi)}{(\xi-z_0)^{n+1}}\,d\xi\right)(z-z_0)^n \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n.\end{aligned}$$

In the last step we have used the higher order Cauchy's integral formula. □

Example 4.1. The function $f(z) = \sin z$ is an entire function. By straight-forward computations, its derivatives are given by:

$$\begin{aligned} f'(z) &= \cos z & f''(z) &= -\sin z \\ f^{(3)}(z) &= -\cos z & f^{(4)}(z) &= \sin z \\ &\vdots & &\vdots \end{aligned}$$

Inductively, it is easy to deduce that $f^{(2k+1)}(0) = (-1)^k$, and $f^{(2k)}(0) = 0$ for any integer $k \geq 0$. Hence, the Taylor series of f about 0 is given by:

$$\begin{aligned}f(z) &= \sum_{k=0}^{\infty} \frac{f^{(2k+1)}(0)}{(2k+1)!} z^{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k+1} \\ &= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ldots\end{aligned}$$

This series converges to $\sin z$ for any $z \in \mathbb{C}$, because $\sin z$ is entire (i.e. holomorphic on every ball $B_R(0)$).

Example 4.2. Consider the function $f(z) = \mathrm{Log}(z)$ which is holomorphic on $\Omega := \mathbb{C}\backslash\{x + 0i : x \leq 0\}$. Note that we can only apply Theorem 4.1 if the ball $B_R(z_0)$ is contained inside Ω.

Let's take $z_0 = 1$ as an example.

$$\begin{aligned} f'(z) &= \frac{1}{z} & f'(1) &= 1 \\ f''(z) &= -\frac{1}{z^2} & f''(1) &= -1 \\ f^{(3)}(z) &= \frac{2}{z^3} & f^{(3)}(1) &= 2 \\ f^{(4)}(z) &= -\frac{2\times 3}{z^4} & f^{(4)}(1) &= -2\times 3 \\ &\vdots & &\vdots \end{aligned}$$

Inductively, we deduce that $f^{(n)}(1) = (-1)^{n-1}\cdot(n-1)!$ for $n \geq 1$.

Therefore, the Taylor series for f about 1 is given by:

$$\begin{aligned} \mathrm{Log}(z) &= \mathrm{Log}(1) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\cdot(n-1)!}{n!}(z-1)^n = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}(z-1)^n \\ &= (z-1) - \frac{1}{2}(z-1)^2 + \frac{1}{3}(z-1)^3 - \frac{1}{4}(z-1)^4 + \dots \end{aligned}$$

Since f is holomorphic on $B_1(1)$ (but not on any larger ball centered at 1), the above Taylor series converges to $\mathrm{Log}(z)$ on $B_1(1)$.

Example 4.3. The Taylor series for some composite functions, such as e^{z^2}, can be derived by substitution instead of deducing the general n-th derivative of the function. For example:

$$\begin{aligned} e^z &= 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \dots \\ e^{z^2} &= 1 + z^2 + \frac{(z^2)^2}{2!} + \frac{(z^2)^3}{3!} + \frac{(z^2)^4}{4!} + \dots \\ &= 1 + z^2 + \frac{z^4}{2!} + \frac{z^6}{3!} + \frac{z^8}{4!} + \dots \end{aligned}$$

Since the series for e^z converges for any $z \in \mathbb{C}$, the series for e^{z^2} converges for any $z \in \mathbb{C}$ as well.

Similarly, by replacing z by $1-z$ in the Taylor series for $\mathrm{Log}(z)$, we get:

$$\mathrm{Log}(1-z) = -z - \frac{1}{2}z^2 - \frac{1}{3}z^3 - \frac{1}{4}z^4 - \dots$$

The series for $\mathrm{Log}(z)$ about 1 converges when $|z-1| < 1$, and so the above series for $\mathrm{Log}(1-z)$ converges when $|(1-z)-1| < 1$, i.e. $|z| < 1$.

Apart from using Theorem 4.1 to find the Taylor series of a given holomorphic function, we can also make use of the geometric series formula directly:

$$\frac{1}{1-w} = 1 + w + w^2 + \dots \quad \text{where } |w| < 1.$$

This method is particularly useful for functions whose n-th derivatives are tedious to compute.

Example 4.4. Consider the function:

$$f(z) = \frac{z-2}{(z+2)(z+3)}.$$

We are going to derive its Taylor series about 0. First, we do partial fractions on the function:

$$f(z) = \frac{5}{z+3} - \frac{4}{z+2}.$$

Then, we try to rewrite each term above in the form of $\frac{a}{1-w}$. Note that:

$$\begin{aligned}
\frac{5}{z+3} &= \frac{5}{3} \cdot \frac{1}{\frac{z}{3}+1} = \frac{5}{3} \cdot \frac{1}{1-\left(-\frac{z}{3}\right)} \\
&= \frac{5}{3}\sum_{n=0}^{\infty}\left(-\frac{z}{3}\right)^n = \sum_{n=0}^{\infty}(-1)^n \frac{5}{3^{n+1}} z^n && \text{(where } |z| < 3) \\
\frac{4}{z+2} &= \frac{4}{2} \cdot \frac{1}{\frac{z}{2}+1} = \frac{2}{1-\left(-\frac{z}{2}\right)} \\
&= 2\sum_{n=0}^{\infty}\left(-\frac{z}{2}\right)^n && \text{(where } |z| < 2) \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n-1}} z^n.
\end{aligned}$$

Hence, for $|z| < 2$, we have:

$$f(z) = \sum_{n=0}^{\infty}(-1)^n \frac{5}{3^{n+1}} z^n - \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n-1}} z^n = \sum_{n=0}^{\infty}(-1)^n\left(\frac{5}{3^{n+1}} - \frac{1}{2^{n-1}}\right) z^n.$$

To derive the Taylor series of f about other center (say 1), we can express $\frac{5}{z+3}$ and $\frac{4}{z+2}$ into:

$$\begin{aligned}
\frac{5}{z+3} &= \frac{5}{(z-1)+4} = \frac{5}{4} \cdot \frac{1}{1-\left(-\frac{z-1}{4}\right)} \\
&= \sum_{n=0}^{\infty}\frac{5}{4}\left(-\frac{z-1}{4}\right)^n && \text{(where } |z-1| < 4) \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n 5}{4^{n+1}}(z-1)^n \\
\frac{4}{z+2} &= \frac{4}{(z-1)+3} = \frac{4}{3} \cdot \frac{1}{1-\left(-\frac{z-1}{3}\right)} \\
&= \sum_{n=0}^{\infty}\frac{4}{3}\left(-\frac{z-1}{3}\right)^n && \text{(where } |z-1| < 3) \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n 4}{3^{n+1}}(z-1)^n.
\end{aligned}$$

Therefore, on $|z-1| < 3$, we have:

$$f(z) = \sum_{n=0}^{\infty}(-1)^n\left(\frac{5}{4^{n+1}} - \frac{4}{3^{n+1}}\right)(z-1)^n.$$

Exercise 4.1. Derive the Taylor series of each function below about the given center z_0:

(a) $f(z) = \sin 2z; \quad z_0 = \frac{2\pi}{3}$

(b) $f(z) = \cos 3z; \quad z_0 = \pi$

(c) $f(z) = e^{-z^3}; \quad z_0 = 0$

(d) $f(z) = \text{Log}(3 - 2z); \quad z_0 = 1$

Exercise 4.2. Find the Taylor series about 0 of the functions below up to the z^4 term:

(a) $f(z) = e^{-z} \cos z$

(b) $f(z) = \text{Log}(1 - e^z)$

Exercise 4.3. Find the Taylor series about z_0 of the function below without using Theorem 4.1. State its radius of convergence.

(a) $f(z) = \dfrac{1}{(z-1)(z-2)}, \quad z_0 = 0$

(b) $f(z) = \dfrac{1}{(z-1)(z-2)}, \quad z_0 = i$

Exercise 4.4. Let α, β and z_0 be three distinct complex numbers. Consider the function

$$f(z) = \frac{1}{(z-\alpha)(z-\beta)}.$$

Find the Taylor series about z_0 of the above function, and state its radius of convergence.

Exercise 4.5. Let α be a fixed non-zero complex number. Consider the principal branch of $(1+z)^\alpha$:

$$(1+z)^\alpha := e^{\alpha \text{Log}(1+z)}.$$

Show that its Taylor series about 0 is given by:

$$(1+z)^\alpha = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\cdot(\alpha-n+1)}{n!} z^n.$$

State its radius of convergence.

4.1.2. Taylor Series with Remainder Term. In Real Analysis, the Taylor Theorem with a remainder term asserts that for any smooth (C^∞) function $f : \mathbb{R} \to \mathbb{R}$, we have:

$$f(x) = \sum_{n=0}^{N-1} \frac{f^{(n)}(a)}{n!}(x-a)^n + \underbrace{\frac{1}{(N-1)!}\int_a^x (x-t)^{N-1} f^{(N)}(t)\, dt}_{=: R_N(x)}$$

The last integral term, commonly denoted as $R_N(x)$, measures how fast the Taylor series converges to $f(x)$ as $N \to \infty$. If $\lim\limits_{N\to\infty} R_N(x) \to 0$ for any x in an interval I, then the Taylor series $\sum\limits_{n=0}^{\infty} \dfrac{f^{(n)}(a)}{n!}(x-a)^n$ converges (pointwise) to $f(x)$ for any $x \in I$. If furthermore, we have:

$$\lim_{N\to\infty} \sup_{x\in I} |R_N(x)| \to 0,$$

then the Taylor series converges uniformly to f on I. However, it is often not easy to show $R_N \to 0$ as the N-th derivative $f^{(N)}$ may not be easy to find.

Back to Complex Analysis, we will soon derive the remainder term for the Taylor series for holomorphic functions. One good thing about the complex version is that the remainder involves only f, but not its derivatives, making it much easier to handle the convergence issue of complex Taylor series. It again thanks to Cauchy's integral formula.

Proposition 4.2. *Let f be a holomorphic function defined on $B_R(z_0)$, then for any $z \in B_R(z_0)$, and any simple closed curve γ in $B_R(z_0)$ enclosing both z and z_0, we have:*

$$f(z) = \sum_{n=0}^{N-1} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n + \underbrace{\frac{1}{2\pi i}\oint_\gamma \frac{f(\xi)}{\xi - z}\left(\frac{z-z_0}{\xi - z_0}\right)^N d\xi}_{=:R_N(z)}$$

Proof. We only outline the proof since it is modified from the proof of Theorem 4.1. Using Cauchy's integral formula, we first have:

$$f(z) = \frac{1}{2\pi i}\oint_\gamma \frac{f(\xi)}{\xi - z}\,d\xi.$$

The key step in the proof of Theorem 4.1 is to write:

$$\frac{1}{\xi - z} = \frac{1}{(\xi - z_0)-(z-z_0)} = \frac{1}{\xi - z_0}\cdot\frac{1}{1-\frac{z-z_0}{\xi-z_0}},$$

so that when $\left|\dfrac{z-z_0}{\xi-z_0}\right| < 1$, we have:

$$\frac{1}{1-\frac{z-z_0}{\xi-z_0}} = \sum_{n=0}^{\infty}\left(\frac{z-z_0}{\xi-z_0}\right)^n.$$

Now, to prove this proposition, we modify the above key step a bit, by considering:

$$\frac{1-\left(\frac{z-z_0}{\xi-z_0}\right)^N}{1-\frac{z-z_0}{\xi-z_0}} = \sum_{n=0}^{N-1}\left(\frac{z-z_0}{\xi-z_0}\right)^n.$$

We leave the rest of the proof for readers (which is a good exercise to test your understanding of the proof of Theorem 4.1). □

Exercise 4.6. Complete the proof of Proposition 4.2.

Exercise 4.7. Consider the remainder term in Proposition 4.2:

$$R_N(z) = \frac{1}{2\pi i}\oint_\gamma \frac{f(\xi)}{\xi - z}\left(\frac{z-z_0}{\xi-z_0}\right)^N d\xi.$$

Let γ be the circle $|\xi - z_0| = R'$ such that $|z-z_0| < R' < R$. Show that:

$$|R_N(z)| \le \frac{R'}{R' - |z-z_0|}\left(\frac{|z-z_0|}{R'}\right)^N \sup_{|\xi - z_0| = R'} |f(\xi)|.$$

Exercise 4.8. Let f be a holomorphic function on $B_R(z_0)$. Using this estimate obtained in Exercise 4.7, deduce that the Taylor series $\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$ converges uniformly to $f(z)$ on any smaller ball $B_r(z_0)$ where $0 < r < R$.

Remark 4.3. The uniform convergence of $\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$ has many remarkable consequences as discussed in MATH 3033/3043. For instance, one can integrate a Taylor series term-by-term.

Exercise 4.9. Consider the Taylor series for $-\mathrm{Log}(1-\xi)$:

$$-\mathrm{Log}(1-\xi) = \xi + \frac{\xi^2}{2} + \frac{\xi^3}{3} + \cdots + \frac{\xi^n}{n} + \cdots \quad \text{where } |\xi| < 1$$

Show that:

$$\frac{z^2}{2} + \frac{z^4}{3 \times 4} + \cdots + \frac{z^{n+1}}{n(n+1)} + \cdots = (1-z)\mathrm{Log}(1-z) + z$$

for any $z \in B_1(0)$.

Exercise 4.10. Show that for any $z \in \mathbb{C}$, we have:

$$\int_0^z e^{-\xi^2}\, d\xi = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{n!(2n+1)}.$$

4.2. Laurent Series

A Laurent series is a "power series" with negative powers of $z - z_0$ as well. The general form of a Laurent series about z_0 is:

$$\begin{aligned}&\sum_{n=1}^{\infty} \frac{a_{-n}}{(z-z_0)^n} + \sum_{n=0}^{\infty} a_n (z-z_0)^n \\ &= \cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots\end{aligned}$$

which can be abbreviated as:

$$\sum_{n=-\infty}^{\infty} a_n (z-z_0)^n.$$

A Laurent series is said to be convergent if both $\displaystyle\sum_{n=1}^{\infty} \frac{a_{-n}}{(z-z_0)^n}$ and $\displaystyle\sum_{n=0}^{\infty} a_n (z-z_0)^n$ converge.

If $a_{-n} = 0$ for any negative $-n$, then the Laurent series is a Taylor series. On the other hand, if $a_{-n} \neq 0$ for some negative $-n$, then the Laurent series is undefined when $z = z_0$. As such, a Laurent series is usually defined on an *annular* region $\{r < |z - z_0| < R\}$ instead of a ball centered at z_0. From now on, we denote such an annular region by:

$$A_{R,r}(z_0) := \{z \in \mathbb{C} : r < |z - z_0| < R\}$$

where $R, r \in [0, \infty]$. Note that:

$$\begin{aligned}A_{R,0}(z_0) &= B_R(z_0) \backslash \{z_0\} \\ A_{\infty,r}(z_0) &= \mathbb{C} \backslash \overline{B_r(z_0)} \qquad \text{for } r > 0 \\ A_{\infty,0}(z_0) &= \mathbb{C} \backslash \{z_0\}.\end{aligned}$$

4.2.1. Examples of Laurent Series. While a Taylor series gives an analytic expression for a holomorphic function on a *ball*, a Laurent series gives an analytic expression for a function that has a singularity at the center of a ball. Before we discuss a general theorem about Laurent series, let's first look at some examples of writing a function as a Laurent series:

Example 4.5. Consider the function $f : \mathbb{C} \backslash \{1, 2\} \to \mathbb{C}$ defined by:

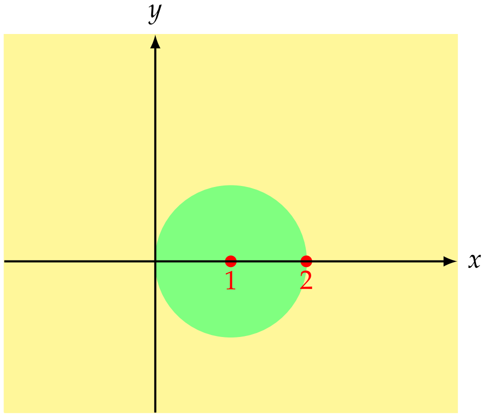
$$f(z) = \frac{1}{(z-1)(z-2)}.$$

It is holomorphic on its domain $\mathbb{C} \backslash \{1, 2\}$. Let's express the above function as a Laurent series about 1:

$$\begin{aligned}\frac{1}{z-2} &= \frac{1}{(z-1)-1} = -\frac{1}{1-(z-1)} \\ &= -\sum_{n=0}^{\infty} (z-1)^n \qquad \text{where } |z-1| < 1.\end{aligned}$$

Hence, on $\in A_{1,0}(1)$, i.e. the green annulus in the figure below, we have:

$$\begin{aligned}f(z) &= \frac{1}{z-1} \cdot \frac{1}{z-2} = -\frac{1}{z-1} \sum_{n=0}^{\infty} (z-1)^n = -\sum_{n=0}^{\infty} (z-1)^{n-1} \\ &= -\frac{1}{z-1} - 1 - (z-1) - (z-1)^2 + \cdots.\end{aligned}$$



However, the green annulus $A_{1,0}(1)$ is not the only annulus centered at 1 on which f is holomorphic. There is another one $A_{\infty,1}(1) = \{1 < |z-1|\}$ centered at 1, i.e. the yellow annulus in the above figure, on which f is also holomorphic. It is also possible to express f as a Laurent series on this yellow annulus:

$$\begin{aligned}\frac{1}{z-2} &= \frac{1}{(z-1)-1} = \frac{1}{z-1}\cdot\frac{1}{1-\frac{1}{z-1}}\\ &= \frac{1}{z-1}\sum_{n=0}^{\infty}\left(\frac{1}{z-1}\right)^n && \text{(where } |z-1|>1\text{)}\\ &= \sum_{n=0}^{\infty}\frac{1}{(z-1)^{n+1}}\end{aligned}$$

Hence, on the yellow annulus $A_{\infty,1}(1)$, the function f can be expressed as the following Laurent series:

$$f(z) = \frac{1}{(z-1)(z-2)} = \frac{1}{z-1}\cdot\frac{1}{z-2} = \sum_{n=0}^{\infty}\frac{1}{(z-1)^{n+2}}.$$

Example 4.6. Find the Laurent series about 0 of the function:

$$f(z) = z^2 e^{\frac{1}{z}}$$

defined on $\mathbb{C}\backslash\{0\}$.

Solution

First recall that the Taylor series for e^w is:

$$e^w = \sum_{n=0}^{\infty}\frac{w^n}{n!} \quad \text{for any } w \in \mathbb{C}.$$

Substitute $w = \frac{1}{z}$, where $z \neq 0$, we get:

$$e^{\frac{1}{z}} = \sum_{n=0}^{\infty}\frac{1}{n!z^n},$$

and hence:

$$\begin{aligned} f(z) &= z^2 e^{\frac{1}{z}} \\ &= z^2 \sum_{n=0}^{\infty} \frac{1}{n! z^n} \\ &= \sum_{n=0}^{\infty} \frac{1}{n! z^{n-2}} \\ &= z^2 + z + \frac{1}{2} + \frac{1}{3! z} + \frac{1}{4! z^2} + \cdots \end{aligned}$$

Exercise 4.11. Express the function:

$$f(z) = \frac{1}{z(z-1)(z-2)}$$

as a Laurent series about 0 in each of the following annuli:

$$A_{1,0}(0), \quad A_{2,1}(0), \quad A_{\infty,2}(0).$$

Also, express the function as a Laurent series about 1 in each of the following annuli:

$$A_{1,0}(1), \quad A_{\infty,1}(1).$$

[Hint: First expand f into partial fractions.]

Exercise 4.12. Find all possible Laurent (or Taylor) series about 1 for the function:

$$f(z) = \frac{1}{z^2 - 2z}.$$

For each series, state the annulus or ball on which it converges.

Exercise 4.13. Find all possible Laurent (or Taylor) series about each z_0 below for the function $f(z) = \frac{1}{z}$.

(a) $z_0 = 0$

(b) $z_0 = 1$

(c) $z_0 = i$

For each series, state the annulus or ball on which it converges.

Exercise 4.14. Show that for any w such that $|w| < 1$, we have:

$$\frac{1}{(1+w)^3} = \sum_{n=2}^{\infty} (-1)^n \frac{n(n-1)}{2} w^{n-2}.$$

[Hint: use Exercise 4.5]

Hence, find all possible Laurent or Taylor series about i for the function:

$$f(z) = \frac{1}{z^3}.$$

For each series, state the annulus or ball on which it converges.

Exercise 4.15. Find the Laurent series about 1 on the annulus $A_{\infty,0}(1)$ for the functions:

$$f(z) = \sin\frac{1}{z-1} \quad \text{and} \quad g(z) = \cos\frac{1}{z-1}.$$

Hence, find the Laurent series about 1 on $A_{\infty,0}(1)$ for:

$$h(z) = \sin\frac{z}{z-1}.$$

Exercise 4.16. What's wrong with the following argument?

$$\frac{z}{1-z} = z\sum_{n=0}^{\infty} z^n = z + z^2 + z^3 + \cdots$$

$$\frac{z}{1-z} = -\frac{1}{1-\frac{1}{z}} = -\sum_{n=0}^{\infty}\frac{1}{z^n} = -1 - \frac{1}{z} - \frac{1}{z^2} - \cdots$$

By subtraction, we get:

$$0 = \cdots + \frac{1}{z^2} + \frac{1}{z} + 1 + z + z^2 + \cdots = \sum_{n=-\infty}^{\infty} z^n.$$

4.2.2. Existence Theorem of Laurent Series. We have learned how to express a function into a Laurent series through examples. Next, we proved a general existence theorem of Laurent series for *any* holomorphic function on *any* annular region.

Theorem 4.4 (Laurent Theorem). *Let f be a holomorphic function defined on an annulus $A_{R,r}(z_0) := \{r < |z - z_0| < R\}$ where $R, r \in [0, \infty]$, then f can be expressed as a Laurent series about z_0 on the annulus $A_{R,r}(z_0)$:*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$$

for some complex numbers c_n's.

Proof. The proof is similar to that of Taylor's series, but is a bit trickier since an annulus is not simply-connected and so Cauchy's integral formula cannot be applied directly.

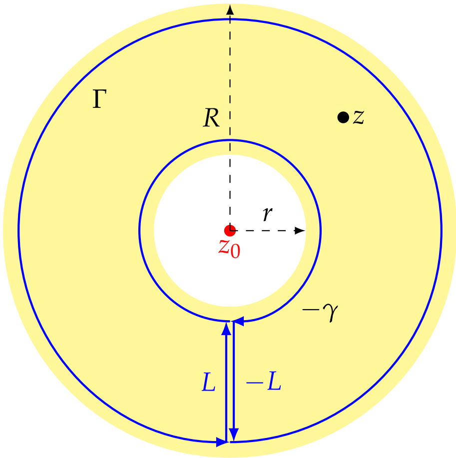
Fix $z \in A_{R,r}(z_0)$, we first consider a simple closed curve Γ in $A_{R,r}(z_0)$ which encloses both z and z_0 (just like in the proof of Taylor's Theorem). However, we cannot apply Cauchy's integral formula on the integral:

$$\frac{1}{2\pi i}\oint_\Gamma \frac{f(\xi)}{\xi - z_0}\,d\xi$$

since f is not holomorphic on $B_R(z_0)$. However, we can construct a "key-hole" contour:

$$C = \Gamma + L - \gamma - L$$

where $-\gamma$ is the clockwise circle, and L is a straight-path as shown in the figure below. We can pick Γ to be the circle with radius slightly smaller than R, and γ with radius slightly bigger than r so that C encloses z.



Under such a construction, the contour $C = \Gamma + L - \gamma - L$ is a simple closed curve and the region enclosed by C becomes simply connected. We can then apply Cauchy's integral formula:

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \oint_C \frac{f(\xi)}{\xi - z}\, d\xi \\ &= \frac{1}{2\pi i} \left(\oint_\Gamma + \oint_L - \oint_\gamma - \oint_L \right) \frac{f(\xi)}{\xi - z}\, d\xi \\ &= \frac{1}{2\pi i} \oint_\Gamma \frac{f(\xi)}{\xi - z}\, d\xi - \frac{1}{2\pi i} \oint_\gamma \frac{f(\xi)}{\xi - z}\, d\xi. \end{aligned} \tag{4.2}$$

The key idea of the proof is to express the integral over Γ as a series of non-negative powers, and the integral over γ as a series of negative powers.

When $\xi \in \Gamma$, we have $|z - z_0| < |\xi - z_0|$, so:

$$\frac{1}{\xi - z} = \frac{1}{(\xi - z_0) - (z - z_0)} = \frac{1}{\xi - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{\xi - z_0}} = \frac{1}{\xi - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\xi - z_0} \right)^n$$

Hence, the first integral becomes:

$$\frac{1}{2\pi i} \oint_\Gamma \frac{f(\xi)}{\xi - z}\, d\xi = \frac{1}{2\pi i} \oint_\Gamma \sum_{n=0}^{\infty} \frac{f(\xi)}{\xi - z_0} \left(\frac{z - z_0}{\xi - z_0} \right)^n d\xi.$$

In order to switch the infinite summation and the integral sign, we justify that the series converges uniformly on $\xi \in \Gamma$. Suppose Γ has radius R', then for any $\xi \in \Gamma$:

$$\left| \frac{f(\xi)}{\xi - z_0} \left(\frac{z - z_0}{\xi - z_0} \right)^n \right| \leq \frac{1}{R'} \left(\frac{|z - z_0|}{R'} \right)^n \sup_\Gamma |f|.$$

Note that $\sup_\Gamma |f|$ is finite by compactness of Γ. Since $|z - z_0| < R'$, the geometric series

$$\sum_{n=0}^{\infty} \frac{1}{R'} \left(\frac{|z - z_0|}{R'} \right)^n \sup_\Gamma |f|$$

converges. By Weierstrass's M-test, the series

$$\sum_{n=0}^{\infty} \frac{f(\xi)}{\xi - z_0} \left(\frac{z - z_0}{\xi - z_0} \right)^n$$

converges uniformly on $\xi \in \Gamma$, so one can switch the summation and integral signs and get:

$$\frac{1}{2\pi i}\oint_\Gamma \frac{f(\xi)}{\xi - z}\,d\xi = \sum_{n=0}^\infty \left(\frac{1}{2\pi i}\oint_\Gamma \frac{f(\xi)}{(\xi - z_0)^{n+1}}\,d\xi\right)(z-z_0)^n. \tag{4.3}$$

The second integral can be handled similarly. The difference is that when $\xi \in \gamma$, we have $|\xi - z_0| < |z - z_0|$ instead. We instead write:

$$\frac{1}{\xi - z} = \frac{1}{(\xi - z_0) - (z - z_0)} = -\frac{1}{z - z_0}\cdot\frac{1}{1 - \frac{\xi - z_0}{z - z_0}} = -\frac{1}{z - z_0}\sum_{n=0}^\infty \left(\frac{\xi - z_0}{z - z_0}\right)^n$$

Hence,

$$\frac{1}{2\pi i}\oint_\gamma \frac{f(\xi)}{\xi - z}\,d\xi = -\frac{1}{2\pi i}\oint_\gamma \sum_{n=0}^\infty \frac{f(\xi)}{z - z_0}\left(\frac{\xi - z_0}{z - z_0}\right)^n d\xi.$$

We leave it as an exercise for readers to argue that the series converges uniformly on $\xi \in \gamma$ so that we can switch the integral and summations signs:

$$\frac{1}{2\pi i}\oint_\gamma \frac{f(\xi)}{\xi - z}\,d\xi = -\sum_{n=0}^\infty \left(\frac{1}{2\pi i}\oint_\gamma f(\xi)(\xi - z_0)^n\,d\xi\right)\frac{1}{(z-z_0)^{n+1}}. \tag{4.4}$$

Combining (4.2), (4.3) and (4.4), we obtain:

$$\begin{aligned} f(z) &= \underbrace{\sum_{n=1}^\infty \left(\frac{1}{2\pi i}\oint_\gamma f(\xi)(\xi - z_0)^{n-1}\,d\xi\right)\frac{1}{(z-z_0)^n}}_{(4.4)} \\ &\quad + \underbrace{\sum_{n=0}^\infty \left(\frac{1}{2\pi i}\oint_\Gamma \frac{f(\xi)}{(\xi - z_0)^{n+1}}\,d\xi\right)(z-z_0)^n}_{(4.3)} \end{aligned}$$

It completes the proof by defining

$$\begin{aligned} c_{-n} &= \frac{1}{2\pi i}\oint_\gamma f(\xi)(\xi - z_0)^{n-1}\,d\xi && \text{for } -n = -1, -2, -3, \cdots \\ c_n &= \frac{1}{2\pi i}\oint_\Gamma \frac{f(\xi)}{(\xi - z_0)^{n+1}}\,d\xi && \text{for } n = 0, 1, 2, 3, \cdots \end{aligned}$$

□

Exercise 4.17. Justify the claim in the above proof that the series:

$$\sum_{n=0}^\infty \frac{f(\xi)}{z - z_0}\left(\frac{\xi - z_0}{z - z_0}\right)^n$$

converges uniformly on $\xi \in \gamma$ (and z, z_0 are considered to be fixed).

Remark 4.5. Although from the proof of Theorem 4.4 one can express the coefficient c_n's of a Laurent series in terms of contour integrals, we do not usually find the coefficients this way since these contour integrals may not be easy to compute.

4.2.3. Laurent Series with Remainders. Similar to Taylor series, one can refine Theorem 4.4 a bit by deriving the remainder terms. Using the remainder terms, one can argue that for a holomorphic function f defined on an annulus $A_{R,r}(z_0)$, the Laurent series converges *uniformly* to f on every smaller annulus $A_{R',r'}(z_0)$ (where $r < r' < R' < R$). This result is remarkable as it allows us to integrate a Laurent's series term-by-term.

Proposition 4.6. *Let f be a holomorphic function on the annulus $A_{R,r}(z_0)$, where $0 \leq r < R \leq \infty$. Then, for each positive integer N and $z \in A_{R,r}(z_0)$, we have:*

$$f(z) = \sum_{n=1}^{N} \left(\frac{1}{2\pi i} \oint_{\gamma} f(\xi)(\xi - z_0)^{n-1}\, d\xi \right) \frac{1}{(z-z_0)^n} + \underbrace{\frac{1}{2\pi i} \oint_{\gamma} \frac{f(\xi)}{z-\xi} \left(\frac{\xi - z_0}{z - z_0} \right)^N d\xi}_{=:r_N(z)}$$

$$+ \sum_{n=0}^{N-1} \left(\frac{1}{2\pi i} \oint_{\Gamma} \frac{f(\xi)}{(\xi - z_0)^{n+1}}\, d\xi \right) (z - z_0)^n + \underbrace{\frac{1}{2\pi i} \oint_{\Gamma} \frac{f(\xi)}{\xi - z} \left(\frac{z - z_0}{\xi - z_0} \right)^N d\xi}_{=:R_N(z)}$$

where Γ and γ are any pair of circles in $A_{R,r}(z_0)$ centered at z_0 such that z is bounded between Γ and γ.

Proof. We leave the proof of Proposition 4.6 as an exercise. It is very similar to the proof of Proposition 4.2 for Taylor series. Readers should first digest the whole proof of Proposition 4.2, then write up a coherent proof for this proposition. □

Exercise 4.18. Prove Proposition 4.6. Using this, show that the Laurent series about z_0 for f converges uniformly on every smaller annulus $A_{R',r'}(z_0)$ where $r < r' < R' < R$. [Hint: show that both remainders $R_N(z)$ and $r_N(z)$ converge uniformly to 0 on $A_{R',r'}(z_0)$ as $N \to \infty$.]

One practical use of uniform convergence is term-by-term integration. For example, consider the function $f(z) = z^2 e^{\frac{1}{z}}$, which can be expressed as a Laurent series:

$$z^2 e^{\frac{1}{z}} = z^2 + z + \frac{1}{2} + \frac{1}{3!z} + \frac{1}{4!z^2} + \cdots$$

Then, to integrate $f(z)$ over the circle $|z| = 1$, we can integrate the Laurent series term-by-term:

$$\begin{aligned} &\oint_{|z|=1} z^2 e^{\frac{1}{z}}\, dz \\ &= \oint_{|z|=1} z^2\, dz + \oint_{|z|=1} z\, dz + \oint_{|z|=1} \frac{1}{2}\, dz + \oint_{|z|=1} \frac{1}{3!z}\, dz + \oint_{|z|=1} \frac{1}{4!z^2}\, dz + \cdots \\ &= 0 + 0 + 0 + \frac{2\pi i}{3!} + 0 + 0 + \cdots = \frac{\pi i}{6}. \end{aligned}$$

Recall that for any simple closed γ enclosing the origin, the contour integral

$$\oint_{\gamma} z^n\, dz$$

is non-zero *only* when $n = -1$.

From the above example, we see the significance of expressing a function as a Laurent series. To compute a contour integral, it often amounts to finding the coefficient c_{-1} of the Laurent series. It leads to the develop of residue theory to be discussed in the next section.

4.3. Residue Calculus

In this section we discuss both theory and applications of an important topic in Complex Analysis: residue calculus. It has many powerful applications on evaluations of some complicated *real* integrals that physicists and engineers often encounter.

4.3.1. Classification of Singularities. A singular point, or singularity, refers to a point z_0 at which a function f fails to be complex differentiable. For instance, 1 and 2 are singularities of the function:

$$f(z) = \frac{1}{(z-1)(z-2)}.$$

It is possible for a function to have infinitely many singularities, such as:

$$g(z) = \frac{1}{\sin z}$$

whose singularities are 0, $\pm\pi$, $\pm 2\pi$, etc.

Some functions even have singularities that form a "cluster". For instance, consider:

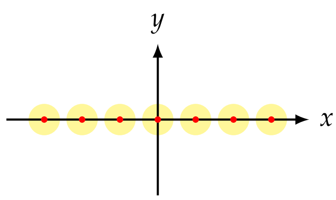
$$h(z) = \frac{1}{\sin\frac{1}{z}}$$

which is singular when $z \in \{\frac{\pi}{n} : n \in \mathbb{Z}\} \cup \{0\}$. The singular set $\{\frac{\pi}{n} : n \in \mathbb{Z}\} \cup \{0\}$ form a cluster around 0, meaning there is no way to find an annulus $A_{R,0}(0)$ centered at 0 such that h is holomorphic on $A_{R,0}(0)$. Hence, it is not possible to analyze the function h by a Laurent series about 0 on $A_{R,0}(0)$.

In order to utilize Laurent series, we focus on those singularities that can be *isolated* from others. We have the following terminology:

Definition 4.7 (Isolated Singularity)**.** A point z_0 is said to be an *isolated singularity* for a function $f(z)$ if there exists $\varepsilon > 0$ such that f is holomorphic on $A_{\varepsilon,0}(z_0) = B_\varepsilon(z_0)\backslash\{z_0\}$.

For the function $g(z) = \dfrac{1}{\sin z}$, all singularities are isolated as depicted in the diagram below:



Around every isolated singularity z_0 of a function $f(z)$, it is possible (thanks to Theorem 4.4) to express the function f as a Laurent series on a small annulus $A_{\varepsilon,0}(z_0)$:

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n.$$

Depending on the *smallest* n such that $c_n \neq 0$, we have the following terminology:

- If $c_{-1} = c_{-2} = c_{-3} = \cdots = 0$, then z_0 is said to be a *removable singularity* of f. For instance, 0 is such a singularity for the function:

$$\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots$$

- If k is a positive integer such that $c_{-k} \neq 0$ while $c_{-(k+1)} = c_{-(k+2)} = \cdots = 0$, then z_0 is said to be a *pole of order* k of f. For instance, 0 is a pole of order 3 for the function:
$$\frac{\sin z}{z^4} = \frac{1}{z^3} - \frac{1}{3!z} + \frac{z}{5!} - \frac{z^3}{7!} + \cdots$$
Moreover, a pole of order 1 is usually called a *simple pole*.
- If $c_{-n} \neq 0$ for infinitely many negative integers $-n$, then z_0 is said to be an *essential singularity*. For instance, 0 is such a singularity for the function:
$$e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots$$

If z_0 is a removable singularity for $f : B_R(z_0) \backslash \{z_0\} \to \mathbb{C}$, then one can define $f(z_0) := c_0$ so that f extends to become a holomorphic function on $B_R(z_0)$. That's why we can z_0 removable. Similarly, if z_0 is a pole of order n for $f : A_{R,0}(z_0) \to \mathbb{C}$, then $(z - z_0)^n f(z)$ extends to become a holomorphic function on $B_R(z_0)$. However, a function with an essential singularity cannot be extended to become a holomorphic function in a similar way (that's why we call it essential).

To determine the order of a pole, we may simply find its Laurent series expansion. However, sometimes it is not easy to do so, such as 0 for the function $\dfrac{1}{\sin z}$. An alternative way to find the order of a pole is to consider the limit:
$$\lim_{z \to z_0} (z - z_0)^k f(z).$$
If k is an integer such that:
$$\lim_{z \to z_0} (z - z_0)^k f(z) \text{ exists and is non-zero,}$$
then the order of the pole z_0 is k. For example, since:
$$\lim_{z \to 0} \frac{z}{\sin z} = 1 \neq 0,$$
0 is a pole of order 1 for the function $\dfrac{1}{\sin z}$. Hence, one can express this function as a Laurent series on a small annulus $A_{\varepsilon,0}(0)$:
$$\frac{1}{\sin z} = \frac{c_{-1}}{z} + c_0 + c_1 z + c_2 z^2 + \cdots$$
Multiplying z on both sides, we get:
$$\frac{z}{\sin z} = c_{-1} + c_0 z + c_1 z^2 + c_2 z^3 + \cdots$$
and by letting $z \to 0$, we can also conclude that $c_{-1} = 1$. Therefore, if γ is a simple close curve enclosing 0 in this small annulus $A_{\varepsilon,0}(0)$, then we have:
$$\oint_\gamma \frac{1}{\sin z}\, dz = \oint_\gamma \left(\frac{1}{z} + c_0 + c_1 z + c_2 z^2 + \cdots \right) dz = \oint_\gamma \frac{1}{z}\, dz + 0 + 0 + \cdots = 2\pi i.$$

Exercise 4.19. Find all isolated singularities of each function below, and classify the nature of these singularities. For poles, state also their orders.

(a) $f(z) = \dfrac{e^z - 1}{z}$

(b) $g(z) = \dfrac{\mathrm{Log}(z)}{(z-3)^5}$

(c) $h(z) = z^{4023} \cos \frac{1}{z}$

4.3.2. Residues. As illustrated in many examples, the coefficient c_{-1} of a Laurent series plays a special role in evaluating a contour integral. It is special in a sense that for an integer n,

$$\oint_{|z-z_0|=\varepsilon} (z-z_0)^n \, dz = \begin{cases} 2\pi i & \text{if } n=-1 \\ 0 & \text{otherwise} \end{cases}$$

Hence, to integrate a Laurent series, one only needs to integrate the term $\frac{c_{-1}}{z-z_0}$, which can be done by Cauchy's integral formula. In view of the special role of c_{-1}, we define:

Definition 4.8 (Residues)**.** Let z_0 be an isolated singularity of $f(z)$ such that the Laurent series about z_0 for f on some annulus $A_{\varepsilon,0}(z_0)$ is given by:

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n,$$

then we denote and define the *residue of f at z_0* by:

$$\operatorname{Res}(f, z_0) := c_{-1}.$$

Example 4.7. Find the residue of the function

$$f(z) = \frac{z^2-2z}{(z+1)^2(z^2+4)}$$

at each of its isolated singularity.

Solution

The denominator has roots -1, $2i$ and $-2i$, hence they are isolated singularities of f. In this solution, we will decompose $f(z)$ into partial fractions. It may not be a pleasant way finding residues, but we will later provide an easier way.

Note that f is a rational function, we can break it into partial fractions:

$$f(z) = \frac{A}{(z+1)^2} + \frac{B}{z+1} + \frac{C}{z-2i} + \frac{D}{z+2i}.$$

We leave it as an exercise for readers to determine the value of A, B, C and D. One should be able to get:

$$f(z) = \frac{\frac{3}{5}}{(z+1)^2} + \frac{-\frac{14}{15}}{z+1} + \underbrace{\frac{\frac{7+i}{25}}{z-2i} + \frac{\frac{7-i}{25}}{z+2i}}_{\text{holomorphic near } -1}.$$

On a small annulus $A_{\varepsilon,0}(-1)$ about -1, the last two terms $\frac{\frac{7+i}{25}}{z-2i} + \frac{\frac{7-i}{25}}{z+2i}$ are holomorphic. Therefore, if one express them as a Laurent series about -1, only non-negative powers of $z+1$ will appear, and the coefficient of $\frac{1}{z+1}$ will not be affected. Therefore, we have:

$$\operatorname{Res}(f, -1) = -\frac{14}{15}.$$

By a similar reason, we have:

$$\operatorname{Res}(f, 2i) = \frac{7+i}{25} \text{ and } \operatorname{Res}(f, -2i) = \frac{7-i}{25}.$$

Exercise 4.20. Determine all isolated singularities of the function
$$f(z) = \frac{z^2+1}{(z+1)(z-1)^2}.$$
and find the residue at each isolated singularity.

It is no doubt that partial fraction decompositions are time-consuming and not fun (it may remind you the computational nightmare you might have encountered in MATH 1014). Fortunately, there is a better way for finding residues for *poles* (does not work for essential singularity).

If we know already that z_0 is a pole of order 1 (i.e. simple pole) of a function $f(z)$, then
$$f(z) = \frac{c_{-1}}{z-z_0} + c_0 + c_1(z-z_0) + c_2(z-z_0)^2 + \cdots$$
It is then easy to see that
$$c_{-1} = \lim_{z\to z_0}(z-z_0)f(z).$$
Therefore, in order to find $\operatorname{Res}(f, z_0)$ for a simple pole z_0, we simply need to compute the above limit.

Now consider the case if z_0 is a pole of order k for f, then its Laurent series about z_0 is given by:
$$f(z) = \frac{c_{-k}}{(z-z_0)^k} + \frac{c_{-(k-1)}}{(z-z_0)^{k-1}} + \cdots + \frac{c_{-1}}{z-z_0} + c_0 + c_1(z-z_0) + c_2(z-z_0)^2 + \cdots$$
Our goal is to find c_{-1}. By multiplying both sides by $(z-z_0)^k$, we can get:
$$(z-z_0)^k f(z) = c_{-k} + c_{-(k-1)}(z-z_0) + \cdots + c_{-1}(z-z_0)^{k-1} + c_0(z-z_0)^k + \cdots$$
By differentiating both sides for $k-1$ times, all terms involving $(z-z_0)^n$ with $n < k-1$ will disappear:
$$\frac{d^{k-1}}{dz^{k-1}}(z-z_0)^k f(z) = c_{-1}(k-1)! + \widetilde{c}_0(z-z_0) + \widetilde{c}_1(z-z_0)^2 + \cdots$$
We have used the fact that $\frac{d^{k-1}}{dz^{k-1}}(z-z_0)^{k-1} = (k-1)!$, and $\widetilde{c}_0, \widetilde{c}_1, \ldots$ are some complex numbers (which we do not need to know their values).

By letting $z \to z_0$, we get:
$$\lim_{z\to z_0} \frac{d^{k-1}}{dz^{k-1}}(z-z_0)^k f(z) = c_{-1}(k-1)!$$
which provides a good way to find c_{-1} without expanding a Laurent series:

Proposition 4.9. *Suppose z_0 is a pole of order $k < \infty$ for a function f, then we have:*
$$\operatorname{Res}(f, z_0) = \frac{1}{(k-1)!} \lim_{z\to z_0} \frac{d^{k-1}}{dz^{k-1}}(z-z_0)^k f(z).$$
In particular, for a simple pole z_0, we have:
$$\operatorname{Res}(f, z_0) = \lim_{z\to z_0}(z-z_0)f(z).$$

Proof. See the preceding paragraph. □

Example 4.8. Find the residue of the function

$$f(z) = \frac{z^2 - 2z}{(z+1)^2(z^2+4)}$$

of each isolated singularity using Proposition 4.9.

Solution

As discussed before, the isolated singularities are -1, $2i$ and $-2i$. Observe that:

$$\lim_{z \to -1} (z+1)^2 f(z) = \lim_{z \to -1} \frac{z^2 - 2z}{z^2 + 4} = \frac{3}{5} \neq 0.$$

Hence -1 is a pole of order 2. From Proposition 4.9, we have:

$$\begin{aligned}
\operatorname{Res}(f, -1) &= \frac{1}{(2-1)!} \lim_{z \to -1} \frac{d^{2-1}}{dz^{2-1}} (z+1)^2 f(z) \\
&= \lim_{z \to -1} \frac{d}{dz} \frac{z^2 - 2z}{z^2 + 4} \\
&= \lim_{z \to -1} \frac{2z^2 + 8z - 8}{(z^2+4)^2} = -\frac{14}{25}.
\end{aligned}$$

Both $2i$ and $-2i$ are simple poles, so we have:

$$\begin{aligned}
\operatorname{Res}(f, 2i) &= \lim_{z \to 2i} (z - 2i) f(z) = \lim_{z \to 2i} \frac{z^2 - 2z}{(z+1)^2(z+2i)} = \frac{7+i}{25} \\
\operatorname{Res}(f, -2i) &= \lim_{z \to -2i} (z + 2i) f(z) = \lim_{z \to -2i} \frac{z^2 - 2z}{(z+1)^2(z-2i)} = \frac{7-i}{25}
\end{aligned}$$

Example 4.9. Find the residue at 0 of each function below:

$$f(z) = \frac{e^z}{\sin z} \qquad g(z) = \frac{e^z - 1}{\sin z} \qquad h(z) = \frac{e^z}{\sin^2 z}$$

Solution

For each function, we first determine whether 0 is a pole, and find out its order.

For $f(z)$, we consider:

$$\lim_{z \to 0} z f(z) = \lim_{z \to 0} \frac{z}{\sin z} \cdot e^z = 1 \cdot e^0 = 1 \neq 0.$$

Hence 0 is a simple pole for f, and $\operatorname{Res}(f, 0) = 1$.

For $g(z)$, note that:

$$\lim_{z \to 0} g(z) = \lim_{z \to 0} \frac{z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots}{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots} = \lim_{z \to 0} \frac{1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots}{1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots} = 1 < \infty.$$

Hence 0 is a removable singularity of $g(z)$, and there is no $\frac{1}{z}$-term in the Laurent series, and so $\operatorname{Res}(g, 0) = 0$.

For $h(z)$:

$$\lim_{z \to 0} z^2 h(z) = \lim_{z \to 0} \left(\frac{z}{\sin z} \right)^2 e^z = 1 \neq 0.$$

Hence 0 is a pole of order 2 for h. By Proposition 4.9, we can find:

$$\begin{aligned}
\operatorname{Res}(h,0) &= \frac{1}{1!}\lim_{z\to 0}\frac{d}{dz}z^2h(z) = \lim_{z\to 0}\frac{d}{dz}\frac{z^2e^z}{\sin^2 z} \\
&= \lim_{z\to 0}\frac{\sin^2 z\,(2ze^z + z^2e^z) - z^2e^z\cdot 2\sin z\cos z}{\sin^4 z} \\
&= \lim_{z\to 0}\left[\frac{z^2e^z}{\sin^2 z} + 2ze^z\left(\frac{\sin z - z\cos z}{\sin^3 z}\right)\right] \\
&= 1 + \lim_{z\to 0} 2ze^z\left(\frac{\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right) - z\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots\right)}{\sin^3 z}\right) \\
&= 1 + \lim_{z\to 0} 2ze^z\left(\frac{(\frac{1}{2!} - \frac{1}{3!})z^3 - (\frac{1}{4!} - \frac{1}{5!})z^5 + \cdots}{\sin^3 z}\right) \\
&= 1 + \lim_{z\to 0} 2ze^z\left(\left(\frac{1}{2!} - \frac{1}{3!}\right)\frac{z^3}{\sin^3 z} - \left(\frac{1}{4!} - \frac{1}{5!}\right)\frac{z^5}{\sin^3 z} + \cdots\right) \\
&= 1 + 0\cdot e^0\cdot\left(\frac{1}{2!} - \frac{1}{3!} + 0 + 0 + \cdots\right) \\
&= 1.
\end{aligned}$$

Exercise 4.21. For each function below, find its residue at each isolated singularity using any method:

$$\frac{z^2-1}{z^3(z^2+1)} \qquad \frac{1}{6z^2+8z+9} \qquad \frac{z^{1997}-1}{z^{2047}-1}$$

$$\frac{1}{e^z-1} \qquad \frac{e^{2zi}}{\sin z} \qquad \frac{e^{2zi}-1}{\sin z}$$

$$\frac{1}{z\sin z} \qquad \frac{z^2}{e^{1/z}} \qquad \frac{\sin z}{z^2(z-\pi)^3}$$

Exercise 4.22. Compute the following residues:

(a) $\operatorname{Res}\left(\dfrac{1}{2\cos z - 2 + z^2}, 0\right)$

(b) $\operatorname{Res}\left(\dfrac{z^{2n}}{(z-1)^n}, 1\right)$

4.3.3. Residue Theorem. The residue $\operatorname{Res}(f,z_0)$ of an isolated singularity z_0 determines the value of a contour integral $\oint_\gamma f(z)\,dz$ where γ is a tiny simple closed curve so that z_0 is the only singularity it encloses. Namely, we have $\oint_\gamma f(z)\,dz = 2\pi i\operatorname{Res}(f,z_0)$.

If a simple closed curve γ encloses more than one isolated singularities $\{z_1,\cdots,z_N\}$, then we may first express the contour integral over γ as the sum of contour integrals:

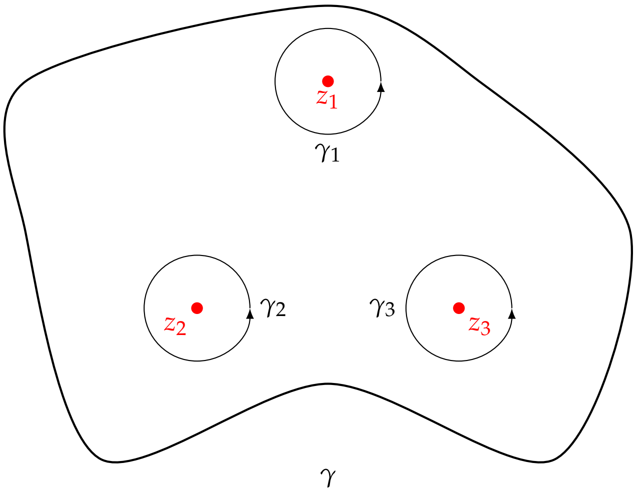
$$\oint_\gamma f(z)\,dz = \oint_{\gamma_1} f(z)\,dz + \cdots + \oint_{\gamma_N} f(z)\,dz$$

where each γ_j is a small simple-closed curve so that z_j is the only singularity it encloses. Then, each γ_j-integral is given by $2\pi i\operatorname{Res}(f,z_j)$, and hence we have the following theorem:

Theorem 4.10 (Residue Theorem). *Let $f : \Omega \to \mathbb{C}$ be a complex-valued functions whose singularities are all isolated. Let γ be a simple closed curve, and $z_1, \cdots, z_N \in \Omega$ be all the singularities enclosed by γ. Then, we have:*

$$\oint_\gamma f(z)\,dz = 2\pi i \sum_{j=1}^{N} \operatorname{Res}(f, z_j).$$

Proof. Let $\varepsilon > 0$ be sufficiently such that each circle $\{|z - z_j| = \varepsilon\}$, denoted by γ_j, encloses z_j as the only singularity of f (see figure below).



Then, by the standard hole-drilling argument, we have:

$$\oint_\gamma f(z)\,dz = \oint_{\gamma_1} f(z)\,dz + \cdots + \oint_{\gamma_N} f(z)\,dz$$

Each γ_j encloses z_j as the only singularity of f. Express f as a Laurent series on $A_{\varepsilon,0}(z_j)$:

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_j)^n.$$

Recall that $\oint_{\gamma_j} (z - z_j)^n\,dz \neq 0$ only when $n = -1$, and by uniform convergence of Laurent series, we get:

$$\oint_{\gamma_j} f(z) = 2\pi i c_{-1} = 2\pi i \operatorname{Res}(f, z_j).$$

Therefore, we have:

$$\oint_\gamma f(z)\,dz = 2\pi i \sum_{j=1}^{N} \operatorname{Res}(f, z_j),$$

completing the proof. □

Example 4.10. Use Residue Theorem to evaluate the contour integral:

$$\oint_{|z|=R} \frac{z^2 - 2z}{(z+1)^2(z^2+4)}\,dz$$

where R is in the range of:

(a) $0 < R < 1$

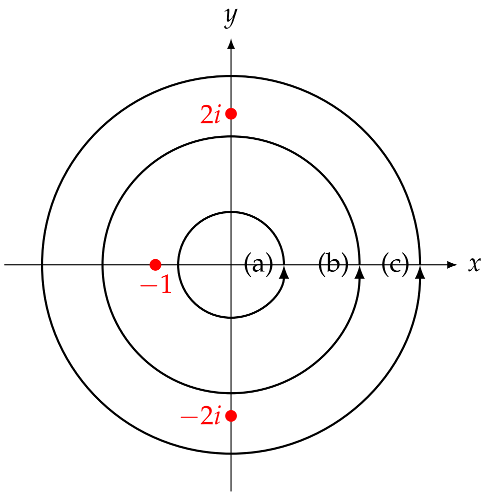
(b) $1 < R < 2$

(c) $2 < R$.

Solution

Denote $f(z) = \dfrac{z^2 - 2z}{(z+1)^2(z^2+4)}$. The singularities of f are -1, $2i$ and $-2i$. We have calculated in Example 4.8 that

$$\operatorname{Res}(f, -1) = -\frac{14}{15} \qquad \operatorname{Res}(f, 2i) = \frac{7+i}{25} \qquad \operatorname{Res}(f, -2i) = \frac{7-i}{25}$$



(a) When $0 < R < 1$, the circle $|z| = R$ does not enclose any singularities, hence

$$\oint_{|z|=R} f(z)\,dz = 0.$$

(b) When $1 < R < 1$, the circle $|z| = R$ encloses the singularity -1 only, hence

$$\oint_{|z|=R} f(z)\,dz = 2\pi i \operatorname{Res}(f, -1) = -\frac{28\pi i}{15}$$

(c) When $R > 2$, the circle $|z| = R$ encloses all three singularities, hence

$$\begin{aligned}\oint_{|z|=R} f(z)\,dz &= 2\pi i\ (\operatorname{Res}(f,-1) + \operatorname{Res}(f,2i) + \operatorname{Res}(f,-2i)) \\ &= 2\pi i \left(-\frac{14}{15} + \frac{7+i}{25} + \frac{7-i}{25}\right) = -\frac{56\pi i}{75}.\end{aligned}$$

Example 4.11. Let N be a positive integer and γ_N be the square contour with vertices $\pm(N+\frac{1}{2}) \pm (N+\frac{1}{2})i$. Use Residue Theorem to show:

$$\sum_{n=1}^{N} \frac{1}{n^2} = \frac{\pi^2}{3} + \frac{1}{2\pi i} \oint_{\gamma_N} \frac{\pi}{z^2} \cot \pi z \, dz.$$

Hence, deduce that:

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}.$$

Solution

Denote $f(z) := \frac{\pi}{z^2} \cot \pi z = \frac{\pi \cos \pi z}{z^2 \sin \pi z}$. Its singularities are the set of all integers n.

First we observe that

$$\lim_{z\to 0} z^3 f(z) = \lim_{z\to 0} \frac{\pi z}{\sin \pi z} \cdot \cos \pi z = 1,$$

so 0 is a pole of order 3 for f. By Proposition 4.9, its residue is given by:

$$\begin{aligned}
\operatorname{Res}(f,0) &= \frac{1}{2!} \lim_{z\to 0} \frac{d^2}{dz^2} z^3 f(z) = \frac{1}{2} \lim_{z\to 0} \frac{d^2}{dz^2} \pi z \cot \pi z \\
&= \lim_{z\to 0} \frac{\pi^2 \left(\pi z \cos \pi z - \sin \pi z\right)}{\sin^3 \pi z} \\
&= \lim_{z\to 0} \frac{\pi^2 \left(\pi z (1 - \frac{\pi^2 z^2}{2!} + \cdots) - (\pi z - \frac{\pi^3 z^3}{3!} + \cdots)\right)}{\sin^3 \pi z} \\
&= \lim_{z\to 0} \pi^2 \cdot \frac{-\frac{\pi^3 z^3}{3} + \text{higher-order terms}}{\sin^3 \pi z} = -\frac{\pi^2}{3}.
\end{aligned}$$

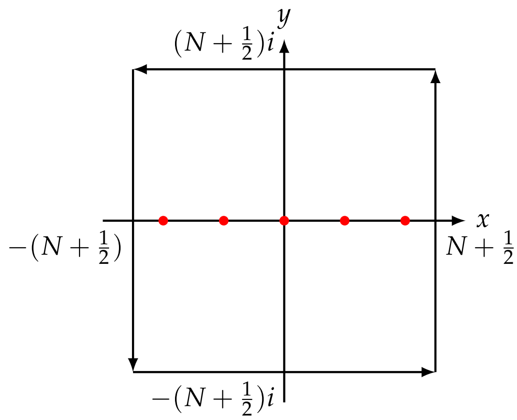
For any non-zero integer n, observe that:

$$\lim_{z\to n} (z-n) f(z) = \lim_{z\to n} \frac{\pi(z-n)\cos \pi z}{z^2 \cdot \underbrace{(-1)^n \sin(\pi(z-n))}_{=\sin \pi z}} = \frac{(-1)^n}{n^2 \cdot (-1)^n} = \frac{1}{n^2}.$$

Hence, n is a simple pole of f (for any $n \neq 0$), and $\operatorname{Res}(f,n) = \frac{1}{n^2}$.

Now consider the contour γ_N. The singularities it encloses are:

$$0, \pm 1, \pm 2 \cdots, \pm N.$$



By Residue Theorem, we have:

$$\begin{aligned}\oint_{\gamma_N} f(z)\,dz &= 2\pi i \sum_{n=-N}^{N} \operatorname{Res}(f,n) \\ &= 2\pi i \left(\operatorname{Res}(f,0) + 2\left(1 + \frac{1}{2^2} + \cdots + \frac{1}{N^2}\right)\right) \\ &= 2\pi i \left(-\frac{\pi^2}{3} + 2\sum_{n=1}^{N} \frac{1}{n^2}\right).\end{aligned}$$

By rearrangement, we have the desired result:

$$\sum_{n=1}^{N} \frac{1}{n^2} = \frac{\pi^2}{6} + \frac{1}{2\pi i}\oint_{\gamma_N} f(z)\,dz.$$

The remaining task is to show:

$$\lim_{N\to\infty} \oint_{\gamma_N} \frac{\pi}{z^2} \cot \pi z\,dz = 0.$$

We do so by estimating the contour integral:

When $z \in \gamma_N$, we have $|z| \geq N + \frac{1}{2} > N$. It is also possible to show that $|\cot \pi z| < 2$ for any $z \in \gamma_N$ (this is left as an exercise). Therefore, on γ_N, we have the bound:

$$\left|\frac{\pi}{z^2} \cot \pi z\right| \leq \frac{2\pi}{N^2}.$$

The length of γ_N is $8N + 4$. By Lemma 3.6, we get:

$$\left|\oint_{\gamma_N} \frac{\pi}{z^2} \cot \pi z\,dz\right| \leq (8N+4) \cdot \frac{2\pi}{N^2} \to 0 \text{ as } N \to \infty,$$

completing the proof.

Exercise 4.23. Complete the detail of the above example that:

$$|\cot \pi z| < 2$$

for any $z \in \gamma_N$. [Hint: Write $z = x + yi$, and find an expression for $\cot \pi z$ in terms of x and y. Then, maximize $|\cot \pi z|$ on each side of the contour γ_N.]

Exercise 4.24. Use Residue Theorem to evaluate the following contour integrals:

(a) $\displaystyle\oint_{|z|=3} \frac{1}{z^2+1}\,dz$

(b) $\displaystyle\oint_{|z|=2} \frac{z^3+3z+1}{z^4-5z^2}\,dz$

(c) $\displaystyle\oint_{|z-i|=2} \frac{e^z+z}{(z-1)^4}\,dz$

(d) $\displaystyle\oint_{|z-i|=2} \frac{\sin z}{(z-i)^{4023}}\,dz$

(e) $\displaystyle\oint_{\gamma} \tan \pi z\,dz$ where γ is the rectangle contour with vertices:

$$(-2,0), \quad (2,0), \quad (2,1), \quad (-2,1).$$

Exercise 4.25. Let γ_N be the square contour with vertices $\pm(N+\frac{1}{2})\pi \pm (N+\frac{1}{2})\pi i$ where N is a positive integer. Show that:

$$\frac{1}{2\pi i}\oint_{\gamma_N} \frac{1}{z^2}\csc z\, dz = \frac{1}{6} + \frac{2}{\pi^2}\sum_{n=1}^{N}\frac{(-1)^n}{n^2}.$$

Hence, show that:

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}.$$

Exercise 4.26. Determine the residues of all isolated singularities of the function:

$$f(z) = \frac{1}{(2z-1)\sin \pi z}.$$

By considering a suitable contour integral of f, show that:

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}.$$

4.3.4. Evaluation of Real Integrals. Residues are often used to evaluate some difficult real integrals that physicists and engineers may encounter.

Example 4.12. Evaluate the real definite integral:

$$\int_0^{2\pi} \frac{1}{a - b\cos\theta}\, d\theta$$

where a and b are real numbers such that $0 < b < a$.

Solution

The key trick is to express the real integral as a complex integral of the circle contour $|z| = 1$, which is parametrized by $z = e^{i\theta}$ where $0 \le \theta \le 2\pi$.

When $z = e^{i\theta}$ is on the contour $\{|z| = 1\}$, we have

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

$$dz = ie^{i\theta}\, d\theta \Longrightarrow d\theta = \frac{1}{ie^{i\theta}}\, dz = \frac{1}{iz}\, dz$$

Therefore, the real integral can be written as a complex integral as:

$$\int_0^{2\pi} \frac{1}{a - b\cos\theta}\, d\theta = \oint_{|z|=1} \frac{1}{a - \frac{b}{2}\left(z + \frac{1}{z}\right)} \cdot \frac{1}{iz}\, dz = 2i\oint_{|z|=1} \frac{1}{bz^2 - 2az + b}\, dz.$$

We can then use residue theory to evaluate the complex integral. The singularities of the integrand are roots of the quadratic equation $bz^2 - 2az + b = 0$, which are:

$$\omega_1 = \frac{a - \sqrt{a^2 - b^2}}{b} \quad \text{and} \quad \omega_2 = \frac{a + \sqrt{a^2 - b^2}}{b}.$$

Note that $a > b$, so both roots are real. We further observe that:

$$\omega_2 > \frac{a + 0}{b} > 1 \quad \text{and} \quad \omega_1\omega_2 = 1,$$

and so $|\omega_1| < 1$. Therefore, ω_1 is the only singularity enclosed by the contour $|z| = 1$. As ω_1 and ω_2 are distinct, they are simple poles, and so the contour integral is given by:

$$\begin{aligned}\oint_{|z|=1}\frac{1}{bz^2-2az+b}\,dz &= \oint_{|z|=1}\frac{1}{b(z-\omega_1)(z-\omega_2)}\,dz\\ &= \oint_{|z|=1}\frac{\frac{1}{b(z-\omega_2)}}{z-\omega_1}\,dz = 2\pi i\left[\frac{1}{b(z-\omega_2)}\right]_{z=\omega_1}\\ &= \frac{2\pi i}{b(\omega_1-\omega_2)} = -\frac{\pi i}{\sqrt{a^2-b^2}}.\end{aligned}$$

Hence, the real integral is given by:

$$\int_0^{2\pi}\frac{1}{a-b\cos\theta}\,d\theta = -2i\cdot\frac{\pi i}{\sqrt{a^2-b^2}} = \frac{2\pi}{\sqrt{a^2-b^2}}.$$

Before we proceed to the next example, let's first prove a useful observation which will come in handy later on.

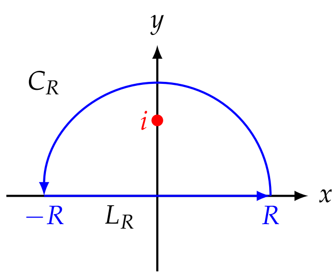
Exercise 4.27. Show that the function e^{iz} is bounded on the upper-half plane, i.e. there exists $M>0$ such that $\left|e^{iz}\right|\le M$ whenever $\operatorname{Im}(z)\ge 0$. On the other hand, show that the function $=\cos z$ is unbounded on the upper-half plane.

Example 4.13. Evaluate the following real integral:

$$\int_{-\infty}^{\infty}\frac{\cos x}{1+x^2}\,dx.$$

Solution

Let's consider the following semi-circle contour:



Denote C_R to be the (open) semi-circle with radius R, L_R to be the straight-path from $-R$ to R, and γ_R to be the closed semi-circular path C_R+L_R. We consider this contour because

$$\int_{-\infty}^{\infty}\frac{\cos x}{1+x^2}\,dx = \lim_{R\to+\infty}\int_{-R}^{R}\frac{\cos x}{1+x^2}\,dx = \lim_{R\to+\infty}\int_{L_R}\frac{\cos z}{1+z^2}\,dz.$$

Note that:

$$\oint_{\gamma_R}\frac{\cos z}{1+z^2}\,dz = \int_{L_R}\frac{\cos z}{1+z^2}\,dz + \int_{C_R}\frac{\cos z}{1+z^2}\,dz.$$

The γ_R-integral can be computed using residues. If we are able to show the C_R-integral tends to 0 as $R\to+\infty$, then one can determine our desired limit $\displaystyle\lim_{R\to+\infty}\int_{L_R}\frac{\cos z}{1+z^2}\,dz$.

Unfortunately, it is not possible to bound $\dfrac{\cos z}{1+z^2}$ as $\cos z$ is unbounded according to Exercise 4.27. One trick to get around with this issue is to consider the following function instead:

$$f(z) = \frac{e^{iz}}{1+z^2}.$$

When $z \in L_R$, we have $z = x + 0i$ and so:

$$f(z) = \frac{e^{ix}}{1+x^2} = \frac{\cos x + i \sin x}{1+x^2} \quad \Longrightarrow \quad \int_{L_R} f(z)\,dz = \int_{-R}^{R} \frac{\cos x + i\sin x}{1+x^2}\,dx.$$

If we are able to find out $\lim_{R\to+\infty} \int_{L_R} f(z)\,dz$, then one can recover the value of $\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}$ by simply taking the real-part of $\lim_{R\to+\infty} \int_{L_R} f(z)\,dz$.

By considering the contour $\gamma_R = C_R + L_R$, we have:

$$\oint_{\gamma_R} f(z)\,dz = \int_{L_R} f(z)\,dz + \int_{C_R} f(z)\,dz.$$

The only singularity enclosed by γ_R is i (when R is sufficiently large), so:

$$\oint_{\gamma_R} f(z)\,dz = 2\pi i \operatorname{Res}(f, i) = 2\pi i \cdot \frac{1}{2ie} = \frac{\pi}{e}.$$

Next we show the C_R-integral converges to 0 as $R \to \infty$. From Exercise 4.27, the term e^{iz} is bounded on the upper-half plane, and so whenever $z \in C_R$, we have:

$$\left|\frac{e^{iz}}{1+z^2}\right| \leq \frac{M}{|1+z^2|} \leq \frac{M}{\left||z|^2 - 1\right|} = \frac{M}{R^2-1},$$

where M is an upper bound of $\left|e^{iz}\right|$ on the upper-half plane. Therefore, by Lemma 3.6, we get the estimate:

$$\left|\int_{C_R} \frac{e^{iz}}{1+z^2}\,dz\right| \leq \pi R \cdot \frac{M}{R^2-1} \to 0 \quad \text{as } R \to +\infty.$$

Therefore, we get:

$$\lim_{R\to\infty} \int_{L_R} f(z)\,dz = \lim_{R\to\infty} \oint_{\gamma_R} f(z)\,dz - \lim_{R\to\infty} \int_{C_R} f(z)\,dz$$

$$\int_{-\infty}^{\infty} \frac{\cos x + i\sin x}{1+x^2}\,dx = \frac{\pi}{e} - 0 = \frac{\pi}{e}.$$

This shows:

$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2} = \frac{\pi}{e}.$$

Before we give another example, we recall some fundamental facts that:

- For any $z \neq 0$, the principal argument $\operatorname{Arg}(z)$ is in $(-\pi, \pi]$.
- $\operatorname{Log}(z) = \ln|z| + i\operatorname{Arg}(z)$ for any $z \neq 0$
- $\operatorname{Log}(z)$ is holomorphic on $\mathbb{C}\backslash(-\infty, 0]$.

Therefore, if we apply Cauchy's integral formula or residue theory for an integral involving $\operatorname{Log}(z)$, then we need to make sure the closed curve γ lies in $\mathbb{C}\backslash(-\infty, 0]$. As such, we cannot apply residue methods with a semi-circle contour as in the previous example. Nonetheless, this kind of semi-circle contour is very useful when dealing with real integrals over $(-\infty, \infty)$.

To get around with this issue, we can define a different branch of logarithm by the following. For any $z \neq 0$, we let

$$\operatorname{Log}_{-\pi/2}(z) := \ln|z| + i\theta(z) \tag{4.5}$$

where $\theta(z)$ is the unique angle in $[-\frac{\pi}{2}, \frac{3\pi}{2})$ such that $z = |z|\, e^{i\theta(z)}$. By doing so, we still have $e^{\mathrm{Log}_{-\pi/2}(z)} = z$. The notable difference from $\mathrm{Log}(z)$ is that now $\mathrm{Log}_{-\pi/2}(z)$ is holomorphic on $\mathbb{C}\backslash\{0 + yi : y \leq 0\}$, the yellow region below.

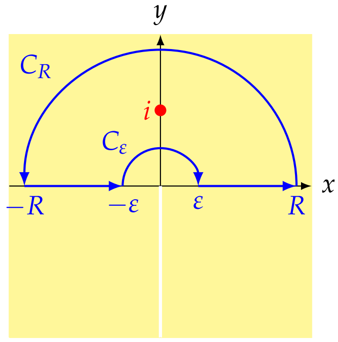


Figure 4.1. Domain of $\mathrm{Log}_{-\pi/2}(z)$

Exercise 4.28. Determine the value of $\mathrm{Log}_{-\pi/2}(z)$ when:

(a) $z = i$

(b) $z = x + 0i$ where $x > 0$

(c) $z = x + 0i$ where $x < 0$

Example 4.14. Let α be a real constant in $(0, 1)$. Evaluate the real integral:

$$I := \int_0^\infty \frac{1}{x^\alpha(1 + x^2)}\, dx.$$

Solution

First observe that for any $x > 0$, we have:

$$x^\alpha = e^{\alpha \ln x} = e^{\alpha \mathrm{Log}_{-\pi/2}(x)}$$

where $\mathrm{Log}_{-\pi/2}$ is the special branch of logarithm defined in (4.5). It prompts us to consider a contour integral of the function:

$$f(z) = \frac{1}{e^{\alpha \mathrm{Log}_{-\pi/2}(z)}(1 + z^2)}.$$

We pick a contour as shown in Figure 4.1, where C_R and C_ε are semi-circles with radii R and ε respectively. Since the closed contour $\gamma_{R,\varepsilon} := [-R, -\varepsilon] + C_\varepsilon + [\varepsilon, R] + C_R$ lies completely inside the domain of $\mathrm{Log}_{-\pi/2}(z)$, by Residue Theorem, we have:

$$\oint_{\gamma_{R,\varepsilon}} \frac{1}{e^{\alpha \mathrm{Log}_{-\pi/2}(z)}(1 + z^2)}\, dz = 2\pi i \,\mathrm{Res}(f, i) = 2\pi i \cdot \left. \frac{1}{e^{\alpha \mathrm{Log}_{-\pi/2}(z)}(z + i)} \right|_{z=i}$$

$$= \frac{2\pi i}{2ie^{\alpha(\ln|i| + \frac{\pi}{2}i)}} = \frac{\pi}{e^{\frac{\alpha\pi}{2}i}}.$$

On the other hand, the $\gamma_{R,\varepsilon}$-integral can break down into:

(4.6)

$$\oint_{\gamma_{R,\varepsilon}} \frac{1}{e^{\alpha \mathrm{Log}_{-\pi/2}(z)}(1 + z^2)}\, dz = \left(\int_{-R}^{-\varepsilon} + \int_{C_\varepsilon} + \int_{\varepsilon}^{R} + \int_{C_R} \right) \frac{1}{e^{\alpha \mathrm{Log}_{-\pi/2}(z)}(1 + z^2)}\, dz$$

When $z = x + 0i \in [\varepsilon, R]$, the integrand is simply:
$$\frac{1}{e^{\alpha \mathrm{Log}_{-\pi/2}(x)}(1+x^2)} = \frac{1}{x^\alpha(1+x^2)}.$$
Hence,
$$\lim_{\varepsilon\to 0}\lim_{R\to\infty}\int_\varepsilon^R \frac{1}{e^{\alpha \mathrm{Log}_{-\pi/2}(z)}(1+z^2)}\,dz = \int_0^\infty \frac{1}{x^\alpha(1+x^2)}\,dx =: I.$$
When $z = x + 0i \in [-R, -\varepsilon]$, the integrand becomes:
$$\frac{1}{e^{\alpha \mathrm{Log}_{-\pi/2}(x)}(1+x^2)} = \frac{1}{e^{\alpha(\ln|x|+\pi i)}(1+x^2)} = \frac{1}{e^{\alpha\pi i}}\cdot\frac{1}{|x|^\alpha(1+x^2)}.$$
Hence,
$$\lim_{\varepsilon\to 0}\lim_{R\to\infty}\int_{-R}^{-\varepsilon} \frac{1}{e^{\alpha \mathrm{Log}_{-\pi/2}(z)}(1+z^2)}\,dz = \frac{1}{e^{\alpha\pi i}}\int_{-\infty}^0 \underbrace{\frac{1}{|x|^\alpha(1+x^2)}}_{\text{even function}}\,dx = \frac{I}{e^{\alpha\pi i}}.$$

We are left to analyze the two semi-circular integrals. We will show that they tend to 0 as $\varepsilon \to 0$ and $R \to \infty$.

When $z \in C_\varepsilon$, we have:
$$\left|\frac{1}{e^{\alpha \mathrm{Log}_{-\pi/2}(z)}(1+z^2)}\right| \leq \left|e^{-\alpha(\ln|z|+i\theta(z))}\right|\cdot = \frac{1}{\left|1-|z|^2\right|} = \frac{e^{-\alpha\ln\varepsilon}}{1-\varepsilon^2} = \frac{\varepsilon^{-\alpha}}{1-\varepsilon^2}.$$
By Lemma 3.6, we get:
$$\left|\int_{C_\varepsilon} \frac{1}{e^{\alpha \mathrm{Log}_{-\pi/2}(z)}(1+z^2)}\,dz\right| \leq \frac{\varepsilon^{-\alpha}}{1-\varepsilon^2}\cdot\pi\varepsilon = \frac{\pi\varepsilon^{1-\alpha}}{1-\varepsilon^2} \to 0 \quad \text{as } \varepsilon \to 0.$$
Similarly when $z \in C_R$, we have:
$$\left|\frac{1}{e^{\alpha \mathrm{Log}_{-\pi/2}(z)}(1+z^2)}\right| \leq \left|e^{-\alpha(\ln|z|+i\theta(z))}\right|\cdot\left|\frac{1}{\left|1-|z|^2\right|}\right| = \frac{R^{-\alpha}}{R^2-1}.$$
By Lemma 3.6, we have the estimate:
$$\left|\int_{C_\varepsilon} \frac{1}{e^{\alpha \mathrm{Log}_{-\pi/2}(z)}(1+z^2)}\,dz\right| \leq \frac{R^{-\alpha}}{R^2-1}\cdot\pi R = \frac{\pi R^{1-\alpha}}{R^2-1} \to 0 \quad \text{as } R \to \infty.$$
Finally, by letting $\varepsilon \to 0$ and $R \to \infty$ on both sides of (4.6), we get:
$$\frac{\pi}{e^{\frac{\alpha\pi}{2}i}} = I + \frac{I}{e^{\alpha\pi i}}.$$
Solving for I, we get:
$$I = \frac{\pi}{e^{\frac{\alpha\pi}{2}i}\left(1+e^{-\alpha\pi i}\right)} = \frac{\pi}{e^{\frac{\alpha\pi}{2}i}+e^{-\frac{\alpha\pi}{2}i}} = \frac{\pi}{2\cos\frac{\alpha\pi}{2}} = \frac{\pi}{2}\sec\frac{\alpha\pi}{2}.$$

Exercise 4.29. Evaluate the following real integrals using residue methods:

(a) $\displaystyle\int_0^{2\pi} \frac{1}{(a+b\cos\theta)^2}\,d\theta$ where $a > b > 0$.

(b) $\displaystyle\int_0^{2\pi} \frac{1}{1-2a\cos\theta+a^2}\,d\theta$ where $a \in \mathbb{R}$ and $a \neq \pm 1$.

(c) $\displaystyle\int_0^{\infty} \frac{x^2}{(x^2+a^2)^2}\,dx$ where $a > 0$.

(d) $\displaystyle\int_0^{\infty} \frac{1}{(x^2+1)^n}\,dx$ where $n \in \mathbb{N}$.

(e) $\displaystyle\int_0^{\infty} \frac{\cos ax}{x^2+b^2}\,dx$ where a and b are positive real numbers

(f) $\displaystyle\int_0^{\infty} \frac{\sin ax}{x(x^2+1)}\,dx$ where a is a positive real number.

(g) $\displaystyle\int_0^{\infty} \frac{\ln x}{x^2+a^2}\,dx$ where $a > 0$.

(h) $\displaystyle\int_0^{\infty} \frac{1}{x^\alpha(1+x^4)}\,dx$ where $\alpha \in (0,1)$.

Exercise 4.30. Show that for any $t \in \mathbb{R}$, we have:

$$\int_{-\infty}^{\infty} \frac{e^{itx}}{x^2+1}\,dx = \pi e^{-|t|}.$$

Exercise 4.31. Show that:

$$\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^{n+1}}\,dx = \frac{(2n-1)!!}{(2n)!!}\pi.$$

Chapter 5

What is the Riemann Hypothesis?

5.1. Analytic Continuation

We will end this course by an introduction to the Riemann Hypothesis, a long-standing unresolved problem in Pure Mathematics, and is a topic of central importance in Complex Analysis, Number Theory, and related fields.

The Riemann Hypothesis concerns about the Riemann zeta function which is *a priori* defined by the following infinite sum:

$$\zeta(z) := \sum_{n=1}^{\infty} \frac{1}{n^z} = \sum_{n=1}^{\infty} \frac{1}{e^{z \ln n}}.$$

Here n^z is regarded as a single-valued function of z. This sum converges absolutely on the domain $\Omega = \{z \in \mathbb{C} : \mathrm{Re}(z) > 1\}$, and converges uniformly on every smaller domain $\Omega_\varepsilon = \{z \in \mathbb{C} : \mathrm{Re}(z) > 1 + \varepsilon\}$. Therefore, Morera's Theorem shows that ζ is holomorphic on Ω.

Although ζ is a priori defined on Ω, we will soon learn that it can be extended to a holomorphic function on $\mathbb{C}\backslash\{1\}$. In other words, there exists a function $\hat{\zeta} : \mathbb{C}\backslash\{1\} \to \mathbb{C}$ such that $\hat{\zeta}(z) = \zeta(z)$ for any $z \in \Omega$, and that $\hat{\zeta}$ is holomorphic on $\mathbb{C}\backslash\{1\}$. This new function $\hat{\zeta}$ is called the *analytic continuation* of ζ.

Such an analytic continuation can be shown to be unique, and it is common to abuse the notations a bit by simply writing ζ (instead of $\hat{\zeta}$) for the analytic continuation of ζ. In this section, we will collect some useful facts about analytic continuations. We will then describe how to extend ζ in the next section.

Definition 5.1 (Analytic Continuations)**.** Given a holomorphic function $f : \Omega \to \mathbb{C}$, a function $\hat{f} : \hat{\Omega} \to \mathbb{C}$ defined on a connected domain $\hat{\Omega} \supset \Omega$ is said to be an *analytic continuation* of f on $\hat{\Omega}$ if:

- $\hat{f}(z) = f(z)$ for any $z \in \Omega$; and
- $\hat{f}$ is holomorphic on $\hat{\Omega}$.

While a (real) differentiable function defined on a smaller domain can be easily extended to a (real) differentiable function defined on a larger domain, it is very difficult to do so for a holomorphic function. One reason is that holomorphic functions are very *rigid*, in a sense that if any two holomorphic functions coincide on an open set, then the two function must be equal elsewhere! As a corollary, if an analytic continuation exists, then it must be unique! Let's state and prove this fact:

Theorem 5.2 (Identity Theorem). *Let $f : \Omega \to \mathbb{C}$ be a holomorphic function on a connected domain Ω. If there exists a non-empty open set $U \subset \Omega$ such that $f(z) = 0$ for any $z \in U$, then $f \equiv 0$ on Ω.*

Proof. Consider the set

$$S := \{z \in \Omega : f^{(n)}(z) = 0 \text{ for any } n \geq 0\}.$$

Since $f(z) = 0$ on U which is an open set, we have $f(z) = f'(z) = f''(z) = \cdots = 0$ for any $z \in U$. This shows $U \subset S$, and so S is non-empty. The proof goes by showing S is both closed and open. Together with the fact that S is non-empty and Ω is connected, it will prove $S = \Omega$ which implies our claim.

To show S is closed, we recall the fact that a holomorphic function f must be infinitely differentiable, and hence $f^{(n)}$ are all continuous functions. The set S can be written as:

$$S := \bigcap_{n=0}^{\infty} \left(f^{(n)}\right)^{-1}(0).$$

The single set $\{0\}$ is closed, and hence the pre-image $\left(f^{(n)}\right)^{-1}(0)$ is closed for each $n \geq 0$. Since the intersection of any family of closed sets is closed, we conclude that S is closed.

To show S is open, we consider Taylor series expansions. For any $z_0 \in \Omega$, we consider the Taylor series about z_0 of f:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

which is defined on an open ball $B_\varepsilon(z_0)$ for some $\varepsilon > 0$ (according to Taylor's Theorem). If $z_0 \in S$, then we will have $f^{(n)}(z_0) = 0$ for any $n \geq 0$, and as such, the above Taylor series shows $f(z) = 0$ for any $z \in B_\varepsilon(z_0)$. In other words, $B_\varepsilon(z_0) \subset S$. This shows S is open.

Finally, S is non-empty, open and closed, and Ω is connected, so $S = \Omega$. □

Corollary 5.3. *Suppose $g : \Omega \to \mathbb{C}$ and $h : \Omega \to \mathbb{C}$ are two holomorphic functions defined on a connected domain Ω, and that g and h coincide on a smaller open set $U \subset \Omega$, then it is necessary that $g \equiv h$ on Ω.*

Proof. Apply $f := g - h$ to Identity Theorem. □

As a result, an analytic continuation $\hat{f}$ of a holomorphic function f, if exists, must be unique. It makes it very difficult to find such an extension!

Exercise 5.1. Why is it necessary for f to be holomorphic in the proof of Identity Theorem? Point out which part of the proof is no longer valid if f is just assumed to smooth (differentiable for infinitely many times).

Example 5.1. Consider the series:

$$f(z) = \sum_{n=0}^{\infty} z^n$$

which converges pointwise on $B_1(0)$, and uniformly on every smaller ball $B_{1-\varepsilon}(0)$ where $\varepsilon > 0$. Therefore, $f : B_1(0) \to \mathbb{C}$ is a holomorphic function on $B_1(0)$.

On the other hand, the infinite sum is:

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z},$$

and the function $\hat{f}(z) = \dfrac{1}{1-z}$ is defined on every $z \in \mathbb{C}\backslash\{1\}$, not only those in $B_1(0)$. Therefore, $\hat{f} : \mathbb{C}\backslash\{1\} \to \mathbb{C}$ is the analytic continuation of f on $\mathbb{C}\backslash\{1\}$.

Exercise 5.2. What's wrong with the following claim?

From $\hat{f}(-1) = f(-1)$ (where f and $\hat{f}$ are defined as in Example 5.1), we have:

$$\sum_{n=0}^{\infty}(-1)^n = \frac{1}{1-(-1)} = \frac{1}{2}.$$

Hence:

$$1 - 1 + 1 - 1 + 1 - 1 + \cdots = \frac{1}{2}.$$

Exercise 5.3. Consider the following function defined by the sum:

$$f(z) = 1 + \frac{1}{z} + \frac{1}{z^2} + \cdots = \sum_{n=0}^{\infty} \frac{1}{z^n}.$$

What is the largest possible domain on which f is holomorphic? Find the analytic continuation of f on the larger domain $\mathbb{C}\backslash\{1\}$. Is it possible to further extend the function to become an entire function on $\mathbb{C}$?

Another common way of extending a holomorphic function is through a *functional equation*. Let's consider the following example. Suppose $f : \Omega \to \mathbb{C}$ is a holomorphic function on $\Omega := \{z \in \mathbb{C} : \mathrm{Re}(z) > 1\}$. If it can be shown that f satisfies an equation such as:

$$f(z+1) = 2f(z) \quad \text{for any } z \in \Omega,$$

then one can define an analytic continuation of it by the following way:

$$\hat{f}(z) := \frac{1}{2} f(z+1).$$

Since $f(z+1)$ is well-defined as long as $z+1 \in \Omega$, or equivalently, $\mathrm{Re}(z) > 0$, the extend function $\hat{f}(z)$ is now defined on a larger domain $\hat{\Omega} := \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$. Note that $\hat{f}(z) = \frac{1}{2}f(z+1)$ is holomorphic on $\{\mathrm{Re}(z) > 0\}$ since f is so on $\{\mathrm{Re}(z) > 1\}$. Also, when $\mathrm{Re}(z) > 1$, we have

$$\hat{f}(z) = \frac{1}{2} f(z+1) = f(z)$$

by the given functional equation. Therefore, $\hat{f}$ is the analytic continuation of f on $\{\mathrm{Re}(z) > 0\}$.

Furthermore, the same functional equation holds for $\hat{f}$. Let's verify this. For any z such that $\mathrm{Re}(z) > 0$, we have:

$$\begin{aligned}\hat{f}(z+1) - 2\hat{f}(z) &= \frac{1}{2}f(z+2) - 2\cdot\frac{1}{2}f(z+1)\\ &= \frac{1}{2}\left(f(z+2) - 2f(z+1)\right) = 0.\end{aligned}$$

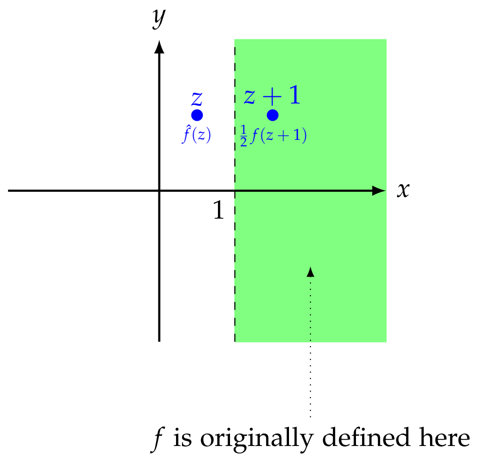
Now that $\hat{f}$ is holomorphic on $\{z : \mathrm{Re}(z) > 0\}$ and satisfies the functional equation

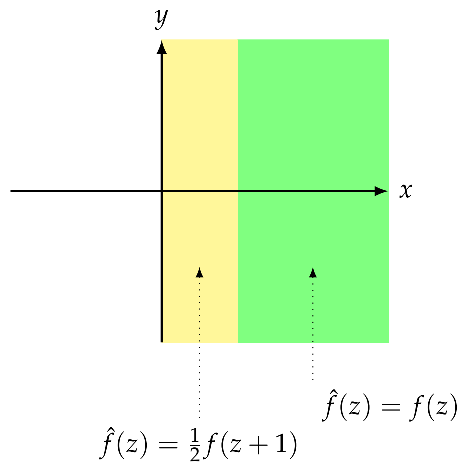
$$\hat{f}(z+1) = 2\hat{f}(z).$$

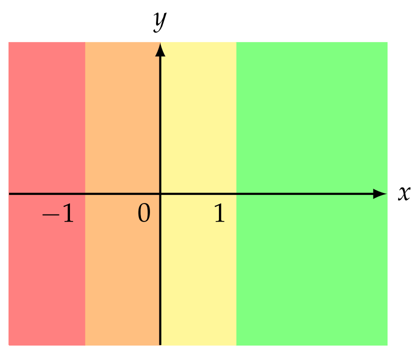
One can then repeat the same procedure as before to extend $\hat{f}$ to a holomorphic function $\hat{\hat{f}}$ defined on $\{z : \mathrm{Re}(z) > -1\}$, which is given by:

$$\hat{\hat{f}}(z) = \frac{1}{2}\hat{f}(z+1), \quad \text{for any } z \in \{\mathrm{Re}(z) > -1\}.$$

Inductively, we can repeat the same procedure over and over again, and extend f to a function $F : \mathbb{C} \to \mathbb{C}$ that is holomorphic on the whole complex plane $\mathbb{C}$.







f can be inductively extended to an entire function F

Exercise 5.4. Given that $f : \Omega \to \mathbb{C}$ is a holomorphic on $\Omega := \{z : \mathrm{Re}(z) > 1\}$, and that it satisfies the relation $f(z+1) = zf(z)$ for any $z \in \Omega$. Show that there is an analytic continuation $\hat{f}$ on $\mathbb{C}\backslash\{0, -1, -2, -3, \cdots\}$. Classify the type of singularities (pole, removable or essential singularity) of each non-positive integer $-n$ for $\hat{f}$.

5.2. Riemann ζ Functions

5.2.1. Analytic Continuation of Γ. In this section we discuss the Γ (Gamma) and ζ (zeta) functions, as well as their analytic continuations. These two functions are closely related. The Gamma function $\Gamma : \Omega \to \mathbb{C}$ is *a priori* defined on $\Omega := \{z : \mathrm{Re}(z) > 0\}$ by:

$$\Gamma(z) := \int_0^\infty t^{z-1}e^{-t}\,dt \quad \text{for } \mathrm{Re}(z) > 0.$$

It is an improper integral. By breaking it down into:

$$\Gamma(z) = \int_0^1 t^{z-1}e^{-t}\,dt + \int_1^\infty t^{z-1}e^{-t}\,dt,$$

one can verify (as an exercise) that the first integral is integrable when $\mathrm{Re}(z) > 0$, and the second integral is integrable for any $z \in \mathbb{C}$.

Exercise 5.5. Show that:

(a) $\displaystyle\int_0^1 t^{z-1}e^{-t}\,dt$ is integrable when $\mathrm{Re}(z) > 0$; and

(b) $\displaystyle\int_c^\infty t^{z-1}e^{-t}\,dt$ is integrable for any $z \in \mathbb{C}$ for any $c > 0$.

Exercise 5.6. Use Morera's Theorem to show that Γ is holomorphic on $\{\mathrm{Re}(z) > 0\}$. Hint: Note that t^{z-1} is holomorphic for each fixed $t > 0$, but not when $t = 0$. Morera's Theorem cannot be directly applied on this integral. To tackle this issue, consider the sequence of functions:

$$f_n(z) := \int_{\frac{1}{n}}^\infty t^{z-1}e^{-t}\,dt.$$

Show that f_n is holomorphic on $\{\mathrm{Re}(z) > 0\}$ for each n, and that f_n converges uniformly to Γ on $\{\mathrm{Re}(z) > 0\}$ as $n \to \infty$.

Using integration-by-parts, one can derive a functional equation for Γ which can be used to extend Γ beyond the domain $\{\mathrm{Re}(z) > 0\}$. For any $\mathrm{Re}(z) > 0$, we consider:

$$\begin{aligned}
\Gamma(z+1) &= \int_0^\infty t^z e^{-t}\,dt = \int_0^\infty t^z\,d(-e^{-t}) \\
&= \left[-t^z e^{-t}\right]_{t=0}^{t=\infty} + \int_0^\infty e^{-t} d(t^z) \\
&= 0 + \int_0^\infty z t^{z-1} e^{-t}\,dt \\
&= z\Gamma(z).
\end{aligned}$$

We leave the part $\left[-t^z e^{-t}\right]_{t=0}^{t=\infty} = 0$ as an exercise for readers:

Exercise 5.7. Show that whenever $\mathrm{Re}(z) > 0$, we have:

$$\lim_{t\to 0^+} t^z e^{-t} = 0 \quad \text{and} \quad \lim_{t\to\infty} t^z e^{-t} = 0.$$

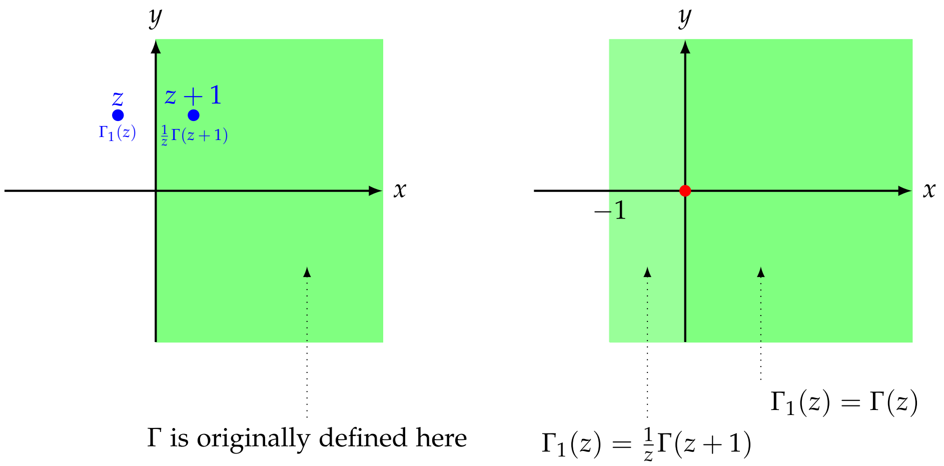
Exercise 5.8. Show that for any positive integer n, we have:

$$\Gamma(n) = (n-1)!$$

From the functional equation $\Gamma(z+1) = z\Gamma(z)$, one can define:

$$\Gamma_1(z) := \frac{1}{z}\Gamma(z+1)$$

for any z such that $z \neq 0$ and $\text{Re}(z+1) > 0$. Then, Γ_1 is an holomorphic function on $\{z : \text{Re}(z) > -1\} \backslash \{0\}$, and when $\text{Re}(z) > 0$, we have $\Gamma_1(z) = \Gamma(z)$. In other words, Γ_1 is an analytic continuation of Γ.



Γ is originally defined here

The functional equation for Γ then induces a new functional equation for Γ_1. Whenever $\text{Re}(z) > -1$, we have:

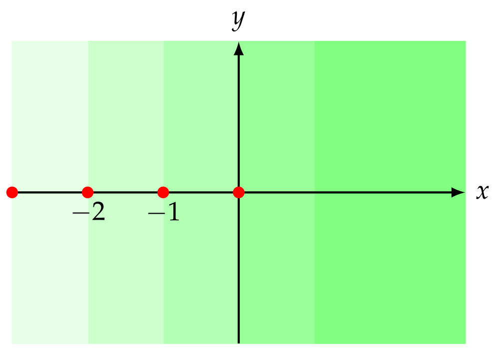
$$\begin{aligned}
\Gamma_1(z+1) &= \frac{1}{z+1}\Gamma(z+2) && \text{(Definition of } \Gamma_1\text{)} \\
&= \frac{1}{z+1} \cdot (z+1)\Gamma(z+1) && \text{(Functional equation for } \Gamma\text{)} \\
&= \Gamma(z+1) = z\Gamma_1(z) && \text{(Definition of } \Gamma_1\text{).}
\end{aligned}$$

Therefore, one can define:

$$\Gamma_2(z) := \frac{1}{z}\Gamma_1(z+1)$$

for any $z \in \mathbb{C}$ such that $z+1$ is in the domain of Γ_1, i.e. $\text{Re}(z) > -2$ and $z \neq -1$. As such, Γ_2 is an analytic continuation of Γ_1 (and hence of Γ) on $\{\text{Re}(z) > -2\} \backslash \{0, -1\}$.

Repeat the above process indefinitely, one can define analytic continuations Γ_m on $\{\text{Re}(z) > -m\} \backslash \{0, -1, -2, \cdots, -(m-1)\}$, and eventually an analytic continuation $\hat{\Gamma}$ of Γ on the domain $\mathbb{C} \backslash \{0, -1, -2, -3, \cdots\}$.



Γ can be inductively extended to $\hat{\Gamma}$

Exercise 5.9. Show that for any integer $m \geq 1$ and z in the domain of Γ_m, we have:

$$\Gamma_m(z) = \frac{\Gamma(z+m)}{z(z+1)\cdots(z+m-1)}$$

Exercise 5.10. Show that each non-positive integer $-n$ is a simple pole of $\hat{\Gamma}$, and that:

$$\operatorname{Res}(\hat{\Gamma}, -n) = \frac{(-1)^n}{n!}.$$

Here is a summary of facts about the Gamma function:

- Γ is a priori defined on $\{z : \operatorname{Re}(z) > 0\}$.
- By the relation $\Gamma(z+1) = z\Gamma(z)$, one can define an analytic continuation $\hat{\Gamma}$ of Γ on $\mathbb{C}\backslash\{0, -1, -2, -3, \cdots\}$.
- Each non-positive integer $-n$ is a simple pole of $\hat{\Gamma}$, with residue equal to $\frac{(-1)^n}{n!}$.

Recall that since the analytic continuation must be unique, some textbooks denote the analytic continuation by simply Γ.

Exercise 5.11. Show that when $\operatorname{Re}(z) > 0$, the Gamma function can be decomposed into:

$$\Gamma(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(z+n)} + \int_1^{\infty} e^{-t} t^{z-1}\, dt.$$

Show also that the infinite sum $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(z+n)}$ converges for any $z \neq 0, -1, -2, -3, \cdots$ and the integral $\displaystyle\int_1^{\infty} e^{-t} t^{z-1}\, dt$ is an entire function of z.

5.2.2. Relation between Γ and ζ. Recall that the Riemann zeta function $\zeta : \{z : \operatorname{Re}(z) > 1\} \to \mathbb{C}$ is defined by the infinite series:

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$$

which converges when $\operatorname{Re}(z) > 1$. The following lemma shows a relation between Γ and ζ.

Lemma 5.4. *For any $z \in \mathbb{C}$ such that $\operatorname{Re}(z) > 1$, we have:*

$$\zeta(z)\Gamma(z) = \int_0^{\infty} \frac{t^{z-1}}{e^t - 1}\, dt. \tag{5.1}$$

Proof. The key step of the proof is the change of variables $t = n\tau$ in the integral that defines Γ:

$$\begin{aligned}
\Gamma(z) &= \int_0^{\infty} t^{z-1} e^{-t}\, dt = \int_0^{\infty} (n\tau)^{z-1} e^{-n\tau}\, d(n\tau) \\
&= n^z \int_0^{\infty} \tau^{z-1} e^{-n\tau}\, d\tau \\
\frac{1}{n^z}\Gamma(z) &= \int_0^{\infty} t^{z-1} e^{-nt}\, dt.
\end{aligned}$$

Here we have used the fact that τ is a dummy variable. Summing up over n, we get:

$$\sum_{n=1}^{\infty} \frac{1}{n^z}\Gamma(z) = \sum_{n=1}^{\infty} \int_0^{\infty} t^{z-1}e^{-nt}\,dt. \tag{5.2}$$

Next we want to switch the integral and summation signs. It has to be justified using LDCT. Consider:

$$\left|t^{z-1}e^{-nt}\right| \leq t^{x-1}e^{-nt}$$

for any $t \in [0, \infty)$. Note that:

$$\sum_{n=1}^{\infty} \int_0^{\infty} t^{x-1}e^{-nt}\,dt = \sum_{n=1}^{\infty} \frac{1}{n^x}\Gamma(x)$$

which converges since $x > 1$. Hence, LDCT shows we can switch the summation and integral signs of (5.2), and it yields:

$$\sum_{n=1}^{\infty} \frac{1}{n^z}\Gamma(z) = \int_0^{\infty} t^{z-1} \sum_{n=1}^{\infty} e^{-nt}\,dt.$$

Observing that $\sum_{n=1}^{\infty} e^{-nt} = \sum_{n=1}^{\infty} \left(e^{-t}\right)^n$ is a geometric series, we get:

$$\sum_{n=1}^{\infty} e^{-nt} = \frac{e^{-t}}{1-e^{-t}} = \frac{1}{e^t - 1}.$$

From (5.2), we get our desired result (5.1). $\square$

5.2.3. Analytic Continuation of ζ. The relation (5.1) will be used to extend ζ beyond the domain $\{\mathrm{Re}(z) > 1\}$. We have already shown that Γ can be extended to almost all of $\mathbb{C}$. If we are able to extend the integral:

$$\int_0^{\infty} \frac{t^{z-1}}{e^t - 1}\,dt$$

beyond $\{\mathrm{Re}(z) > 1\}$, then ζ can also be extended accordingly.

First break down the integral into two part:

$$\int_0^{\infty} \frac{t^{z-1}}{e^t - 1}\,dt = \int_0^{1} \frac{t^{z-1}}{e^t - 1}\,dt + \int_1^{\infty} \frac{t^{z-1}}{e^t - 1}\,dt.$$

The second integral is well-defined for any $z \in \mathbb{C}$. To see this, we first note that $\left|t^{z-1}\right| = t^{x-1} \ll e^{t/2}$ as $t \to \infty$, and hence $\frac{t^{x-1}}{e^t - 1} \ll \frac{e^{t/2}}{e^t - 1} \sim e^{-t/2}$. The function $e^{-t/2}$ is integrable over $[1, \infty)$. By comparison, the integral $\int_1^{\infty} \left|\frac{t^{z-1}}{e^t - 1}\right| dt$ is finite for any $z \in \mathbb{C}$ (not only those with $\mathrm{Re}(z) > 1$). By Morera's Theorem, the integral is an entire function of z.

Next we handle the first integral $\int_0^1 \frac{t^{z-1}}{e^t - 1}\,dt$. The key trick is to consider the denominator $\frac{1}{e^t - 1}$, and expand it as a series. Consider the function:

$$f(w) = \frac{1}{e^w - 1} - \frac{1}{w}.$$

Although it is not defined when $w = 0$, we can see that 0 is a removable singularity:

$$\begin{aligned}\lim_{w\to 0} f(w) &= \lim_{w\to 0}\left(\frac{1}{w + \frac{w^2}{2!} + \frac{w^3}{3!} + \cdots} - \frac{1}{w}\right) \\ &= \lim_{w\to 0}\frac{w - w - \frac{w^2}{2!} - \frac{w^3}{3!} - \cdots}{w(w + \frac{w^2}{2!} + \frac{w^3}{3!} + \cdots)} \\ &= \lim_{w\to 0}\frac{-\frac{1}{2} - \frac{w}{6} - \cdots}{1 + \frac{w}{2} + \frac{w^2}{6} + \cdots} \\ &= -\frac{1}{2}.\end{aligned}$$

Therefore, by declaring that $f(0) = -\frac{1}{2}$, it becomes a holomorphic function defined on $B_{2\pi}(0)$ (why 2π?). Consider its Taylor series about 0:

$$\begin{aligned} f(w) &= -\frac{1}{2} + f'(0)w + \frac{f''(0)}{2!}w^2 + \frac{f^{(3)}(0)}{3!}w^3 + \cdots \\ \frac{1}{e^w - 1} - \frac{1}{w} &= \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}w^n.\end{aligned}$$

Substitute $w = t \in [0,1]$, then we get:

$$\frac{1}{e^t - 1} = \frac{1}{t} + \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}t^n.$$

Recall from Exercise 4.8 that the Taylor's series $\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}w^n$ converges uniformly on every ball $B_{2\pi-\varepsilon}(0)$ slightly smaller than $B_{2\pi}(0)$, say $B_2(0)$. In particular, since $[0,1] \subset B_2(0)$, the convergence of the series $\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}t^n$ is also uniform on $[0,1]$. When z is a fixed complex number such that $\mathrm{Re}(z) > 1$, we have $\left|t^{z-1}\right| \le t^{x-1} \le 1$. Therefore, the series $\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}t^{n+z-1}$ also converges uniformly on $t \in [0,1]$ regarding z as fixed. Using the fact, one can write the first integral as:

$$\begin{aligned}\int_0^1 \frac{t^{z-1}}{e^t - 1}\,dt &= \int_0^1 t^{z-1}\left(\frac{1}{t} + \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}t^n\right)dt \\ &= \int_0^1\left(t^{z-2} + \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}t^{z+n-1}\right)dt \\ &= \left[\frac{t^{z-1}}{z-1}\right]_{t=0}^{t=1} + \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}\left[\frac{t^{z+n}}{z+n}\right]_{t=0}^{t=1} \\ &= \frac{1}{z-1} + \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}\cdot\frac{1}{z+n}.\end{aligned}$$

Here we have integrated term-by-term thanks to uniform convergence of the series.

Although the integral $\int_0^1 \frac{t^{z-1}}{e^t-1}\,dt$ on the LHS is defined only when $\mathrm{Re}(z) > 1$, the RHS series is defined whenever $z \neq 1, 0, -1, -2, -3, \cdots$. Furthermore, the RHS series is holomorphic on $\Omega := \mathbb{C}\backslash\{1, 0, -1, -2, -3, \cdots\}$. To show this, it suffices to prove $\displaystyle\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}\cdot\frac{1}{z+n}$ converges uniformly on any small ball $B_r(z_0)\subset\Omega$. Note that the singularities $\{1,0,-1,-2,-3,\cdots\}$ are isolated, points in $B_r(z_0)$ must be well away from the singularities. There exists $\delta>0$ such that $|z+n|\geq\delta$ for any $z\in B_r(z_0)$ and $n=0,1,2,3,\cdots$. As a result, we have:

$$\left|\frac{f^{(n)}(0)}{n!}\cdot\frac{1}{z+n}\right|\leq\left|\frac{f^{(n)}(0)}{n!}\right|\cdot\frac{1}{\delta}.$$

By Weierstrass's M-test, the series $\displaystyle\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}\cdot\frac{1}{z+n}$ converges uniformly on any small ball $B_r(z_0)\subset\Omega$. By Morera's Theorem, it defines a holomorphic function on any small ball $B_r(z_0)\subset\Omega$, and so is holomorphic on Ω.

Combining the result (5.1), we have so far established that on $\{\mathrm{Re}(z)>1\}$:

$$\zeta(z)=\frac{1}{\Gamma(z)}\left(\underbrace{\frac{1}{z-1}+\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}\cdot\frac{1}{z+n}}_{\text{extendable to }\Omega}+\underbrace{\int_1^{\infty}\frac{t^{z-1}}{e^t-1}\,dt}_{\text{entire}}\right)$$

Since Γ has an analytic continuation $\hat{\Gamma}$ on $\mathbb{C}\backslash\{0,-1,-2,-3,\cdots\}$. From the above relation, we can then define an analytic continuation of ζ on $\mathbb{C}\backslash\{1,0,-1,-2,-3,\cdots\}$ as:

$$\begin{aligned}\hat{\zeta}(z)&=\frac{1}{\hat{\Gamma}(z)}\left(\frac{1}{z-1}+\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}\cdot\frac{1}{z+n}+\int_1^{\infty}\frac{t^{z-1}}{e^t-1}\,dt\right)\\&=\frac{1}{(z-1)\hat{\Gamma}(z)}+\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}\frac{1}{(z+n)\hat{\Gamma}(z)}+\frac{1}{\hat{\Gamma}(z)}\int_1^{\infty}\frac{t^{z-1}}{e^t-1}\,dt.\end{aligned}\tag{5.3}$$

It appears (5.3) has singularities at every $1,0,-1,-2,-3,\cdots$, yet we can show $0,-1,-2,-3,\cdots$ are all removable. It is because $\hat{\Gamma}$ has a simple pole at every of $\{0,-1,-2,-3,\cdots\}$, so they are *zeros* of $1/\hat{\Gamma}$. Therefore, $\frac{1/\hat{\Gamma}}{z+n}$ has a removable singularity at $-n$. Precisely, for any integers $m,n\in\{0,1,2,3,\cdots\}$ we have:

$$\lim_{z\to-m}\frac{1}{(z+n)\hat{\Gamma}(z)}=\begin{cases}\frac{1}{\mathrm{Res}(\hat{\Gamma},-n)} & \text{if } m=n\\ 0 & \text{if } m\neq n\end{cases}$$

This shows $\{0,1,2,3,\cdots\}$ are all removable singularities of $\hat{\zeta}$ since the following limit is finite for any $m=0,1,2,3,\cdots$

$$\lim_{z\to-m}\hat{\zeta}(z)=\frac{f^{(m)}(0)}{m!}\frac{1}{\mathrm{Res}(\hat{\Gamma},-m).}$$

Therefore, $\zeta(z)$ can be holomorphically defined on $\mathbb{C}\backslash\{1\}$ by declaring that

$$\hat{\zeta}(-m):=\frac{f^{(m)}(0)}{m!}\frac{1}{\mathrm{Res}(\hat{\Gamma},-m)}$$

for any $m=0,1,2,\cdots$. Note that 1 is a simple pole of $\hat{\zeta}$.

5.2.4. Special Values of $\hat{\zeta}$. We will determine the value of $\hat{\zeta}$ at some special $z \in \mathbb{C}$. When $z = -m$ where $-m$ is a non-positive integer, then we have already discussed that

$$\hat{\zeta}(-m) = \frac{f^{(m)}(0)}{m!} \frac{1}{\operatorname{Res}(\hat{\Gamma}, -m)}.$$

Here f is the function:

$$f(w) = \frac{1}{e^w - 1} - \frac{1}{w}.$$

We have already figured out that $\operatorname{Res}(\hat{\Gamma}, -m) = \dfrac{(-1)^m}{m!}$, so $\hat{\zeta}(-m) = (-1)^m f^{(m)}(0)$. However, it is not straight-forward to find a general expression for $f^{(m)}(0)$, but by direct computations one can verify that the first few terms of $f^{(m)}(0)$ are given as follows:

$$f(0) = -\frac{1}{2} \qquad f'(0) = \frac{1}{12} \qquad f''(0) = 0 \qquad f^{(3)}(0) = -\frac{1}{120}.$$

Therefore, the extended Riemann zeta function $\hat{\zeta}$ takes the following values:

$$\hat{\zeta}(0) = -\frac{1}{2} \qquad \hat{\zeta}(-1) = -\frac{1}{12}$$
$$\hat{\zeta}(-2) = 0 \qquad \hat{\zeta}(-3) = \frac{1}{120}$$

To many people's surprise, the fact that $\hat{\zeta}(-1) = -\frac{1}{12}$ is used in String Theory! However, many "muggles" misunderstand the meaning of it, and misinterpret it as $\sum_{n=1}^{\infty} \frac{1}{n^{-1}} = -\frac{1}{12}$, which is mathematically wrong as $\hat{\zeta}(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$ only when $\operatorname{Re}(z) > 1$. It would lead to the following awkward and non-sense expression:

$$1 + 2 + 3 + 4 + \cdots = -\frac{1}{12}.$$

Similarly, some "amateurs" mix up $\hat{\zeta}(0) = -\frac{1}{2}$ with $\sum_{n=1}^{\infty} \frac{1}{n^0} = -\frac{1}{2}$, and $\hat{\zeta}(-2) = 0$ with $\sum_{n=1}^{\infty} \frac{1}{n^{-2}} = 0$, both would lead to awkward expressions:

$$1 + 1 + 1 + 1 + \cdots = -\frac{1}{2}.$$
$$1^2 + 2^2 + 3^2 + 4^2 + \cdots = 0.$$

5.2.5. Riemann Hypothesis. Finally, we are ready to understand the statement of the Riemann Hypothesis. It is a conjecture about the zeros of the (extended) Riemann zeta function $\hat{\zeta}$. To begin, let's first recall that for any negative integer $-m$, we have:

$$\hat{\zeta}(-m) = (-1)^m f^{(m)}(0),$$

where $f(w) = \dfrac{1}{e^w - 1} - \dfrac{1}{w}$. It is not difficult to show that $f^{(m)}(0) = 0$ for any even integer m:

Exercise 5.12. Show that

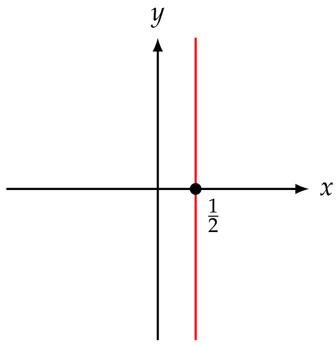
$$g(w) := f(w) + \frac{1}{2}$$

is an odd function, and hence deduce that $f^{(m)}(0) = 0$ for any even integer $m \geq 2$.

Therefore, we have $\hat{\zeta}(-2) = \hat{\zeta}(-4) = \hat{\zeta}(-6) = \cdots = 0$. These negative even integers $\{-2, -4, -6, \cdots\}$ are called *trivial zeros* of $\hat{\zeta}$.

Any complex number z_0 which is not a negative even integer is called a *non-trivial zero* of $\hat{\zeta}$ whenever $\hat{\zeta}(z_0) = 0$. The Riemann Hypothesis is concerned with the locations of these non-trivial zeros. It is conjectured by Bernhard Riemann in 1859 that:

"All non-trivial zeros z_0 of $\hat{\zeta}$ must have real part equal to $\frac{1}{2}$."



The zeros of $\hat{\zeta}$ have deep connections with the distribution of prime numbers. The renowned *Prime Number Theorem* asserts that:

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\ln x} = 1$$

where $\pi(x)$ is the number of positive prime numbers less than or equal to x. A corollary of the theorem is that the n-th prime number p_n is approximately equal to $n \ln n$. The proof of Prime Number Theorem relies surprisingly on the fact that there is no zero of $\hat{\zeta}$ with real part equal to 1. If the Riemann Hypothesis is proven to be true, then the Prime Number Theorem can be substantially improved, and many mysteries about the distribution of primes will be revealed.

As of today (January 20, 2017), this conjecture remains unsolved, and is one of the most important open problem in Pure Mathematics nowadays. In 2000, the Clay Mathematics Institute compiled a list of 7 problems, called Millennium Prize Problems. For each problem in the list, the institute promises to award US$1,000,000 to the first person who solves or disproves it. Riemann Hypothesis is one of the problems in the list. The other 6 problems are: P versus NP Problem, Hodge Conjecture, Poincaré Conjecture, Yang-Mills Existence and Mass Gap, Navier-Stokes Existence and Smoothness, and Birch Swinnerton-Dyer Conjecture. The only Millennium Prize Problem that was solved is the Poincaré Conjecture, which concerns about simply-connected 3-manifolds (MATH 4033 stuff), by Grigori Perelman in 2002-03 using the idea of Ricci flow developed by Richard Hamilton in 1982.

*** End of MATH 4023 ***

**** I hope you have learned a lot and/or enjoyed the course. ****

Appendix A

Results from MATH 3033/3043

In this appendix we list some important concepts and theorems from MATH 3033/3043 that we will use frequently in this course. Proofs are all omitted since they are essentially the same as in the real case. This appendix is intended to be brief (no worked example here). For detail, please consult Chapter 10 of MATH 3033, or Chapter 4 in MATH 3043.

Definition A.1 (Uniform Convergence)**.** A sequence of functions $f_n(z)$ is said to converge to $f(z)$ *uniformly on* Ω if

$$\sup\{|f_n(z) - f(z)| : z \in \Omega\} \to 0 \quad \text{as } n \to \infty.$$

A (pointwise) convergent series $\sum_{n=1}^{\infty} f_n(z)$ is said to *converge uniformly on* Ω if the N-th partial sum $\sum_{n=1}^{N} f_n(z)$ converges uniformly on Ω as $N \to \infty$. In other words:

$$\sup\left\{\left|\sum_{n=1}^{N} f_n(z) - \sum_{n=1}^{\infty} f_n(z)\right| : z \in \Omega\right\} \to 0 \quad \text{as } N \to \infty.$$

It is sometimes difficult to show a series converges uniformly from the definition. Fortunately, we have the following useful test:

Theorem A.2 (Weierstrass' M-test)**.** *Consider a series* $\sum_{n=1}^{\infty} f_n(z)$ *defined on* Ω*. If there exists a sequence of real numbers* $M_n \in \mathbb{R}$*, independent of* z*, such that:*

- $|f_n(z)| \leq M_n$ *for any* $z \in \Omega$ *and any* n*, and*
- *the series* $\sum_{n=1}^{\infty} M_n$ *converges,*

then $\sum_{n=1}^{\infty} f_n(z)$ *converges uniformly on* Ω*.*

There are many nice consequences if a series or sequence converges uniformly, namely we can switch the integral, limit and summation signs quite freely:

Proposition A.3. *Suppose $f_n(z)$ converges uniformly on Ω to the limit function $f(z)$, then:*

- *If f_n are continuous on Ω for all n, then f is also continuous on Ω.*
- *For any $\alpha \in \Omega$, we have*
$$\lim_{z\to\alpha}\lim_{n\to\infty} f_n(z) = \lim_{n\to\infty}\lim_{z\to\alpha} f_n(z).$$
- *Let $[a,b]$ be a bounded interval in $\mathbb{R}$, and $f_n(t)$'s be integrable functions on $[a,b]$ If $f_n(t)$ converges uniformly to $f(t)$ on $[a,b]$, then*
$$\lim_{n\to\infty}\int_a^b f_n(t)\,dt = \int_a^b \lim_{n\to\infty} f_n(t)\,dt.$$
- *Let γ be a curve in $\mathbb{C}$ of finite length, and $f_n(z)$'s be integrable functions on γ. If $f_n(z)$ converges uniformly to $f(z)$ on Ω, then*
$$\lim_{n\to\infty}\int_\gamma f_n(z)\,dz = \int_\gamma \lim_{n\to\infty} f_n(z)\,dz.$$

Analogous results hold for uniform convergence series. For instance, if $\sum_{n=1}^{\infty} f_n(z)$ converges uniformly on Ω, then for any curve γ in Ω of finite length, we have:
$$\int_\gamma \sum_{n=1}^{\infty} f_n(z)\,dz = \sum_{n=1}^{\infty}\int_\gamma f_n(z)\,dz.$$

In the above proposition, the conditions that $[a,b]$ is a finite interval, and γ is a curve of finite length are necessary. While we mostly encounter curves of finite lengths for contour integrals, we will occasionally come across real intervals of *unbounded intervals*. In such case, uniform convergence is not sufficient to guarantee switching of the integral and summation signs! Fortunately, there is another tool to deal with improper integrals, namely Lebesgue Dominated Convergence Theorem (LDCT), which stems from measure theory:

Theorem A.4 (Lebesgue Dominated Convergence Theorem)**.** *Let $f_n(t) : (a,b) \to \mathbb{C}$ be a sequence of measurable functions (including continuous functions) defined on a possibly infinite interval $(a,b) \subset \mathbb{R}$. Suppose:*

- *$f_n(t) \to f(t)$ pointwise on every $t \in (a,b)$, and*
- *there exists an integrable function $h : (a,b) \to \mathbb{R}$ independent of n such that*
$$|f_n(t)| \leq h(t) \quad \text{for any } t \in (a,b) \text{ and any } n,$$

then we have
$$\lim_{n\to\infty}\int_a^b f_n(t)\,dt = \int_a^b \lim_{n\to\infty} f_n(t)\,dt.$$

Consider a series $\sum_{n=1}^{\infty} g_n(t)$ where $g_n : (a,b) \to \mathbb{C}$ are measurable. Suppose
$$\sum_{n=1}^{\infty}\int_a^b |g_n(t)|\,dt < \infty,$$
then we have:
$$\int_a^b \sum_{n=1}^{\infty} g_n(t)\,dt = \sum_{n=1}^{\infty}\int_a^b g_n(t)\,dt.$$

Recall from MATH 3033/3043 that even if $f_n(x)$ converges uniformly on (a,b) to $f(x)$, the derivatives $f_n'(x)$ may not converge to f'. Sometimes, the limit of f_n' may not even be differentiable. Likewise, even when the sum $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on (a,b), term-by-term differentiation

$$\frac{d}{dx}\sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} f_n'(x)$$

may not hold. We will not pursue a full discussion about term-by-term differentiation here, but we would like to remind you one fact that you can always do term-by-term differentiation for a convergent *power series*.

Proposition A.5. *Suppose the power series* $f(z) = \sum_{n=0}^{\infty} c_n(z-z_0)^n$ *converges on* $B_r(z_0)$, *then we have:*

$$f'(z) = \sum_{n=0}^{\infty} \frac{d}{dz} c_n(z-z_0)^n = \sum_{n=1}^{\infty} nc_n(z-z_0)^{n-1}$$

at every $z \in B_r(z_0)$.

When using Morera's Theorem, we often consider a double integral of the form:

$$\oint_T \int_a^b f(z,t)\,dtdz.$$

It we can switch the two integral signs, and it happens that $f(z,t)$ is a holomorphic function for each fixed $t \in [a,b]$, then we have:

$$\oint_T \int_a^b f(z,t)\,dtdz = \int_a^b \oint_T f(z,t)\,dzdt = \int_a^b 0\,dt = 0.$$

The question is whether we can switch the two integral signs. It thanks for the following (special case) of Fubini's Theorem

Theorem A.6 (Fubini's Theorem: special case)**.** *Suppose* $f(z,t) : \Omega \times I \to \mathbb{C}$ *is a continuous function, where* $\Omega \subset \mathbb{C}$ *and* I *is an interval (possibly infinite) in* $\mathbb{R}$. *Let* γ *be a curve in* Ω. *If one of the following is finite:*

$$\int_\gamma \int_I |f(z,t)|\,dt\,|dz| \quad \text{or} \quad \int_I \int_\gamma |f(z,t)|\,|dz|\,dt.$$

Then, we have:

$$\int_\gamma \int_I f(z,t)\,dtdz = \int_I \int_\gamma f(z,t)\,dzdt.$$

Here $|dz|$ *means* $\sqrt{(dx)^2 + (dy)^2}$.