Math 1060Q Lecture 16

Jeffrey Connors

University of Connecticut

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We shall cover some very useful trigonometric idenities

- Pythagorean Identities
- Angle addition/subtraction identities
- Double angle identities
- Half-angle identities
- Product identities

The Pythagorean identity can be manipulated to involve co-functions.

Recall that for any angle θ, $\cos^2(\theta) + \sin^2(\theta) = 1$. If we divide through by $\sin^2(\theta)$, we see

$$\frac{\cos^2(\theta)}{\sin^2(\theta)} + 1 = \frac{1}{\sin^2(\theta)} \Rightarrow \cot^2(\theta) + 1 = \csc^2(\theta).$$

Similarly, dividing through by $\cos^2(\theta)$ yields

$$1 + \frac{\sin^2(\theta)}{\cos^2(\theta)} = \frac{1}{\cos^2(\theta)} \Rightarrow 1 + \tan^2(\theta) = \sec^2(\theta).$$

The main thing is to be able to apply trig. formulas.

Example L16.1: Let $f(x) = \csc(x) - \cot(x)$ and $g(x) = \csc(x) + \cot(x)$. Find and simplify $h = f \cdot g$.

Solution: We see that

$$h(x) = (\csc(x) - \cot(x))(\csc(x) + \cot(x)) = \csc^2(x) - \cot^2(x).$$

We apply now the identity

$$\cot^2(x) + 1 = \csc^2(x) \Rightarrow \csc^2(x) - \cot^2(x) = 1 = h(x).$$

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Angle addition/subtraction formulas

$$\sin(x_1 \pm x_2) = \sin(x_1)\cos(x_2) \pm \cos(x_1)\sin(x_2)$$

$$\cos(x_1 \pm x_2) = \cos(x_1)\cos(x_2) \mp \sin(x_1)\sin(x_2)$$

Here, x_1 and x_2 are just two angles. For example, if $x_1 = \pi/2$ and $x_2 = \pi/4$, we verify that

$$\sin(\pi/2 - \pi/4) = \sin(\pi/2)\cos(\pi/4) - \cos(\pi/2)\sin(\pi/4)$$

$$1 \cdot \frac{1}{\sqrt{2}} - 0 \cdot \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}.$$

We know this is correct, since

$$\sin(\pi/2 - \pi/4) = \sin(\pi/4) = \frac{1}{\sqrt{2}}.$$

We can use these identities to calculate $\sin(x)$ or $\cos(x)$ exactly for more angles now.

Example L16.2: Find $\sin(\pi/12)$.

Solution: Rewrite $\pi/12 = \pi/3 - \pi/4$ and plug into the identity:

$\sin(\pi/12) = \sin(\pi/3 - \pi/4) = \sin(\pi/3)\cos(\pi/4) - \cos(\pi/3)\sin(\pi/4)$.

Inserting the appropriate values, we see that

$$\sin(\pi/12) = \frac{\sqrt{3}}{2}\frac{1}{\sqrt{2}} - \frac{1}{2}\frac{1}{\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}}.$$

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We also encounter double-angles frequently, in which case these formulas may be used.

$$\sin(2x) = 2\sin(x)\cos(x)$$
$$\cos(2x) = \cos^2(x) - \sin^2(x)$$

Note that the latter identity may be put into a couple of other forms by way of the Pythagorean identity:

$$\cos(2x) = 1 - 2\sin^2(x)$$

$$\cos(2x) = 2\cos^2(x) - 1$$

One application is to solve an equation.

Example L16.3: Solve $\sin(2x) = \cos(x)$.

Solution: We apply the appropriate double-angle formula:

$$2\sin(x)\cos(x) = \cos(x) \Rightarrow 2\sin(x)\cos(x) - \cos(x) = 0.$$

Factor out $\cos(x)$...

$$(2\sin(x) - 1)\cos(x) = 0 \Rightarrow \cos(x) = 0 \quad \text{or} \quad \sin(x) = \frac{1}{2}.$$

There are infinitely many solutions; $\cos(x) = 0$ holds for all x of the form $x = (2k-1)\pi/2$ with k any integer. Also, $\sin(x) = 1/2$ for all $x = 2k\pi + \pi/6$ and $x = 2k\pi + 5\pi/6$.

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Half-angle identities

$$\sin^2(x/2) = \frac{1-\cos(x)}{2}$$
$$\cos^2(x/2) = \frac{1+\cos(x)}{2}$$

Often, these are used by taking the square-root:

$$\sin(x/2) = \pm\sqrt{\frac{1-\cos(x)}{2}}$$

$$\cos(x/2) = \pm\sqrt{\frac{1+\cos(x)}{2}}$$

Note that the correct sign depends on which quadrant the angle $x/2$ is in.

We can also calculate $\sin(x)$, $\cos(x)$ for more angles x now by leveraging half-angle identities.

Example L16.4: Find the exact value of $\cos(7\pi/8)$.

Solution: Write $7\pi/8 = (7\pi/4)/2$ and apply the formula

$$\cos\left(\frac{1}{2}\cdot\frac{7\pi}{4}\right) = -\sqrt{\frac{1+\cos(7\pi/4)}{2}}.$$

We take the negative of the square-root because the angle $7\pi/8$ is in the second quadrant, which makes $\cos(7\pi/8)$ a negative number. Now insert $\cos(7\pi/4) = 1/\sqrt{2}$:

$$\cos\left(\frac{1}{2}\cdot\frac{7\pi}{4}\right) = -\sqrt{\frac{1+1/\sqrt{2}}{2}}.$$

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Product identities are used to rewrite products of $\sin(x)$ and $\cos(x)$ as sums/differences or vice-versa

$$\sin(x_1)\cos(x_2) = \frac{1}{2}\left(\sin(x_1 - x_2) + \sin(x_1 + x_2)\right)$$

$$\cos(x_1)\cos(x_2) = \frac{1}{2}\left(\cos(x_1 - x_2) + \cos(x_1 + x_2)\right)$$

$$\sin(x_1)\sin(x_2) = \frac{1}{2}\left(\cos(x_1 - x_2) - \cos(x_1 + x_2)\right)$$

In this example we switch from a product to a sum of sinusoidal functions

Example L16.5: Rewrite $\cos(x)\sin(2x)$ as a sum/difference of sinusoids.

Solution: Upon application of the formula,

$$\sin(2x)\cos(x) = \frac{1}{2}\left(\sin(2x - x) + \sin(2x + x)\right)$$
$$= \frac{1}{2}\sin(x) + \frac{1}{2}\sin(3x).$$

Practice!

Problem L16.1: Calculate the exact value of $\cos(7\pi/12)$.

Problem L16.2: Calculate $\sin(-\pi/12)$.

Problem L16.3: Use the half-angle formulas to find $\sin(7\pi/8)$.

Problem L16.4: Write $\sin(4x)\cos(3x)$ as a sum/difference of sinusoids.

Problem L16.5: Solve $2\sin^2(x) + \cos(x) = 1$ for x.