# Trig without Tears,

# or

# How to Remember Those Trig Identities


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# Contents: Trig without Tears

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this page: `http://oakroadsystems.com/math/trig0.htm`

Was this page useful? You might want to look at my other math pages.

# Part 1 of Trig without Tears

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## Introduction

Trigonometry is a fascinating subject −− or, at least, it can be. It has lots of obvious practical uses, some of which are actually taught in the usual trig course. And the computations aren't difficult, now that we have calculators.

> Would you believe that when I studied trig, back when dinosaurs ruled the earth (actually, in the 1960s), to solve any problem we had to look up function values in long tables in the back of the book, and then multiply or divide those five−place decimals *by hand*? The "better" books even included tables of logs of the trig functions, so that we could save work by adding and subtracting five−place decimals instead of multiplying and dividing them. My *College Outline Series* trig book covered all of plane and spherical trigonometry in 188 pages −− but then needed an additional 138 pages for the necessary tables!

So calculators have freed us from tedious computation. But there's still one big stumbling block in the way many trig courses are taught: all those identities. They're just too much to memorize. (Many students despair of understanding what's going on, so they just try to memorize everything and hope for the best at exam time.) Is it $\tan^2 A + \sec^2 A = 1$ or $\tan^2 A = \sec^2 A + 1$? (Actually, it's neither −− see (14)!)

Fortunately, you don't *have* to memorize them. This paper shows you the few that you do need to memorize, and how you can produce the others as needed. I'll present some ideas of my own, and a wonderful insight by W.W. Sawyer. (You can read some ideas of mine on the pros and cons of memorizing, if you like.)

### Other math sites

There are a number of math sites out there: some are good, some less good. I've collected a dozen or so that you may find useful.

### About this document

I believe my notation should be self explanatory. But if you're puzzled, please take a look at my notes on notation.

By the way, I love explaining things but sometimes I go on a bit too long. So I've put some interesting but nonessential notes in a separate page and inserted hyperlinks to them at appropriate points. If you follow them (and I hope you will), your browser's "back" command to return to the main text. Much as it pains me to say so, if you're pressed for time you can still get all the essential points by ignoring those side notes. But you'll miss some of the fun.

I'd be happy to hear your comments on the organization of this document, or anything else about it.

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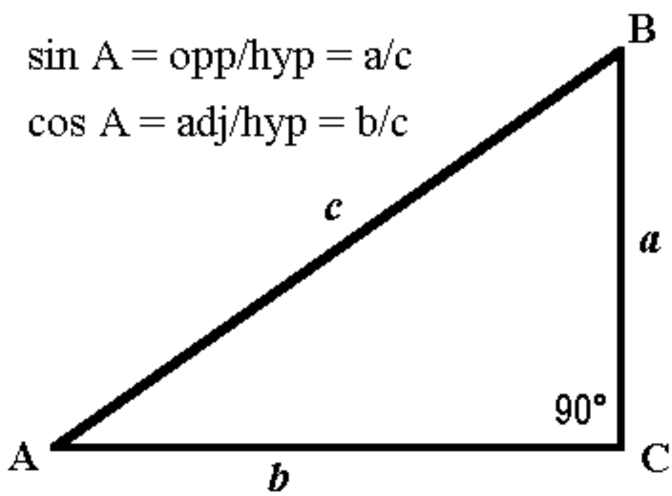
# The Basic Two: Sine and Cosine

A picture is worth a thousand words (which is why it takes a thousand times as long to download). The trig functions are nothing more than lengths of various sides of a right triangle combined in various ways.




This is one of the conventional ways of showing a right triangle. A key point is that the lower−case letters a, b, c are the sides opposite to the angles marked with the corresponding capital letters A, B, C.

The two fundamental definitions are marked in the diagram. These you must commit to memory, and in fact they should become second nature to you, so that you recognize them no matter how the triangle is turned around. Always, always, the sine of an angle is equal to the opposite side divided by the hypotenuse (opp/hyp in the diagram). The cosine is equal to the adjacent side divided by the hypotenuse (adj/hyp).

***memorize:***

---

sine = (opposite side) / hypotenuse
cosine = (adjacent side) / hypotenuse (1)

---

I'll number the important facts and results. The very few that you just have to memorize will be marked "memorize".

Please ***don't*** memorize the others. The whole point of this page is to teach you how to derive them as needed without memorizing them. If you can't think how to derive one, the boxes should make it easy to find it. But then, please work through the explanation. I truly believe that if you once thoroughly understand how all these identities hang together, you'll never have to memorize them again. (It's worked for me since I first studied trig in 1965.)

What is the sine of B in the diagram? Remember opp/hyp: the opposite side is b and the hypotenuse is c, so *sin* B = b/c. What about *cos* B? Remember adj/hyp: the adjacent side is a, so the cosine of B is a/c.

Do you notice that the sine of one angle is the cosine of the other? Since A+B+C = 180° for any triangle, and C is 90°, A+B must equal 90°, so A = 90°−B and B = 90°−A. When two angles add to 90°, each is the **co**mplement of the other, and the sine of each of the **co**sine of the other.

$\sin A = \cos(90° - A)$
$\cos A = \sin(90° - A)$ (2)

## Expressions for lengths of sides

The definitions of sine and cosine can be rearranged a little bit to let you write down the lengths of the sides. For example, when we say that b/c = *cos* A , you can multiply through by c and get b = c *cos* A. Can you write another expression for length b, one that uses a sine instead of a cosine? Remember that opposite over hypotenuse equals the sine, so b/c = *sin* B. Multiply through by c and you have b = c *sin* B.

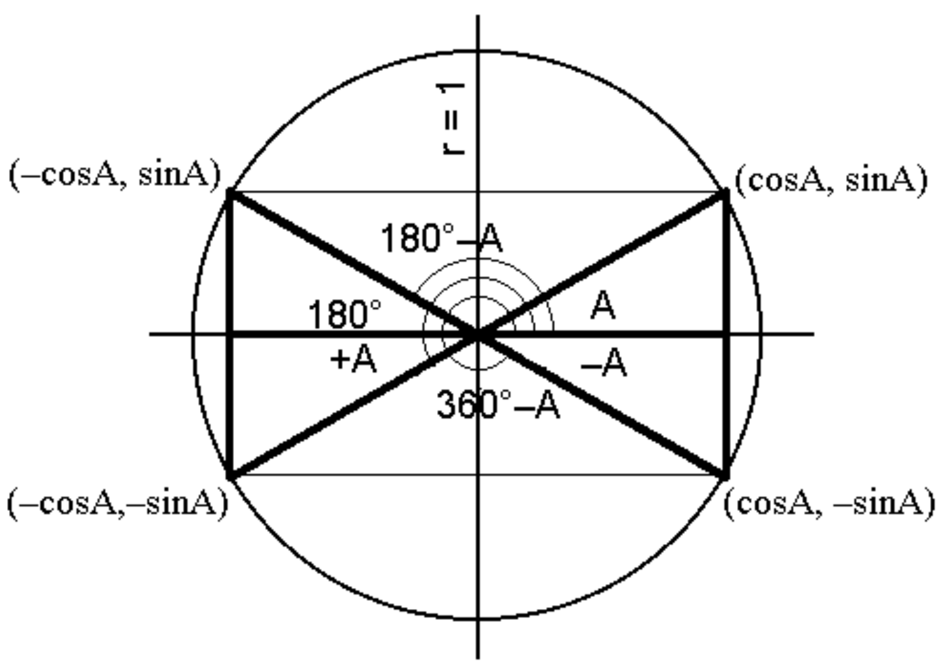
Can you see how to write down two expressions for the length of side a? Please work from the definitions and verify that a = c *sin* A = c *cos* B.

One important special case comes up frequently. If the hypotenuse c = 1, then you can see from the paragraphs just above that a = *sin* A and b = *cos* A. In other words, in a unit right triangle the opposite side will equal the sine and the adjacent side will equal the cosine of the angle. Try to draw this on your own and then compare with my diagram.

Can you see from your drawing what happens if you write down the Pythagorean theorem for a triangle whose hypotenuse is 1? We'll explore that later.

## Negative angles and supplementary angles

Some people have trouble remembering whether sine or cosine of a negative angle is negative. This diagram should help.




Here you see four identical triangles with their angles A at the origin, arranged so that they're mirror images of each other. The hypotenuses form radii of the circle. Just to make things easy, I've drawn a circle of radius 1. That means that the hypotenuse of each triangle is 1, so the other two sides will be *sin* A and *cos* A. (Why? Remember that sine and cosine are defined in terms of sides divided by the hypotenuse. If the hypotenuse is 1, the sides will be equal to the sine and cosine.)

From the diagram, you can see at once what the values of the functions are for angles of −A, 180°−A, and 180°+A.

> If you don't immediately see why, for instance, $\cos(-A) = \cos A$ but $\sin(-A) = -\sin A$, let me give you a guided tour. Do you see that the two triangles on the right−hand side of the origin must be identical? They have equal angles A and 90°, and the side between those angles is shared, so they meet the side−angle−side requirement you learned in geometry. That means their vertical sides must be the same length. What is that length? By definition, *sin* A = opposite/hypotenuse. But the hypotenuse is 1, so the opposite sides must be equal to *sin* A. But one triangle's vertical side goes up from the axis, so the *sin* A is positive; but the other's goes down, so *sin*(−A) is negative.

The relations are summarized below. *Don't* memorize them! Just draw a diagram whenever you need them −− it's easiest if you use a hypotenuse of 1. Soon you'll find that you can quickly visualize the triangles in your mind and you won't even need to draw a diagram.

---

$$\sin(-A) = -\sin A \qquad \cos(-A) = \cos A$$
$$\sin(180°-A) = \sin A \qquad \cos(180°-A) = -\cos A \qquad (3)$$
$$\sin(180°+A) = -\sin A \qquad \cos(180°+A) = -\cos A$$

---

You should also see, since 360° brings you all the way around the circle, that an angle of 360°+A is the same as angle A, so function values are unchanged when you add 360° (or a multiple of 360°) to the angle. Also, if you move in the opposite direction for angle A, that's the same angle as 360°+(−A) or 360°−A, so the function values of −A and 360°−A are the same.

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## The Other Four: Tangent, Cotangent, Secant, Cosecant

The other four functions have no real independent life of their own; they're just combinations of the first two. You could do all of trigonometry without ever knowing more than sines and cosines. But knowing something about the other four, especially the tangent, can often save you some steps in a calculation −− and your teacher will expect you to know about them for exams.

I find it easiest to memorize (sorry!) the definition of the tangent in terms of the sine and cosine:

***memorize:***

---

*tan* A = (*sin* A) / (*cos* A) (4)

---

You'll use the tangent (*tan*) very much more than the last three functions. (I'll get to them in a minute.)

There's an alternative way to remember the meaning of the tangent. Remember from the diagram that *sin* = opposite/hypotenuse and *cos* = adjacent/hypotenuse. Plug those into (4), the definition of *tan*, and you have *tan* = (opposite/hypotenuse) / (adjacent/hypotenuse) or

---

*tan* = (opposite side) / (adjacent side) (5)

---

Notice this is *not* marked "memorize": you don't have to memorize it because it flows directly from the definition (4), and in fact the two statements are equivalent. I've chosen to present them in this order to minimize the jumble of opp, adj, and hyp among *sin*, *cos*, and *tan*. However, if you prefer you can memorize (5) and then derive the equivalent identity (4) whenever you need it.

The other three trig functions −− cotangent, secant, and cosecant −− are defined in terms of the first three. They're much less often used, but they do simplify some problems in calculus.

***memorize:***

---

*cot* A = 1 / (*tan* A)
*sec* A = 1 / (*cos* A) (6)
*csc* A = 1 / (*sin* A)

---

Guess what! That's the last trig identity you have to memorize.

> You'll probably find that you end up memorizing certain other identities without even intending to, just because you use them frequently. But (6) are the last ones that you'll have to sit down and make a point of memorizing just on their own.

Unfortunately, those definitions (6) aren't the easiest thing in the world to remember. Does the secant equal 1 over the sine or 1 over the cosine? Here are two helpful hints: Each of those definitions has a co−function on one and only one side of the equation, so you won't be tempted to think *sec* A = 1/(*sin* A). And secant and cosecant go together just like sine and cosine, so you won't be tempted to write *cot* A = 1/(*sin* A).

You can immediately notice an important relation between tangent and cotangent. Each is the co−function of the other, just like sine and cosine:

---

*tan* A = *cot*(90°−A)
*cot* A = *tan*(90°−A) (7)

---

If you want to prove this, take the definition of *tan* and use (2) to substitute *cos* (90°−A) for *sin* A and *sin* (90°−A) for *cos* A. Tangent and **co**tangent are co−functions just like sine and **co**sine. By doing the same sort of substitution, you can show that secant and **co**secant are also co−functions:

---

$$\sec A = \csc(90°-A)$$
$$\csc A = \sec(90°-A) \qquad (8)$$

---

The formulas for negative angles of tangent and the other functions drop right out of the definitions (4) and (6), since you already know the formulas (3) for sine and cosine of negative angles. For instance, $\tan(-A) = \sin(-A)/\cos(-A) = -\sin A / \cos A$.

---

$$\tan(-A) = -\tan A$$
$$\cot(-A) = -\cot A$$
$$\sec(-A) = \sec A \qquad (9)$$
$$\csc(-A) = -\csc A$$

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# Part 2 of Trig without Tears

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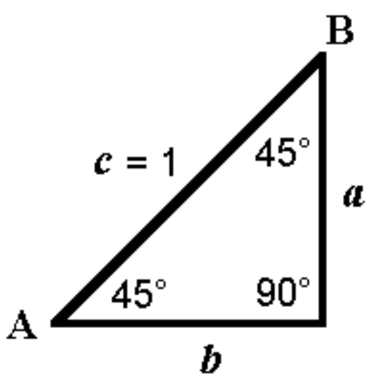
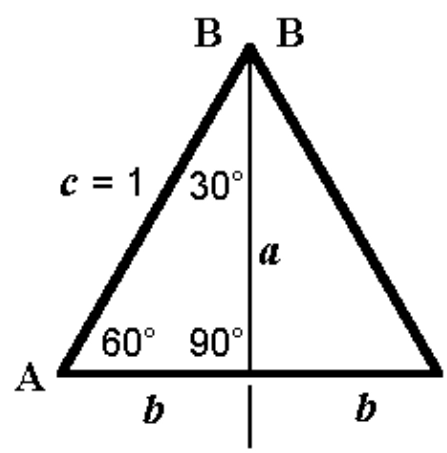
## Trig Functions of Special Angles

Certain angles come up frequently, and it's worth knowing the values of the functions of those angles. These special angles are 30°, 45°, and 60° (pi/6, pi/4, and pi/3). The function values for those angles are easy to derive. Even if you choose to memorize them because they're used so often, you should still know how to derive them at need in case you aren't quite sure of your memory.

The method for all of them is the same: draw a right triangle whose hypotenuse (side c) is 1 and whose angle A is the desired angle. Take a look at these two diagrams:




Look first at the diagram at the left. It's a 45−45−90° triangle, which means sides a and b are equal. By the Pythagorean theorem, $a^2 + b^2 = c^2$. Since a = b and c = 1, we have $2a^2 = 1$ or a = *sqrt*(2)/2. But a = *sin* 45°, so *sin* 45° = *sqrt*(2)/2. Also, b = *cos* 45° and b = a, so *cos* 45° = *sqrt*(2)/2. By definition (4) or (5) of the tangent, *tan* 45° = a/b = 1.

| | |
|---|---|
| *sin* 45° = *cos* 45° = *sqrt*(2)/2<br>*tan* 45° = 1 | (10) |

Now look at the other diagram. I've drawn two 30−60−90° triangles back to back, so that the two 30° angles are next to each other. Since 2×30° = 60°, the big triangle is a 60−60−60° equilateral triangle. Each of the small triangles has hypotenuse 1, so the length 2b is also 1, which means that b = ½. But b = *cos* 60°, so *cos* 60° = ½. We can find a, which is *sin* 60°, using the Pythagorean theorem: $(½)^2 + a^2 = c^2 = 1$, so *sin* 60° = *sqrt*(3)/2.

Since we know the sine and cosine of 60°, we can easily use (2) *sin* A = *cos*(90°−A) to get the cosine and sine of 30°. Therefore *cos* 30° = *sin* 60° = *sqrt*(3)/2, and *sin* 30° = *cos* 60° = ½.

As before, we can use definition (4) of the tangent to find the tangents of 30° and 60° from the sines and cosines:

$$\tan 30° = (\sin 30°) / (\cos 30°) = ½ / (\mathrm{sqrt}(3)/2) = 1 / \mathrm{sqrt}(3) = \mathrm{sqrt}(3)/3$$

and

$$\tan 60° = (\sin 60°) / (\cos 60°) = (\mathrm{sqrt}(3)/2) / ½ = \mathrm{sqrt}(3).$$

The values of the trig functions of 30° and 60° can be summarized like this:

---

$$\sin 30° = ½,\ \sin 60° = \mathrm{sqrt}(3)/2$$
$$\cos 30° = \mathrm{sqrt}(3)/2,\ \cos 60° = ½ \qquad (11)$$
$$\tan 30° = \mathrm{sqrt}(3)/3,\ \tan 60° = \mathrm{sqrt}(3)$$

---

Incidentally, the sines and cosines of 30°, 45°, and 60° display a pleasing pattern:

---

$$\sin 30°, 45°, 60° = \mathrm{sqrt}(1)/2,\ \mathrm{sqrt}(2)/2,\ \mathrm{sqrt}(3)/2$$
$$\cos 30°, 45°, 60° = \mathrm{sqrt}(3)/2,\ \mathrm{sqrt}(2)/2,\ \mathrm{sqrt}(1)/2 \qquad (12)$$

---

It's not surprising that the cosine pattern is a mirror image of the sine pattern, since $\sin(90°-A) = \cos A$.

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## The "Squared" Identities

As I said earlier, I think the problem with the identities is that students are expected to memorize all of them. But really you don't have to, because they're all just forms of the two or three basic identities.

For example, let's start with the really basic identity:

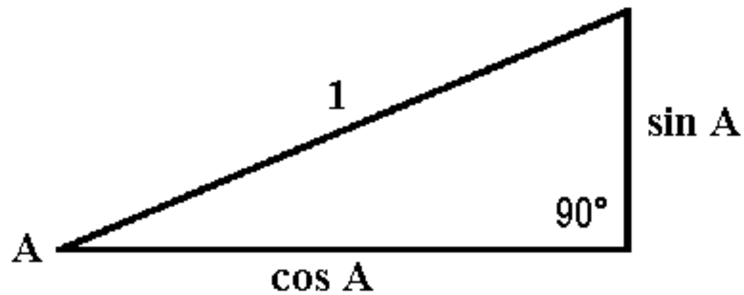
---

$$\sin^2 A + \cos^2 A = 1 \qquad (13)$$

---

That one's easy to remember: it involves only the basic sine and cosine, and you can't get the order wrong unless you try.

But you don't even have to remember it, since it's really just another form of the Pythagorean theorem. (You do remember *that*, I hope?) Just think about the basic triangle with a hypotenuse of one unit.




First convince yourself that the figure is right, and that there's no magic to it. The basic definition is *sin* A = opposite/hypotenuse; but if the hypotenuse is 1 then *sin* A equals the opposite side. Similar reasoning shows that *cos* A equals the adjacent side. Now write down the Pythagorean theorem for this triangle. Voilà!

What's nice is that you can get the other "squared" identities from this one, and you don't have to memorize any of them.

For example, what about the riddle we started with, the relation between $tan^2$ and $sec^2$? Take $sin^2A + cos^2A = 1$. If you want $tan^2$, remember the definition (4) of the tangent. For $tan^2$ you need $(sin/cos)^2$ or $sin^2/cos^2$. So divide (13) through by $cos^2A$ to get

$$[(sin\ A)/(cos\ A)]^2 + [(cos\ A)/(cos\ A)]^2 = [1/(cos\ A)]^2$$

or, using (4) *tan* A = *sin* A /*cos* A and (6) *sec* A = 1/*cos* A,

$$tan^2A + 1 = sec^2A \qquad (14)$$

You should be able to work out the third identity (involving $cot^2$ and $csc^2$) easily enough. You can either start with (14) above and use the co−function rules (7) and (8), or start with (13) and divide by something appropriate. Either way, check to make sure that you get

$$cot^2\ A + 1 = csc^2\ A \qquad (15)$$

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## Sums and Differences of Angles

Formulas for *cos*(A+B), *sin*(A−B), and so on are important but hard to remember. It's possible to derive them by strictly trigonometric means, but such means are lengthy, too hard to reproduce when you're in the middle of an exam or of some long calculation.

This brings us to W.W. Sawyer's marvelous idea, as expressed in chapter 15 of *Mathematician's Delight* (1943; reprinted by Penguin Books, 1991). He shows how you can derive the sum and difference formulas by ordinary algebra and one simple formula.

The ordinary algebra is simply the rules for combining powers:

$$x^a x^b = x^{a+b}$$
$$(x^a)^b = x^{ab} \qquad (16)$$

## Euler's formula

You may already know the "simple formula" that I mentioned above. It's

***memorize:***

$$\cos x + i \sin x = e^{ix} \qquad (17)$$

The formula is not Sawyer's, by the way; it's commonly called Euler's formula. I don't even know whether the idea of using Euler's formula to get the sine and cosine of sum and difference is original with Sawyer. But I'm going to give him credit, since his explanation is simple and clear and I've never seen it explained in this way anywhere else.

I've marked Euler's formula (17) "memorize". Although it's not hard to derive (and Sawyer does it in a few steps by means of power series), you have to start *somewhere*. And that formula has so many other applications that it's well worth committing to memory. For instance, you can use it to get the roots of a complex number and the logarithm of a negative number.

## Sine and cosine of a sum

Okay, back to Sawyer's idea. What happens if you substitute $x$ = A+B in (17) above? You get

$$\cos(\text{A+B}) + i \sin(\text{A+B}) = e^{i\text{A}+i\text{B}}$$

Hmmm, this looks interesting, because it involves exactly what we're looking for, $\cos$(A+B) and $\sin$(A+B). Can you simplify the right−hand side? Use (16) and then (17) to rewrite it:

$$e^{i\text{A}+i\text{B}} = e^{i\text{A}}\, e^{i\text{B}} = (\cos \text{A} + i \sin \text{A})(\cos \text{B} + i \sin \text{B})$$

Now multiply that out and set it equal to the original left−hand side:

$$\cos(\text{A+B}) + i \sin(\text{A+B}) = (\cos\text{A} \cos\text{B} - \sin\text{A} \sin\text{B}) + i(\sin\text{A} \cos\text{B} + \cos\text{A} \sin\text{B})$$

But if two complex numbers a+b$i$ and c+d$i$ are equal, the real and imaginary parts must be separately equal (a=c and b=d). So the above equation is actually two equations, neither of them involving $i$:

$$\cos(\text{A+B}) = \cos\text{A} \cos\text{B} - \sin\text{A} \sin\text{B}$$
$$\sin(\text{A+B}) = \sin\text{A} \cos\text{B} + \cos\text{A} \sin\text{B} \qquad (18)$$

In just a few short steps, the formulas for *cos*(A+B) and *sin*(A+B) flow right from Euler's equation (17) for $e^{ix}$. No more need to memorize which one has the minus sign and how all the sines and cosines fit on the right−hand side: all you have to do is a couple of substitutions and a multiply.

## Sine and cosine of a difference

What about the formulas for the differences of angles? You can write them down at once from (18) by substituting −B for B and using (3). Or, if you prefer, you can get them by substituting $x$ = A−B in (17) above. Either way, you get

$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$
$$\sin(A-B) = \sin A \cos B - \cos A \sin B \qquad (19)$$

## Some geometric proofs

I personally find the algebraic reasoning given above very easy to follow, though you do have to remember Euler's formula. If you prefer geometric derivations of *sin*(A±B) and *cos*(A±B), there is a good set at http://saturn.math.uaa.alaska.edu/~smiley/trigproofs.html. (Thanks to Phil Kenny for the URL.)

## Tangent of a sum or difference

Sometimes (not very often) you have to deal with the tangent of the sum or difference of two angles. I have only a vague idea of the formula, but it's easy enough to work out "on the fly":

$$\tan(A+B) = \sin(A+B) / \cos(A+B) = (\sin A \cos B + \cos A \sin B) / (\cos A \cos B - \sin A \sin B)$$

What a mishmash! There's no way to factor that and remove common terms −− or is there? Suppose you start with a vague idea that you'd like to know *tan*(A+B) in terms of *tan* A and *tan* B rather than all those sines and cosines. The numerator and denominator contain sines and cosines, so if you divide by cosines you'd expect to end up with sines or perhaps sines over cosines. But sine/cosine is tangent, so this seems a promising line of attack. Since you've got cosines of angles A and B to contend with, try dividing the numerator and denominator of the fraction by *cos*A *cos*B. This gives

$$\tan(A+B) = (\sin A/\cos A + \sin B/\cos B) / (1 - \sin A \sin B/\cos A \cos B)$$

Hmmm, looks like this is the right track. Simplify it using the definition (4) of *tan* $x$, and you have

$$\tan(A+B) = (\tan A + \tan B) / (1 - \tan A \tan B) \qquad (20)$$

And if you replace B with −B, you have the formula for *tan*(A−B). (Take a minute to review why *tan*(−x) = −*tan*(x).)

$$\tan(A-B) = (\tan A - \tan B) / (1 + \tan A \tan B) \qquad (21)$$

# Part 3 of Trig without Tears

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## Double Angles and Half Angles

Once you have the formulas (18) and (20) for sums of angles, you can easily write down the formulas for double angles, simply by replacing B with A so that you have A+A or 2A.

### Sine and cosine of a double angle

Start with the sine. You have

$$sin(2A) = sin(A+A) = sinA\ cosA + cosA\ sinA = 2\ sinA\ cosA$$

The cosine formula is just as easy:

$$cos(2A) = cos(A+A) = cosAcosA - sinA\ sinA = cos^2A - sin^2A$$

Though this is valid, it's not satisfying. It would be nice if we had a formula for *cos*(2A) in terms of just a sine or just a cosine. Fortunately, we can use $sin^2 + cos^2 = 1$ to eliminate either the sine or the cosine from that formula:

$$cos(2A) = cos^2A - sin^2A = cos^2A - (1 - cos^2A) = 2\ cos^2A - 1$$
$$cos(2A) = cos^2A - sin^2A = (1 - sin^2A) - sin^2A = 1 - 2\ sin^2A$$

On different occasions you'll have occasion to use all three forms of the formula for *cos*(2A). Don't worry too much about where the minus signs and 1s go; just remember that you can always transform any of them into the others by using good old $sin^2 + cos^2 = 1$.

$$sin(2A) = 2\ sinA\ cosA$$
$$cos(2A) = cos^2A - sin^2A = 2\ cos^2A - 1 = 1 - 2\ sin^2A \qquad (22)$$

There's a very cool second proof of these formulas, using Sawyer's marvelous idea.

### Tangent of a double angle

To get the formula for *tan*(2A), you can either take (20) and put B = A to get *tan* (A+A), or use (22) for *sin*(2A) / *cos*(2A) and then divide the result through by $cos^2A/cos^2A$. Either way, you get

$$tan(2A) = 2\ tanA / (1 - tan^2A) \qquad (23)$$

## Sine and cosine of half angles

What about the formulas for sine, cosine, and tangent of half an angle? Since A = (2A)/2, we'd expect the double−angle formulas (22) and (23) to be some use. And indeed they are, though you have to pick carefully.

For instance, $\sin(2A)$ isn't much help. Put B = A/2 and you have

$$\sin B = 2 \sin(B/2) \cos(B/2)$$

It's true enough, but there's no easy way to solve for $\sin(B/2)$ or $\cos(B/2)$.

There's much more help in the formulas (22) for $\cos(2A)$. Put B = A/2 and you get

$$\cos B = \cos^2(B/2) - \sin^2(B/2) = 2\cos^2(B/2) - 1 = 1 - 2\sin^2(B/2)$$

Use just the first and last parts of that:

$$\cos B = 1 - 2\sin^2(B/2)$$

Rearrange a bit:

$$\sin^2(B/2) = (1 - \cos B) / 2$$

and take the square root

$$\sin(B/2) = \pm \mathit{sqrt}[(1 - \cos B) / 2]$$

You need the plus or minus sign because $\sin(B/2)$ may be positive or negative, depending on B. For any given B (or B/2) there will be only one correct sign, which you already know from the diagram. For instance, if B = 280°, then B/2 = 140°, and you know that $\sin 140°$ is positive.

To find $\cos(B/2)$, start with a different piece of the $\cos(2A)$ formula (22):

$$\cos(2A) = 2\cos^2 A - 1$$

and put B = A/2 to get

$$\cos B = 2\cos^2(B/2) - 1$$

Rearrange and solve for $\cos(B/2)$:

$$\cos^2(B/2) = 1 + \cos B$$
$$\cos(B/2) = \pm \mathit{sqrt}[(1 + \cos B) / 2]$$

You have to pick the correct sign for $\cos(B/2)$ depending on the value of B/2, just as you did with $\sin(B/2)$. But of course the sign of the sine is not always the sign of the cosine.

| | |
|---|---|
| $\sin(B/2) = \pm \mathit{sqrt}[(1 - \cos B)/2]$<br>$\cos(B/2) = \pm \mathit{sqrt}[(1 + \cos B)/2]$ | (24) |

## Tangent of a half angle

Finally, you can find $\tan(B/2)$ in the usual way, dividing sine by cosine from (24):

$$\tan(B/2) = \sin(B/2)/\cos(B/2) = \pm \mathit{sqrt}[(1 - \cos B)/(1 + \cos B)]$$

In the sine and cosine formulas you can't avoid the square root, but in the tangent formula you can eliminate it. Multiply top and bottom by (1 + $\cos$ B) and then use (13) good old $\sin^2 + \cos^2 = 1$:

$$\mathit{sqrt}[(1 - \cos B) / (1 + \cos B)] = \mathit{sqrt}[(1 - \cos B)(1 + \cos B) / (1 + \cos B)^2] =$$
$$\mathit{sqrt}(1 - \cos^2 B) / (1 + \cos B) = \mathit{sqrt}(\sin^2 B) / (1 + \cos B) = \sin B / (1 + \cos B)$$

If instead you multiply top and bottom by 1 − $\cos$ B, you get

$$\mathit{sqrt}[(1 - \cos B) / (1 + \cos B)] = \mathit{sqrt}[(1 - \cos B)^2 / (1 + \cos B)(1 - \cos B)] =$$
$$(1 - \cos B) / \mathit{sqrt}(1 - \cos^2 B) = (1 - \cos B) / \mathit{sqrt}(\sin^2 B) = (1 - \cos B) / \sin B$$

We can summarize the half−angle tangent formulas like this:

$$\tan(B/2) = (1 - \cos B) / \sin B = \sin B / (1 + \cos B) \qquad (25)$$

You may wonder what happened to the plus or minus sign in *tan*(B/2). Fortuitously, it drops out. Since *cos* B is always between −1 and +1, (1 − *cos* B) and (1 + *cos* B) are both positive for any B. And the sine of any angle always has the same sign as the tangent of the corresponding half−angle.

Don't take my word for that last statement, please. There are only four possibilities, and they're easy enough to work out in a table:

| **B/2** | > 0°, < 90° | > 90°, < 180° | > 180°, < 270° | > 270°, < 360° |
|---|---|---|---|---|
| ***tan*(B/2)** | + | − | + | − |
| **B** | > 0°, < 180° | > 180°, < 360° | > 360°, < 540°<br>same as 0° to 180° | > 540°, < 720°<br>same as 180° to 360° |
| ***sin* B** | + | − | + | − |

Of course, you can ignore the whole matter of the sign of the sine and just assign the proper sign when you do the computation.

Another question you may have about formula (25): what happens if *cos* B = −1, so that (1 + *cos* B) = 0? Don't we have division by zero then? Well, take a little closer look at those circumstances. The angles B for which *cos* B = −1 are ±180°, ±540°, and so on. But in this case the half angles B/2 are ±90°, ±270°, and so on: angles for which the tangent is not defined anyway. So the problem of division by zero never arises.

And in the other formula, *sin* B = 0 is not a problem. Excluding the cases where *cos* B = −1, this corresponds to B = 0°, ±360°, ±720°, etc. But the half angles B/2 are 0°, ±180°, ±360°, and so on. For all of them, *tan*(B/2) = 0, as you can verify from the second half of formula (25).

# Part 4 of Trig without Tears

revised 18 Dec 2000


## Solving Triangles

So far we've talked about only right triangles. But in the real world, triangles don't always have 90° angles. In many contexts, you'll know some of the sides and angles of an oblique triangle and need to find one or more of the other sides or angles. This is known as "solving" the triangle.

For instance, suppose you have a triangle where one side has a length of 180, an adjacent angle is 42°, and the opposite angle is 31°. You're asked to find the other angle and the other two sides.

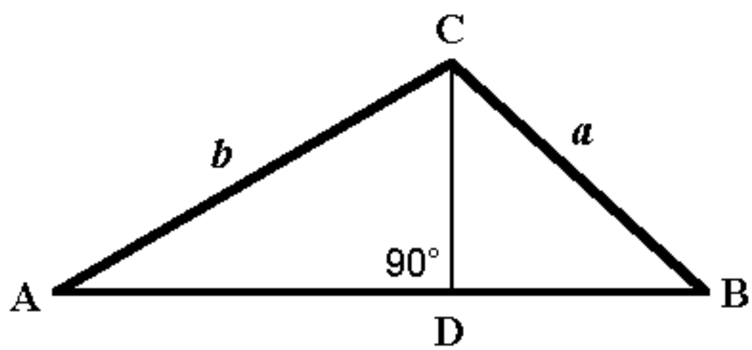
It's always a good idea to draw a rough sketch, like this one. Not only does it help you to organize your solution process better, but it can help you check your work. For instance, since the 31° angle is the smallest, you know that the opposite side must also be the shortest. If you were to come up with an answer of, say, 110 for one of the other sides, you'd know at once that you had made a mistake somewhere.


How would you go about solving that triangle? It's not immediately obvious, I agree. But maybe we can get some help from some useful general techniques in problem solving:

- Can you draw a diagram?
- Can you use what you already know to solve a piece of this problem, or a related problem?
- If you have a specific case, can you solve a more general problem? (Sometimes it works the other way, too, where taking a specific example points out a good technique for solving a general problem.)

We've already got the diagram, but let's see if those other techniques will be helpful. (By the way, they're not original with me, but are from a terrific book on problem−solving techniques that I think you should know about.

"Can you use what you already know to solve a piece of this problem?" For example, if this were a right triangle you'd know right away how to write down the lengths of sides in terms of sines or cosines. But it's not a right triangle, alas. Is there any way to turn it into a right triangle? Not exactly, but if you construct a line at right angles to one side and passing through the opposite vertex, you'll have two right triangles. Maybe solving those right triangles will show how to solve the triangle you're really interested in.




This diagram shows the same triangle after I drew just such a perpendicular. I've also used another principle ("Can you solve a more general problem?") and replaced the specific numbers with the usual letters for sides and angles. Dropping perpendicular CD in the diagram divides the big triangle (which you don't know how to solve) into two right triangles ACD and BCD, with a common side CD. And you *can* solve those right triangles.

We're going to use this simple diagram to develop two important tools for solving triangles: the law of sines and the law of cosines. Just drawing this one perpendicular line will show you how to solve not just the triangle we started with, but *any* triangle. (Some trig courses teach other laws like the law of tangents and the law of segments. I'm going to ignore them because you can solve triangles just fine without them.)

[to document contents]

## Law of sines

The law of sines is simple and beautiful and easy to derive. It's useful when you know two angles and any side of a triangle, or sometimes when you know two sides and one angle.

Let's start by writing down things we know that relate the sides and angles of the two right triangles. You remember how to write down the lengths of the legs of a right triangle? The leg is always equal to the hypotenuse times *either* the cosine of the adjacent angle *or* the sine of the opposite angle. (If that looks like just empty words to you, or even if you're not 100% confident about it, please go back and review the diagram and text until you feel confident.)

In the diagram above, look at triangle ADC at the left: the right angle is at D and the hypotenuse is b. We don't know how much of original angle C is in this triangle, so we can't use C to find the lengths of any sides. What can we write down using angle A? By using its cosine and sine we can write the lengths of both legs of the triangle:

$$AD = b \cos A \qquad \text{and} \qquad CD = b \sin A$$

By the same reasoning, in the other triangle you have

$$DB = a \cos B \qquad \text{and} \qquad CD = a \sin B$$

This is striking: you see two different expressions for the length CD. But things that are equal to the same thing are equal to each other. So

$$b \sin A = a \sin B$$

Divide through by $\sin A$ and you have the solution, $b = a \sin B / \sin A$. In this case, plugging in the numbers tells you that $b = 180 \times \sin 42° / \sin 31°$, or about 234.

What about the third angle, C, and the third side, c? Well, when you have two angles of a triangle you can find the third one easily by using $A+B+C = 180°$, or $C = 180°-A-B$. This gives $C = 107°$.

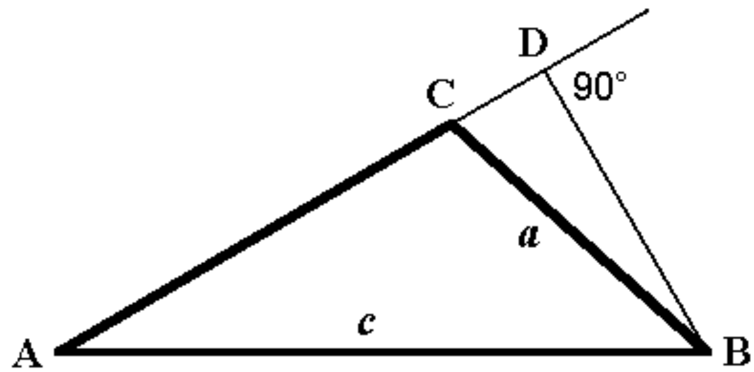
As for the third side, there are a couple of ways to go. You wrote expressions above for AD and DB, and you know that $c = AD+DB$, so you could compute $c = b \cos A + a \cos B$. But that's two multiplies and an add, a bit more complicated than the one multiply and one divide to find side b. I'm lazy, and I like to reduce the amount of tapping I do on my calculator. Is there an easier way, even if just slightly easier?

Yes, there is. Go back a step, to a *sin* B = b *sin* A. Divide through by *sin*A *sin*B to get

$$a / \sin A = b / \sin B$$

But there's nothing special about the two angles A and B. You could just as well have dropped a perpendicular from A to BC or from B to AC. (For this particular triangle, C > 90°, so the other perpendiculars would be outside the original triangle rather than inside, but all the algebra would still be the same.)

Once again, this is the same triangle. But here I've dropped a perpendicular from B to the extension of side AC. Here the two right triangles ABD and CBD are overlapping, but they still share the side BD. By the way, the angle in triangle CBD is not C but 180°−C, the supplement of C. Angle C belongs to the original triangle ABC.




You can write the length of the common side as BD = a *sin*(180−C) = c *sin*A. But *sin*(180−C) = *sin* C, so you have a *sin* C = c *sin* A.

It's nice that the derivation doesn't take into account obtuse versus acute triangles. As you see, when an obtuse angle is involved some dropped perpendicular will lie outside the original triangle, and in that case the derivation uses 180° minus an angle of the original triangle. But since $\sin x = \sin(180°-x)$, you end up with the same form of the law whether the perpendicular is inside or outside the triangle, whether all three angles are acute or one is obtuse.

Divide through by *sin*A *sin*C and you have

$$a / \sin C = c / \sin A$$

But from above, you already know that

$$a / \sin A = b / \sin B$$

Once again, things that are equal to the same thing are equal to each other, so you have the standard form of the law of sines:

$$a / \sin A = b / \sin B = c / \sin C \qquad (26)$$

This is very simple and beautiful: for any triangle, if you divide any of the three sides by the sine of the opposite angle, you'll get the same result. This law is valid for any triangle.

> The law of sines is sometimes given upside down:
>
> $$\sin A / a = \sin B / b = \sin C / c$$
>
> Of course that's the same law, just as 2/3 = 6/9 and 3/2 = 9/6 are the same statement. Work with it either way and you'll come up with the same answers.

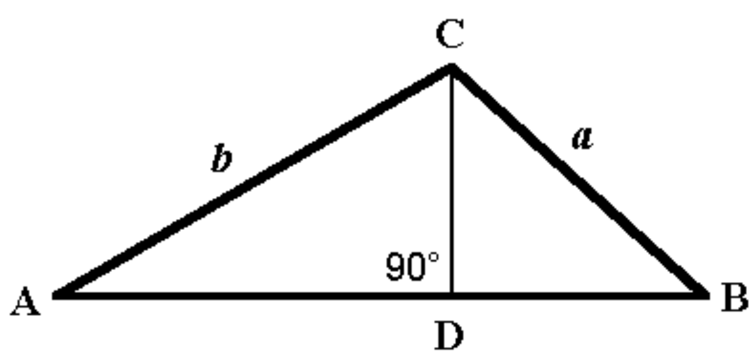
You can derive the law of sines at need, so I don't specifically recommend memorizing it. But it's so simple and beautiful that it's pretty hard not to memorize if you use it at all. It's also pretty hard to remember it wrong: there are no alternating plus and minus signs or combinations of different functions.

Be careful with the law of sines in the angle−side−side case, where you know two sides and an angle other than the one between them. In the other cases you use the law of sines to find lengths of sides, and you get a unique solution. But with angle−side−side, you use the law of sines to find the sine of an angle. Since $\sin A = \sin(180° - A)$, there could be two angles A for any given value of *sin* A. So in the angle−side−side case, there may be two different triangles that fit the facts. Please see the discussion and example below, after the table in the section on solving triangles.

[to document contents]

## Law of cosines

The law of sines is fine when you can relate sides and angles. But suppose you know three sides of the triangle and have to find the three angles? The law of sines is no good for that because it relates two sides and their opposite angles. If you don't know any angles, you have an equation with two unknowns and you can't solve it.




But a triangle *can* be solved when you know all three sides; you just need a different tool. And knowing me, you can be sure I'm going to help you develop one! It's called the law of cosines.

Suppose the three sides are a = 180, b = 238, c = 340.

You may remember that when we first looked at this picture, we pulled out information using both the sine and the cosine of the two angles. We used the sine information to develop the law of sines, but we never went anywhere with the cosine information, which was

$$AD = b \cos A \qquad \text{and} \qquad DB = a \cos B$$

Let's see where that can lead us. You remember that the way we came up with the law of sines was to write two equations that featured the length of the construction line CD, and then combine the equations to eliminate CD. Can we do anything like that here?

Well, we know the other two sides of those right triangles, so we can write an expression for the height CD using the Pythagorean theorem −− actually, two expressions, one for each triangle.

$$a^2 = (CD)^2 + (BD)^2$$

$$b^2 = (CD)^2 + (AD)^2$$

Solving each of them for $(CD)^2$ and setting them equal, we have

$$a^2 - (BD)^2 = b^2 - (AD)^2$$

Substitute the known values of BD and AD in terms of angles and sides of the original triangle, and you have

$$a^2 - b^2 \cos^2 A = b^2 - a^2 \cos^2 B$$

Bzzt! No good! That uses two sides and two angles, but we need an equation in three sides and *one* angle, so that we can solve for that angle. Let's back up a step, to

$$a^2 - (BD)^2 = b^2 - (AD)^2,$$

and see if we can go in a different direction.

Maybe the problem is in treating BD and AD as separate entities when actually they're parts of the same line. Since BD+AD = c, we can write BD = c−AD = c − b *cos*A. Notice this brings in the third side, c, and stops using one of the angles B. Substituting, we now have

$$a^2 - (c - b \cos A)^2 = b^2 - (b \cos A)^2$$

This looks worse than the other one, but actually it's better because it's what we're looking for: an equation for the three sides and one angle. We can solve it with a little algebra:

$$a^2 - c^2 + 2bc \cos A - b^2 \cos^2 A = b^2 - b^2 \cos^2 A$$
$$a^2 - c^2 + 2bc \cos A = b^2$$
$$2bc \cos A = b^2 + c^2 - a^2$$
$$\cos A = (b^2 + c^2 - a^2) / 2bc$$

We were a long time getting there, but finally we made it. now we can plug in the lengths of the sides and come up with a value for *cos* A, which in turn will tell us angle A. Do the same thing to find the second angle (or use the law of sines, since it's less work), then subtract the two known angles from 180° to find the third angle.

If we had substituted AD = c−BD instead of the other way around, we would have obtained the same law but for a different angle:

$$\cos B = (c^2 + a^2 - b^2) / 2ac$$

And if we'd picked one of the other two perpendiculars to start the whole process, we'd have got the law of cosines for angle C:

$$\cos C = (a^2 + b^2 - c^2) / 2ab$$

Just for fun, let's plug in the known sides and find angle C.

$$\cos C = (180^2 + 238^2 - 340^2) / 2 \times 180 \times 238 = -0.309944$$

Don't be surprised at the negative number. Remember from the diagram that *cos* A 0 when A is between 90° and 180°. Because the cosine has unique values all the way from 0° to 180°, you never have to worry about multiple solutions of a triangle when you use the law of cosines. In this case, C is about 108°.

There's another well−known form of the law of cosines, which may be a bit easier to remember. Start with the above form, multiply through by 2ab, and isolate c on one side to get

$$c^2 = a^2 + b^2 - 2ab \cos C$$

Notice the arrangement: side c is opposite angle C in the triangle, and they're at opposite ends of this equation. Sides a and b are adjacent to angle C both in the triangle and in the equation. I have more thoughts on remembering the law of cosines.

Depending on how you're using it, you may need the law of cosines in either of the two forms that we've obtained, the first form for finding an angle and the second form for finding a side. Here's a summary of both forms:

$$\cos C = (a^2 + b^2 - c^2) / 2ab$$
$$c^2 = a^2 + b^2 - 2ab \cos C \qquad (27)$$

[to document contents]

## Detective work: solving all types of triangles

If you've got the law of sines and the law of cosines under your belt, you can solve any triangle that *can* be solved. (Some sets of givens lead to an impossible situation, like a "triangle" with sides 3−4−9.)

In this section I'll run down the various possibilities and give you some pointers. I'll just reprint the law of sines and both forms of the law of cosines here so we'll have them in one place:

$$a / \sin A = b / \sin B = c / \sin C \qquad (28)$$

$$\cos C = (a^2 + b^2 - c^2) / 2ab \qquad (29)$$

$$c^2 = a^2 + b^2 - 2ab \cos C \qquad (30)$$

Now whenever you have to solve a triangle, think about what you have and then think about which formula you can use to get what you need. (When you have two angles, you can always find the third by A+B+C = 180°, so I'm not going to mention that.)

I'm ***not*** presenting the following table for you to memorize. Instead, what I hope to do is show you that between the law of sines and the law of cosines you can solve any triangle, and that you simply pick which law to use based on which one has just one unknown and otherwise uses information you already have. You should see from working with the formulas that

- if you have a side and its opposite angle (plus any other angle or side) you can use the law of sines to find the next piece of information; but
- if you have two sides and the included angle, or three sides and no angles, you need to use a form of the law of cosines.

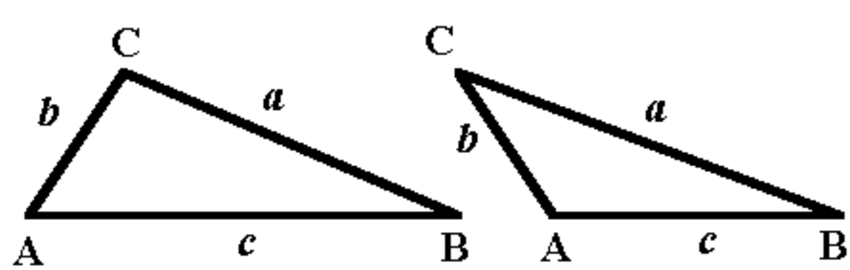
The table is just an exhaustive elaboration of those two principles, so you probably don't even need to read it! <grin>

| **If you know this...** | **You can solve the triangle this way...** |
| --- | --- |

| | |
|---|---|
| three angles | There's not enough information. Without at least one side you have the shape of the triangle, but no way to scale it correctly. For example, the same angles could give you a triangle with 7/12/13, 35/60/65, or any multiple. |
| two angles and a non−included side (angle−angle−side) | If the side is not between the angles it must be opposite one of them, so you can use the law of sines (28) to find a second side, then find the third angle and use the law of sines again to find the third side. |
| two angles and the included side (angle−side−angle) | Always use the law of sines (28) if you can, because it uses fewer operations than the law of cosines. You need an opposite angle to a known side, so first find the third angle and then use (28) twice to find the other two sides. |
| two sides and the included angle (side−angle−side) | This is tailor made for the second form (30) of the law of cosines, because the right−hand side works with two sides and the included angle. Use it to find the third side. Now you have three sides and an angle opposite one of them, so you can use either the law of sines (28) or the law of cosines (29) to find the second angle. |
| two sides and a non−included angle (angle−side−side) | If the angle is not between the two known sides then it must be opposite one of them. Whenever you have an angle and an opposite side, the law of sines (28) is likely to be your easiest route. You can use it twice, once to get a second angle and once to get the third side.<br><br>But...<br>This case may have no solutions, one solutions, or two solutions. See more details after the table. |
| three sides | With the first form (29) of the law of cosines you use all the sides to compute one angle. Use that angle and its opposite side in the law of sines (28) to find the second angle. |

### Special note: angle−side−side

This case can be tricky, as the diagram shows. Suppose you know angle B and sides a and b. The given facts fit two different triangles. Why? because when you use the law of sines to find *sin* A, there are two possible solutions for angle A, one being 180° minus the other. The same is true for angle C.




If the known angle is >= 90°, the other two angles must be < 90°, so you have a unique solution to the triangle. But if the known angle is < 90°, like angle B in the picture, you have enough information only to narrow the triangle down to two possibilities.

Has the law of sines failed? No, the problem is that the two angles C in the picture have the same sine, and the two angles A have the same sine. (Remember (3): *sin*(180°−A) = *sin* A.) To solve this triangle you need some more information: specifically, you need to know which is the largest angle, either because you're told it explicitly or because it's implied by other facts you know. For instance, if you know angle B and sides a and b, and b > a (which is not true in the picture), then you would know angle B > angle A, so A 90°; once you got A you could find C = 180°−A−B and then find side c. But in this particular case all you can do is give both possible solutions, because there's not enough information to choose between them.

Short and sweet advice: always draw a picture. If you can draw two pictures that both fit all the available facts, you have two legitimate solutions. If only one picture fits all the facts, it will show you which angle (if any) is > 90°.

[to document contents]

# Part 5 of Trig without Tears

revised 18 Dec 2000


## Inverse Functions

Sometimes you have a sine or cosine or tangent and need to find the associated angle. For instance, this happens whenever you solve a triangle. When you have a sine function value and find the corresponding angle, you are finding the arc sine or inverse sine of that value, and similarly for the other functions.

Different books use different notation: *sin* with a superscript −1, or *arcsin.* I prefer the "arc" forms because the superscript −1 looks too much like an exponent.

[to document contents]

### Principal values

What is *arcsin*(0.5)? You probably recognize that 0.5 is ½, and it must be a sine of one of the special angles. In fact (11), sin(30°) = ½. So you can say that *arcsin*(0.5) = 30° or pi/6.

But wait, there's more! You know from equations (3) that
$sin(30°) = sin(150°) = sin(390°) = sin(−210°)$
and so on; they all equal ½.
In fact,
$sin(30° + 360°k) = sin(150° + 360°k) = 0.5$ for all integer $k$
So which of this infinite number of values is *the* arc sine?

To make an arc sine *function*, we have to restrict the range so that each number has at most one arc sine. (I don't say "one and only one arc sine." The sine of any angle is between −1 and +1 inclusive; therefore only those numbers have arc sines.) The arc sine is defined so that its range is the interval [−pi/2;+pi/2], which is the same as [−90°;+90°]. The capital letter (*Arcsin*) distinguishes this function from the multi−valued relation (*arcsin*). So we could say
*arcsin*(0.5) = pi/6+2*k**pi or 5pi/6+2*k**pi for all integer *k*
but
*Arcsin*(0.5) = pi/6

Why the particular range −pi/2 to +pi/2? To start with, it seems tidy that any arc function of a positive number should fall in Quadrant I, [0;+pi/2]. So the only real question is arc functions of negative numbers. If we prefer the numerically smallest values for the arc sine function, then *Arcsin*(−0.5) = −30° = −pi/6 fits that rule, and a negative number's arc sine (and arc tangent, too) will be in Quadrant IV, [−pi/2;0].

What about the arc cosine? The cosine is positive in both Quadrant I and Quadrant IV, so the arc cosine of a negative number must fall in Quadrant II or Quadrant III. Thomas (*Calculus and Analytic Geometry,* 4th edition) resolves this in a neat way. Remember (2) that
*cos* A = *sin*(pi/2 − A)
It makes a nice symmetry to write
*Arccos x* = pi/2 − *Arcsin x*
And that is how Thomas defines the inverse cosine function. Since the range of *Arcsin* is the closed

interval [−pi/2;+pi/2], the range of *Arccos* is [0;pi].

Thomas defines the arc secant and arc cosecant functions using the reciprocal relationships (6):

*sec x* = 1/(*cos x*) ==> *Arcsec x* = *Arccos*(1/*x*)

*csc x* = 1/(*sin x*) ==> *Arccsc x* = *Arcsin*(1/*x*)

This means that *Arcsec* and *Arccsc* have the same ranges as *Arccos* and *Arcsin*.

The arc cotangent could be defined as

*cot x* = 1/(*tan x*) ==> *Arccot x* = *Arctan*(1/*x*)

which makes *Arccot*'s range the messy union of two open intervals (−pi/2;0) and (0;+pi/2). But most authors define it as

*cot x* = *tan*(pi/2 − *x*) ==> *Arccot x* = pi/2 − *Arctan x*

which gives the single open interval (0;pi) as the range.

Here are the domains and ranges of all six inverse trig functions:

| function | derived from | domain | range |
|---|---|---|---|
| *Arcsin* | inverse of sine function | [−1;+1] | [−pi/2;+pi/2] |
| *Arccos* | *Arccos x* = pi/2 − *Arcsin x* | [−1;+1] | [0;pi] |
| *Arctan* | inverse of tangent function | all reals | (−pi/2;+pi/2) |
| *Arccot* | *Arccot x* = pi/2 − *Arctan x* | all reals | (0;pi) |
| *Arcsec* | *Arcsec x* = *Arccos*(1/*x*) | reals except (−1;+1) | [0;pi] |
| *Arccsc* | *Arccsc x* = *Arcsin*(1/*x*) | reals except (−1;+1) | [−pi/2;+pi/2] |

Remember that the *relations* are many−valued, not limited to the above ranges of the *functions*. If you see the capital A in the function name, you know you're talking about the function; otherwise you have to depend on context.

[to document contents]

## Functions of arc functions

Sometimes you have to evaluate expressions like

*cos*( *Arctan x* )

That looks scary, but actually it's a piece of cake. You can simplify any trig function of any inverse function in a few easy steps, using the method in the following examples. (A summary of the method follows the examples.)

### Example 1: *cos*(*Arctan x*)

It may be helpful to read the expression out in words: "the cosine of *Arctan x*." Doesn't help much? Well, remember what *Arctan x* is. It's the (principal) angle whose tangent is *x*. So what you have to find reads as "the cosine of the angle whose tangent is *x*." And that suggests your plan of attack: first identify that angle, then find its cosine.

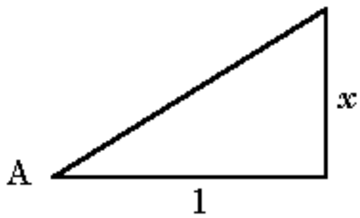
Let's give a name to that "angle whose". Call it A:

A = *Arctan* $x$

from which you know that

*tan* A = $x$

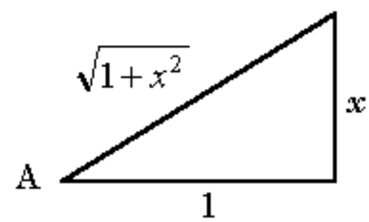
So now all you have to do is find *cos* A, and that's easy if you draw a little picture.




Start by drawing a right triangle, and mark one acute angle as A.

Using the definition of A, write down the lengths of two sides of the triangle. Since *tan* A = $x$, and the definition of tangent is opposite side over adjacent side, the simplest choice is to label the opposite side x and the adjacent side 1. Then, by definition, *tan* A = $x$/1 = $x$.

The next step is to find the third side. Here you know the two legs, so you use the theorem of Pythagoras to find the hypotenuse, sqrt(1+$x^2$). (For some problems, you'll know one leg and the hypotenuse, and you'll use the theorem to find the other leg.)




Once you have all three sides' lengths, you can write down the value of any function of A. In this case you need *cos* A, which is adjacent side over hypotenuse.

*cos* A = 1/sqrt(1+$x^2$)

But *cos* A = *cos*(*Arctan* $x$), and therefore

*cos*( *Arctan* $x$ ) = 1/sqrt(1+$x^2$)

and there's your answer.

### Example 2: *cos*(*Arcsin u*)

Read this as "the cosine of the angle A whose sine is $u$". To start, draw your triangle, and label A. (Please take a minute and make the drawing.) You know that

*sin* A = $u$ = opposite/hypotenuse

and so you label the opposite side $u$ and the hypotenuse 1.

Next, solve for the third side, which is sqrt(1−$u^2$), and write that down. Now you need *cos* A, which is the adjacent side over the hypotenuse, which is sqrt(1−$u^2$)/1, so

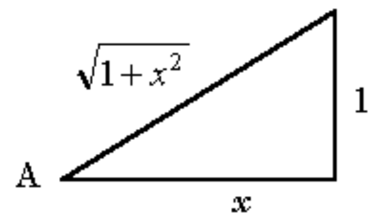
*cos*( *Arcsin u* ) = sqrt(1−$u^2$)

There you go: quick and painless.

### Example 3: *cos*(*Arctan*(1/*x*))

This looks similar to Example 1, but as you'll see there's an additional wrinkle. (Thanks to Brian Scott, who raised the issue in an article he posted 12 Dec 2000 to alt.algebra.help.)

Proceed in the regular way: draw your triangle, and since A = *Arctan*(1/*x*), or *tan* A = 1/*x*, you make 1 the length of the opposite side and *x* the length of the adjacent side. The hypotenuse is then sqrt(1+x²).




Now you can write down *cos* A, which is adjacent over hypotenuse:

*cos* A = *x* / sqrt(1+*x*²)

But suppose *x* is negative, say −sqrt(3)? Then A = *Arctan*(−1/sqrt(3)) = −pi/6, and *cos*(−pi/6) = +sqrt(3)/2. But the above formula *x*/sqrt(1+*x*²) yields −sqrt(3)/sqrt(1+3) = −sqrt(3)/2, which has the wrong sign.

What went wrong? The trouble is that *Arctan* always yields values in (−pi/2;+pi/2), which is Quadrants IV and I. But the cosine is always positive on that interval, so *cos*(*Arctan x*) must yield a positive result. Remember also (3) that *cos*(−A) = *cos* A. To ensure this, use the absolute value sign, and the final answer is

*cos*( *Arctan*(1/*x*) ) = | *x* | / sqrt(1+*x*²)

Why doesn't every example have this problem? The earlier examples involved only the square of a variable, which is naturally nonnegative. Only here, where we have an odd power, does it matter.

### *Summary of the method*

This will work for any trig function of any arc function (36 permutations).

1. Draw a right triangle and label one acute angle A.
2. Label two sides for the given function value of A, the "angle whose".
3. Find the third side using Pythagoras' theorem.
4. Read off the desired function value.
5. If there are any odd powers of variables, check signs (Example 3).

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## Arc functions of functions

(for the hard−core trig fan)

You may be wondering about the inside−out versions, taking the arc function of a function. Some of these expressions can be solved algebraically, on a restricted domain anyway, but some cannot. (I am grateful to David Cantrell for help with analysis of these problems in general and Example 6 in particular.)

We can say at once that there will be no pure algebraic equivalent to an arc function of a trig function. Why? The six trig functions are all periodic, and therefore any function of any of them must also be periodic. But no algebraic functions are periodic, except trivial ones like f(x) = 2, and therefore no function of a trig function can be represented by purely algebraic operations. As we will see, some can be represented if we add non−algebraic functions like mod and floor.

### Example 4: *Arccos*(*sin u*)

This is the angle whose cosine is *sin u*. To come up with a simpler form, set *x* equal to the desired expression, and solve the equation by taking cosine of both sides:

*x* = *Arccos*( *sin u* )

*cos x* = *sin u*

This could be solved if we could somehow transform it to *sin*(something) = *sin*(*u*) or *cos*(*x*) = *cos*(something else). In fact, we can use equation (2) to do that. It tells us that

*sin u* = *cos*(90°−*u*),

and combining that with the above we have

*cos x* = *cos*( 90° − *u* )

Now if *x* is in Quadrant I, the interval [0;pi/2], then *u* will be in Quadrant I also, and we can write

*x* = 90°−*u* (or pi/2−*u*)

and therefore

*Arccos*( *sin u* ) = 90°−*u* (or pi/2−*u*) for *u* in Quadrant I

But this solution does not work for all quadrants. For instance,

*Arccos*(*sin* 5pi/6) = *Arccos*(½) = pi/6,

which is certainly not equal to pi/2 − 5pi/6. Try graphing *Arccos*(*sin*(x)) and pi/2−x and you'll see the problem: one is a sawtooth and the other is a straight line.

Sparing you the gory details, pi/2−*u* is right only in Quadrants IV and I. We have to "decorate" it rather a lot to make it match *Arccos*(*sin u*) in the other quadrants, and also to account for the repetition of values every 2pi. The first modification is not too hard: On the interval [−pi/2;+3pi/2], the absolute−value expression |pi/2−*u*| matches the sawtooth graph of *Arccos*(*sin u*). The repetition every 2pi is harder to reflect, but this manages it:

*Arccos*(*sin u*) = | pi/2 − *u* + 2pi*floor[(*u*+pi/2)/2pi] |

where "floor" means the greatest integer less than or equal to. Messy, eh? (Note also that "floor" is not an algebraic function.) It could be made a bit shorter with mod (which is also not algebraic):

*Arccos*(*sin u*) = | pi − mod(*u*+pi/2,2pi) |

where mod(a,b) is the nonnegative remainder when a is divided by b.

### Example 5: *Arcsec*(*cos u*)

This one, the angle whose secant is *cos u*, has a very odd solution. Try the above solution method and you get

*x* = *Arcsec*( *cos u* )

*sec x* = *cos u*

But *sec x* = 1/(*cos x*), so we have

1/( *cos x* ) = *cos u*

Now here's the thing: the cosine's values are all numerically <=1. So the only way one cosine can be the reciprocal of another is if they're both equal to 1 or both equal to −1. That occurs only when *x* = *u* = a multiple of pi. If *u* = 0, ±2pi, ±4pi, etc., then *cos u* = 1 and *Arcsec*(1) = 0. On the other hand, if *u* = ±pi, ±3pi, etc., then *cos* = −1 and *Arcsec*(−1) = pi. Therefore the solution is

*Arcsec*( *cos u* ) = 0 when *u* = 2*k**pi for integer *k*

*Arcsec*( *cos u* ) = pi when *u* = (2*k*+1)pi for integer *k*

The graph of *Arcsec*(*cos u*) is rather curious, single points at the ends of an infinite sawtooth: ..., (−3pi,pi), (−2pi,0), (−pi,pi), (0,0), (pi,pi), (2pi,0), (3pi,pi), ...

### Example 6: *Arctan*(*sin u*)

Proceeding in the regular way, we have

*x* = *Arctan*( *sin u* )

*tan x* = *sin u*

The most likely approach is the one from Example 4: try to transform the above into *tan*(*x*) = *tan*(something) or *sin*(something else) = *sin*(*u*). If there is any trig identity or combination that can be used to do that, it is unknown to me. I suspect strongly that *Arctan*(*sin u*) can't be converted to an algebraic expression, even with the use of mod or floor, but I can't prove it.

### Summary for arcfunctions of functions

There's no nice regular method for these, as there is for functions of arcfunctions. When analyzing arcfunc1(func2), you can usually come up with something if func1 and func2 are cofunctions or reciprocals, and probably not otherwise.

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# Notes to Trig without Tears

revised 18 Dec 2000


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## Notation

I've tried to use standard notation (and standard HTML). In this section I list the usages that might be confusing.

1. The following special characters have been part of RFC 1866 since at least 1995, according to Michael Hannah's *HTML Special Character Entity Names* (internal document at Sandia Labs), so most browsers should render them. If you have an older browser that doesn't, I apologize.
   - the multiplication sign, × (`×`)
   - the plus−or−minus sign, ± (`±`)
   - the superscript 2 (squared sign), ² (`²`)
   - the fraction one half, ½ (`½`)
   - the degree sign, ° (`°`)
2. I've made some compromises since many common math characters can't be displayed in a standard way.
   - pi is written that way, since the Greek letter is not available on most computers.
   - *sqrt* is the square−root sign. The only way I could get it in formulas would be to turn them into graphics, and to save bandwidth I've elected not to do that.
   - For the same reason, fractions other than ½ are written using the slash, a/b.
3. Please watch carefully for minus signs (−). In many fonts the minus sign is a tiny hyphen, easy to miss. Microsoft Windows offers a true minus sign (–, `–`), but computers that don't use that character set will show a different symbol, or nothing.

### Interval Notation

In talking about the domains and ranges of functions, it is handy to use interval notation. Thus instead of saying that x is between 0 and pi, we can use the *open interval* (0;pi) if the endpoints are not included, or the *closed interval* [0;pi] if the endpoints are included.

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## The Problem with Memorizing

"Dad sighed. 'Kip, do you think that table was brought down from on high by an archangel?'" Robert A. Heinlein, in *Have Space Suit −− Will Travel* (1958)

It's not just that there are so many trig identities; they seem so *arbitrary*. Of course they're not really arbitrary, since all can be proved; but when you try to memorize all of them they seem like a jumble of symbols where the right ones aren't more obviously right than the wrong ones. For example, is it $\sec^2 A = 1 + \tan^2 A$ or $\tan^2 A = 1 + \sec^2 A$ ? I doubt you know off hand which is right; I certainly don't remember. Who can remember a dozen or more like that, and remember all of them accurately?

Too many teachers expect students to memorize the trig identities and be able to parrot them on demand, much like a series of Bible verses. In other words, even if they're originally taught as a series of connected propositions, they're remembered and used as a set of unrelated facts. And that, I think, is the problem. The trig identities were *not* brought down by an archangel; they were developed by mathematicians, and it's well within your grasp to re−develop them when you need to. With effort, we can remember a few key facts about anything. But it's much easier if we can fit them into a context, so that they work together as a whole.

Why bother? Well, of course it will make your life easier in trig class. But you'll also need the trig identities in later math classes, especially calculus, and in physics and engineering classes. In all of those, you'll find the going much easier if you're thoroughly grounded in trigonometry as a unified field of knowledge instead of a collection of unrelated facts. This is why it's easier to remember almost any song than an equivalent length of prose: the song gives you additional cues in the form rhythm, common patterns of emphasis, and usually rhymes at the ends of lines. With prose you have only the general thought to hold it together, so that you must memorize it as essentially a series of words. With the song there are internal structures that help you, even if you're not aware of them.

If you're memorizing Lincoln's Gettysburg Address, you might have trouble remembering whether he said "recall" or "remember" at a certain point; in a song, there's no possible doubt which of those words is right because the wrong one won't fit in the rhythm.

### On the other hand ...

I'm not against all memorization. Some things *have* to be memorized because they're a matter of definition. Others you may *choose* to memorize because you use them very often, you're confident you can memorize them correctly, and the derivation takes more time than you're comfortable with. Still others you may not set out to memorize, but after using them many times you find you've memorized them without trying to −− much like a telephone number that you dial often.

I'm not against all memorization; I'm against *needless* memorization used as a substitute for thought. If you decide in particular cases that memory works well for you, I won't argue. But I do hope you see the need to be able to re−derive things on the spot, in case your memory fails. Have you ever dialed a friend's telephone number and found you couldn't quite remember whether it was 6821 or 8621? If you can't remember a phone number, you have to look it up in the book. My goal is to free you from having to look up trig identities in the book.

Thanks to David Dixon at `margot@cnwl.igs.net` for an illuminating exchange of notes on this topic. He made me realize that I was sounding more more anti−memory than I intended to, and in consequence I've added this note. But he may not necessarily agree with what I say here.

I wrote this paper to show you how to make the trig identities "fit" as a coherent whole, so that you'll have no more doubt about them than you do about the words of a song you know well. The difference is that you won't need to do it from memory. And you'll gain the sense of power that comes from mastering your subject instead of groping tentatively and hoping to strike the right answer by good luck.

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## Proof of Euler's Formula

Euler's formula (17) is easily proved by means of power series. Start with the formulas $\cos x = \text{SUM}\,[(-1)^n x^{2n} / (2n)!\,] = 1 - x^2/2! + x^4/4! - x^6/6! + \ldots$

$\sin x = \text{SUM}[\,(-1)^n x^{2n+1} / (2n+1)!\,] = x - x^3/3! + x^5/5! - x^7/7! + \ldots$
$e^x = \text{SUM}[\,x^n / n!\,] = 1 + x + x^2/2! + x^3/3! + \ldots$ where $n = 0, 1, 2, \ldots$ (These are how the function values are actually calculated, by the way. If you want to know the value of $e^2$, you just substitute 2 for $x$ in the formula and compute until the additional terms fall within your desired accuracy.)

Now we have to prove Euler's formula, which makes an assertion about the value of $e^{ix}$, where $i$ = sqrt(−1). Use the third formula to find $e^{ix}$, by substituting $ix$ for $x$ in the formula. I'll write out eight terms so that you can see the pattern. This will involve powers of $i$, which I'll simplify using $i^2 = -1$. Finally I'll group the real and imaginary terms separately.

$$e^{ix} = \text{SUM}[\,x^n / n!\,] =$$
$$1 + (ix) + (ix)^2/2! + (ix)^3/3! + (ix)^4/4! + (ix)^5/5! + (ix)^6/6! + (ix)^7/7! + \ldots =$$
$$1 + ix - x^2/2! - ix^3/3! + x^4/4! + ix^5/5! - x^6/6! - ix^7/7! + \ldots =$$
$$[1 - x^2/2! + x^4/4! - x^6/6! + \ldots] + i[x - x^3/3! + x^5/5! - x^7/7! + \ldots]$$

Those should look familiar, because the first group of terms is just *cos x* and the second group is just *sin x*. So we have $e^{ix} = \cos x + i \sin x$ just as advertised!

You may wonder where the series for *cos x*, *sin x*, and $e^x$ come from. The answer is that they are the Taylor series expansions of the functions. Look up "Taylor series" in any decent calculus book and you'll find the derivation.

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# Roots of a Complex Number

One of the applications of Euler's formula (17) is finding any root of any complex number. (Sawyer doesn't do this, or at least not in the same book.)

### Square root of *i*

For instance, you know the square roots of −1 are $i$ and $-i$, but what's the square root of $i$? Simply use (17), (16), and (17) again to solve.

Start by putting $i$ into $e^{ix}$ form.

$$\cos x = 0 \text{ and } \sin x = 1 \text{ when } x = 90° \text{ or pi/2}$$

Therefore by (17)

$$i = 0 + 1i = \cos(\text{pi}/2) + i \sin(\text{pi}/2) = e^{i\,\text{pi}/2}.$$

And by (16)

$$\text{sqrt}(e^{i\,\text{pi}/2}) = (e^{i\,\text{pi}/2})^{1/2} = e^{(i\,\text{pi}/2)\times(1/2)} = e^{i\,\text{pi}/4}$$

Now apply (17) again:

$$e^{i\,\text{pi}/4} = \cos(\text{pi}/4) + i \sin(\text{pi}/4) = (1+i)/\text{sqrt}(2).$$

The other square root is minus that, as usual.

### Roots of other numbers

You can find any root of any other complex number in a similar way, but usually with one preliminary step. For instance, suppose you want the cube roots of $3+4i$. When you ask what angle has a cosine of 3 and a sine of 4, of course you come up empty. What you need to do is separate the number into two parts, (i) a scale factor and (ii) a part that will translate into the cosine of some angle plus $i$ times of the sine of that angle. The first part is simply the absolute value of the complex number (*sqrt*($a^2+b^2$) for a+b*i*); the second part will then automatically be usable as a sine and cosine.

For our example, 3+4*i*, the absolute value is *sqrt*($3^2+4^2$) = 5, so rewrite the number as 5×(0.6+0.8*i*). Now you need to find an angle whose cosine is 0.6 and whose sine is 0.8. It doesn't come out exactly, but it's about 53.13°, which I'll call ø. So far we have

$$3+4i = 5\times(0.6+0.8i) = 5e^{i\text{ø}} \quad \text{for ø about } 53.13°$$

To take a cube root of that, remember that the cube root of *x* is the same as $x^{(1/3)}$. Therefore

$$\text{cube root of } (3+4i) = (5e^{i\text{ø}})^{1/3} = 5^{1/3} \times e^{i\text{ø}/3} = 5^{1/3} \times [\cos(\text{ø}/3) + i \sin(\text{ø}/3)]$$

Angle ø/3 is about 17.71°; the sine and cosine of that are about 0.30 and 0.95. The cube root of 5 is about 1.71. So we have

$$\text{cube root of } (3+4i) = \text{about } 1.71 \times (\cos 17.71 + i \sin 17.71) = \text{about } 1.63 + 0.52i$$

### Other roots

You may have noticed that I talked about the "cube root***s*** of 5" and "***a*** cube root". Even with the square root of *i*, I waved my hand and said that the "other" square root was minus the first one, "as usual".

You already know that every positive real has two square roots. In fact, every complex number has *n n*th roots.

How can you find them? Look back at Euler's formula,

$$e^{ix} = \cos x + i \sin x$$

What happens if you add 2pi or 360° to *x*? You have

$$e^{i(x+2\text{pi})} = \cos (x+2\text{pi}) + i \sin (x+2\text{pi})$$

Well, taking sine or cosine of 360° plus an angle is exactly the same as taking sine or cosine of the original angle. So the right−hand side is equal to $\cos x + i \sin x$, which is equal to $e^{ix}$. Therefore

$$e^{i(x+2\text{pi})} = e^{ix}$$

In fact, you can keep adding 2pi or 360° to *x* as long as you like, and never change the value of the result. Symbolically,

$$e^{i(x+2\text{pi } n)} = e^{ix} \quad \text{for all integer } n$$

When you take an *n*th root, you simply use that identity. So the three cube roots of $e^{i\text{ø}}$ are

$$e^{i\text{ø}/3},\ e^{(i\text{ø}+360)/3}, \text{ and } e^{(i\text{ø}+720)/3}$$

or

$$e^{i\text{ø}/3},\ e^{(i\text{ø}/3)+120}, \text{ and } e^{(i\text{ø}/2)+240}$$

Compute those as $\cos x + i \sin x$ in the usual way, and then multiply by the (principal) cube root of 5. I get these three roots:

$$\text{cube roots of 5} = \text{about } 1.63+0.52i,\ -1.26+1.15i,\ -0.36-1.67i$$

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## Logarithm of a Negative Number

Another application flows from a famous special case of Euler's formula. Substitute *x* = pi or 180° in (17). Since *sin* 180 = 0, the imaginary term drops out. And *cos* 180 = −1, so the formula

$$-1 = e^{i\,\text{pi}}$$

is the result.

It's also interesting to take the natural log of both sides:

$$\ln(-1) = \ln(e^{i\,\text{pi}})$$

which gives

$$ln(-1) = i\ \text{pi}$$

It's easy enough to find the logarithm of any other negative number. Since

$$ln(ab) = ln\ a + ln\ b$$

then for all *a* you have

$$ln(-a) = ln[a \times -1] = ln\ \ a + ln(-1) = ln\ a + i\ \text{pi}$$

I don't honestly know whether all of this has any practical application. But if you've ever wondered about the logarithm of a negative number, now you know.

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## Cool Proof of Double−Angle Formulas

I can't resist pointing out something cool. You can also use Sawyer's marvelous idea. From Euler's formula (17) for $e^{ix}$ you can immediately obtain the formulas for *cos*(2A) and *sin*(2A) without going through the formulas for sums of angles. Here's how.

Remember that $x^{ab} = (x^a)^b$. One important special case is that $x^{2b} = (x^b)^2$. Use that with Euler's formula (17).

$$cos(2A) + i\ sin(2A) = e^{(2A)i} = (e^{iA})^2 = (cos\ A + i\ sin\ A)^2 =$$

$$cos^2A + 2i\ sinA\ cosA + i^2sin^2A = cos^2A + 2i\ sinA\ cosA - sin^2A$$

Now set the original expression equal to the final expression, and collect real and imaginary parts:

$$cos(2A) + i\ sin(2A) = (cos^2A - sin^2A) + i \times (2sinA\ cosA)$$

Since the real parts on left and right must be equal, you have the formula for *cos*(2A). Since the imaginary parts must be equal, you have the formula for *sin*(2A). That's all there is to it.

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## Great Book on Problem Solving

I have to recommend a terrific little book, *How To Solve It* by G. Polya. Most teachers aren't very good at teaching you how to solve problems and do proofs. They show you how *they* do them, and expect you to pick up their techniques by a sort of osmosis. But most of them aren't very good at explaining the thought process that goes into doing a geometrical proof, or solving a dreaded "story problem".

Polya's book does a great job of teaching you how to solve problems. He shows you the kinds of questions you should ask yourself when you see a problem. In other words, he teaches you how to get yourself over the hum, past the floundering that most people do when they see an unfamiliar problem. And he does it with lots of examples, so that you can develop confidence in your techniques and compare your methods with his. The techniques I've mentioned above are just three out of the many in his book.

There's even a handy checklist of questions you can ask yourself whenever you're stuck on a problem.

*How To Solve It* was published in 1945 and republished in 1979, and it's periodically in and out of print. If you can't get it from your bookstore, go to the library and borrow a copy. You won't be sorry.

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# Musings on the Law of Cosines

"I beg your pardon. I never promised you a rose garden." Dionne Warwick, in the song of that name

The law of cosines doesn't have the nice, neat form of the law of sines, unfortunately. It might even look like just a chaotic jumble of symbols to you. Unfortunately, the law that's tougher to remember than most is also tougher to derive than most: you have to drop a perpendicular from any vertex of the triangle to the opposite side, write the Pythagorean theorem for the two right triangles formed, combine the two equations to eliminate the term for the common side, and express one of the partial sides (like BD in the diagram) in terms of the whole side and the other partial (c−AD in the diagram).

But don't be overly intimidated. For one thing, don't think that there are three laws of cosines: all three are just the same law, written from the perspective of each of the three sides (or angles) in turn. So whether you decide to memorize or derive, you have only one law of cosines to deal with.

Try to focus on the geometry that's involved. The first form lets you solve for an angle if you know the three sides. The side opposite the desired angle occurs only one place in the formula, and it occurs with the only minus sign in the formula, as $-c^2$. The other two sides are interchangeable as far as the formula is concerned, so really it's best to think of the cosine of the angle in terms of the two adjacent sides and, treated differently, the opposite side.

To remember the second form,

$$c^2 = a^2 + b^2 - 2ab \cos C$$

notice that it looks a bit like the Pythagorean theorem, with a "correction term". Since you know the Pythagorean theorem, the only thing new to remember is that correction term. If you try to remember this form, it may help to think of it as just like the triangle: on one side you have one side, and on the other side you have the opposite angle and its two adjacent sides.

Any way you slice it, the law of cosines is awkward. But you have to be able to use it, because there's no other way to solve some triangles. You'll have to decide for yourself whether you'd rather memorize it (if you're sure you can memorize it correctly) or know how to derive it.

[to document contents]

# Revision History

- 2000−12−18:
  - refine the analysis of arcfunction(function(x))
  - rewrite the discussion of principal values of the arc functions
  - correct wrong analysis of *Arcsec*(*cos u*)
  - add a section on interval notation
- 2000−12−15:
  - add a new chapter on the arc functions, to show how to find functions of arc functions
  - de−list Eric's Treasure Trove from "Other math sites," since it no longer exists
  - supply the hundreds of degree signs I previously left implicit
- 2000−08−22: as suggested by Peter Karp, add a PDF version with embedded graphics

- 2000−06−05: Split table of contents into a separate page; add Oak Road Systems navigation bar to every page; fix a broken link in the history
- 2000−05−19: Move these trig pages to a separate directory as part of a general site overhaul; add navigation arrows to every page; update link to Eric's Treasure Trove.
- 2000−03−10: at Peter Karp's suggestion, package all parts of this document (with graphics) for easy download in a ZIP file
- 1999−12−17: add a link to some geometric derivations of *sin*(A±B) and *cos*(A±B)
- (intervening changes suppressed)
- 1997−02−19: new document

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