Chapter 6
Section 6.1 - 6.4

Main Topic # 1: [Basic Trig Identities] Today we will simplify expressions involving the six trig functions using what we call "identities". The most common identities are the following:

The Pythagorean Identities

Alternate
notation
$\sin^2(\theta)+\cos^2(\theta)=1$

$$\Big(\sin(x)\Big)^2 + \Big(\cos(x)\Big)^2 = 1$$
$$1 + \Big(\cot(x)\Big)^2 = \Big(\csc(x)\Big)^2$$
$$\Big(\tan(x)\Big)^2 + 1 = \Big(\sec(x)\Big)^2$$

The Odd or Even Identities

Odd:

$$\sin(-x) = -\sin(x) \qquad \csc(-x) = -\csc(x)$$
$$\tan(-x) = -\tan(x) \qquad \cot(-x) = -\cot(x)$$

Even:

$$\cos(-x) = \cos(x) \qquad \sec(-x) = \sec(x)$$

Main Topic # 2: [Verifying Identities] The major concept of today is the process of verifying equations to be true, we give the following strategy:

The Strategy (the book's strategy)

(i) Work on one side of the equation (usually the more complicated side), keeping in mind the expression on the other side as your goal.
(ii) Some expressions can be simplified quickly if they are rewritten in terms of sines and cosines only.
(iii) To convert one rational expression into another, multiply the numeratopr and denominator of the first by either the numerator or the denominator of the desired expression.
(iv) If the numerator of a rational expression is a sum or difference, convert the rational expression into a sum or difference of two rational expressions.
(v) If a sum or difference of two rational expressions occurs on one side of the equation, then find a common denominator and combine them into one rational expression.

Learning Outcome # 1: [Simplifying expressions]

Problem 1. Simplify the expression $\frac{\tan(x)}{\sec(x)}$.

$$\frac{\left[\frac{\sin(x)}{\cos(x)}\right]}{\left[\frac{1}{\cos(x)}\right]} = \frac{\sin(x)}{\cancel{\cos(x)}} \cdot \frac{\cancel{\cos(x)}}{1}$$
$$= \sin(x)$$

Problem 2. Simplify the expression $\sin(x) + \cot(x)\cos(x)$.

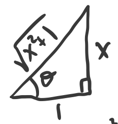
$$\sin(x) + \frac{\cos(x)}{\sin(x)}\cdot\cos(x) = \sin(x) + \frac{\cos^2(x)}{\sin(x)}$$

$$= \frac{\sin^2(x)+\cos^2(x)}{\sin(x)} = \frac{1}{\sin(x)} = \csc(x)$$

Problem 3. Find an equivalent algebraic expression for $\sin(\underbrace{\arctan(x)}_{\theta})$.

opp/adj

$\frac{x}{1}$



$$\sin(\theta) = \frac{\pm x}{\sqrt{x^2+1}}$$

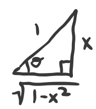
$$x^2+1=h^2$$

$$\sqrt{x^2+1}=h$$

Learning Outcome # 2: [Finding equivalent statements using composition of inverses]

Problem 4. Find an equivalent algebraic expression for $\cos(\underbrace{\arcsin(x)}_{\theta})$.

$\theta \rightarrow \sin(\theta) = \frac{x}{1}$



$$\cos(\theta) = \frac{\pm\sqrt{1-x^2}}{1} = \pm\sqrt{1-x^2}$$

Learning Outcome # 3: [Verifying equality]

Problem 5. Verify the following identity $(\csc(\theta) + \cot(\theta))(1 - \cos(\theta)) = \sin(\theta)$.

$$\left[\frac{1}{\sin(\theta)}+\frac{\cos(\theta)}{\sin(\theta)}\right]\cdot\left(1-\cos(\theta)\right)$$

$$= \frac{1+\cos(\theta)}{\sin(\theta)} - \left(\frac{1+\cos(\theta)}{\sin(\theta)}\right)\cdot\cos(\theta)$$

$$= \frac{1+\cancel{\cos(\theta)}-\cancel{\cos(\theta)}-\cos^2(\theta)}{\sin(\theta)} = \frac{1-\cos^2(\theta)}{\sin(\theta)} = \frac{\sin^2(\theta)}{\sin(\theta)} = \sin(\theta)$$

Problem 6. Verify the following identity $\frac{\sec(\theta)+\csc(\theta)}{1+\tan(\theta)} = \csc(\theta)$

$$\frac{\frac{1}{\cos(\theta)}+\frac{1}{\sin(\theta)}}{1+\frac{\sin(\theta)}{\cos(\theta)}} = \frac{\frac{\sin(\theta)+\cos(\theta)}{\cos(\theta)\sin(\theta)}}{\frac{\cos(\theta)+\sin(\theta)}{\cos(\theta)}} = \frac{\cancel{(\sin(\theta)+\cos(\theta))}}{\cancel{\cos(\theta)}\sin(\theta)}\cdot\frac{\cancel{\cos(\theta)}}{\cancel{(\sin(\theta)+\cos(\theta))}}$$

$$= \frac{1}{\sin\theta} = \csc(x)$$

Main Topic # 3: [More Identities] Another important identity is the Sum and Difference formulas

Sum and Difference Formula

$$\begin{aligned}
\sin(A+B) &= \sin(A)\cos(B) + \cos(A)\sin(B)\\
\cos(A+B) &= \cos(A)\cos(B) - \sin(A)\sin(B)\\
\sin(A-B) &= \sin(A)\cos(B) - \cos(A)\sin(B)\\
\cos(A-B) &= \cos(A)\cos(B) + \sin(A)\sin(B)
\end{aligned}$$

* Typo *
Make note

And yet another important set of identities follows from these above ones and they deal with twice and angle

Double Angle Formula

There is only one formula for sine

$$\sin(2A) = 2\sin(A)\cos(A)$$

There are 3 different ones for cosine and we have know which one to use when

$$\begin{aligned}
\cos(2A) &= \Big(\cos(A)\Big)^2 - \Big(\sin(A)\Big)^2\\
&= 2\Big(\cos(A)\Big)^2 - 1\\
&= 1 - 2\Big(\sin(A)\Big)^2
\end{aligned}$$

Next we consider taking half the angle

Half Angle Formulas

$$\begin{aligned}
\sin\left(\frac{A}{2}\right) &= \pm\sqrt{\frac{1-\cos(A)}{2}}\\
\cos\left(\frac{A}{2}\right) &= \pm\sqrt{\frac{1+\cos(A)}{2}}
\end{aligned}$$

take the positive or negative square root depending on the quadrant of the angle $\frac{A}{2}$. For example, if $A/2$ is in the first quadrant, then the positive root would be used.

Learning Outcome # 4: [Using the Sum and Difference Formula]

Problem 7. Use sum or difference formula to evaluate the following exactly. There may be more than one way to evaluate each.

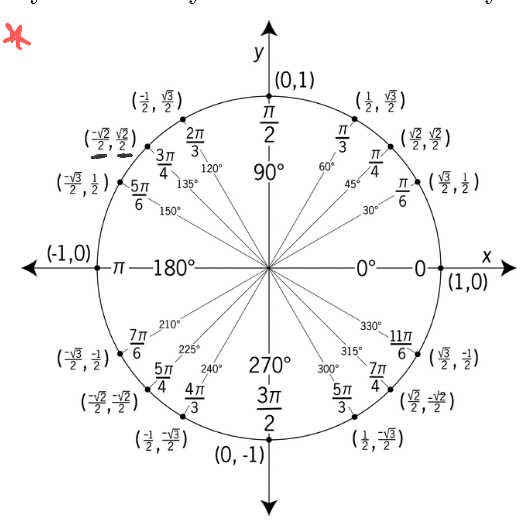
* how can we add or subtract to get angle? *

(a) $\cos(7\pi/12)$

$$\cos\left(\frac{1\pi}{3} + \frac{1\pi}{4}\right) = \cos\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right) = \frac{1}{2}\cdot\frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{2}(1-\sqrt{3})}{4}$$

(b) $\sin(13\pi/12)$

$$\sin\left(\frac{1\pi}{3} + \frac{3\pi}{4}\right) = \sin\left(\frac{\pi}{3}\right)\cos\left(\frac{3\pi}{4}\right) + \cos\left(\frac{\pi}{3}\right)\sin\left(\frac{3\pi}{4}\right) = \frac{\sqrt{3}}{2}\cdot\frac{(-\sqrt{2})}{2} + \frac{1}{2}\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{2}(1-\sqrt{3})}{4}$$



Learning Outcome # 5: [Verifying the Double angle formula]

Problem 8. Using the fact that $2\theta = \theta + \theta$, find an identity for $\sin(2\theta)$.

$$\sin(2\theta) = \sin(\theta+\theta) = \sin(\theta)\cos(\theta) + \underbrace{\cos(\theta)\sin(\theta)}_{\sin(\theta)\cos(\theta)}$$

$$= 2\sin(\theta)\cos(\theta)$$

Learning Outcome # 6: [Using the Half-angle Formula]

Problem 9. Use the half-angle formulas to evaluate the following exactly.

(a) $\sin(\pi/12)$

(How can we tell if it's + or − ?) [see next page!]

$$\sin\left(\frac{\left(\frac{\pi}{6}\right)}{2}\right) = \pm\sqrt{\frac{1-\cos\left(\frac{\pi}{6}\right)}{2}} = \pm\sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}}$$

$\frac{\pi/6}{2} = \pi/12$

$$= \pm\sqrt{\frac{\frac{2-\sqrt{3}}{2}}{2}} = \pm\frac{\sqrt{2-\sqrt{3}}}{2}$$

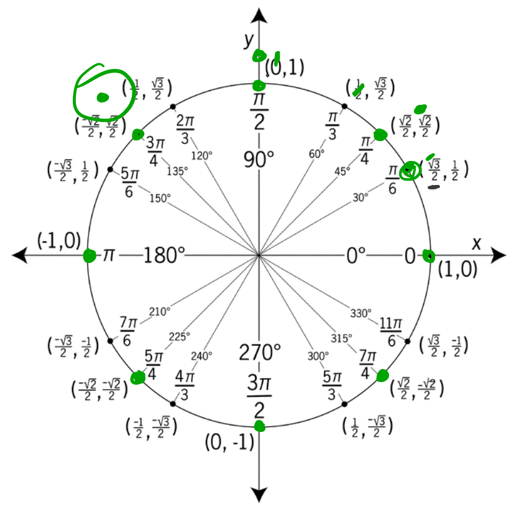
$$= \frac{\sqrt{2-\sqrt{3}}}{2}$$

(b) $\cos(11\pi/8)$

$$\cos\left(\frac{\left(\frac{11\pi}{4}\right)}{2}\right) = \pm\sqrt{\frac{1+\cos\left(\frac{11\pi}{4}\right)}{2}} = \pm\sqrt{\frac{1+\cos\left(\frac{3\pi}{4}\right)}{2}} = \pm\sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}}$$

$\frac{11\pi/4}{2} = 11\pi/8$

$$= \pm\frac{\sqrt{2-\sqrt{2}}}{2} = \frac{-\sqrt{2-\sqrt{2}}}{2}$$



Problem 10. Suppose angles A and B are in the first quadrant, and $\sin(A) = \frac{1}{4}$ and $\sin(B) = \frac{12}{13}$.

(a) Find $\cos(A)$ and $\cos(B)$ exactly.

$$\sin^2(A) + \cos^2(A) = 1$$

in first quadrant

$$\cos(A) = \pm\sqrt{1-\sin^2(A)} = \sqrt{1-\frac{1}{16}} = \sqrt{\frac{15}{16}} = \frac{\sqrt{15}}{4}$$

$$\cos(B) = \pm\sqrt{1-\sin^2(B)} = \sqrt{1-\left(\frac{12}{13}\right)^2} = \sqrt{\frac{169-144}{169}} = \frac{5}{13}$$

(b) Find $\sin(A+B)$ and $\sin(A-B)$ exactly.

$$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B) = \left(\frac{1}{4}\right)\left(\frac{5}{13}\right) + \left(\frac{\sqrt{15}}{4}\right)\left(\frac{12}{13}\right) = \frac{5+12\sqrt{15}}{52}$$

$$\sin(A-B) = \sin(A)\cos(B) - \cos(A)\sin(B) = \left(\frac{1}{4}\right)\left(\frac{5}{13}\right) - \left(\frac{\sqrt{15}}{4}\right)\left(\frac{12}{13}\right) = \frac{5-12\sqrt{15}}{52}$$

Problem 11. Verify the following identity using formulas you already know.

$$\sin(3\theta) = 3\sin(\theta) - 4\sin^3(\theta)$$

$$2\theta + \theta = 3\theta$$

$$\sin(\overset{A}{2\theta} + \overset{B}{\theta}) = \sin(2\theta)\cos(\theta) + \cos(2\theta)\sin(\theta) = 2\sin(\theta)\cos(\theta)\cos(\theta) + (1-2\sin^2(\theta))\sin(\theta)$$

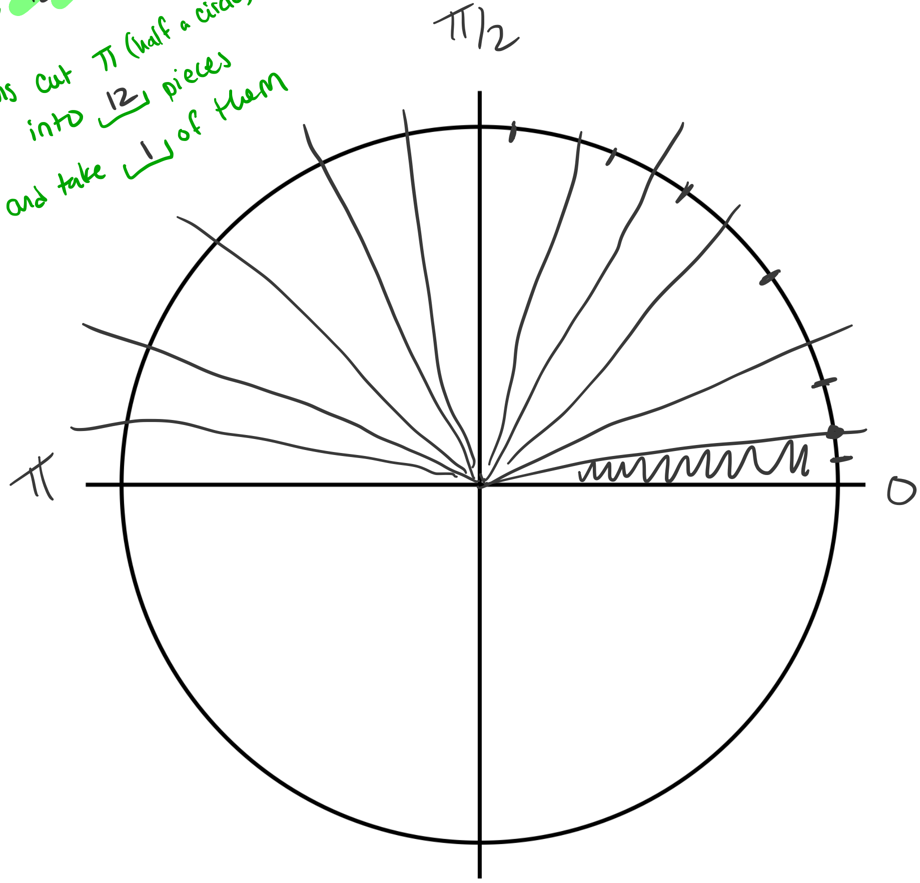
$$= \sin(\theta)\left[2\cos^2(\theta) + 1 - 2\sin^2(\theta)\right]$$

$$= \sin(\theta)\left[2\left(1-\sin^2(\theta)\right) + 1 - 2\sin^2(\theta)\right]$$

$$= \sin(\theta)\left[2 - 2\sin^2(\theta) + 1 - 2\sin^2(\theta)\right]$$

Where is $\frac{\pi}{12}$?

$\frac{\pi}{12}$ means cut π (half a circle) into $\underbrace{12}$ pieces and take $\underbrace{1}$ of them



where is $\frac{11\pi}{8}$?

$\frac{11\pi}{8}$ means cut 2π (whole circle) into $\underbrace{16}$ pieces and take $\underbrace{11}$ of them