Additional Lecture Notes for Complex Analysis

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Chapter 1

Introduction

1.1 Why study complex analysis?

1.1.1 Polynomials

The equation

$$x^2 + 3x + 2 = 0$$

has the real roots $x = 1$ and $x = 2$, but the equation

$$x^2 + 1 = 0$$

has no real roots. For this latter equation we make up the number i so that $i^2 = -1$, and then this is a root for the equation. We make up complex numbers as $a + bi$, $a, b \in \mathbb{R}$, so that, according to Gauss and D'Alembert, every non-constant polynomial equation with constant coefficients has at least one complex root.

Operations with polynomials with complex coefficients and variables mirror those of polynomials with real coefficients and variables. In particular we can define the derivative of a polynomial by the same formula. But do we need it? In the case of real coefficients and variables, we know that between any real zeros of a polynomial lies a real zero of the derivative, and this is for example important to show that the zeros of certain special polynomials, such as the Legendre polynomials, are real. Does a such a result hold in complex? The answer is yes, according to Lucas' theorem: the zeros of the derivative lie in the convex hull of the zeros of the polynomial.

Moreover, for a complex number z we can solve the equation $x^2 = z$ and call the answer $\sqrt{z}$ (actually one of the two answers). For example $i = \sqrt{-1}$. Then

$$-1 = \sqrt{-1} \cdot \sqrt{-1} = \sqrt{(-1)(-1)} = \sqrt{1} = 1.$$

What is going on? The answer lies in understanding the definition of the function $z \mapsto \sqrt{z}$, and for that we need integrals.

Moreover, when you look at algebraic curves:

$$P(x, y) = 0$$

they don't always contain points when you work over real numbers. But they do contain points, and are much nicer, when you work over complex numbers. For example, the group structure on an elliptic curve $y^2 = x^2 + ax + b$, $a, b \in \mathbb{R}$ looks quite mysterious when working over $\mathbb{R}$, but when you work over complex numbers the elliptic curve is a torus and the group structure is the standard Lie group structure of the torus.

1.1.2 Differential equations

The solutions to the differential equation

$$y'' - y = 0$$

are $y(x) = a_1 e^x + a_2 e^{-x}$, but the solutions to the differential equation

$$y'' + y = 0$$

are $y(x) = a_1 \cos x + a_2 \sin x$. But they are also $b_1 e^{ix} + b_2 e^{-ix}$, if we were to solve the characteristic equation and apply the standard formula. How are the two related? The answer is obtained by passing to complex variables. We can extend the definition of the exponential to complex numbers: $z \mapsto e^z$ and then define

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

In fact, these formulas explain the addition formulas in trigonometry:

$$\cos(a+b) = \cos a \cos b - \sin a \sin b, \quad \sin(a+b) = \sin a \cos b + \cos a \sin b.$$

They are just consequences of $e^{a+b} = e^a e^b$.

1.1.3 Elliptic integrals

If you integrate polynomials you get polynomials, when you integrate rational function you almost always get rational functions, and when you integrate irrational functions you sometimes get irrational functions.

We have

$$\int \frac{1}{x} dx = \ln x,$$

and it is certainly more interesting to study the inverse function of this, which is e^x. Similarly

$$\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x,$$

and it is more interesting to study the inverse function $\sin x$. And we have seen that it is the complex setting where the two functions are related.

A similar situation is when we work with elliptic integrals, which are of the form

$$\int R(x,y) dx$$

where R is a rational function and $y = \sqrt{P(x)}$ with P a polynomial of degree 3 or 4 without multiple roots. This situation was considered by Lagrange, and studied intensively by Abel and Jacobi. They were the first to have the idea to pass to complex coordinates. Riemann perfected this idea, and considered arbitrary polynomials $P(x,y)$ that defined y in terms of x. For the polynomial equation $P(z,w) = 0$ Riemann introduced a complex surface (which we now call a Riemann surface) on which $w(z)$ is univalent. Riemann's programme was to study integrals of rational functions $R(z,w)$ along paths in Σ. Interesting enough, topology plays a major role in the computation, and the Cauchy theorem that we will study later is a good illustration of this phenomenon.

Chapter 2

Holomorphic functions

2.1 Polynomials and power series

2.1.1 Differentiation of Polynomials

We can of course define formally

$$\frac{d}{dz}z^n = nz^{n-1}, n = 0, 1, 2, 3, ...$$

And we can also define

$$\frac{d}{dz}z^n = \lim_{h\to 0}\frac{(z+h)^n - z^n}{h},$$

provided that we have a good definition of limits in the complex plane.

Definition. We say that for $f : D \subset \mathbb{C} \to \mathbb{C}$, we have $\lim_{w\to z} f(w) = L$ if for every $\epsilon > 0$ there is $\delta > 0$ such that if $0 < |w - z| < \delta$, then $|f(w) - L| < \epsilon$.

So we have a well defined notion of differentiation in the complex variable z for a polynomial in z. This differentiation satisfies all the nice rules (sum, product, quotient, chain) that differentiation with respect to a real variable satisfies.

How does this relate to the derivatives of two-variable functions? Let $z = x + iy$. If $P(z)$ is a polynomial in the complex variable z we can think of P as being a polynomial in the real variables x, y having complex coefficients.

Example 1. $P(z) = z^2 + 3z + 1$ can be thought of as $P(x, y) = x^2 - y^2 + 2ixy + 3x + 3iy + 1$.

How does $\frac{d}{dz}$ relate to $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$? It turns out that

$$\frac{d}{dz} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right).$$

Example 2. $\frac{d}{dz}(z^2 + 3z + 1) = 2z + 3$. And

$$\frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(x^2 - y^2 + 2ixy + 3x + 3iy + 1) = \frac{1}{2}(2x + 2iy + 3 + 2iy + 2x + 3) = 2x + 2iy + 3,$$

which is the same thing in the other system of coordinates.

In this setting, an interesting question arises. If I give you a two variable polynomial, say $Q(x,y) = ix^3 + 3ixy^2 - 3x^2y + y^3 + (3+2i)x^2 - 3ixy$, is this actually a polynomial in z? To answer this question, we introduce a second "variable" $\bar{z}$. For this to be a true variable, we have to pass to the complexification of the real 2-dimensional plane, but let us not worry about this and just work formally.

Because $z = x + iy$ and $\bar{z} = x - iy$, you can also solve for $x = \frac{z+\bar{z}}{2}$ and $y = \frac{z-\bar{z}}{2i}$. So every polynomial in x and y can be written uniquely as a polynomial in $z, \bar{z}$. The fact that the polynomial is in z only means that when you "differentiate" with respect to $\bar{z}$ you get 0. We have

$$\frac{d}{d\bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

So, the polynomial $Q(x,y)$ is actually a polynomial in z if $\frac{d}{d\bar{z}}Q(x,y) = 0$.

How do we know that the two formulas for differentiation with respect to z and $\bar{z}$ are correct? We can check easily that

$$\begin{aligned}
1 = \frac{d}{dz}z &= \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(x+iy) = \frac{1}{2}(1+1) = 1,\\
0 = \frac{d}{dz}\bar{z} &= \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(x-iy) = \frac{1}{2}(1-1)) = 0,\\
1 = \frac{d}{d\bar{z}}z &= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(x+iy) = \frac{1}{2}(1-1) = 0,\\
0 = \frac{d}{d\bar{z}}\bar{z} &= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(x-iy) = \frac{1}{2}(1+1)) = 1,
\end{aligned}$$

and for a general polynomial use the rule for the derivative of the sum and the derivative of the product.

So a 2-variable polynomial with complex coefficients $P(x,y)$ is actually a polynomial in z if and only if $\frac{d}{d\bar{z}}P(x,y) = 0$. Separate the real and the complex parts of the polynomial, say $P(x,y) = Q(x,y) + iR(x,y)$, where Q, R have real coefficients. Then

$$\begin{aligned}
\frac{d}{d\bar{z}}P(x,y) &= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(Q+iR)\\
\frac{1}{2}\left(\frac{\partial Q}{\partial x} - \frac{\partial R}{\partial y}\right) &+ \frac{i}{2}\left(\frac{\partial Q}{\partial y} + \frac{\partial R}{\partial x}\right).
\end{aligned}$$

Settting the real and imaginary parts equal to zero we obtain that the necessary and sufficient condition that the 2-real variables polynomial complex coefficients $P = Q + iR$ to be a polynomial in z is that

$$\frac{\partial Q}{\partial x} = \frac{\partial R}{\partial y} \text{ and } \frac{\partial Q}{\partial y} = -\frac{\partial R}{\partial x}.$$

2.1.2 Power series

One possible generalization of polynomials, dictated by the necessity to define e^z, $\sin z$, and $\cos z$, is given by power series:

$$\sum_{n=0}^{\infty} a_n z^n, \quad a_n \in \mathbb{C} \text{ for all } n.$$

There are questions to be addressed here. When does the series converge, and when is the resulting function in z differentiable? Can we differentiate term-by-term?

This story can be read in the book, Chapter III, section 1. I just want to emphasize the theorem about the radius of convergence:

Theorem 1. For the power series $\sum_{n=0}^{\infty} a_n(z-a)^n$, let $R = (\limsup |a_n|^{1/n})^{-1}$.
(a) If $|z-a| < R$ the series converges absolutely,
(b) For $r < R$ the series converges uniformly on the closed disk $|z-a| \leq r$,
(c) If $|z-a| > R$ the series diverges.

It is the uniform convergence that allows term-by-term differentiation. Note that the radii of convergence of a series and of the series obtained by term-by-term differentiation are the same, because $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

Example 3. The series

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad \sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}, \quad \cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$

converge and are differentiable everywhere, and we have

$$\frac{d}{dz} e^z = e^z, \quad \frac{d}{dz} \sin z = \cos z, \quad \frac{d}{dz} \cos z = -\sin z.$$

Proposition 1. Let $\sum_{n=0}^{\infty} a_n(z-a)^n$ have the radius of convergence R. If the limit $\lim_{n\to\infty} |a_n/a_{n+1}|$ exists, then this limit equals R.

Proof. This is a consequence of the discrete version of l'Hospital's theorem:

Theorem 2. (Cesáro-Stolz) If $(x_n)_n$ and $(y_n)_n$ are two sequences of real numbers with $(y_n)_n$ strictly positive, increasing, and unbounded, and if

$$\lim_{n\to\infty} \frac{x_{n+1} - x_n}{y_{n+1} - y_n} = L$$

then the limit

$$\lim_{n\to\infty} \frac{x_n}{y_n}$$

exists and is equal to L.

Apply this theorem to $x_n = \ln |a_n|$ and $y_n = n$, and don't forget to exponentiate. $\square$

2.2 Holomorphic functions

2.2.1 The definition of holomorphic functions

I have a problem with the definitions from the text book. So here is how I like to define things:

Definition. Let G be an open set in $\mathbb{C}$. A function $f : G \to \mathbb{C}$ is called (complex) differentiable at $z \in \mathbb{C}$ if

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$$

exists and is finite. The limit is denoted by $f'(z)$ and is called the derivative of f at z.

Definition. Let G be an open set in $\mathbb{C}$. A function $f : G \to \mathbb{C}$ is called holomorphic on G if it is (complex) differentiable at every point in G.

Definition. Let G be an open set in $\mathbb{C}$. A function $f : G \to \mathbb{C}$ is called analytic on G if it is infinitely (complex) differentiable at every point in G and in a neighborhood of every point it coincides with the Taylor series at that point.

The big result is that **every holomorphic function is analytic**. Surprisingly, the trickiest part to prove is that the derivative of a holomorphic functions is continuous. To avoid having to rephrase the statements later, let us for the moment add to the condition that a function is holomorphic the fact that the derivative is continuous, and later remove from the definition this redundant condition.

Note that as a direct consequence of this assumption we obtain that the real and the imaginary parts of f are C^1 functions.

Theorem 3. Let $f = u+iv : G \subset \mathbb{R}^2 \to \mathbb{C}$ be such that u and v have continuous partial derivatives. Then f is holomorphic if and only if u and v satisfy the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Using the operator $\frac{d}{d\bar{z}}$ we have that f is holomorphic if and only if

$$\frac{df}{d\bar{z}} = 0.$$

The proof is in the book at pages 40-42.

Let D be a connected open set, which is usually called a domain.

Proposition 2. If $f : \mathbb{D} \to \mathbb{C}$ is holomorphic and $f'(z) = 0$ for all $z \in D$, then f is constant.

Proof. Write $f = u + iv$. For every w with $|w| = 1$, we have

$$f'(z) = \lim_{h\to 0} \frac{f(z+hw) - f(z)}{hw} = \lim_{h\to 0} \frac{u(z+hw) + iv(z+hw) - u(z) - iv(z)}{h} \cdot \frac{1}{w} = 0.$$

This means that the directional derivatives of u and v at any point are zero.

Let $z, w \in D$ such that the line segment $[z, w]$ is in D. We will show that $f(z) = f(w)$. Restrict u to $[z, w]$. Then we have a one variable function whose derivative is identically equal to zero on an interval. It follows that $u(z) = u(w)$. Similarly $v(z) = v(w)$.

Finally, if we fix $z_0 \in D$ then the set $\{w \,|\, f(w) = f(z_0)\}$ is both open and closed in D. So it must be a connected component of D. It is therefore equal to D. $\square$

Proposition 3. The derivative satisfies:

1. $(f + g)' = f' + g'$,
2. $(fg)' = f'g + fg'$,
3. $(f/g)' = (f'g - fg')/g^2$,
4. $(f \circ g)' = (f' \circ g)g'$ (for a proof of this see page 34 in the book).

Proposition 4. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$. Then

1. the two series have the same radius of convergence,
2. $f' = g$.

Proof. Pages 35–37. $\square$

Proposition 5. If we define

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!},$$

then

1. $e^{z+w} = e^z e^w$ for all $z, w \in \mathbb{C}$;
2. $e^z = e^{\mathrm{Re}z}(\cos \mathrm{Im}z + i \sin \mathrm{Im}z)$;
3. For every $w \in \mathbb{C}$ there is $z \in \mathbb{C}$ such that $e^z = w$ if and only if $w \neq 0$;
4. $e^{z+2k\pi i} = e^z$, for all $z \in \mathbb{C}$, $k \in \mathbb{Z}$.

Define $\cos z = (e^{iz} + e^{-iz})/2$, $\sin z = (e^{iz} - e^{-iz})/2i$.

Proposition 6. If $f : G_1 \to G_2$ is an invertible holomorphic between open sets G_1 and G_2, and if $f'(z) \neq 0$ for all z, then f^{-1} is holomorphic as well, and

$$(f^{-1})' = \frac{1}{f' \circ f^{-1}}.$$

Proof. Page 40. (Note that our f is the g from the book, and the f from the book is our f^{-1}). $\square$

Let

$$G_k = \mathbb{C} \backslash \{z \,|\, \mathrm{Im}z = 2k,\ \mathrm{Re}z \leq 0\}, \quad k \in \mathbb{Z}.$$

For every k,

$$e^z : \{z \,|\, (2k-1)\pi < \mathrm{Im}z < (2k+1)\pi\},$$

is invertible, so we can define a branch of the natural logarithm by

$$\log : G_k \to \{z \,|\, (2k-1)\pi < \mathrm{Im}z < (2k+1)\pi\}, \quad \log(z) = \ln|z| + i(\arg z + 2k\pi).$$

For $k = 0$ we have the principal branch of the logarithm.

We can define the set Σ by gluing the top part of the slit $\{\mathrm{Re}z \leq 0\}$ of G_k to the bottom part of the slit $\{\mathrm{Re}z \leq 0\}$ of G_{k+1}. This set is a Riemann surface (we will return to this), and we obtain a one-to-one and onto holomorphic function

$$\log : \Sigma \to \mathbb{C} \backslash \{0\}.$$

For every $z, w \in \mathbb{C}$, we define

$$z^w = e^{w \log z}.$$

2.2.2 Holomorphic maps as transformations, Möbius transformations

Definition. A C^1 path in $\mathbb{C}$ is a C^1 function $\gamma : [a,b] \subset \mathbb{R} \to \mathbb{C}$.

The angle between two C^1 paths γ_1 and γ_2 that intersect at $z_0 = \gamma_1(t_0) = \gamma_2(t_0)$ is

$$\arg \gamma_1'(t_0) - \arg \gamma_2'(t_0).$$

Theorem 4. Assume that $f : G \to \mathbb{C}$ is holomorphic and has continuous derivative, and that $f'(z_0) \neq 0$. Then f preserves angles at z_0.

Proof. The chain rule

$$(f \circ \gamma)'(t_0) = f'(\gamma(t_0))\gamma'(t_0) = f'(z_0)\gamma'(t_0),$$

yields

$$\arg(f \circ \gamma)'(t_0) = \arg f'(z_0) + \arg \gamma'(t_0),$$

So f rotates the tangent to every C^1 path through z_0 by the same angle $\arg f'(z_0)$. □

A map is called conformal if it preserves angles at every point, and also at every point a,

$$\lim_{z \to a} \frac{|f(z) - f(a)|}{|z - a|}$$

exists.

Example 4. Let $f : \mathbb{C} \to \mathbb{C}^2$, $f(z) = z^2$. Then $f(x,y) = x^2 - y^2 + 2ixy$. So f maps the hyperbolas $x^2 - y^2 = c$, $2xy = d$ into the lines $x = c$, $y = d$. The hyperbolas $x^2 - y^2 = d$ and $xy = c$ intersect at $90°$ angles. This is easy knowing the equation of the tangent at (x_0, y_0) for the two hyperbolas:

$$\begin{aligned} xx_0 - yy_0 &= d \\ \frac{1}{2}(xy_0 + yx_0) &= c. \end{aligned}$$

Now compute the slopes at (x_0, y_0) to be x_0/y_0, respectively $-y_0/x_0$ (which are slopes of perpendicular lines). And the lines $x = c$ and $y = d$ are also perpendicular.

Remark 1. Holomorphic maps preserve orientation as long as they have nonzero derivative. This means that if we view a holomorphic map as a map from $\mathbb{R}^2 \to \mathbb{R}^2$, then it maps any pair of linearly independent vectors (the tangents to two trajectories that cross at a point) to two other vectors that have the same orientation. This is because both vectors are rotated by the same angle. This means that if a region D whose boundary is a smooth curve Γ that is defined by a closed path (loop) that, when traversed has the region on the left, and if f is holomorphic on D and extends continuously to $D \cup \Gamma$, and is such that it has nonzero derivative and is a one-to-one map onto the image, then $f(D)$ is to the left of $f \circ \Gamma$.

Definition. A mapping of the form $S(z) = \frac{az+b}{cz+d}$ with $ad - bc \neq 0$ is called a Möbius transformation.

Extend it to the one-point compactification of the plane, which is the Riemann sphere: $\mathbb{C} \cup \{\infty\}$.

$$S : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}.$$

The new function is differentiable at ∞, meaning that if you replace z by $1/z$ in the expression of the function, then you get a function that is differentiable at 0. So we have a holomorphic function on the entire Riemann sphere.

Remark: $c = 0, b = 0$ defines a dilation, $c = 0$, $a = d$ defines a translation, and $a = 0, b = 1, c = 1, d = 0$ is the inversion about the unit circle. This should be compared with geometric inversion which is the map

$$z \mapsto \frac{1}{\bar{z}},$$

and which should be interpreted as the reflection over the unit circle (in the Poincaré model of Lobachevskian geometry it is actually the reflection over a line). The dilation $z \mapsto re^{i\theta}z$ is geometrically the composition of the dilation by ratio r and center 0, and the rotation about 0 by θ.

It is known that in geometry, for translations, dilations, rotations, and inversions:

$$\{\text{lines}, \text{circles}\} \to \{\text{lines}, \text{circles}\}.$$

The following proposition shows that the same is true for Möbius transformations.

Proposition 7. Every Möbius transformation is the composition of translations, inversion, and dilations.

Proof. The case $c = 0$ is easy. For $c \neq 0$, scale the a, b, c, d such that $ad - bc = -c$. Now take the composition

$$\begin{aligned} z \mapsto z + \alpha \mapsto \frac{1}{z+\alpha} &\mapsto \frac{1}{\beta} \cdot \frac{1}{z+\alpha} \\ &\mapsto \frac{1}{\beta z + \alpha\beta} + \gamma. \end{aligned}$$

Then $\beta = c$, $\alpha = \frac{d}{c}$, $\gamma = \frac{a}{c}$, and we are done. $\square$

A Möbius transformation can have at most 2 fixed points, so it is completely determined by the images of three points.

The map

$$S_{z_2,z_3,z_4}(z) = \frac{z - z_3}{z - z_4} : \frac{z_2 - z_3}{z_2 - z_4}$$

is the unique Möbius transformation that satisfies $S_{z_2,z_3,z_4}(z_2) = 1$, $S_{z_2,z_3,z_4}(z_3) = 0$, $S_{z_2,z_3,z_4}(z_4) = \infty$.

The map $S^{-1}_{w_2,w_3,w_4} \circ S_{z_2,z_3,z_4}$ is the unique Möbius transformation for which $w_1 \mapsto z_1$, $w_2 \mapsto z_2$, and $w_3 \mapsto z_3$.

Corollary 1. For any pair $\{C, C'\} \in \{\text{circles}, \text{lines}\}$ there is a Möbius transformation that maps C to C'.

Notation:

$$(z, z_2, z_3, z_4) := S_{z_2,z_3,z_4}(z).$$

Remark 2. z_1, z_2, z_3, z_4 lie on a circle or line if and only if $(z_1, z_2, z_3, z_4) \in \mathbb{R}$, because $0, 1, \infty$ lie on a line, and Möbius transformations map lines and circles to lines and circles. If the four points lie on a circle and if $(z_1, z_2, z_3, z_4) = -1$ then the quadrilateral formed by the points is called harmonic.

Proposition 8. For any Möbius transformation T,

$$(z_1, z_2, z_3, z_4) = (Tz_1, Tz_2, Tz_3, Tz_4).$$

Proof. Page 48. □

Now we look at geometric inversion in more detail. This is the reflection over a line or a circle. The reflection over the x-axis is just $z \mapsto \bar{z}$. We define the reflection over an arbitrary circle passing through z_2, z_3, z_4 as $z \mapsto z^*$, where

$$(z*, z_2, z_3, z_4) = \overline{(z, z_2, z_3, z_4)}.$$

We call z^* the *symmetric* of z with respect to the circle.

Geometrically, if the circle over which you are reflecting has radius r and center z_0, then z and z^* lie on the same ray originating at z_0 and $|z - z_0||z^* - z_0| = R^2$. Here is a simpler proof than the one in the book (page 51). Because the cross-ratio is invariant under translation and dilation, and that geometric inversion is well behaved under translation and dilation, we may assume that the circle is the unit circle centered at the origin. Using Proposition 8 we have

$$(z^*, z_2, z_3, z_4) = \overline{(z, z_2, z_3, z_4)} = \left(\bar{z}, \frac{1}{z_2}, \frac{1}{z_3}, \frac{1}{z_4}\right) = \left(\frac{1}{\bar{z}}, z_2, z_3, z_4\right).$$

And this is geometric inversion over the unit circle.

Symmetry Principle. A Möbius transformation maps a pair of points that are symmetric with respect to a circle, or a line, to a pair of points that are symmetric with respect to the image.

Orientation Principle. Möbius transformations preserve orientation.

All holomorphic maps preserve orientation as long as they have nonzero derivative.

Example 5. Let us find all Möbius transformations that map the upper half plane $\{z \mid \operatorname{Im} z > 0\}$ onto itself.

Let us first focus on the case $d \neq 0$. Clearly such a Möbius transformation

$$\phi(z) = \frac{az+b}{cz+d}$$

maps the real axis to itself. So for $t \in \mathbb{R}$,

$$\phi(t) = \frac{az+b}{cz+d} = \frac{(at+b)(\bar{c}t+\bar{d})}{(ct+d)(\bar{c}t+\bar{d})} = \frac{a\bar{c}t^2 + (a\bar{d}+b\bar{c})t + b\bar{d}}{|ct+d|^2} \in \mathbb{R}$$

Hence $a\bar{c}t^2 + (a\bar{d}+b\bar{c})t + b\bar{d} \in \mathbb{R}$ for all $t \in \mathbb{R}$. Setting $t = 0$ we deduce that $b\bar{d} \in \mathbb{R}$. But then $[a\bar{d}t + (a\bar{d}+b\bar{c})]t \in \mathbb{R}$ for all $t \in \mathbb{R}$, so $a\bar{c}t + (a\bar{d}+b\bar{c}) \in \mathbb{R}$. Again setting $t = 0$ we obtain $a\bar{d}+b\bar{c}$, and finally $a\bar{c} \in \mathbb{R}$. Since $a\bar{c}, b\bar{d} \in \mathbb{R}$, there are $s, t \in \mathbb{R}$ such that $a = sc$ and $b = td$. But then

$$a\bar{d} + b\bar{c} = sc\bar{d} + t\bar{c}d = sc\bar{d} + t\overline{c\bar{d}} \in \mathbb{R}.$$

Set $w = c\bar{d}$. Then $sw + t\bar{w}$ should be real. This can happen if either w is real, or if $s = t$. But if $s = t$, then ϕ is constant, which is not allowed (the condition $ad - bd$ is violated. So $c\bar{d} \in \mathbb{R}$, meaning that $c = ud$, $u \in \mathbb{R}$. Normalizing so that $d \in \mathbb{R}$ we see that every such Möbius transformation is of the form

$$\phi(z) = \frac{az+b}{cz+d}, \quad a, b, c, d \in \mathbb{R}.$$

The case where $d = 0$ can be treated similarly, and the same conclusion is reached. We thus conclude that these are precisely the Möbius transformations that map the real axis to itself. But do they all map the upper half plane to itself?

The upper half plane is mapped to either the upper half plane, or the lower half plane $\{z \mid \operatorname{Im} z < 0\}$. It is mapped to the upper half plane when the imaginary part of $\phi(i)$ is positive. We have

$$\phi(i) = \frac{ai + b}{ci + d} = \frac{(ai + b)(-ci + d)}{(ci + d)(-ci + d)} = \frac{ac + bd + (ad - bc)i}{|c|^2 + |d|^2}.$$

The imaginary part of this number is positive when $ad - bc > 0$. Thus the Möbius transformations that map the upper half plane to itself are

$$z \mapsto \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbb{R}, \quad ad - bc > 0.$$

Chapter 3

Complex integration

3.1 Cauchy's theorem and integral formula

3.1.1 Line integrals and the Fundamental Theorem of Calculus

It is easy to integrate one variable functions or real variable, but what does it mean to integrate a function of complex variable? Should we integrate over a curve in the plane, or over a domain in the plane? Surprisingly, the answer is: both! The integration over curves in the plane grew out of the work of Abel, Jacobi, and Riemann on elliptic integrals, and this is the natural way to find antiderivatives.

If $\gamma(t) = (x(t), y(y))$, and we are given a function $f = (u, v) : \mathbb{R}^2 \to \mathbb{R}^2$, then we can define the line integral of f on γ by

$$\int_\gamma (udx + vdy) = \int_\gamma f = \int_a^b f(\gamma(t))\gamma'(t)dt.$$

Only the curve $\gamma([a, b])$ and the direction in which it is traveled matter, and not how it is parametrized in the same way as in one variable definite integrals we can change the variable without changing the value of tte integral (and because of that).

If $udx + vdy$ is the differential of a function F, that is

$$u = \frac{\partial F}{\partial x}, \quad v = \frac{\partial F}{\partial y},$$

and if the path starts at $p = \gamma(a)$ and ends at $q = \gamma(b)$, then

$$\int_\gamma (udx + vdy) = F(q) - F(p).$$

This is known as the Fundamental theorem of calculus (Leibnitz-Newton). The Fundamental theorem of calculus is the natural way to find antiderivatives, as we will see below.

Now we can pass to complex variables and make the following definition:

Definition. Let γ be parametrized by $z(t)$.

$$\int_\gamma f(z)dz = \int_\gamma (udx - vdy) + i \int_\gamma (vdx + udy).$$

Note that this is the same as

$$\int_\gamma f = \int_a^b f(\gamma(t))\gamma'(t)dt,$$

where now we work with complex numbers.

To obtain this definition, you basically write $f = u + iv$, and $dz = dx + idy$ substitute in the formula on the left, and multiply out.

Lemma 1. Let γ be the unit circle, traveled counterclockwise. Then

$$\int_\gamma \frac{1}{z} dz = 2\pi i.$$

Proof. Parametrize $z = e^{i\theta}$. □

Corollary 2. Let γ be a circle of center a, traveled counterclockwise. Then

$$\int_\gamma \frac{1}{z-a} dz = 2\pi i.$$

But if the curve γ does not cross the ray $\{z \in \mathbb{C} \,|\, \mathrm{Im} z \leq 0\}$, then because

$$\frac{d}{dz} \log z = \frac{1}{z},$$

we have

$$\int_\gamma \frac{1}{z} dz = \log \gamma(b) - \log \gamma(a),$$

and this is zero if the curve is closed. Can you prove Lemma 1 using this? Indeed, we can. We have the following result.

Lemma 2. Assume that $\gamma : [a, b] \to \mathbb{C}$ is a rectifiable curve in the domain of the holomorphic function f (whose first derivative is continuous). Then

$$\int_\gamma \frac{df}{dz} dz = f(\gamma(b)) - f(\gamma(a)).$$

Proof. Write $f(z) = u(x, y) + iv(x, y)$. Then

$$\begin{aligned}\int_\gamma \frac{df}{dz} dz &= \int_\gamma \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(u + iv)(dx + idy) \\ &= \frac{1}{2}\int_\gamma \frac{\partial u}{\partial x}dx + i\frac{\partial u}{\partial x}dy + i\frac{\partial v}{\partial x}dx - \frac{\partial v}{\partial x}dy - i\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial y}dy + \frac{\partial v}{\partial y}dx + i\frac{\partial v}{\partial y}dy\end{aligned}$$

Using the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

we obtain that this is equal to

$$\int_\gamma \frac{\partial}{\partial x}(u + iv)dx + \frac{\partial}{\partial y}(u + iv)dy = (u + iv)(\gamma(b)) - (u + iv)(\gamma(a)),$$

where for the last step we used the fundamental theorem of calculus. □

3.1.2 Green's Formula

We will look at this fact in a different perspective, but for that we need integrals over 2-dimensional domains. For that we need

Theorem 5. (Stokes' Theorem)

$$\int_{\partial D} \omega = \int_D d\omega.$$

If $D \subset \mathbb{C}$ is a domain whose boundary is a finite union of rectifiable curves that are oriented so that the domain is to the left, and if $udx + vdy$ is such that u, v are differentiable with continuous partial derivatives, then

$$\int_{\partial D} udx + vdy = \iint \frac{\partial u}{\partial y} dy \wedge dx + \frac{\partial v}{\partial x} dx \wedge dy = \iint \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx \wedge dy,$$

where for the last step we used $dx \wedge dx = dy \wedge dy = 0$ and $dx \wedge dy = -dy \wedge dx$. This is the well known Green's formula.

Let us see what Green's formula becomes when switching to complex integration.

$$\begin{aligned}
&\int_{\partial D} f(z)dz = \int_{\partial D} (u+iv)(dx+idy) = \int_{\partial D} (udx - vdy) + i\int_{\partial D} (udy + vdx) \\
&= \iint_D \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx \wedge dy + i \iint \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dx \wedge dy \\
&= i \iint_D \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} \right) (u+iv) dx \wedge dy = 2i \iint \frac{df}{d\bar{z}} dx \wedge dy.
\end{aligned}$$

Now we can also do the computation

$$dz \wedge d\bar{z} = (dx + idy) \wedge (dx - idy) = dx \wedge dx - idx \wedge dy + idy \wedge dx - dy \wedge dy = -2idx \wedge dy.$$

We obtain the complex form of Green's formula:

Theorem 6. (Green's formula) Let $D \subset \mathbb{C}$ be a domain whose boundary is a finite union of rectifiable curves that are oriented so that the domain is to the left, and let $\bar{D} = D \cup \partial D$. Let $f : \bar{D} \to \mathbb{C}$ be a function that is continuous on $\bar{D}$ and has continuous partial derivatives in D. Then

$$\int_{\partial D} f(z)dz = -\iint \frac{df}{d\bar{z}} dz \wedge d\bar{z}.$$

An example of such an open set is shown in Figure 3.1.

3.1.3 Cauchy's theorem and Cauchy's formula

Here are several corollaries to Green's formula, which are probably the most important results in this course.

Theorem 7. (Cauchy's Theorem) Let $D \subset \mathbb{C}$ be a domain whose boundary is a finite union of rectifiable curves that are oriented so that the domain is to the left, and let $\bar{D} = D \cup \partial D$. Let $f : \bar{D} \to \mathbb{C}$ be a function that is continuous on $\bar{D}$ and holomorphic (with continuous derivatives) in D. Then

$$\int_{\partial D} f(z)dz = 0.$$

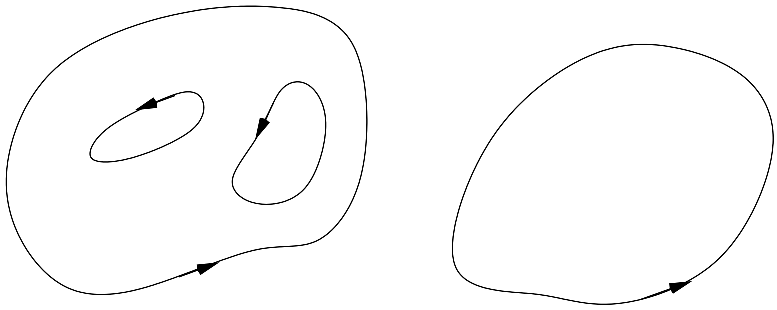

Figure 3.1: Open set in $\mathbb{C}$.

Theorem 8. (The Cauchy-Pompeiu Formula) Let $D \subset \mathbb{C}$ be a domain whose boundary is a finite union of rectifiable curves that are oriented so that the domain is to the left, and let $\bar{D} = D \cup \partial D$. Let $f : \bar{D} \to \mathbb{C}$ be a function that is continuous on $\bar{D}$ and has continuous partial derivatives in D. Let $a \in D$. Then

$$f(a) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{z-a} dz + \frac{1}{2\pi i} \iint_D \frac{\frac{df}{d\bar{z}}}{z-a} dz \wedge d\bar{z}.$$

Proof. Let $\epsilon > 0$ be small, and take out from D the open disk $B(a, \epsilon)$, as shown in Figure 3.2. The boundary of this is the circle $|z - a| = \epsilon$, which we can parametrize clockwise (so that $D \backslash B(0, \epsilon)$ is to the left) by $a + \epsilon e^{-it}$, $0 \leq t \leq 2\pi$. We write Green's formula for

$$g(z) = \frac{f(z)}{z-a},$$

and notice that, using the product rule,

$$\frac{dg}{d\bar{z}} = \frac{\frac{df}{d\bar{z}}}{z-a}.$$

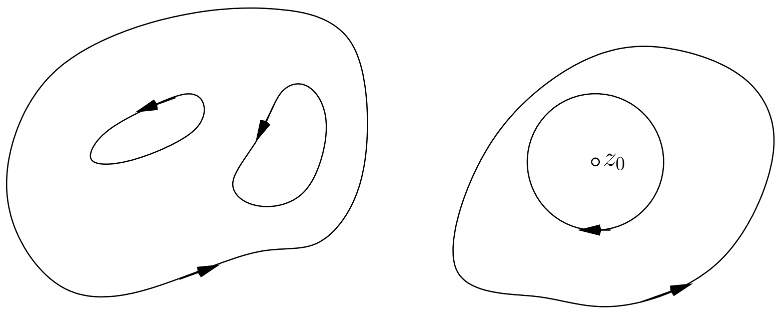

Figure 3.2: Open set in $\mathbb{C}$ with a disk removed.

We have

$$\int_{\partial D} \frac{f(z)}{z-a} dz + \int_0^{2\pi} \frac{f(a+\epsilon e^{-it})}{a+\epsilon e^{-it}-a} (-i\epsilon e^{-it}) dt = -\iint_{D \backslash B(a\epsilon)} \frac{\frac{df}{d\bar{z}}}{z-a} dz \wedge d\bar{z}.$$

This can be rewriten as

$$\int_0^{2\pi} \frac{f(a+\epsilon e^{-it})}{\epsilon e^{-it}} i\epsilon e^{-it} dt = \int_{\partial D} \frac{f(z)}{z-a} dz + \iint_D \frac{\frac{df}{d\bar{z}}}{z-a} dz \wedge d\bar{z},$$

or

$$i\int_0^{2\pi} f(a+\epsilon e^{-it})dt = \int_{\partial D}\frac{f(z)}{z-a}dz + \iint_D \frac{\frac{df}{d\bar{z}}}{z-a}dz\wedge d\bar{z}.$$

Because f is continuous, when $\epsilon \to 0$, $f(a+\epsilon e^{-it}) \to f(a)$, and so $\int_0^{2\pi} f(a+\epsilon e^{-it})dt \to 2\pi f(a)$. Therefore

$$f(a) = \frac{1}{2\pi i}\int_{\partial D}\frac{f(z)}{z-a}dz + \frac{1}{2\pi i}\iint_D \frac{\frac{df}{d\bar{z}}}{z-a}dz\wedge d\bar{z}.$$

□

Here at the last step we have used a result which is good to emphasize.

Lemma 3. Let γ be a rectifiable path, and assume that the continuous functions F_n converge uniformly to F on γ. Then

$$\int_\gamma F(z)dz = \lim_{n\to\infty}\int_\gamma F_n(z)dz.$$

Proof. This follows from

$$\begin{aligned}\left|\int_\gamma F(z)dz - \int_\gamma F_n(z)dz\right| &= \left|\int_\gamma (F(z)-F_n(z))dz\right| \leq \int_\gamma |F(z)-F_n(z)||dz| \\ &\leq \sup|F(z)-F_n(z)|\int_\gamma |dz| = \sup|F(z)-F_n(z)|\text{length}(\gamma),\end{aligned}$$

where $|dz|$ is the measure given by the arclength. □

Theorem 9. (Cauchy's Formula) Let $D \subset \mathbb{C}$ be a domain whose boundary is a finite union of rectifiable curves that are oriented so that the domain is to the left, and let $\bar{D} = D \cup \partial D$. Let $f : \bar{D} \to \mathbb{C}$ be a function that is continuous on $\bar{D}$ and holomorphic (with continuous derivatives) in D. Then

$$f(a) = \frac{1}{2\pi i}\int_{\partial D}\frac{f(z)}{z-a}dz.$$

Proof. This follows from the previous result, because for holomorphic functions the double integral is zero. □

Remark 3. So the Cauchy-Riemann equations state that the integrand in the double integral from Green's formula is zero. In other words, by writing a smooth function as $f(z,\bar{z})$, the condition that f does not depend on $\bar{z}$ is about that integrand being zero. This is the beautiful miracle of complex analysis.

This theorem has a shocking consequence.

Theorem 10. Let $f : D \to \mathbb{C}$ be a holomorphic function (with continuous derivative) in the open set D, and assume that the open disk $B(a,R)$ lies inside D. Then on $B(a,R)$, f coincides with a power series $\sum_{n=0}^\infty a_n z^n$, whose radius of convergence is at least R. Consequently, f is analytic.

Proof. Let $0 < r < R$, so that $\bar{B}(a,r) \in D$. Set $\gamma(t) = a + re^{it}$, $0 \leq t \leq 2\pi$. Using the Cauchy formula we can write

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}dw, \qquad \text{for } z \in B(a,r).$$

The Taylor series expansion of $g_w(z) = f(w)/(w-z)$ around a is

$$\frac{f(w)}{w-z} = \frac{f(w)}{(w-a)-(z-a)} = \sum_{n=0}^{\infty} \frac{f(w)}{(w-a)^{n+1}}(z-a)^n.$$

Note that

$$\frac{|f(w)|}{|w-a|^{n+1}}|z-a|^n \leq \frac{\sup_{w\in\gamma}|f(w)|}{r}\left(\frac{|z-a|}{r}\right)^n,$$

so by the Weierstrass M-test, the series converges uniformly for $w \in \gamma$. So the integral commutes with the series expansion

$$\int_\gamma \sum_{n=0}^{\infty} \frac{f(w)}{(w-a)^{n+1}}(z-a)^n dw = \sum_{n=0}^{\infty}\left(\int_\gamma \frac{f(w)}{(w-a)^{n+1}}dw\right)(z-a)^n,$$

Adding a factor of $\frac{1}{2\pi i}$ you obtain that

$$f(z) = \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-a)^{n+1}}dw\right)(z-a)^n,$$

and the latter is just a power series expansion about a. Write this as $f(z) = \sum_n a_n z^n$.

For the moment it seems that a_n depends on r (that is, it depends on γ). But, since the power series converges uniformly, the summation commutes with differentiation, and consequently

$$a_n = \frac{f^{(n)}(a)}{n!},$$

and the latter is clearly independent of γ. □

Remark 4. This explanation should be addressed to those who have some knowledge of algebraic topology. This observation that Cauchy's theorem and Cauchy's formula are consequences of Green's theorem places these two theorems in their correct perspective. They are usually associated with the concept of homotopy (e.g. integrals of holomorphic functions on homotopic curves are equal), when in fact they should correctly be associated with the concept of homology (e.g. integrals of holomorphic functions on homologous curves are equal), and it is this setting that can be generalized (e.g. integrals of holomorphic 1-forms on homologous curves on a Riemann surface are equal).

Theorem 11. (Cauchy's estimate) Let f be holomorphic (with continuous derivative) in $B(a,R)$, and suppose $|f(z)| \leq M$ in $B(a,R)$. Then

$$|f^n(a)| \leq \frac{n!M}{R^n}.$$

Proof. For $r < R$,

$$\begin{aligned}|f^{(n)}(a)| = n!\left|\frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-a)^{n+1}}dw\right| &\leq \frac{n!}{2\pi}\int_\gamma \frac{|f(w)|}{|w-a|^{n+1}}|dw| \\ &\leq \frac{n!}{2\pi}\cdot\frac{M}{r^{n+1}}2\pi r = \frac{n!M}{r^n}.\end{aligned}$$

Letting $r \to R$, we obtain the conclusion. □

3.2 Every complex differentiable function is analytic

Now let us assume that we are given a holomorphic function f (with continuous derivative) on an open set G that is homeomorphic to a disk (meaning that it has no holes, which is phrased mathematically as saying that it is simply connected). Then we can find an antiderivative of f as follows. First, fix a point $a \in G$. Then for every point $z \in G$, connect a to z by a polygonal line γ. Define the antiderivative of f as

$$F(z) = \int_\gamma f(w)dw.$$

Of course we can make this construction for any continuous function f, but it only yields an antiderivative when f is holomorphic (with continuous derivative). This is because of Cauchy's theorem. Otherwise different paths give different derivatives, and because of that the entire definition falls apart. In fact, from Cauchy's theorem we only need one condition: that

$$\int_T f(w)dw = 0$$

for all triangles T included in G.

Theorem 12. (Morera's Theorem) Let G be an open set and let $f : G \to \mathbb{C}$ be a continuous function such that $\int_T f(w)dw = 0$ for every triangular path T in G. Then f is analytic in G.

Proof. The condition of f to be analytic is local, so we can just prove it is analytic in the neighborhood of every point. Then we can work inside a disk. Let a be the center of the disk, and define the antiderivative of f as above. Note that the antiderivative does not depend on the path. Indeed, if γ_1 and γ_2 are two such paths, then

$$\int_{\gamma_1 \cup \bar{\gamma_2}} f(w)dw = 0,$$

(where $\bar{\gamma_2}$ is γ_2 traced backwards, because $\gamma_1 \cup \bar{\gamma_2}$ can be decomposed into nonskew polygons, and these can be decomposed into triangles, and on the boundary of each triangle is zero, so on each polygon it is zero. Now, for two points z and z_0, we have that

$$F(z) - F(z_0) = \int_{[z_0,z]} f(w)dw,$$

because to define $F(z)$ we can use a path whose last segment is $[z_0, z]$. Then

$$\left| \frac{F(z) - F(z_0)}{z - z_0} - f(z_0) \right| \le \sup_{w \in [z,z_0]} |f(w) - f(z_0)|,$$

and the latter goes to 0 as $z \mapsto z_0$ since f is continuous. Therefore F is holomorphic with continuous derivative (which is f), and so it is analytic. Its derivative, f, is also analytic. □

Theorem 13. (Goursat's Theorem) Let G be an open set and let $f : G \to \mathbb{C}$ be a complex differentiable function. Then f is analytic in G.

Proof. All we have to do is show that it satisfies the hypothesis of Morera's theorem. Again, it suffices to check analyticity on a disk. We argue by contradiction, assuming there is f that is complex differentiable, but does not satisfy the hypothesis of Morera's theorem. We can assume that there is a function f such that there is a triangle of perimeter 1 (just rescale the complex plane to make the perimeter 1) on which the integral of f is 1 (if not, just multiply f by the appropriate variable). Divide the triangle by midlines into 4 equal triangles. On one of them the integral has the absolute value at least $1/4$, because the integral on the big triangle is the sum of the integrals on the small triangles. Repeat. At the nth step there is a triangle of perimeter $1/2^n$ on which the integral has the absolute value at least $1/4^n$. Repeating the process we get a decreasing sequence of closed triangular regions with diameters going to 0. By Cantor's theorem, the intersection consists of one point, call it z_0.

Now for every $\epsilon > 0$, there is $\delta > 0$ such that if $|z - z_0| < \delta$,

$$|f(z) - f(z_0) - f'(z_0)(z - z_0)| \leq \epsilon|z - z_0|.$$

Note that $f(z_0)+f'(z_0)(z-z_0)$ is analytic, so its integral on any triangle is zero. Then for a triangle that lies within the disk of radius δ around z_0,

$$\left|\int_T f(z)dz\right| = \left|\int_T f(z) - f(z_0) - f'(z_0)(z - z_0)dz\right| \leq \epsilon \int_T |z - z_0||dz| \leq \epsilon\frac{1}{4^{n+1}}.$$

But for a triangle from the decreasing sequence that lies in the disk of radius δ, the first integral is greater than $1/4^n$. And this is absurd. Hence the conclusion. □

3.3 The winding number and the generalization of Cauchy's Formula

Definition. The winding number (also known as the index) of a rectifiable curve with respect to a point is $(\theta(1) - \theta(0))/2\pi$, where the point has coordinates (x_0, y_0) and the curve is parametrized in polar coordinates centered at (x_0, y_0) as $(x_0 + r(t)\cos\theta(t), y_0 + r(t)\sin\theta(t))$.

We denote the winding number of γ with respect to a by $n(\gamma; a)$.

Proposition 9.

$$n(\gamma; a) = \frac{1}{2\pi i}\int_\gamma (z - a)^{-1}dz.$$

As a corollary of Theorem 9 we obtain

Theorem 14. (General form of Cauchy's formula) Let G be an open subset of the plane and let $f : G \to \mathbb{C}$ be an analytic function. If $\gamma_1, \gamma_2, \ldots, \gamma_n$ are closed rectifiable curves in G such that $n(\gamma_1; w) + n(\gamma_2; w) + \cdots + n(\gamma_n; w) = 0$ for all $w \in \mathbb{C}\backslash G$, then for $a \in G - \cup_{k=1}^m \gamma_k$,

$$f(a)\sum_{k=1}^m n(\gamma_k; a) = \sum_{k=1}^m \frac{1}{2\pi i}\int_{\gamma_k}\frac{f(z)}{z - a}dz.$$

Let $S_1 = \{z \in \mathbb{C} \,|\, |z| = 1\}$.

Definition. Two loops $\gamma_0, \gamma_1 : S^1 \to G$ are homotopic if there is a continuous map $H : S^1 \times [0, 1] \to G$ such that $H|S^1 \times \{j\} = \gamma_j$, $j = 0, 1$.

Definition. Two paths $\gamma_0, \gamma_1 : [0,1] \to G$ are homotopic relative to the endpoints if there is a continuous map $H : [0,1] \times [0,1] \to G$ such that $H|[0,1] \times \{j\} = \gamma_j$, $j = 0, 1$, and $H|\{0\} \times [0,1]$ and $H|\{1\} \times [0,1]$ are constant.

Theorem 15. (General form of Cauchy's formula) If two loops are homotopic or two paths are homotopic relative to the endpoints, then the integrals of a holomorphic function on them are equal.

For people who know algebraic topology, we have the following reformulations, which are more appropriate generalizations, given the relationship to Green's theorem and, implicitly to de Rham cohomology. One should note that the homological formulation generalizes to Riemann surfaces, and for compact surfaces it relates to Hodge theory and the construction of Jacobian varieties.

Theorem 16. Let G be an open subset of the plane and let $f : G \to \mathbb{C}$ be an analytic function. Let γ be a collection of finitely many rectifiable curves in G whose homology class in $H_1(G, \mathbb{Z})$ is zero. Let $a \in G$ and assume that the homology class of γ in $H_1(\{\mathbb{C}\}\backslash\{a\}, \mathbb{Z})$ is n under the isomorphism $H_1(\{\mathbb{C}\}\backslash\{a\}, \mathbb{Z}) \simeq \mathbb{Z}$ that maps a circle centered at z and oriented counterclockwise to 1. Then

$$nf(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}dz.$$

Theorem 17. Let G be an open subset of the plane and let $f : G \to \mathbb{C}$ be an analytic function. Let γ be a collection of finitely many rectifiable curves in G whose homology class in $H_1(G, \mathbb{Z})$ is zero. Then $\int_\gamma f(z)dz = 0$.

Example 6. Let γ consist of the circles $\{z \in \mathbb{C} \,|\, |z-1-i| = 2\}$, $\{z \in \mathbb{C} \,|\, |z-2| = 3\}$, and $\{z \in \mathbb{C} \,|\, |z-3 = 2\}$, the first two oriented counterclockwise, the third oriented clockwise. Let us find

$$\int_\gamma \frac{1}{z^2+1}dz.$$

We have

$$\int_\gamma \frac{1}{z^2+1}dz = \frac{1}{2i}\left(\int_\gamma \frac{dz}{z-i} + \int_\gamma \frac{dz}{z+i}\right) = \frac{1}{2i}(2\pi i + 4\pi i) = 3\pi.$$

Chapter 4

Zeros and poles; classification of singularities

4.1 Zeros of a holomorphic function; the fundamental theorem of algebra

Definition. A number $a \in \mathbb{C}$ is called a zero of a holomorphic function f if $f(a) = 0$.

Theorem 18. (Liouville's theorem) If $f : \mathbb{C} \to \mathbb{C}$ is a bounded holomorphic function, then f is constant.

Proof. Page 77 $\square$

Theorem 19. (The Gauss-d'Alembert fundamental theorem of algebra) Every nonconstant polynomial with complex coefficients has at least one complex zero.

Proof. page 77 $\square$

Definition. If $f : G \to \mathbb{C}$ is holomorphic and $a \in G$ is so that $f(a) = 0$, we say that a is a zero of multiplicity m (where m is a positive integer) if there is a holomorphic function $g : G \to \mathbb{C}$ such that $g(a) \neq 0$ and $f(z) = (z - a)^m g(z)$.

Because of the power series expansion about every point, every zero of a non-identically zero holomorphic function has some multiplicity. Indeed, we have the following result:

Theorem 20. Let G bea connected open set and let $f : G \to \mathbb{C}$ be a holomorphic function. The following are equivalent

(a) f is identically equal to zero;

(b) there is a point a in G at which all derivatives of f are zero;

(c) the set of zeros of f has a limit point in G.

Proof. page 78 $\square$

Example 7. For $f(z) = \sin z$, 0 is a zero of multiplicity 1. Because of periodicity, every zero of $\sin z$ is of the first order.

Example 8. The situation should be contrasted with the real case. The function $f(x) = x^{3/2}$ does not have a zero of integer multiplicity at zero. Note that at zero it is not analytic, while it is analytic at every other point. You see that $f(z) = z^{3/2}$ is not holomorphic everywhere, you have to remove $\{z \,|\, \mathrm{Re} z \leq 0\}$ from its domain.

Note that holomorphic functions have only isolated zeros, and consequently, if two holomorphic functions coincide on a set that has an accumulation point, then they are equal.

4.2 Poles; Meromorphic functions

The concept of zero of a holomorphic function comes naturally associated with that of a pole. In short, a is a zero of f if it is a pole of $1/f$ and it is a pole of f if it is a zero of $1/f$. For example 0 is a zero of second order of $f(z) = z^2$, and so it is a pole of second order of $g(z) = z^{-2}$. It is not just the elegance of formulation that brings these two notions together, they also appear together in various situations, for example every meromorphic function on a compact Riemann surface has as many zeros as poles (multiplicity counted). We will see this later.

There is a problem with identifying poles, in that if a is a pole of f then naturally $f(a)$ is not defined. So then $1/f(a)$ is not a priori defined, so then how can a be a zero of $f(a)$. The fact is that $1/f$ can be extended to a as well. For this we need the notion of removable singularity. For that we first need the notion of isolated singularity.

Definition. A function f has an isolated singularity at a if its domain is an open set in $\mathbb{C}\backslash\{a\}$ that contains a set of the form $\{z \,|\, < 0|z-a| < R\}$.

Definition. A function f has a removable singularity at a if a is an isolated singularity of f and there is a holomorphic function g on some $B(a, R)$, such that $f(z) = g(z)$ for $0 < |z-a| < R$.

Theorem 21. An isolated singularity a if f is removable if and only if $\lim_{z\to a}(z-a)f(z) = 0$.

Proof. pages 103-104 □

In particular, if $\lim_{z\to a} f(z)$ exists, the singularity is removable. This is not an obvious fact, and not that in the real setting, 0 is not a "removable singularity" of $f(x) = |x|$, despite the fact that f is analytic everywhere but at zero and $\lim_{x\to 0} f(x) = 0$.

Example 9. For $f(z) = \frac{\sin z}{z}$, 0 is a removable singularity.

Based on this, we can define

Definition. An isolated singularity a is a pole for f if $\lim_{z\to a} |f(z)| = \infty$.

An alternative definition is that a is a removable singularity for $1/f$ and $(1/f)(a) = 0$. A corollary of the above discussion is the following.

Proposition 10. If $f : G\backslash\{a\} \to \mathbb{C}$ is holomorphic (where G is open) with a pole at $z = a$, then there is a holomorphic function $g : G \to \mathbb{C}$ such that $g(a) \neq 0$ and

$$f(z) = \frac{g(z)}{(z-a)^m}, \text{ for all } z \in G.$$

Proof. Find a disk $B(a, R)$ on which f has no zeros. Then $1/f$ is analytic on $B(a, R)\backslash\{a\}$, and has a removable singularity at a. Write $(1/f) = (z-a)^m h(z)$ with $h(a) \neq 0$. Let $g = 1/h$. Then $f(z) = \frac{g(z)}{(z-a)^m}$ on $B\backslash\{a\}$, so $g(z) = (z-a)^m f(z)$ on this set. But this formula allows us to extend g to the entire G. We are done. □

Definition. If f has a pole at a and m is the smallest integer so that $f(z)(z-a)^m$ has a removable singularity at a, then m is called the order of the pole.

Proposition 11. If f is defined on $B(a,R)\backslash\{a\}$ and has a pole of order m at a, then on $B(a,R)\backslash\{a\}$

$$f(z)=\sum_{k=-m}^{\infty}a_k(z-a)^k.$$

The part $\sum_{k=-m}^{-1}a_k(z-a)^k$ is called the singular part of f.

I have said before that zeros and poles should be studied together, and we have seen before that poles are defined using zeros. But there is a more profound reason why poles and zeros should be studied together. For that we should add the point at infinity. First, a definition.

Definition. A meromorphic function on G is a function that is holomorphic on G except of some poles.

Now let us add the point at infinity to the range. We say that

$$f: G\to\mathbb{C}\cup\{\infty\}$$

is holomorphic if f is holomorphic in some neighborhood of every point where f takes a finite value and $1/f$ is holomorphic in some neighborhood of every point where f takes the value ∞.

Then a meromorphic function is just a holomorphic function $f: G\to\{\infty\}$. Now let us consider $f:\mathbb{C}\cup\{\infty\}\to\mathbb{C}\cup\{\infty\}$. We say that f is holomorphic at ∞ if $f(1/z)$ is holomorphic at 0. Now we have included the concept of a meromorphic function in that of holomorphic functions on Riemann surfaces. This is yet another reason why to study Riemann surfaces.

Example 10. Let $p(z)$ be a polynomial of nth degree; then it has n zeros multiplicities counted. Let us look at the order of the pole at ∞. We write

$$p(z)=a_nz^n+a_{n-1}z^{n-1}+\cdots+a_0.$$

Then after changing $z\to 1/z$, the order of the pole of $p(z)$ at ∞ is the same as the order of the pole of

$$p(1/z)=\frac{a_n}{z^n}+\frac{a_{n-1}}{z^{n-1}}+\cdots+a_0.$$

at 0. And this is n. As a corollary, the number of the zeros on the Riemann sphere is equal to the number of the poles.

Let me explain the big picture. If we have two Riemann surfaces X and Y, and $f: X\to Y$ holomorphic, then for $w\in Y$ the equation $f(z)=w$ has the same number of solutions regardless of w, with multiplicities counted. The number of solutions just counts the number of times f "wraps" X around Y. A particular case is $X=Y=\mathbb{C}\cup\{\infty\}$, $w=0,\infty$. Then this just says that for a meromorphic function, the number of zeros equals the number of poles, multiplicities counted. We will return to this when we talk about Riemann surfaces. For the moment let us prove a particular case. For it we need the logarithmic derivative:

$$\frac{d}{dz}\log f=\frac{f'}{f}.$$

Note that $\frac{d}{dz}\log(fg)=\frac{d}{dz}\log f+\frac{d}{dz}\log g$ and $\frac{d}{dz}\log(f/g)=\frac{d}{dz}\log f-\frac{d}{dz}\log g$.

Theorem 22. (The Argument Principle) Let $f : G \to \mathbb{C}$ be meromorphic with zeros and poles counted with multiplicity. If γ is a closed, rectifiable, curve (consisting of maybe several closed curves) that lies in G and bounds an open subset D in G to its left and does not pass through the zeros and poles of f, then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \text{ number of zeros in } D - \text{ number of poles in } D.$$

Proof. Let γ bound the surface D. Let the zeros be $z_1, z_2, \ldots, z_m$ and poles $p_1, p_2, \ldots, p_{n'}$. Let $B(z_j, \epsilon_j)$, $j = 1, 2, \ldots, m$, and $B(p_k, \delta_k)$, $k = 1, 2, \ldots, n$ be disjoint disks in D that don't contain other zeros or poles inside or on the boundary, let their boundaries be γ_j, $j = 1, 2, \ldots, m$, γ'_k, $k = 1, 2, \ldots, n'$ oriented counterclockwise. Then $D\backslash(\cup_j B(z_j, \epsilon_j) \cup \cup_k B(p_k, \delta_k))$ is bounded by $\gamma \cup (\cup_j \bar{\gamma}_j) \cup (\cup_k \bar{\gamma}'_k)$. In this set f'/f is holomorphic, and is continuous on its closure, so by Cauchy's theorem its integral is zero. Hence

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \sum_j \frac{1}{2\pi i}\int_{\gamma_j} \frac{f'(z)}{f(z)}dz + \sum_k \frac{1}{2\pi i}\int_{\gamma'_k} \frac{f'(z)}{f(z)}dz.$$

Note that if f has a zero of order m_j at z_j, then there is a holomorphic function on $B(z_j, \epsilon)$ such that $f(z) = (z - z_j)^{m_j} g(z)$ where g is holomorphic and nonzero in $B(z_j, \epsilon)$. Then

$$\frac{f'(z)}{f(z)} = \frac{m_j}{z - z_j} + \frac{g'(z)}{g(z)}.$$

Consequently,

$$\frac{1}{2\pi i}\int_{\gamma_j} \frac{f'(z)}{f(z)}dz = \frac{1}{2\pi i}\int_{\gamma_j} \frac{m_j}{z - z_j}dz + \frac{1}{2\pi i}\int_{\gamma_j} \frac{g'(z)}{g(z)}dz = m_j + 0 = m_j.$$

Similarly if f has a pole of order m_k at p_k, then there is a holomorphic function on $B(p_k, \epsilon)$ such that $f(z) = (z - z_k)^{-m_k} g(z)$ where g is holomorphic and nonzero in $B(p_k, \epsilon)$. Then

$$\frac{f'(z)}{f(z)} = -\frac{m_k}{z - z_j} + \frac{g'(z)}{g(z)}.$$

Consequently,

$$\frac{1}{2\pi i}\int_{\gamma_j} \frac{f'(z)}{f(z)}dz = -\frac{1}{2\pi i}\int_{\gamma'_k} \frac{m_k}{z - p_k}dz + \frac{1}{2\pi i}\int_{\gamma_j} \frac{g'(z)}{g(z)}dz = -m_k + 0 = -m_k.$$

Adding we obtain the formula.

For a slightly different proof, see page 123 in the book. □

Theorem 23. (Rouché's Theorem) Suppose f and g are meromorphic in a neighborhood of $\bar{B}(a, R)$ with no zeros or poles on the circle $\gamma = \{z \mid |z - a| = R\}$. If Z_f, Z_g, P_f, P_g are the number of zeros, respectively poles of f and g, inside γ counted with multiplicity, and if

$$|f(z) + g(z)| < |f(z)| + |g(z)| \text{ for all } z \in \gamma,$$

then

$$Z_f - P_f = Z_g - P_g.$$

Proof. Page 125. □

The Rouche's theorem has a beautiful consequence.

Theorem 24. Let a be a zero of multiplicity m of f such that f has no other zeros or poles in $B(a, \epsilon)$. Then there is $\delta > 0$ such that for every $\alpha \in B(a, \delta)$, $f(z) = \alpha$ has m zeros (multiplicities counted) in $B(a, \epsilon)$.

Proof. Let $R < \epsilon$. Since f is continuous, $t \mapsto |f(a + Re^{it})|$ has a minimum δ on $a + R^{it}$, $0 \leq t \leq 2\pi$. For $|\alpha| < \delta$, apply Rouche's theorem to f and $g = f - \alpha$. □

As a corollary we obtain:

Theorem 25. (Open Mapping Theorem) A holomorphic function maps open sets to open sets.

Proof. Let $w \in f(G)$. Then by the previous theorem there is $\delta > 0$ such that $B(w, \delta) \in f(G)$. Done. □

We conclude this section with a result about meromorphic functions on the Riemann sphere.

Theorem 26. Let $f : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ be a holomorphic function (in other words $f : \mathbb{C} \cup \{\infty\} \to \mathbb{C}$ is meromorphic). Then there exist polynomials $P(z)$ and $Q(z)$ such that

$$f(z) = \frac{P(z)}{Q(z)} \text{ for all } z.$$

Proof. Around every finite pole α_k, $k = 1, 2, \ldots, p$,

$$f(z) = \sum_{n \geq -m_k} a_{n,k}(z - \alpha_k)^n = R_k(z) + \sum_{n \geq 0} a_{n,k}(z - \alpha_k)^n = R_k(z) + g_k(z),$$

where R_k is rational, and $g_k(z)$ is in a neighborhood of α_k. Also, at infinity

$$f(z) = \sum_{n=-\infty}^{m_\infty} a_{n,\infty} z^n = R_\infty(z) + \sum_{n \leq 0} z^n = R_\infty(z) + g_\infty(z),$$

where $R_\infty(z)$ is a polynomial and g_k is holomorphic in a neighborhood of ∞. Then

$$f(z) - \sum_{k=1}^{p} R_k(z) - R_\infty(z)$$

is holomorpic on $\mathbb{C} \cup \{\infty\}$ and takes values in $\mathbb{C}$, so by Liouville's theorem, it is a constant function. Let c be its value. Then

$$f(z) = \sum_{k=1}^{p} R_k(z) + R_\infty(z) + c,$$

which is a rational function. □

4.3 Essential singularities

Definition. An isolated singularity is essential if it is neither a zero nor a pole.

Example 11. 0 is an essential singularity of $f(z) = e^{1/z}$. Indeed,

$$e^{1/z} = \sum_{n=-\infty}^{0} \frac{1}{n!} z^n.$$

This example prompts us to look at Laurent series. And as we will see, Laurent series cover all singularities.

Definition. A Laurent series is a series of the form

$$\sum_{n=-\infty}^{\infty} a_n z^n.$$

Theorem 27. (Laurent series development) Let f be analytic in the annulus $R_1 < |z-a| \leq R_2$ $(0 \leq R_1 < R_2 \leq \infty)$. Then

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n,$$

where the convergence is absolute and uniform in any annulus $r_1 \leq |z-a| \leq r_r$ with $R_1 < r_1 < r_2 < R_2$, and the coefficients are given by

$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)^{n+1}} dz, \quad \gamma(t) = a + re^{it}, \quad 0 \leq t \leq 2\pi, R_1 < r < R_2.$$

Proof. Denote $A = \{z \,|\, R_1 < |z-a| < R_2\}$. Choose ρ_1, ρ_2 such that $R_1 < \rho_1 < \rho_2 < R_2$, and define $\gamma_1(t) = a + \rho_1 e^{it}$, $\gamma_2 = a + \rho_2 e^{it}$, $0 \leq t \leq 2\pi$. Let $\gamma = \bar{\gamma}_1 \cup \gamma_2$ (where the bar means that we reverse orientation of the path). Then Cauchy's formula yields

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w-z} dw, \text{ for } \rho_1 < |z-a| < \rho_2.$$

On $r_1 < |z-a| < r_2$, define

$$f_1(z) = \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(w)}{w-z} dz, \quad f_2(z) = \frac{1}{2\pi i} \int_{\gamma_2} \frac{f(w)}{w-z} dz,$$

Note that f_1 and f_2 are holomorphic in $\rho_1 < |z-a| < \rho_2$ and $f = -f_1 + f_2$. Note that for $w \in \gamma_1$, $|w-a| < |z-a|$, while for $w \in \gamma_2$, $|w-a| > |z-a|$. Thus for $w \in \gamma_1$,

$$\frac{1}{w-z} = -\frac{1}{(z-a)-(w-a)} = -\sum_{n=0}^{\infty} \frac{(w-a)^n}{(z-a)^{n+1}} = -\sum_{n=-\infty}^{-1} \frac{(z-a)^n}{(w-a)^{n+1}},$$

and for $w \in \gamma_2$,

$$\frac{1}{w-z} = \frac{1}{(w-a)-(z-a)} = \sum_{n=0}^{\infty} \frac{(z-a)^n}{(w-a)^{n+1}}.$$

The two series converge uniformly and absolutely for $\rho_1 < r_1 \le |z-a| \le r_2 < \rho_2$, and using Fubini (summation commutes with integration) we have

$$f(z) = \sum_{n=-\infty}^{-1} \left(\frac{1}{2\pi i} \int_{\gamma_1} \frac{f(w)}{(w-a)^{n+1}} dw \right) (z-a)^n + \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{\gamma_1} \frac{f(w)}{(w-a)^{n+1}} dw \right) (z-a)^n,$$

as desired. (Note that in the first sum the two minus signs in the first sum, one from $-f_1$ and one from the series expansion, cancel.) $\square$

So a singularity is removable if for all negative indices n, $a_n = 0$, is a pole if for all but finitely many negative indices n, $a_n = 0$, and essential if for infinitely many negative indices n, $a_n \neq 0$.

Theorem 28. (Casorati-Weierstrass theorem) If the holomorphic function f has an essential singularity at a then for every $\delta > 0$, $f(\{z \,|\, 0 < |z-a| < \delta\})$ is dense in $\mathbb{C}$.

Proof. Assume that for some holomorphic function f and some annulus $A = \{z \,|\, 0 < |z-a| < \delta\}$ this is not true. Then there is a number $c \in \mathbb{C}$ and some $\epsilon > 0$ such that $|f(z) - c| > \epsilon$ for all $z \in A$. Then $1/(f-c)$ is bounded in A, so a is a removable singularity for it. This means that either a is a removable singularity for $f-c$ or a is a pole of $f-c$ (if when you remove the singularity $1/(f-c)$ is extended to have value 0 at a). Consequently a is either a removable singularity or a pole of f itself, a contradiction. The conclusion follows. $\square$

4.4 Residues

4.4.1 How to find residues

If f has an isolated singularity at a, let its Laurent series expansion around a be

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n.$$

The residue of f is $\mathrm{Res}(f;a) = a_{-1}$.

Theorem 29. (Residue Theorem) Let f be analytic in the open set G except for the isolated singularities $\alpha_1, \alpha_2, \dots, \alpha_m$. If γ is a closed rectifiable curve in G which does not pass through any of the singularities of f, and if γ bounds an open set in G to its left then

$$\frac{1}{2\pi i} \int_\gamma f(z) dz = \sum_{k=1}^{n} n\mathrm{Res}(f;\alpha_k).$$

More generally, if $\gamma = 0 \in H_1(G, \mathbb{R})$, then

$$\frac{1}{2\pi i} \int_\gamma f(z) dz = \sum_{k=1}^{n} n(\gamma; a_k)\mathrm{Res}(f;\alpha_k).$$

Proof. page 112, 118 $\square$

Methods to find the residue:

- If f has a pole of order m at a. Let $g(z) = (z-a)^m f(z)$. Then

$$\mathrm{Res}(f;a) = \frac{1}{(m-1)!} g^{(m-1)}(a).$$

- If f has a simple pole at a,

$$\mathrm{Res}(f;a) = \lim_{z\to a}(z-a)f(z).$$

- If f has a double pole at a, then $a_{-2} = \lim_{z\to a}(z-a)^2 f(z)$ and

$$\mathrm{Res}(f;a) = \lim_{z\to a}(z-a)\left(f(z) - \frac{a_{-2}}{(z-a)^2}\right)$$

  Alternatively, $\mathrm{Res}(f;a)$ is the value at a of the function

$$\frac{d}{dz}[(z-a)^2 f(z)].$$

- If f, g are analytic, and g has a simple zero at a, then

$$\mathrm{Res}\left(\frac{f}{g};a\right) = \frac{f(a)}{g'(a)}.$$

4.4.2 Computations of integrals using residues

Example 12. Let us compute

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2}.$$

Let $R>0$, and consider the contour $\gamma = \gamma_1 \cup \gamma 2$, where $\gamma_1(t) = t$, $-R \le t \le R$, and $\gamma_2(t) = Re^{it}$, $0 \le t \le \pi$. Note that γ bounds a semidisk, and inside γ there is only one isolated singularity i, which is a pole of order 1. Thus

$$2\pi i a_{-1} = \int_\gamma \frac{1}{1+z^2}dz = \int_{\gamma_1} \frac{dz}{1+z^2} + \int_{\gamma_2} \frac{dz}{1+z^2}.$$

Set $f(z) = 1$, and $g(z) = 1+z^2$. Then

$$\mathrm{Res}\left(\frac{f}{g};i\right) = \frac{f(i)}{g'(i)} = \frac{1}{2i}.$$

Also, note that

$$\lim_{R\to\infty}\int_{\gamma_2}\frac{dz}{1+z^2} = \lim_{R\to\infty}\int_0^\pi \frac{Re^{it}}{1+R^2e^{2it}}dt = \lim_{R\to\infty}\frac{1}{R}\int_0^\pi \frac{e^{it}}{\frac{1}{R}+e^{2it}}dt = 0.$$

Hence

$$\int_{-\infty}^{\infty}\frac{dx}{1+x^2} = \int_\gamma \frac{dz}{1+z^2} = 2\pi i \frac{1}{2i} = \pi.$$

Example 13. Evaluate

$$\int_0^{2\pi} \frac{1}{a+\cos\theta} d\theta, \quad a > 1.$$

Set $z = e^{i\theta}$, then $\cos\theta = \frac{z+z^{-1}}{2}$, and $d\theta = \frac{1}{ie^{i\theta}} dz = \frac{dz}{iz}$. The integral becomes

$$\frac{2}{i} \int_\gamma \frac{1}{z^2+2az+1} dz,$$

where γ is the unit circle. The only singularity inside the unit circle is $-a+\sqrt{a^2-1}$, which is a simple pole whose residue is obtained by setting $f=1$, $g = z^2+az+1$, and then

$$\mathrm{Res}\left(\frac{f}{g}; -a+\sqrt{a^2-1}\right) = \frac{f(-a+\sqrt{a^2-1})}{g'(-a+\sqrt{a^2-1})} = \frac{1}{2\sqrt{a^2-1})}.$$

So the value of the integral is

$$\frac{2}{i} \int_\gamma \frac{1}{z^2+2az+1} dz = \frac{2}{i} 2\pi i \frac{1}{2\sqrt{a^2-1}} = \frac{2\pi}{\sqrt{a^2-1}}.$$

As a corollary we obtain

$$\int_0^{2\pi} \frac{1}{w+\cos\theta} d\theta = \frac{2\pi}{\sqrt{w^2-1}}, \quad w \in \mathbb{C}\backslash[-1,1]$$

where $\sqrt{w^2-1}$ is defined so that it is positive on the real axis. Indeed, both sides are holomorphic functions and they coincide on the set $\mathbb{R}\backslash[-1,1]$ which contains accumulation points, so they must coincide everywhere.

Example 14. Let us compute

$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2} dx.$$

We compute instead

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{1+x^2} dx.$$

Now let $f(z) = \frac{e^{iz}}{1+z^2}$, and consider the contour $\gamma_1 \cup \gamma_2$, where $\gamma_1(t) = t$, $-R \le t \le R$ and $\gamma_2(t) = Re^{it}$, $0 \le t \le \pi$. Note that on $\gamma_2(t)$, $|f(z)| \le 1/(R^2-1)$, so

$$\int_{\gamma_2} f(z) dz \to 0, \text{ when } R \to \infty$$

On the other hand,

$$\int_{\gamma_1 \cup \gamma_2} f(z) dz = 2\pi i \mathrm{Res} f(i) = 2\pi i \frac{e^{-1}}{2i} = \frac{\pi}{e}.$$

Letting $R \to \infty$ we obtain that

$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2} dx = \frac{\pi}{e}.$$

Example 15. Compute

$$\int_0^1 \frac{dx}{\sqrt{x(1-x)}} dx.$$

Note that this is an improper integral. Let $\epsilon, \theta > 0$ be small, and consider the contour which is the union of the curves:

$$\begin{aligned}
&\gamma_1(t) = t + \epsilon \sin\theta, \epsilon\cos\theta \le t \le 1 - \epsilon\cos\theta, \\
&\gamma_2(t) = 1 + \epsilon e^{-it}, -\pi + \theta \le t \le \pi - \theta, \\
&\gamma_3(t) = 1 - t - \epsilon \sin\theta, \epsilon\cos\theta \le t \le 1 - \epsilon\cos\theta, \\
&\gamma_4(t) = \epsilon e^{-it}, \theta \le t \le 2\pi - \theta.
\end{aligned}$$

Then $\gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$ is a closed contour. Consider the analytic function

$$f : \mathbb{C}\backslash[0,1] \to \mathbb{C}, f(z) = \frac{1}{\sqrt{z(1-z)}},$$

Note that f is analytic at infinity, meaning that at infinity it has a Laurent series expansion

$$\sum_{n \le -1} a_n z^n.$$

By letting $\epsilon, \theta \to 0$, we obtain that

$$\int_0^1 \frac{dx}{\sqrt{x(1-x)}} + (-1)\int_1^0 \frac{dx}{\sqrt{x(1-x)}} = 2\pi i \mathrm{Res}(f;\infty).$$

Note that the -1 factor in front of the second integral comes from that fact that we pick a phase of -1 in the square root as we go around the zero. Note also that

$$\frac{1}{\sqrt{z(1-z)}} = \frac{1}{z}\left(\frac{1}{z} - 1\right)^{-1}.$$

The residue at ∞ is

$$\lim_{z\to\infty} z\frac{1}{z}\left(\frac{1}{z} - 1\right)^{-1/2}.$$

This is $-i$ because we pick the branch of the square root which has $\sqrt{-1} = i$. Thus the integral is

$$\int_0^1 \frac{dx}{\sqrt{x(1-x)}} = \frac{1}{2} 2\pi i(-i) = \pi.$$

Remark 5. Given this example, we point out that the residue at infinity of $f(z)$ is the residue at zero of $-\frac{1}{z^2} f(1/z)$. This is because if γ_1 is oriented clockwise, and we let $\gamma_2(z) = 1/\gamma_1(z)$, then γ_2 is oriented counterclockwise. The residue of f at ∞ is

$$\frac{1}{2\pi i}\int_{\gamma_1} f(z)dz$$

and by changing the variable $z = 1/w$,

$$\frac{1}{2\pi i}\int_{\gamma_1} f(z)dz = \frac{1}{2\pi i}\int_{\gamma_2} f(1/w)(-1/w^2)dw,$$

which is the residue at 0 of $f(1/z)(-1/z^2)$.

Example 16. Compute

$$\int_1^\infty \frac{dx}{x\sqrt{x^2-1}}.$$

First, note that the integral converges absolutely, namely that

$$\lim_{a\to 1+}\lim_{b\to\infty}\int_a^b \frac{dx}{x\sqrt{x^2-1}}$$

exists. Now consider a contour that runs around -1 clockwise, runs parallel and close to $[-1,-R]$, funs clockwise on $|z|=R$ until close to R, runs parallel and close to $[R,1]$, runs clockwise around 1, runs parallel and close to $[1,R]$, then runs counterclockwise from R to $-R$ on $|z|=R$, and finally returns to -1 running parallel to $[-R,-1]$.

Now use the branch cut of $\sqrt{z}$ that removes the positive real semiaxis. This choice allows us to define $f(z)$ on $\mathbb{C}\backslash((-\infty,-1]\cup[1,\infty))$. The only singularity of f inside the contour is $z=0$, and f runs counterclockwise around it. We compute $\mathrm{Res}(f;0)=-i$.

Consequently the residue theorem gives us that the integral of f on the contour is 2π. The integral on the countour is four times the integral that we are computing, so the integral we are computing is $\pi/2$.

Proposition 12. The Gaussian integral formula holds:

$$\int_{-\infty}^{\infty} e^{-t^2}dt=\sqrt{\pi}.$$

Proof. (cf. R. Remmert, *Theory of Complex Functions*, Springer) Set

$$f(z)=\frac{e^{-z^2}}{1+e^{-2az}},\quad a=(1+i)\sqrt{\frac{\pi}{2}}=\sqrt{\pi}e^{i\pi/4}.$$

It is not hard to see that $f(z)-f(z+a)=e^{-z^2}$, and that the poles of f are located at $\frac{a}{2}+na$, $n\in\mathbb{Z}$. Moreover, we can see that $g(z)=1+e^{-2az}$ has a simple zero at $a/2$ because $g'(a/2)=2a$. Hence the residue of f at $z=a/2$ is

$$\lim_{z\to a}\frac{z-a/2}{e^{-2az}-(-1)}e^{-z^2}=\frac{1}{g'(a/2)}e^{-a^2/4}=\frac{e^{-i\pi/4}}{2\sqrt{\pi}e^{i\pi/4}}=\frac{1}{2\sqrt{\pi}}e^{-i\pi/2}=-\frac{i}{2\sqrt{\pi}}.$$

Let $r,s>0$, and integrate f on the parallelogram with corners $-r,s,s+a,-r+a$. For sufficiently large r,s this parallelogram f has only one residue at $a/2$, and it is $-i/(2\sqrt{\pi})$. Note also that the integral of f on the short sides of the parallelogram converges to zero. And integrating f on the long sides is the same as integrating the Gaussian on $(-r,s)$. Passing to the limit with $r,s\to\infty$ we obtain

$$\int_{-\infty}^{\infty} e^{-t^2}dt=2\pi i\mathrm{Res}_{a/2}f=2\pi i\frac{(-i)}{2\sqrt{\pi}}=\sqrt{\pi}.$$

□

Corollary 3. The Fresnel integral formulas hold:

$$\int_0^\infty \cos t^2dt=\int_0^\infty \sin t^2dt=\frac{\sqrt{2\pi}}{4}.$$

Proof. Consider the contour γ consisting of the union of the curves $\gamma_1(t) = t$, $0 \leq t \leq R$, $\gamma_2(t) = Re^{it}$, $0 \leq t \leq \pi/4$ and $\gamma_3(t) = (R-t)e^{i\pi/4}$, $0 \leq t \leq R$. Note that $\int_\gamma e^{-z^2}dz = 0$, because the integrand is holomorphic inside the contour. Also, $\lim_{R\to\infty} \int_{\gamma_2} e^{-z^2}dz = 0$. So

$$\int_0^\infty e^{-t^2}dt = \int_0^\infty e^{-it^2}e^{i\pi/4}dt.$$

Consequently

$$\frac{\sqrt{\pi}}{2} = \int_0^\infty (\cos t^2 - i\sin t^2)\frac{1+i}{\sqrt{2}}dt.$$

Equating the real and imaginary parts and solving the system implies that each of the Fresnel integrals equals $\frac{\sqrt{2\pi}}{4}$. $\square$

4.5 Computation of integrals - a different perspective

I find it easier to understand the integrals from the previous section in the context of Riemann surfaces (for the definition of a Riemann surface, see Chapter 6.

First, the idea that we integrate functions is not quite correct. We integrate forms:

- we integrate 1-forms over curves;
- we integrate 2-forms over surfaces;
- in general, we integrate n-forms over n-dimensional manifolds.

Why is a form better than a function? Because it carries with it the information over the integration measure, which is a must when the curve, surface, etc. lives inside a more general space than the plane (such as a manifold).

Let us discuss just the case of 1-form on a Riemann surface X. In local coordinates, a 1-form looks like $f(z)dz$. But when you change from one coordinates system to another, that is from one chart $\phi_a : U_a \cap U_b \to \mathbb{C}$ to $\phi_b : U_a \cap U_b \to \mathbb{C}$, whith the change of coordinates $\phi = \phi_b \circ \phi_a^{-1}$, then the form changes by $f(\phi(z))\phi'(z)dz$. So the 1-form is defined everywhere (with maybe some singularities), and it has a concrete formula in every local chart, with this formula changing from one system of coordinates to another by

$$f(z)dz \mapsto f(\phi(z))d\phi(z) = f(\phi(z))\phi'(z)dz.$$

This formula is chosen so that the integral does not change when changing the local coordinates, since by the first substitution we have

$$\int_\gamma f(z)dz = \int_\gamma f(\phi(z))\phi'(z)dz.$$

In this formula we think of γ as the physical curve on X (and not of its formula in local coordinates, which changes when you change coordinates). Because of this we have a well defined integral of the form on any (compact) curve in X: simply decompose the curve into pieces such that each piece lies in a coordinate chart, integrate on each piece in local coordinates, and then sum the results.

Example 17. Compute

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+4)^2}.$$

We are supposed to compute the integral of the function

$$f : \mathbb{C} \to \mathbb{C}, \quad f(z) = \frac{1}{(z^2+4)^2}$$

over the real axis. We can add the point at infinity to the plane, and now we work on the Riemann sphere. Thus we have to integrate the 1-form

$$\alpha = f(z) = \frac{dz}{(z^2+4)^2}$$

over a circle of the Riemann sphere that passes through the point at infinity. There is one problem, the form is not yet defined at the point at infinity. Let us show that we can define it there, too. To work in local coordinates, use the chart

$$\phi_\infty : (\mathbb{C} \cup \{\infty\}) \backslash \{0\} \to \mathbb{C}, \phi_\infty(z) = \begin{cases} \frac{1}{z} & \text{if } z \in \mathbb{C}, \\ 0 & \text{if } z = \infty. \end{cases}$$

In local coordinates near ∞,

$$\omega = \frac{1}{((1/z)^2+4)^2} \frac{d}{dz}(1/z)dz = -\frac{z^2}{(4z^2+1)^2} dz.$$

This clearly can be extended to $z = 0$. So our 1-form is defined on the entire Riemann sphere. On each side it bounds a disk, and the value of the integral equals $2\pi i$ times the sum of the residues on one side. And these residues can be computed in local coordinates. We compute the integral as the sum of the residues that are on the left side of the countour, and this is the residue at $z = 2i$. This residue is

$$\frac{d}{dz}[(z-2i)^2 \frac{1}{(z-2i)^2(z+2i)^2}]|_{z=2i} = -\frac{2}{(z+2i)^3}|z = 2i = -\frac{2}{(4i)^3} = \frac{1}{32i}.$$

Hence

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+4)^2} = 2\pi i \frac{1}{32i} = \frac{\pi}{16}.$$

Chapter 5

The maximum modulus principle

5.1 The Maximum modulus theorem

Theorem 30. (The Maximum Modulus Theorem) Let G be a bounded open set in $\mathbb{C}$ and let f be a nonconstant function that is continuous on $\bar{G}$ and holomorphic in G. Then there is $a \in \partial G$ such that $|f(z)| \leq |f(a)|$ for all $z \in G$.

Proof. We will give two proofs:
1. Because f is continuous on $\bar{G}$, it has a maximum on $\bar{G}$. But this maximum cannot be in G because $f(G)$ is open in $\mathbb{C}$, so $|f(G)|$ is open in $\mathbb{R}$.
2. Because f is continuous on $\bar{G}$, it has a maximum on $\bar{G}$. Assume that there is a maximum $a \in G$. Choose $\epsilon > 0$ such that $\overline{B(a, \epsilon)} \in G$. Then for $\gamma(t) = a + \epsilon e^{it}$, using Cauchy's formula we have

$$f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - a} dz = \frac{1}{2\pi} \int_0^{2\pi} f(a + \epsilon e^{it}) dt.$$

So

$$|f(a)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(a + \epsilon e^{it})| dt \leq |f(a)|$$

with equality only if f is constantly equal on γ. But then the zeros of the holomorphic function $f(z) - f(a)$ contain a circle, so this function is identically zero, a contradiction. The conclusion follows. $\square$

Corollary 4. If f is analytic on the open set G and f has a maximum in G, then f is constant on G.

Corollary 5. Let f be a holomorphic function on $G \subset \mathbb{C} \cup \infty$. If there is $M > 0$ such that for every point a on the boundary of G in $\mathbb{C} \cup \infty$, $\limsup_{z \to a} |f(z)| \leq M$, then $|f(z)| \leq M$ for all $z \in G$.

As a corollary, we have the following result.

Theorem 31. (Schwarz's Lemma) Suppose $f : B(0, 1) \to \mathbb{C}$ is analytic with
(a) $|f(z)| \leq 1$ for all z;
(b) $f(0) = 0$.
Then $|f'(0)| \leq 1$ and $|f(z)| \leq |z|$ for all $z \in B(0, 1)$. Moreover if $|f'(0)| = 1$ or if $|f(z)| = |z|$ for some z, then there is $\theta \in \mathbb{R}$ such that $f(z) = e^{i\theta} z$.

Proof. Page 131. □

Theorem 32. The only analytic maps that map bijectively the open unit disk onto itself are of the form

$$f(z) = e^{i\theta}\frac{z-a}{-\bar{a}z+1}, \quad \theta \in \mathbb{R}, |a| < 1.$$

Proof. We have seen as a homework exercise that the above are the only Möbius transformations satisfying the property from the statement, and moreover that these maps extend to the closed unit disk, mapping the boundary to the boundary.

Consider the Möbius transformation

$$\phi_a(z) = \frac{z-a}{-\bar{a}z+1}.$$

For f an automorphism of the unit disk onto itself, as specified in the statement, let $g = f \circ \phi_{-a}$. Then g is also an automorphism of the unit disk and it satisfies $g(0) = 0$. If there is a z such that $|g(z)| = |z|$, then we are done, because by the Schwarz's lemma $g(z) = e^{i\theta}z$, and so f is a Möbius transformation. If $|g(z)| < z$ on $|z| < 1$, then, again by Schwarz's lemma,

$$|z| = |g^{-1}(g(z))| \leq |g(z)| < |z|,$$

impossible. We are done. □

Theorem 33. (Phragmén-Lindelöf) Let G be a simply connected region and let f be an analytic functoin on G. Suppose that there is an analytic function $\phi : G \to \mathbb{C}$ which never vanishes and is bounded on G. If M is a constant and if the boundary of G in the Riemann sphere can be partitioned into two sets A and B such that
(a) for every $a \in A$, $\limsup_{z\to a} |f(z)| \leq M$;
(b) for every $b \in B$ and for every $\eta > 0$, $\limsup_{z\to b} |f(z)||\phi(z)|^\eta \leq M$;
then $|f(z)| \leq M$ for all $z \in G$.

Proof. Let $|\phi(z)| \leq \kappa$ for all $z \in G$. Because G is simply connected, there is analytic branch of $\log \phi(z)$ on G. Hence $g(z) = \exp(\eta \log \phi(z))$ is an analytic branch of $\phi(z)^\eta$, and $|g(z)| = |\phi(z)|^n$. Define $F : G \to \mathbb{C}$, $F(z) = f(z)g(z)\kappa^{-\eta}$. Then $|F(z)| \leq |f(z)|$ for all $z \in G$. But then by the Maximum Modulus Theorem $|F(z)| \leq \max(M, \kappa^{-\eta}M)$ for all $z \in G$. Then

$$|f(z)| \leq |\kappa/\phi(z)|^\eta \max(M, \kappa^{-\eta}M).$$

Now let $\eta \to 0$ to get the conclusion. □

Chapter 6

Convergence and compactness in spaces of holomorphic functions

6.1 Constructing topologies on spaces of continuous and holomorphic functions

6.1.1 Some introductory remarks

We will first define a topology on the space of continuous functions on an open set, then we will embed the space of holomorphic functions in the space of continuous functions and consider the induced topology.

This topology will turn both the space of continuous functions and the space of holomorphic functions on an open set into a Fréchet space, and in particular into a metric space. The choice of the topology is standard, it is motivated by the choice of the topology on the space of continuous functions on a compact set, and by our discussion on uniform convergence of power series, which is a particular example of convergence of holomorphic functions.

Sorry folks, but we do need some point set topology and again we need some real, and even functional analysis. I have lecture notes for both on my web page in case you need more information. The main idea is that we treat functions as points in an infinite dimensional space and we have a notion of distance between two functions which allows us to address problems of convergence of sequences of functions.

We mostly care about holomorphic functions whose domain and range are subsets of $\mathbb{C}$, but we also care about maps between more general Riemann surfaces such as maps from the Riemann sphere into itself (like Möbius transformations) and maps from a subset of $\mathbb{C}$ to a Riemann surface.

The notion of convergence is a direct extension of that for power series, so we will see uniform convergence on closed bounded sets. It also extends the notion of convergence of a sequence of continuous functions on a closed bounded interval, which is usually defined using the sup norm (the L^∞ norm).

Since this course is suppose to help you build your skills for advancing in mathematics, I feel obliged to tell you the correct general framework in which convergence is phrased for spaces of functions.

6.1.2 Elements of topology; Riemann surfaces

Recall the notion of convergence of a sequence of complex numbers: $\lim z_n = z^*$ if for every $\epsilon > 0$, there is $N \in \mathbb{N}$ such that for $n \geq N$, $z_n \in B(z^*, \epsilon)$. The open disks of radius ϵ allow us to define the notion of neighborhood, namely what it means for z_n to be near z^*. In this class we have already encountered the notion of an open set in $\mathbb{C}$ as a set that can be written as a union of open disks. And it is not hard to see that we can rephrase the notion of convergence to say that $\lim z_n = z^*$ if for every open set G containing z^*, there is $N \in \mathbb{N}$ such that for $n \geq N$, $z_n \in G$. Note that arbitrary unions of open sets are open, and also finite intersections of open sets are open. The collection of all open sets is called a topology on $\mathbb{C}$. Together with open sets come closed sets, which are by definition the complements of open sets.

Definition. A topology on a set X is a collection of open sets such that

- X and $\emptyset$ are open;
- if $U_1, U_2, \ldots, U_n$ are open, then $\cap_{j=1}^n U_n$ is open;
- if U_α, $\alpha \in A$ are open, then $\cup_{\alpha \in A} U_\alpha$ is open.

Now we want to treat functions as points in a space and have a similar notion of convergence for functions (or maps), and for that we need open sets in the space of functions. In order for this to be a nice notion of convergence, the open sets should satisfy the same two properties: arbitrary unions of open sets are open, and also finite intersections of open sets are open. What kind of functions do we have in mind? First, let us discuss the playground where these function live.

Definition. A Riemann surface is a set X that can be written as a union of subsets $\{U_\alpha\}_{\alpha \in A}$ for which there exist bijective maps $\phi_\alpha : U_\alpha \to G_\alpha$ (with G_α an open set of $\mathbb{C}$) such that for every α, β such that $U_\alpha \cap U_\beta \neq \emptyset$,

$$\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

is holomorphic.

The maps $\phi_\alpha : U_\alpha \to G_\alpha$ are called coordinate charts, and together they form an atlas on X. In this class we only consider those Riemann surfaces that admit an atlas with *countably many charts*.

In other words, a Riemann surface is a set that locally looks like an open subset of $\mathbb{C}$.

Example 18. The Riemann sphere is a Riemann surface. Indeed, we let $U_0 = \mathbb{C}$, and $U_\infty = (\mathbb{C}\backslash\{0\}) \cup \{\infty\}$, with

$$\phi_0 : U_0 \to \mathbb{C}, \quad \phi_0(z) = z,$$
$$\phi_\infty : U_\infty \to \mathbb{C}, \quad \phi_\infty(z) = \begin{cases} \frac{1}{z} & \text{if } z \neq \infty \\ \infty & \text{if } z = \infty. \end{cases}$$

Then

$$\phi_0 \circ \phi_\infty^{-1} : \mathbb{C}\backslash\{0\} \to \mathbb{C}, \quad \phi_0 \circ \phi_\infty^{-1}(z) = \frac{1}{z}.$$

Example 19. The Riemann surface of the logarithm also satisfies these properties. For every point $w \neq 0$ we can define an open set U_w on this surface that is in bijection with the set of the form $G_a = \mathbb{C}\backslash\{\lambda a \,|\, \lambda \geq 0\}$. Compositions of maps $\phi_\beta \circ \phi_\alpha^{-1}$ are defined on $\mathbb{C}$ without two rays starting at the origin, and on each of the two connected components they are the identity map.

We declare any set of the form $\phi^{-1}(G)$ with $G \subset G_\alpha$ open in $\mathbb{C}$ to be open in X, and we consider arbitrary unions and finite intersections of such sets to be open. This defines a topology on X. The complements of open sets are called closed sets.

Now we the concepts of open and closed sets at hand, we can talk about compactness. Let X be a topological space, namely a set endowed with a collection of open sets that have the property that both X and the empty set are open, the finite intersecion of open sets is open, and the infinite union of open sets is open.

Definition. We say that K is compact in X if every family of open sets whose union contains K has a finite subfamily whose union still covers K.

This is phrased by saying that every open cover has a finite subcover.

Definition. We say that K is sequentially compact in X if every sequence in K contains a convergent subsequence.

We recall that a distance (or metric) on a set X is a function $d : X \to [0, \infty)$ such that $d(x, y) = 0$ if and only if $x = y$, $d(x, y) = d(y, x)$ and $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in X$. The set X endowed with this function is called a metric space. The topology on a metric space is the smallest topology in which all balls $B(x, \epsilon) = \{y \,|\, d(x, y) < \epsilon\}$ are open. Every open set is actually a union of balls (no need to worry about intersections).

Theorem 34. For metric spaces the two notions of compactness are equivalent.

Example 20. A set $K \subset \mathbb{C}$ is compact if and only if it is closed and bounded. This is the content of the Heine-Borel theorem.

Example 21. The Riemann sphere $\mathbb{C} \cup \{\infty\}$ is compact. Indeed, if $\{G_\alpha\}_\alpha$ is a family of open sets whose union is the Riemann sphere, then for some α_0, $\infty \in G_{\alpha_0}$. But G_{α_0} is open if and only if $\phi_\infty(G_{\alpha_0})$ is open in $\mathbb{C}$. It is not hard to see that this forces $(\mathbb{C} \cup \{\infty\}) \backslash G_{\alpha_0}$ to be both closed and bounded. So there are finitely many of the G_α that cover it, which together with G_{α_0} form a finite cover of the Riemann sphere.

For $\epsilon > 0$, we say that a set K in a metric space has an ϵ-net if there are $x_1, x_2, \ldots, x_n \in K$ such that every point x in K is at distance less than ϵ from one of $x_1, x_2, \ldots, x_n$.

Proposition 13. A closed set is compact if and only if for every $\epsilon > 0$ there is an ϵ-net.

Now we turn to maps and functions. We are interested in holomorphic maps between Riemann surfaces.

Definition. Let X and Y be Riemann surfaces defined respectively by the atlases $\phi_\alpha : U_\alpha \to G_\alpha$ and $\psi_\mu : V_\mu \to D_\mu$. A map $f : X \to Y$ is called holomorphic if $\psi_\mu \circ f \circ \phi_\alpha^{-1}$ is holomorphic whenever the composition makes sense.

Example 22. Recall holomorphic maps $f : G \subset \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$, which we have called meromorphic functions.

Holomorphic maps are a subset of the much larger set of continuous functions.

Definition. A function $f : X \to Y$ is called continuous if the preimage through f of every open set is open.

Now we have two spaces: that of continuous functions: $C(X, Y)$ and of holomorphic functions: $H(X, Y)$. It is not hard to see that $H(X, Y) \subset C(X, Y)$. We introduce now a standard topology on $C(X, Y)$ that will induce a topology on $H(X, Y)$.

Definition. The compact-open topology on $C(X,Y)$ is the smallest topology for which all sets of continuous functions of the form

$$\{f : X \to Y \mid f(K) \subset V\}, \quad K \text{ compact in } X, V \text{ open in } Y$$

are open.

Note that finite intersection of sets of this form are not necessarily of this form, but they are nevertheless postulated to be open. The induced topology on $H(X,Y)$ is the smallest topology such that all sets of holomorphic functions of the same form are open.

Proposition 14. Let X, Y be Riemann surfaces.
(i) Assume that X' is a Riemann surface such that $X' \subset X$, and let us consider the subset S of $C(X',Y)$ consisting of the restrictions of functions in $C(X,Y)$ to X'. Then the topology of S induced by the compact-open topology of $C(X,Y)$ (in which the open sets are the images of open sets in $C(X,Y)$) is finer than the topology induced by the inclusion of S into $C(X',Y)$. (This means that the first topology has more open sets than the second).
(ii) Assume that Y' is a Riemann surfaces such that $Y' \subset Y$. Then the compact-open topology of $C(X,Y')$ coincides with the topology induced by the inclusion $C(X,Y') \subset C(X,Y)$.

Proof. For (i) note that a compact set in X' is also compact in X. For (ii) note that a set is open in Y' if and only if it is open in Y. □

6.1.3 The case $f : X \to Y$ where Y is a subset of the Riemann sphere

All Riemann surfaces of interest in this class are naturally metric spaces, but the plane and the Riemann sphere have metrics that are easy to describe.

The plane is a one dimensional $\mathbb{C}$-vector space and it has a norm given by the absolute value. We recall that a norm on a $\mathbb{C}$-vector space V is a function $|\cdot| : V \to [0,\infty)$ such that $|v| = 0$ if and only if $v = 0$; $|\lambda v| = |\lambda||v|$ for every scalar $\lambda \in \mathbb{C}$ and $|v + w| \leq |v| + |w|$. The norm on $\mathbb{C}$ defines a distance function by $d(z,w) = |z - w|$.

We can define on the Riemann sphere the following distance

$$\rho : \mathbb{C} \cup \{\infty\} \to [0,1], d(z,w) = \begin{cases} \dfrac{|z-w|}{1+|z-w|}, & \text{if } z, w \in \mathbb{C} \\ 1 & \text{if } z = \infty \text{ or } w = \infty. \end{cases}$$

- If X is a compact Riemann surface, such as the Riemann sphere, and $Y = \mathbb{C}$ we can turn $C(X,Y)$ and implicitly $H(X,Y)$ into normed vector spaces by using the norm

$$\|f\| = \sup_{z \in X} |f(z)|.$$

  This norm introduces a notion of distance: $\rho(f,g) = \|f - g\|$, and the distance introduces a topology on $C(X,Y)$ (and hence on $H(X,Y)$) where the open sets are unions of balls of the form $B(f,r) = \{g \mid \|f - g\| < r\}$. This turns out to coincide with the compact-open topology. This topology turns $C(X,Y)$ into a Banach space, meaning that every Cauchy sequence is convergent. We will see that $H(X,Y)$ is also a Banach space.

- If X is compact and Y is the Riemann sphere, or any open subset of it, then $C(X,Y)$ and $H(X,Y)$ are metric spaces with the metric

$$\rho(f,g) = \sup_{z \in X} d(f(z), g(z)).$$

Again it is standard that $C(X,Y)$ is a complete metric space, meaning that every Cauchy sequence is convergent. But we will see that $H(X,Y)$ is also complete.

- If X can be written as a countable union of compact subsets: $X = \cup_{n=1}^{\infty} K_n$, and $Y = \mathbb{C}$, then $C(X,Y)$ and $H(X,Y)$ each are endowed with a countable collection of what are called seminorms (like norms except that $\|f\|$ can be zero without necessarily f being zero). These are defined by

$$\|f\|_n = \sup_{z \in K_n} |f(z)|.$$

There is a metric on $C(X,Y)$ and $H(X,Y)$ defined by

$$\rho(f,g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|f-g\|_n}{1+\|f-g\|_n}.$$

Again $C(X,Y)$ is complete, and we will see that $H(X,Y)$ is also complete. These two linear spaces are therefore what we call Fréchet spaces.

- If X can be written as a countable union of compact subsets: $X = \cup_{n=1}^{\infty} K_n$, and Y is the Riemann sphere or an open subset of it, then $C(X,Y)$ and $H(X,Y)$ are endowed with a metric defined by

$$\rho(f,g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \sup_{z \in K_n} d(f(z), g(z)).$$

Again $C(X,Y)$ is complete, and we will see that $H(X,Y)$ is also complete.

Here we consider the case where X is compact, or can be covered in a nice way by a sequence of compact sets (such as when X is $\mathbb{C}$ or an open subset of $\mathbb{C}$) and Y a metric space, such as $\mathbb{C}$, an open subset of $\mathbb{C}$, or the Riemann sphere. As we have seen above, $C(X,Y)$ admits a metric. But it also has the compact-open topology. In this section we will prove that the two topologies: the one induced by the metric, and the compact-open topology coincide. We thus conclude that the compact-open topology is metrizable.

Proposition 15. Let G be an open subset of $\mathbb{C}$ (which can be $\mathbb{C}$ itself. There is a sequence $\{K_n\}_n$ of compact subsets of G such that
(a) $G = \cup_{n=1}^{\infty} K_n$;
(b) $K_n \subset \mathrm{int} K_{n+1}$;
(c) Every component of $(\mathbb{C} \cup \{\infty\}) \backslash K_n$ contains a component of $(\mathbb{C} \cup \{\infty\}) \backslash G$.

Here $\mathrm{int} K$ is the interior of K namely the largest open set contained in K.

Proof. Page 143. □

Corollary 6. If $\{K_n\}_n$ is such a sequence of compact sets, then for every compact set $K \subset G$ there is an n such that $K \subset K_n$.

Proof. Indeed, the interiors of K_n cover K so finitely many of them cover K. As they are nested, one of them covers K. □

Here is a first consequence of this construction.

Proposition 16. The compact-open topology on $C(X,Y)$ is generated by countably many open sets. In other words, there are countably many open subsets of $C(X,Y)$ such that any other open set is an arbitrary union of finite intersections of such sets.

Proof. Let $\phi_\alpha : U_\alpha \to G_\alpha$, $\alpha \in A \subset \mathbb{N}$ and $\psi_\mu : V_\mu \to D_\mu$, $\mu \in B \subset \mathbb{N}$ be atlases of X and Y, respectively. For each G_α, consider the compact sets K_n^α provided by Proposition 15. For each μ, let B_n^μ be a countable collection of open subsets of D_μ such that any other open set of D_μ is a union of some of the B_n^μ (in other words a countable basis). Then the family of open sets in $C(X,Y)$,

$$\{f : X \to Y \,|\, f(K) \subset V\},$$

where K ranges over all finite unions of sets of the form $\phi_\alpha^{-1}(K_n^\alpha)$ and V ranges over all arbitrary unions (which must be countable unions) of sets of the form $\psi_\mu^{-1}(B_n^\mu)$ is a countable union of sets with this property.

Indeed, if K' is a compact set in X, we will show that K' lies inside of such a compact K. Indeed, being compact it lies in finitely many charts, $\phi_{\alpha_1}, \phi_{\alpha_2}, \ldots, \phi_{\alpha_k}$. Let us prove the property by induction on k. If $k = 1$, we are done by the Proposition. If not, the set $K \backslash (U_{\alpha_1} \cup \cdots \cup U_{\alpha_{k-1}})$ is compact (being a closed subset of a compact set). It therefore lies in the interior of some $\phi^{-1}(K_n^{\alpha_k})$. But $K \backslash \mathrm{int}\phi^{-1}(K_n^{\alpha_k})$ is compact, and lies in $k-1$ charts, thus it is inside a compact of the desired form. Add $\phi^{-1}(K_n^{\alpha_k})$ to this compact to conclude that K' itself lies inside such a compact K.

On the other hand, it is clear that any open set of Y is in the union of open sets of the described form, because this is how we define the topology of Y. □

Corollary 7. Every point in $C(X,Y)$ has a countable system of neighborhoods, meaning that for every f in $C(X,Y)$ there is a countable family of open sets $\mathcal{O}_n, n \geq 1$ such that for every open set $\mathcal{O}$ containing f, there is n such that $\mathcal{O}_n \subset \mathcal{O}$.

The sequence of compact sets constructed above allows us to define a metric ρ on $C(G,Y)$ when G is an open subset of $\mathbb{C}$ and Y is a metric space. We will now prove that the metric topology and the compact-open topology coincide.

Lemma 4. For very $\epsilon > 0$ there is $\delta > 0$ and $K \subset G$ compact such that for $f, g \in C(G,Y)$

$$\sup\{d(f(z), g(z)) \,|\, z \in K\} < \delta \text{ implies } \rho(f,g) < \epsilon.$$

Conversely, if $\delta > 0$ and a compact set K are given, there is $\epsilon > 0$ such that for $f, g \in C(G,Y)$,

$$\rho(f,g) < \epsilon \text{ implies } \sup\{d(f(z), g(z)) \,|\, z \in K\} < \delta.$$

Proof. Page 144. □

Proposition 17. (a) A set $\mathcal{O} \subset C(G,Y)$ is open if and only if for each f in $\mathcal{O}$, there is $K \subset G$ compact and $\delta > 0$ such that

$$\{g \,|\, d(f(z), g(z)) < \delta, z \in K\}.$$

(b) A sequence $\{f_n\}_n$ in $C(G,Y)$ converges to f if and only if it converges to f uniformly on all compact subsets of G.
Consequently, the compact-open topology is metrizable.

Note that the metric depends on the sequence of compacts that cover X, but the topology that it defines is independent of it.

An important observation is that the sets

$$B_K(f,\epsilon) = \{g \mid \sup_{z\in K} d(f(z),g(z)) < \epsilon\}$$

play the same role that the ϵ-neighborhoods (i.e. $B(z,\epsilon)$ play in $\mathbb{C}$).

Now let X be an arbitrary Riemann surface, and Y a Riemann surface that is a subset of the Riemann sphere, and thus Y is a metric space. We can then define a metric on $C(X,Y)$ as well. Let $\phi_m : U_m \to G_m$, $m \in \mathbb{N}$ be an atlas of X. For each G_m, consider the compact sets K_n^m provided by Proposition 15. Define the compacts

$$\mathbf{K}_m = \cup_{j=1}^m \phi_j^{-1}(K_m^m).$$

One can show that for every compact subset K of X, there is $m \in \mathbb{N}$ such that $K \subset \mathbf{K}_m$. Then the metric

$$\rho(f,g) = \sum_{m=1}^{\infty} \frac{1}{2^m} \frac{\sup_{z\in \mathbf{K}_m} d(f(z),g(z))}{1+\sup_{z\in \mathbf{K}_m} d(f(z),g(z))}$$

defines the same topology as the compact-open topology. The proof is similar to that of Lemma 4.

Theorem 35. The space $C(X,Y)$ is a complete metric space.

Proof. If $\{f_n\}_n$ is Cauchy, then its restriction to any compact set K is Cauchy. Then the sequence $f_n|K$ is pointwise convergent, and being on a compact set, it is uniformly convergent. But then its limit is a continuous function. □

Definition. A set $\mathcal{F} \subset C(G,Y)$ is normal if its closure is compact.

Proposition 18. A set $\mathcal{F} \subset C(G,Y)$ is normal if and and only if for every compact set $K \subset G$ and $\delta > 0$ there are functions $f_1, f_2, \ldots, f_n \in \mathcal{F}$ such that for every $f \in \mathcal{F}$ there is at least one k, $1 \le k \le n$ with

$$\sup_{z\in K} d(f(z), f_k(z)) < \delta.$$

Proof. Since the closure of $\mathcal{F}$ is compact, for every $\epsilon > 0$, $\mathcal{F}$ has an ϵ-net. Now use Lemma 4. □

Definition. A set $\mathcal{F} \subset C(G,Y)$ is equicontinuous at a point z_0 in G if for every $\epsilon > 0$ there is $\delta > 0$ such that for $|z - z_0| < \delta$, $\rho(f(z), f(z_0)) < \epsilon$ for every $f \in \mathcal{F}$.

Theorem 36. (Arzela-Ascoli) A set $\mathcal{F} \subset C(G,Y)$ is normal if and only if the following two conditions are satisfied:
(a) for each $z \in G$, $\{f(z) \mid f \in \mathcal{F}\}$ has compact closure in Y.
(b) $\mathcal{F}$ is equicontinuous at every point in G.

Proof. Pages 149-150. □

All the above considerations apply when X has such a sequence of compact sets (and this is the case for the Riemann surface of a function $w(z)$ defined implicitly by a polynomial equation $R(z,w) = 0$). If X itself is compact, the sup norm already defines a metric on $C(X,Y)$, and in this metric $C(X,Y)$ is complete. As such, $C(X,Y)$ is a Banach space.

6.2 Spaces of analytic functions

6.2.1 Closeness of $H(X,Y)$

Let us consider the general case of X, Y Riemann surfaces, and let us look at the subspace $H(X,Y)$ of $C(X,Y)$. As we have seen above, the topology of $C(X,Y)$, and hence of $H(X,Y)$, is generated by countably many open sets. Consequently, every point in $H(X,Y)$ has a countable system of neighborhoods.

Theorem 37. $H(X,Y)$ is a closed subspace of $C(X,Y)$.

Proof. Because in the topology of $C(X,Y)$ every point has a countable system of neighborhoods, it suffices to show that every sequence in $H(X,Y)$ that converges in $C(X,Y)$ converges in $H(X,Y)$. So let us assume that the sequence of holomorphic maps $f_n : X \to Y$ converges to a continuous function f in the compact-open topology, and let us show that f is holomorphic.

Let $p \in X$ be an arbitrary point. We want to show that f is holomorphic in a neighborhood of p. Choose charts $\phi : U \subset X \to \mathbb{C}$ and $\psi : V \subset Y \to \mathbb{C}$ such that $f(U) \subset V$, and choose a neighborhood W of p such that $K = \bar{W}$ is a compact subset of U. Let $\mathcal{O} = \{g \mid g(K) \subset V\}$. Then there is N such that for $n \geq N$, $f_n \in \mathcal{O}$, that is $f_n(K) \subset V$. Set $W' = \phi(W)$. Because f_n converges to f in the compact-open topology, $f_n|K$ converges to $f|K$ in the compact-open topology. But then the sequence of holomorphic functions $h_n = \psi \circ f_n \circ \phi^{-1} : W' \to \mathbb{C}$, $n \geq 1$ converges in the compact-open topology to $h = \psi \circ f \circ \phi^{-1} : W' \to \mathbb{C}$. By Lemma 4 $\{h_n\}$ converges uniformly to h. We will show that h is holomorphic.

Let T be a triangle in W'. Then by Cauchy's theorem,

$$\int_T h_n(z)dz = 0.$$

Then

$$0 = \lim_{n\to\infty} \int_T h_n(z)dz = \int_T \lim_{n\to\infty} h_n(z)dz = \int_T h(z)dz.$$

So $\int_T h(z)dz = 0$ for any triangle T in W'. By Morera's theorem, h is holomorphic in W'. Thus, by definition, f is holomorphic in W. We are done. □

Remark 6. Note that the situation is quite the opposite from real analysis. The analytic functions on an interval of the real axis are dense in the space of continuous functions, as a consequence of Weierstrass' Theorem which shows that polynomials are dense in the space of continuous functions in the compact-open topology.

Theorem 38. If Y is a Riemann surface that is a subset of the Riemann sphere, then $H(X,Y)$ is a complete metric space.

Proof. By Theorem 35, $C(X,Y)$ is a complete metric space. By Theorem 37, $H(X,Y)$ is a closed subspace of $C(X,Y)$. Thus $H(X,Y)$ is a complete metric space. □

In all that follows G is an open subset of $\mathbb{C}$.

Theorem 39. If $f_n \to f$ in the compact-open (i.e. uniformly in the metric) topology of $H(G,\mathbb{C})$ then $f_n' \to f'$ in the compact-open topology of $H(G,\mathbb{C})$.

Proof. Let $\bar{B}(a,r) \subset G$. Choose $R > r$ such that $\bar{B}(a,R) \subset G$. If $\gamma(t) = a + Re^{it}$, then Cauchy's integral formula gives

$$f_n'(z) - f'(z) = \frac{1}{2\pi i}\int_\gamma \frac{f_n(w) - f(w)}{(w-z)^2} dz.$$

Hence

$$|f_n'(z) - f'(z)| \leq \frac{R \sup_{w\in\gamma} |f_n(w) - f(w)|}{(R-r)^{k+1}}, \quad |z-a| \leq r.$$

Since f_n converges uniformly to f on the curve γ (which is compact), f_n' converges uniformly to f' on $|z-a| \leq r$. By Theorem 38, f' is holomorphic. □

Corollary 8. If $f_n : G \to \mathbb{C}$ converges to f in the compact-open topology, then $f_n^{(k)}$ converges to $f^{(k)}$ in the compact-open topology. In other words, if $f_n : G \to \mathbb{C}$ converges to f uniformly on compact sets, then $f_n^{(k)}$ converges to $f^{(k)}$ uniformly on compact sets. In particular, if $f_n : G \to \mathbb{C}$ and $\sum_{n=1}^{\infty} f_n(z)$ converges uniformly on compact sets to $f(z)$ then

$$f^{(k)}(z) = \sum_{n=1}^{\infty} f_n^{(k)}(z).$$

Note that this last fact is a generalization of the result about differentiating a power series.

Theorem 40. (Hurwitz's Theorem) Let $G \subset \mathbb{C}$ be open and suppose that the sequence f_n in $H(G,\mathbb{C})$ converges to f. If f is not identically equal to zero, and if $\bar{B}(a,R) \subset G$ and $f(z) \neq 0$ for $|z-a| = R$, then there is an integer N such that for $n \geq N$, f and f_n have the same number of zeros in $B(a,R)$.

Proof. Page 152. □

Corollary 9. If $f_n \in H(G,\mathbb{C})$ converges to $f \in H(G,\mathbb{C})$, and if each f_n never vanishes, then f is either identically equal to zero, or it is never zero. Indeed, if f had an isolated zero a, then by Hurwitz's Theorem, for sufficiently large n, f_n would have an isolated zero in some neighborhood of a. But this does not happen. So either f is never zero or it is identically equal to zero.

Let X be a Riemann surface. Let $f_\infty : X \to \mathbb{C} \cup \{\infty\}$, $f_\infty(z) = \infty$.

Proposition 19. $H(X,\mathbb{C}) \cup \{f_\infty\}$ is a closed subset of $H(X, \mathbb{C} \cup \{\infty\})$. In other words a sequence of holomorphic functions converges in the compact-open topology of the space of meromorphic functions either to a holomorphic function or to a function that is constantly equal to ∞.

Proof. Let us assume that $f \neq f_\infty$. We will show that f is holomorphic. It suffices to show that f is holomorphic in every chart, so let $\phi : U \to G$ be a chart. Define $g = f \circ \phi^{-1}$ and $g_n = f_n \circ \phi^{-1}$, $n \geq 1$. Because $f_n \to f$ in the compact-open topology of $H(X, \mathbb{C} \cup \{\infty\})$, $g_n \to g$ in the compact-open topology of $H(G, \mathbb{C} \cup \{\infty\})$.

The inversion $z \mapsto 1/z$ is an automorphism of the Riemann sphere, so it maps compact sets to compact sets and open sets to open sets. Since $g_n \to g$ in the compact-open topology, $1/g_n \to 1/g$ in the compact-open topology of $H(G, \mathbb{C} \cup \{\infty\})$. Since all function g_n are analytic, the holomorphic function $1/g_n$ is never zero.

Assume that $(1/g)(a) = 0$. Because zeros and poles are isolated, there is a disk $B(a,R)$ such that a is the only zero of $1/g$ in $\overline{B(a,R)}$, and $1/f$ has no poles in $\overline{B(a,R)}$. Because $1/g_n$ converges to $1/g$ in the metric of $H(G, \mathbb{C}\cup\{\infty\})$ (as the metric and the compact-open topology are the same), there is N such that for $n \geq N$, $1/g_n$ has no poles in $\overline{B(a,R)}$ either.

We therefore have a sequence of holomorphic functions $1/g_n$ on $B(a,R)$ that converges to the holomorphic function $1/g$ in the topology of $H(B(a,R), \mathbb{C}\cup\{\infty\})$. But notice that the metrics $d(z,w) = |z-w|$ and $d'(z,w) = \frac{|z-w|}{1+|z-w|}$ induce the same topology on $\mathbb{C}$. Consequently, if we work only with holomorphic functions, the compact-open topology of $H(B(a,R), \mathbb{C})$ and the topology induced by the compact-open topology on $H(B(a,R), \mathbb{C}\cup\{\infty\})$ are the same. Consequently, $1/g_n$ converges to $1/g$ in $H(B(a,R), \mathbb{C})$. Now we can apply Hurwitz' Theorem and concluded that for large n, $1/g$ and $1/g_n$ have the same number of zeros in $B(a,R)$. But this contradicts the fact that $1/g_n$ has no zeros. Hence $1/g$ has no zeros either, showing that g is holomorphic. Hence f is holomorphic in the local chart U, and consequently it is holomorphic everywhere. □

Remark 7. Convergence in $H(X, \mathbb{C})$ and $H(X, \mathbb{C}\cup\infty)$ are not the same thing. For example $f_n(z) = n$ does not converge in the first topology, but it converges to f_∞ in the second. What we have used in the proof is that if $f_n \to f$ in $H(X, \mathbb{C}\cup\{\infty\})$ and if $f_n, f \in H(X, \mathbb{C})$ then $f_n \to f$ in $H(X, \mathbb{C})$. At the heart of this lies the fact that if $f \in H(X, \mathbb{C})$, then in $H(X, \mathbb{C}\cup\{\infty\})$ f has a countable system of neighborhoods that is also a countable system of neighborhoods in $H(X, \mathbb{C})$.

6.2.2 Theta functions

There are holomorphic functions on $\mathbb{C}$ that are periodic: e^z, $\sin z$, $\cos z$. But are there any double periodic functions? Liouville's theorem proves that this is impossible, because a double periodic function is bounded. The closest that we can get is to have two complex numbers a, b that are not a real multiple of each other and a function f such that $f(z+a) = f(z)$ and $f(z+b) = \mu(z)f(b)$ where $\mu(z)$ is a correction factor that is as simple as possible. Theta functions are of this form. They were introduced by Jacobi in relation to elliptic integrals. More precisely, he has shown that the inverse function of an elliptic integral, a so called elliptic function, can be written as a rational expression in theta functions. Theta functions were further studied and generalized by Riemann in his treatise on elliptic functions and Riemann surfaces.

Definition. For every τ with $\mathrm{Im}\tau > 0$, the Riemann theta function is defined by

$$\theta(z,\tau) = \sum_{n=-\infty}^{\infty} \exp(\pi i n^2 \tau + 2\pi i n z), \quad z \in \mathbb{C}.$$

Theorem 41. (a) For fixed τ, the series $\sum_{n=-\infty}^{\infty} \exp(\pi i n^2\tau + 2\pi i n z)$ converges uniformly and absolutely on every compact subset of $\mathbb{C}$ and therefore defines a holomorphic function of the variable z on $\mathbb{C}$.
(b) For fixed z, the series converges uniformly and absolutely on every compact subset of the upper half plane and therefore defines a holomorphic function of the variable τ on the upper half-plane.

Proof. We can prove (a) and (b) simultaneously, by noticing that if $|\mathrm{Im} z| \leq \alpha$ and $\mathrm{Im}\tau \geq \beta > 0$ then

$$|\exp(\pi i n^2\tau + 2\pi i n z)| \leq \exp(-\pi n^2\beta + 2\pi n\alpha) = [\exp(-|n|\pi\beta + 2\pi\alpha)]^{|n|}.$$

If we choose $N > 2\alpha/\beta$, then $r = \exp(-|n|\pi\beta + 2\pi\alpha) < 1$, so for $|n| \geq N$, $[\exp(-|n|\pi\beta + 2\pi\alpha)]^{|n|} < r^{|n|}$, showing that the series that defines theta functions is bounded from above by a power series, hence converges uniformly. So for fixed τ, the series converges uniformly on $A_\alpha = \{z \mid -\alpha \leq \mathrm{Im} z \leq \alpha\}$, for all $\alpha \geq 0$, and since every compact in $\mathbb{C}$ is contained in some set A_α, as a function of z, the series converges uniformly on compacts. For fixed z, the series converges uniformly on $B_\beta = \{\tau \,|\, \mathrm{Im}\tau \geq \beta\}$, for all $\beta > 0$, and since every compact in $\mathrm{Im}\tau > 0$ is contained in some B_β, as a function of τ the series converges uniformly on compacts. □

Proposition 20. Theta functions satisfy the identities

$$\theta(z+1,\tau) = \theta(z,\tau), \quad \theta(z+\tau) = e^{-\pi i\tau - 2\pi i z}\theta(z,\tau).$$

Proof. The first identity follows from

$$\exp(\pi i n^2\tau + 2\pi i n(z+1)) = e^{2\pi i}\exp(\pi i n^2\tau + 2\pi i n z) = \exp(\pi i n^2\tau + 2\pi i n z).$$

For the second identity, we have

$$\theta(z+\tau,\tau) = \sum_{n=-\infty}^{\infty}\exp(\pi i n^2\tau + 2\pi i n(z+\tau)) = \sum_{n=-\infty}^{\infty}\exp(\pi i(n+1)^2\tau - \pi i\tau + 2\pi i(n+1)z - 2\pi i z)$$

$$= \sum_{m=-\infty}^{\infty}\exp(\pi i m^2\tau - \pi i\tau + 2\pi i m z - 2\pi i z) = e^{-\pi i\tau - 2\pi i z}\theta(z,\tau). \quad \square$$

Let now $x, t \in \mathbb{R}$ and consider the function of 2 real variables $\theta(x, it)$.

Proposition 21. The function $\theta(x, it)$ satisfies the heat equation

$$\frac{\partial}{\partial t}\theta(x,it) = \frac{1}{4\pi}\frac{\partial^2}{\partial x^2}\theta(x,it).$$

Proof. We compute

$$\theta(x,it) = \sum_{n=-\infty}^{\infty}\exp(-\pi n^2 t)\exp(2\pi i n x) = 1 + 2\sum_{n=1}^{\infty}\exp(-\pi n^2 t)\cos(2\pi n x).$$

Hence

$$\frac{\partial}{\partial t}\theta(x,it) = 2\sum_{n=1}^{\infty}(-\pi n^2)\exp(-\pi n^2 t)\cos 2\pi n x;$$

$$\frac{\partial^2}{\partial x^2}\theta(x,it) = 2\sum_{n=1}^{\infty}(-4\pi^2 n^2)\exp(-\pi n^2 t)\cos 2\pi n x.$$

The proposition is proved. □

And now let us apply the Argument Principle to find the number of zeros of the theta function in the fundamental domain, which is the parallelogram with vertices $0, 1, 1+\tau, \tau$, which includes the sides from 0 to 1 and from 0 to τ, but misses the other sides. Choose a parallelogram P that is the translate of this parallelogram and does not pass through the zeros, and let $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ be the sides $0 \to 1$, $1 \to 1+\tau$, $1+\tau \to \tau$, $\tau \to 0$.

The number of zeros of $\theta(z,\tau)$ inside this parallelogram is

$$\frac{1}{2\pi i}\int_P \frac{\theta'(z,\tau)}{\theta(z,\tau)}dz = \frac{1}{2\pi i}\left(\int_{\gamma_1}\frac{\theta'(z,\tau)}{\theta(z,\tau)}dz + \int_{\gamma_2}\frac{\theta'(z,\tau)}{\theta(z,\tau)}dz + \int_{\gamma_3}\frac{\theta'(z,\tau)}{\theta(z,\tau)}dz + \int_{\gamma_4}\frac{\theta'(z,\tau)}{\theta(z,\tau)}dz\right).$$

Because $\theta(z+1,\tau)=\theta(z,\tau)$,

$$\int_{\gamma_2}\frac{\theta'(z,\tau)}{\theta(z,\tau)}dz+\int_{\gamma_4}\frac{\theta'(z,\tau)}{\theta(z,\tau)}dz=0.$$

Because $\theta(z+\tau,\tau)=e^{-\pi i\tau-2\pi iz}\theta(z,\tau)$,

$$\theta'(z+\tau,\tau)=(-2\pi i)e^{-\pi i\tau-2\pi iz}\theta(z,\tau)+e^{-\pi i\tau-2\pi iz}\theta'(z,\tau)=(-2\pi i)\theta(z+\tau,\tau)+e^{-\pi i\tau-2\pi iz}\theta'(z,\tau),$$

so

$$\begin{aligned}\int_{\gamma_1}\frac{\theta'(z,\tau)}{\theta(z,\tau)}dz+\int_{\gamma_3}\frac{\theta'(z,\tau)}{\theta(z,\tau)}dz&=\int_{\gamma_1}\frac{\theta'(z,\tau)}{\theta(z,\tau)}dz-\int_{\gamma_1}\frac{\theta'(z+\tau,\tau)}{\theta(z+\tau,\tau)}dz\\ =\int_{\gamma_1}\frac{\theta'(z,\tau)}{\theta(z,\tau)}dz-\int_{\gamma_1}\frac{e^{-\pi i\tau-2\pi iz}\theta'(z,\tau)}{e^{-\pi i\tau-2\pi iz}\theta(z,\tau)}dz&-(-2\pi i)\int_{\gamma_1}\frac{e^{-\pi i\tau-2\pi iz}\theta(z,\tau)}{e^{-\pi i\tau-2\pi iz}\theta(z,\tau)}dz=2\pi i.\end{aligned}$$

Thus

$$\frac{1}{2\pi i}\int_P\frac{\theta'(z,\tau)}{\theta(z,\tau)}dz=1$$

showing that the theta function has a single zero in a fundamental domain. In fact because the theta function is even, this zero can only be $0,1/2,\tau/2$, or $(1+\tau)/2$. It is $(1+\tau)/2$.

6.3 The Weierstrass factorization theorem

6.3.1 Convergence of products and the Weierstrass factorization theorem

Definition. If $(z_n)_n$ is a sequence of complex numbers then

$$\prod_{n=1}^{\infty}z_n=\lim_{N\to\infty}\prod_{n=1}^{N}z_n,$$

if the latter exists.

Note that the product exists and is equal to zero if one of the numbers is zero. But the product can be zero even if none of the numbers is zero. However, if $\prod_n z_n\neq 0$, then $\lim_{n\to\infty}z_n=1$.

Proposition 22. If $\mathrm{Re}z>0$ for all $n\geq 1$, then $\prod_{n=1}^{\infty}z_n$ converges to a nonzero complex number if and only if $\sum_{n=1}^{\infty}\log z_n$ converges.

Proof. page 165. □

Lemma 5. If $|z|<1/2$ then

$$\frac{1}{2}|z|\leq\log(1+z)|\leq\frac{3}{2}|z|.$$

Proposition 23. If $\mathrm{Re}z_n>-1$ then the series $\sum\log(1+z_n)$ converges absolutely if and only if the series $\sum z_n$ converges absolutely.

Proof. page 165 □

Definition. If Re$z_n > 0$ for all n the $\prod_n z_n$ converges absolutely if $\sum_n \log z_n$ converges absolutely.

Corollary 10. If Re$z > 0$ the $\prod z_n$ converges absolutely if and only if $\sum(z_n - 1)$ converges absolutely.

Lemma 6. Let $(f_n)_n$ be a sequence of functions that converges uniformly to f. Assume that there is a such that Re$f \leq a$. Then $\exp f_n$ converges uniformly to $\exp f$.

Proof. The proof is at Page 166. The idea is that if f_n converges uniformly to f then $f_n - f$ converges uniformly to zero. But then $\exp(f_n - f)$ converges uniformly to 1. Because $|\exp f|$ is bounded $\exp f \exp(f_n - f)$ converges uniformly to $\exp f$, that is $\exp f_n$ converges uniformly to f. □

Lemma 7. Let K be a compact subset of a Riemann surface and let $(g_n)_n$ be a sequence of continuous functions from K into $\mathbb{C}$ such that $\sum(g_n - 1)$ converges absolutely and uniformly on K. Then the product

$$f = \prod_{n=1}^{\infty} g_n$$

converges absolutely and uniformly on K. Also, there is n_0 such that $f(x) = 0$ if and only if $g_n(x) = 0$ for some n, $1 \leq n \leq n_0$.

Proof. Page 167. □

Theorem 42. Let X be a Riemann surface and let $(f_n)_n$ be a sequence in $H(X, \mathbb{C})$ such that no f_n is identically equal to zero. If $\sum(f_n - 1)$ converges absolutely and uniformly on compact subsets of X then $\prod_n f_n$ converges in $H(X, \mathbb{C})$ to an analytic function f. If a is a zero of f then a is a zero of only a finite number of the functions f_n and the multiplicity of the zero of f at a is the sum of the multiplicities of the zeros of f_n at a.

Proof. page 167 □

Definition. An elementary function is a function $E_p(z)$ of the form

$$E_p(z) = (1 - z) \exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}\right), \quad p \geq 1.$$

Lemma 8. If $|z| \leq 1$ then $|E_p(z) - 1| \leq |z|^{p+1}$.

Proof. Page 168. □

Theorem 43. Let $(a_n)_n$ be a sequence in $\mathbb{C}$ with If p_n is a sequence of positive integers such that

$$\sum_{n=1}^{\infty} m_n \left(\frac{r}{|a_n|}\right)^{p_n+1} < \infty, \quad \text{for all } r > 0,$$

then

$$f(z) = \prod_{n=1}^{\infty} E_{p_n}(z/a_n)$$

converges uniformly on compacts to a holomorphic function on $\mathbb{C}$ whose only zeros are at the points a_n and have orders equal to the number of times a_n occurs in the sequence. Moreover, the sequence $p_n = n - 1$ satisfies the condition.

Proof. Page 169 Note that $E_{p_n}(z/a_n) = 0$ if and only if $z = a_n$. For every $R > 0$, there is n_0 such that for $n \geq n_0$, $|a_n| > r$. Then for $|z| \leq r$, by Lemma 8,

$$|E_{p_n}(z/a_n) - 1| \leq \left(\frac{r}{|a_n|}\right)^{p_n+1} \quad \text{for } n \geq n_0.$$

It follows that $\sum_n (E_{p_n}(z/a_n) - 1)$ converges absolutely and uniformly in $\overline{B(0,r)}$. The conclusion now follows from Theorem 42.

For $p_n = n - 1$, just choose n_0, such that for $n \geq n_0$, $|a_n| \geq R > r$. Then $\sum_n |1 - E_{p_n}(z/a_n)|$ is dominated by a geometric series, done. □

Theorem 44. (The Weierstrass Factorization Theorem) Let f be an entire function and let $(a_n)_n$ be the nonzero zeros of f with multiplicities m_n, respectively. Then there is a nonnegative integer m (if $m \geq 1$, then m is the order of zero at zero), an entire function g, and a sequence of positive integers $(p_n)_n$ such that

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} \left[E_{p_n}\left(\frac{z}{a_n}\right)\right]^{m_n}.$$

Proof. page 170. □

Example 23. The factorization of the sine function:

$$\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) = \pi z \prod_{n=-\infty}^{\infty} \left(1 - \frac{z}{n}\right).$$

The proof is at page 175.

6.3.2 The Gamma function

6.3.3 Riemann's zeta function and the Riemann hypothesis

Riemann's zeta function is a fundamental tool in analytic number theory. Analytic number theory studies properties of integer numbers using the tools of analysis. The path to constructing the Riemann zeta function starts with Euler, who considered the s-series

$$\sum_{n=1}^{\infty} \frac{1}{n^s}$$

as a function of the real variable $s > 1$. He noticed

Theorem 45. (Euler's Theorem)

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right).$$

Note that $\lim_{z\to 1} 1/\zeta(z) = 0$. From this we obtain two immediate consequences:

Proposition 24. (a) (Euclid) There are infinitely many primes.
(b) The sum of the reciprocals of primes

$$\sum_{p \text{ prime}} \frac{1}{p}$$

diverges.

Proof. The first of these facts is obvious, if the product $\prod(1-1/p^z)$ were finite, we could take the limit as $z \to 1$ and obtain a finite number.

For the second part, note that the product $\prod(1-1/p)$ equals zero. Then apply Corollary 10 to conclude that $\sum(1-1/p-1) = -\sum 1/p$ does not converge absolutely, so this sum diverges. Consequently $\sum 1/p$ diverges. □

Riemann's genius idea was to extend Euler's construction to the entire complex plane. This he published in a memoir in 1859. For $\mathrm{Re}(z) > 1$, he defined Riemann's zeta function as

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z},$$

exactly like Euler. Its absolute convergence that is uniform on compacts follows from the convergence of p-series, with $p = \mathrm{Re}\ z$. Then he extended the function to the entire plane.

Definition. The Riemann zeta function $\zeta(z)$ is a meromorphic function which has a single pole of order 1 at $z = 1$ and which for $\mathrm{Re}\ z > 1$ is given by the formula

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z},$$

Riemann's goal was to understand the distribution of prime numbers among the positive integers. In 1848 Chebyshev (also spelled as Tchebychev) has found an estimate for the the number $\pi(n)$ of primes less than or equal to n. Chebyshev's prime number theorem shows that there exist constants A and B such that for every n,

$$A\frac{n}{\ln N} < \pi(N) < B\frac{n}{\ln n}.$$

In other words $\pi(n)$ is of order $n/\ln n$. Chebyshev actually found that $A = 0.4$ and $B = 0.48$ work. The procedure of extending the zeta function to a meromorphic function on the entire plane, as it is outlined in detail in Conway's book, is based on Chebyshev's work

Here are the steps:

1. A formula for $\mathrm{Re}\ z > 1$. Using the change of variable $t \mapsto nt$, we can write

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt = \int_0^\infty e^{-nt}n^{z-1}t^{z-1}ndt = n^z\int_0^\infty e^{-nt}t^{z-1}dt.$$

Thus

$$\begin{aligned}\zeta(z)\Gamma(z) &= \sum_{n=1}^{\infty} n^{-z}\Gamma(z) = \sum_{n=1}^{\infty}\int_0^\infty e^{-nt}t^{z-1}dt \\ &= \int_0^\infty \sum_{n=1}^{\infty} e^{-nt}t^{z-1}dt = \int_0^\infty \frac{t^{z-1}}{e^t-1}dt.\end{aligned}$$

2. A formula for Re $z > 0$. The above formula works for Re $z > 1$. We want to write a formula that works for Re $z > 1$. We need to "correct" the issues that arise in this integral when t is close to 0.

$$\zeta(z)\Gamma(z) = \int_0^1 \left(\frac{1}{e^t - 1} - \frac{1}{t}\right) t^{z-1} dt + (z-1)^{-1} + \int_1^\infty \frac{t^{z-1}}{e^t - 1} dt.$$

3. A formula for $0 <$ Re $z < 1$. We can restrict the formula to this strip and then rewrite it in a more compact form

$$\zeta(z)\Gamma(z) = \int_0^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t}\right) t^{z-1} dt.$$

4. A formula for $-1 <$ Re $z < 1$. Again we can rewrite this formula as

$$\zeta(z)\Gamma(z) = \int_0^1 \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2}\right) t^{z-1} dt - \frac{1}{2z} + \int_1^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t}\right) t^{z-1} dt,$$

then extend the definition of the left-hand side to Re $z > -1$ using the right-hand side.

5. A formula for -1Re $z < 0$. In this strip the above formula can be written as

$$\begin{aligned}\zeta(z)\Gamma(z) = \int_0^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2}\right) t^{z-1} dt &= 2\int_0^\infty \left(\sum_{n=1}^\infty \frac{1}{t^2 + 4n^2\pi^2}\right) t^z dt \\ &= 2(2\pi)^{z-1}\zeta(1-z)\int_0^\infty \frac{t^z}{t^2+1} dt.\end{aligned}$$

Theorem 46. (Riemann's functional equation) For $-1 <$ Re $z < 0$,

$$\zeta(z) = 2(2\pi)^{z-1}\Gamma(1-z)\zeta(1-z)\sin\left(\frac{1}{2}\pi z\right).$$

We can use this to extend the zeta function to the entire half-plane Re $z < 0$. As a corollary, we obtain

Theorem 47. The zeta function can be extended to a meromorphic function in the entire complex plane with only a simple pole at $z = 1$ whose residue is 1. For $z \neq 1$, ζ satisfies Riemann's functional equation.

This is a good example on how analytic functions are extended. There is a general procedure, analytic continuation, which we will study later. Sometimes it produces extensions to the whole complex plane, sometimes we obtain multivalued functions and then the disambiguation is done using a Riemann surface (as it was done when trying to extend the square root to the plane). You can understand better the above construction if you think about the following much simpler example:

The function $f(z) = \sum_{n=0}^\infty z^n$ is defined on $|z| < 1$. It can be extended to the entire plane by the formula $f(z) = 1/(1-z)$, and then on $|z-2| < 1$ it has the series expansion $f(z) = \sum_{n=0}^\infty (-1)^{n+1}(z-2)^n$.

From the function equation we can deduce that $0, -2, -4, -6, \ldots$ (the nonpositive even numbers) are zeros of the zeta function. These are called the *trivial zeros*.

The Riemann hypothesis. The nontrivial zeros (which are the zeros different from $0, -2, -4, -6, ...$) of $\zeta(z)$ have real part equal to $1/2$.

The Riemann hypothesis is useful for estimating the number $\pi(n)$ of primes less than or equal to n. If the Riemann hypothesis were true, then a stronger fact could be proved, namely that

$$\pi(n) = \int_0^n \frac{dx}{\ln x} + O(n^{1/2}\ln n).$$

Here $O(n^{1/2}\ln n)$ is a quantity for which there exists a constant C, not depending on n, such that $O(n^{1/2}\ln n) < Cn^{1/2}\ln n$. This improves the observation of Gauss (1849) that

$$\pi(n) \approx Li(n) = \int_0^n \frac{dt}{\ln t}.$$

Here $Li(n)$ is called the logarithmic integral (not to be confused with the dilogarithm).

Here are some known values of the zeta function:

$$\zeta(2) = \frac{\pi^2}{6}, \zeta(4) = \frac{\pi^4}{90}, ..., \zeta(2n) = (-1)^{n+1}\frac{B_{2n}(2\pi)^{2n}}{2(2n)!},$$

where B_{2n} is the $2n$th Bernoulli number. Also,

$$\zeta(-1) = -\frac{1}{12}, \zeta(-3) = \frac{1}{120}, ..., \zeta(-n) = (-1)^n\frac{B_{n+1}}{n+1}.$$

6.4 Compactness in $H(X,Y)$

Recall that on the spaces of continuous and holomorphic maps between two Riemann surfaces X and Y we have put the compact-open topology. We have concluded that if Y admits a metric (such as when Y is an open subset of the Riemann sphere but there are other situations as well), then the compact-open topology on both continuous and holomorphic functions is metrizable. This is the setting in which we work now. Recall also that we require X to have a countable dense subset.

We want to analyze compact families (sets) of maps, or rather families of functions whose closure is compact. Since in metric spaces compactness can be phrased in terms of convergence, we make the following definition:

Definition. A set $\mathcal{F}$ in $C(X,Y)$ or $H(X,Y)$ is normal if each sequence in $\mathcal{F}$ has a convergent subsequence.

We need a characterization of normal families for which we have to introduce another notion

Definition. A set $\mathcal{F} \subset C(X,Y)$ is equicontinuous at a point $z_0 \in X$ if for every$\epsilon > 0$ there is $\delta > 0$ such that for $|z - z_0| < \delta$, $d(f(z), f(z_0)) < \epsilon$ for every $f \in \mathcal{F}$.

Theorem 48. (Arzela-Ascoli) A set $\mathcal{F} \in C(X,Y)$ is normal if and only if the following two conditions are satisfied:
(a) for each $z \in X$, $\{f(z) \,|\, f \in \mathcal{F}\}$ has compact closure in Y;
(b) $\mathcal{F}$ is equicontinuous at every point in X.

Now we can proceed with characterizing normal families of holomorphic functions. Yet another definition:

Definition. A set $\mathcal{F} \subset H(X, \mathbb{C})$ is locally bounded if for every $a \in X$ there are $M > 0$ and an open set U containing a such that

$$|f(z)| \leq M, \text{ for } z \in U.$$

Lemma 9. A set $\mathcal{F}$ in $H(X, \mathbb{C})$ is locally bounded if and only if for each compact set $K \subset X$ there is a constant M such that $|f(z)| \leq M$ for all $f \in \mathcal{F}$ and $z \in K$.

Proof. Every open neighborhood of a point contains inside a compact neighborhood, so one implication is obvious. For the other implication, let K be compact, and for $a \in K$ let U_a, M_a be the corresponding open neighborhood and constant. Then by compactness, finitely many of the U_a's cover K, let them be $U_{a_1}, U_{a_2}, \ldots ,U_{a_n}$. Then $M = \max(M_{a_1}, M_{a_2}, ..., M_{a_n})$ has the desired property for the set K. $\square$

Theorem 49. (Montel) A family $\mathcal{F} \in H(X, \mathbb{C})$ is normal if and only if $\mathcal{F}$ is locally bounded.

Proof. If $\mathcal{F}$ is normal but not locally bounded, then there is K such that $\sup\{|f(z)| \,|\, z \in K, f \in \mathcal{F}\} = \infty$. Choose $\{f_n\}$ a sequence such that $\sup\{|f_n(z)| \,|\, z \in K\} > n$. Choose a convergent subsequence of f_n, say $f_{n_k} \to f$. Then convergence in the compact-open toplogy implies $\sup\{|f_{n_k}(z) - f(z)| \,|\, z \in K\} \to 0$. Hence $\sup\{|f(z)| \,|\, z \in K\} = \infty$, a contradiction because f is continuous on K and hence has a maximum.

For the converse, we check the conditions from the Arzela-Ascoli theorem. Since both conditions are local, we can restrict ourselves to a local coordinate chart, thus we can work in some closed disk about $a \in \mathbb{C}$, $\bar{B}(a, r)$, where $|f(z)| \leq M$ for all $f \in \mathcal{F}$. Let z be such that $|z - a| \leq \frac{1}{2}r$. By Cauchy's formula for $\gamma = a + re^{it}$, $0 \leq t \leq 2\pi$,

$$\begin{aligned} |f(a) - f(z)| &= \frac{1}{2\pi} \left| \int_\gamma \left(\frac{f(w)}{w-a} - \frac{f(w)}{w-z} \right) dw \right| \\ &= \frac{1}{2\pi} \left| \int_\gamma \frac{f(w)(a-z)}{(w-a)(w-z)} dw \right| \leq \frac{2M}{r} |a - z|. \end{aligned}$$

Let $\delta < \min(\frac{1}{2}r, \frac{r}{4M}\epsilon)$. Then if $|z - a| < \delta$, $|f(a) - f(z)| < \epsilon$, showing that $\mathcal{F}$ is equicontinuous at a, which is condition (b) from Arzela-Ascoli. Condition (a) is immediate from local boundedness, choosing $K = \{a\}$. Thus, by the Arzela-Ascoli theorem, $\mathcal{F}$ is normal. $\square$

Corollary 11. A subset of $H(X, \mathbb{C})$ is compact if and only if it is closed and locally bounded.

Now we look at meromorphic functions. We have seen before $H(X, \mathbb{C} \cup \infty) = M(X) \cup \{f_\infty\}$ is closed in $C(X, \mathbb{C} \cup \infty)$. Also, we can put on the Riemann sphere the following metric, so that we are in agreement with Conway:

$$d(z, z') = \frac{2|z - z'|}{[(1 + |z|^2)(1 + |z'|^2)]^{1/2}}, \quad d(z, \infty) = \frac{2}{(1 + |z|^2)^{1/2}}$$

This metric is equivalent to the one defined before.

Definition. For every $f \in M(X)$ define

$$\mu(f) : X \to \mathbb{C}, \quad \mu(f)(a) = \lim_{z \to a} \frac{2|f'(z)|}{1 + |f(z)|^2}.$$

This definition makes sense regardless of whether a is a pole or not. If a is not, then the right-hand side is the limit of a continuous function.

Theorem 50. A family $\mathcal{F} \subset H(X, \mathbb{C} \cup \{\infty\})$ is normal if and only if $\mu(\mathcal{F}) = \{\mu(f) \,|\, f \in \mathcal{F}\}$ is locally bounded.

Proof. Here is a rewrite of the proof from Conway. Again we apply the Arzela-Ascoli theorem, and the only difficult part is to check that if $\mu(\mathcal{F})$ is locally bounded then $\mathcal{F}$ is equicontinuous. Certainly, we can work in local coordinates, thus we can think X to be an open subset of $\mathbb{C}$. Assume $\mu(f) < M$ in some open disk containing z. Let $\epsilon > 0$. We want to find $\delta > 0$ such that $|z - z'| < \delta$ implies $|f(z) - f(z')| < \epsilon$ for all $f \in \mathcal{F}$.

Fix $f \in \mathcal{F}$. Suppose that z, z' are not poles of f, and choose a polygonal path $w_0 = z, w_1, \ldots, w_n = z'$ that avoids poles. Using the triangle inequality we have

$$d(f(z), f(z')) \leq \sum_{k=1}^{n} d(f(w_k), f(w_{k-1})) = \sum_{k=1}^{n} \frac{2}{[(1+f(w_k)^2)(1+f(w_{k-1})^2)]^{1/2}} |f(w_k) - f(w_{k-1})|$$
$$\leq \sum_{k=1}^{n} \frac{2}{[(1+f(w_k)^2)(1+f(w_{k-1})^2)]^{1/2}} \left| \frac{f(w_k) - f(w_{k-1})}{w_k - w_{k-1}} - f'(w_{k-1}) \right| |w_k - w_{k-1}|$$
$$+ \sum_{k=1}^{n} \frac{2}{[(1+f(w_k)^2)(1+f(w_{k-1})^2)]^{1/2}} |f'(w_{k-1})||w_k - w_{k-1}|$$
$$\leq \sum_{k=1}^{n} 2 \left| \frac{f(w_k) - f(w_{k-1})}{w_k - w_{k-1}} - f'(w_{k-1}) \right| |w_k - w_{k-1}| + M \sum_{k=1}^{n} \frac{2(1+|f(w_k)|^2)}{[(1+f(w_k)^2)(1+f(w_{k-1})^2)]^{1/2}} |w_k - w_{k-1}|$$

Now, for this *particular* f we have some freedom of choice. We can choose $w_1, w_2, \ldots, w_{n-1}$ so that

$$\sum_{k=1}^{n} |w_k - w_{k-1}| \leq 2|z - z'|$$
$$\left| \frac{f(w_k) - f(w_{k-1})}{w_k - w_{k-1}} - f'(w_{k-1}) \right| < \alpha$$
$$\frac{2(1+|f(w_k)|^2)}{[(1+f(w_k)^2)(1+f(w_{k-1})^2)]^{1/2}} \leq 1 + \alpha,$$

for some $\alpha > 0$. Then

$$d(f(z), f(z')) \leq 4\alpha|z - z'| + 2(1+\alpha)M|z - z'| = (4\alpha + 2\alpha + 2\alpha M)|z - z'|.$$

Letting $\alpha \to 0$, we obtain $d(f(z), f(z')) \leq 2M|z - z'|$.

If z' is a pole, then

$$d(f(z), \infty) = \lim_{w \to z'} d(f(z), f(w)) \leq 2M \lim_{w \to z'} |z - w| = 2M|z - z'|.$$

So we can choose $\delta = \epsilon/2M$ and we are done. □

6.4.1 The Riemann mapping theorem

The category of Riemann surfaces has as objects the Riemann surfaces and as morphisms the holomorphic maps between such surfaces. The isomorphisms of this category are the *biholomorphic* maps, namely the maps who admit holomorphic inverses.

Definition. Two Riemann surfaces that admit a biholomorphic map between them are called conformally equivalent.

Lemma 10. If $f : X \to Y$ is, holomorphic, one-to-one, and onto, then it is biholomorphic.

Proof. By the Open Mapping Theorem, f is an open map so its inverse, f^{-1} is continuous. Now let us focus on one point z_0 and work in local coordinates. Then $f'(z_0) \neq 0$, or else $f(z) - f(z_0)$ has a zero of order 2 at z_0 so it is not injective, by Theorem 24. Then, by the continuity of f', $f'(z) \neq 0$ in a neighborhood of U of z_0. Let $V = f(U)$. Then $f : U \to V$ is holomorphic, invertible, and with nonzero derivative. By Proposition 6, f^{-1} is holomorphic on V. Varying z_0, and thus $f(z_0)$, we obtain that f^{-1} is holomorphic, so f is biholomorphic. □

We now prove that if $X \subset \mathbb{C}$ and X simply connected (meaning that any loop in X can be continuously deformed, in X, to a point) then X is conformally equivalent to either the entire complex plane $\mathbb{C}$, or to the open unit disk D. Moreover, the former happens only when $X = \mathbb{C}$, and by Liouville's theorem, the complex plane is not conformally equivalent to the unit disk.

Theorem 51. (The Riemann Mapping Theorem) Let G be a simply connected region which is not the whole plane and let $a \in \mathbb{G}$. Then there is a unique biholomorphic map $f : G \to D$ such that $f(a) = 0$ and $f'(a) > 0$, where D is the open unit disk.

Proof. Pages 160-162. Here are the main ideas to keep in mind:

- *You first show that there are one-to-one analytic functions $f : G \to D$ such that $f(a) = 0$ and $f'(a) > 0$.* You do this by choosing a point $b \in \mathbb{C} \backslash G$, and then taking $z \to \sqrt{z-b}$. The image of this function lies in the exterior of a disk, and via a Möbius transformation, it can be mapped inside the unit disk. Let now $\mathcal{F}$ be the set of such functions.
- *You show that $\mathcal{F} \cup \{0\}$ is closed.* For this you use Hurwitz's theorem to prove that if a sequence of one-to-one functions converges, then the limit is either one-to-one or constant.
- *You show that there is a function in $\mathcal{F}$ that maximizes $f'(a)$.* By Montel's theorem $\mathcal{F} \cup \{0\}$ is compact, so the continuous functional $f \mapsto f'(a)$ has a maximum. This maximum is not 0, so there is a function in $\mathcal{F}$ for which this maximum is attained.
- *You show that if f is not onto, then f is not a maximum of the functional.* For this you pick a point $\omega \in D \backslash f(G)$, and construct

$$\phi(z) = \frac{z - \omega}{1 - \bar{\omega} z}, \quad h = \sqrt{\phi \circ f}, \quad \psi(z) = \frac{|h'(a)|}{h'(a)} \frac{z - a}{1 - \bar{a} z},$$

then finally set $g = \psi \circ h$. You show that $g'(a) > f'(a)$.

□

Here is a brute force example.

Example 24. Let us find a biholomorphic map f between $G_1 = \{z \,|\, |z-1| < 1\}$ and $G_2 = \{z \,|\, 3\pi/4 < \arg z < 5\pi/4\}$ such that $f(1) = -1$ and $f'(1) > 0$.

Note that one can map G_2 by $f_1(z) = z^2$ to the half-plane $G_3 = \{z \,|\, \mathrm{Re}\, z > 0\}$. The rotation $f_2(z) = iz$ maps it further to the upper half-plane.

We now find a Möbius transformation f_3 from G_1 to the upper half-plane. For that we map $0 \mapsto 1$, $1 + i \mapsto 0$, $2 \mapsto \infty$. The formula is

$$f_3(z) = \frac{z - (1+i)}{z - 2} : \frac{0 - (1+i)}{0 - 2} = \frac{(1-i)z - 2}{z - 2}.$$

Now the map

$$g(z) = (f_2 \circ f_1)^{-1} \circ f_3(z) = \sqrt{-if_3(z)} = \sqrt{\frac{(-1-i)z+2i}{z-2}}$$

maps G_1 to G_2. But

$$g(1) = \sqrt{1-i} = \sqrt{\sqrt{2}e^{7i\pi/4}} = \sqrt[4]{2}e^{7i\pi/8} \neq -1.$$

So we need to adjust, in order to both map 1 to -1 and to make sure that the derivative is positive. Let us instead find a function $h : G_2 \to G_1$ such that $h'(-1) > 0$ and $h(-1) = 1$. Then we take $f = h^{-1}$, since this works by Proposition 6. The function h is obtained by composing an automorphism of G_1 with

$$g^{-1}(z) = \frac{2z^2+2i}{z^2+1+i}.$$

Note that

$$g^{-1}(-1) = \frac{4i}{1+3i} = \frac{6}{5} + \frac{2}{5}i.$$

Let us now find $\phi : G_1 \to G_1$ such that $\phi(\frac{6}{5} + \frac{2}{5}i) = 1$. Note that 1 is the center of the circle G_1. If this were the unit disk, and we were mapping $a = \frac{1}{5} + \frac{2}{5}i$ to the origin, we would simply use

$$\phi_a(z) = \frac{z-a}{1-\bar{a}z}.$$

Let t_1 be the translation $t_1(z) = z - 1$. Then we must use

$$\phi(z) = t_1^{-1} \circ \phi_a \circ t_1(z) = \frac{(\frac{4}{5}+\frac{2}{5}i)z - \frac{4}{5}i}{(-\frac{1}{5}+\frac{2}{5}i)z + \frac{6}{5} - \frac{2}{5}i} = \frac{(4+2i)z-4i}{(-1+2i)z+6-2i}.$$

Then

$$\phi \circ g^{-1}(z) = \frac{4z^2+2i}{(2+i)z^2+2+i}.$$

maps G_2 to G_1 and -1 to 1. We have

$$\frac{d}{dz}(\phi \circ g^{-1})(z) = \frac{2}{2+i}\frac{d}{dz}\left(2 + \frac{-2+i}{z^2+1}\right) = 4\frac{-2+i}{2+i} \cdot \frac{z}{(z^2+1)^2}.$$

The derivative at -1 is therefore $\frac{2+i}{2-i} = \frac{3}{5} + \frac{4}{5}i$. The only place where we can insert a phase factor is in front of ϕ_a. So let α be a complex number whose absolute value is 1. Set $\psi_a = \lambda\phi_a$ and let

$$\phi(z) = t_1^{-1} \circ \psi_a \circ t_1(z) = \frac{(5\lambda - 1 + 2i)z + 6 - 6\lambda - 4i}{(-1+2i)z+6-2i}.$$

Then

$$h(z)\phi \circ g^{-1}(z) = \frac{(2\lambda+2)z^2 + (3-3\lambda) + 2\lambda i}{(2+i)z^2 + (2+i)}.$$

This maps G_2 to G_1 and -1 to 1. Its derivative is

$$\frac{d}{dz}h(z) = \frac{1}{2+i}\frac{d}{dz}\left(2\lambda + 2 + \frac{1-5\lambda+2\lambda i}{z^2+1}\right) = 2 \cdot \frac{1-5\lambda+2\lambda i}{2+i} \cdot \frac{z}{(z^2+1)^2}.$$

The derivative at -1 is

$$-\frac{1}{4+2i}(1-5\lambda+2\lambda i) = \frac{1}{20}(2i-4)(1-5\lambda+2\lambda i) = \frac{1}{10}[(8-9i)\lambda + (i-2)].$$

We want this number to be positive. Set $\lambda = \cos\theta + i\sin\theta$. Then the real part of this number is $8\cos\theta + 9\sin\theta + 2$, which we want to be positive, and the imaginary part is $-9\cos\theta + 8\sin\theta + 1$, which we want to be zero. Divide by $\sqrt{8^2+9^2} = \sqrt{145}$, and choose $\alpha \in (0, \pi/2)$ such that $\sin\alpha = 8/\sqrt{145}, \cos\alpha = 9/\sqrt{145}$. Then we should have

$$\begin{aligned}\sin\alpha\cos\theta + \sin\theta\cos\alpha + 2/\sqrt{145} > 0\\ \cos\alpha\cos\theta - \sin\alpha\sin\theta = 1/\sqrt{145}.\end{aligned}$$

Now choose $\beta \in (0, \pi/2)$ such that $\cos\beta = 1/\sqrt{145}$. Then we should have $\cos(\theta - \alpha) = \cos\beta$. We make the choice $\theta = \alpha + \beta$. Then the first expression is $\sin(2\alpha + \beta) + 2/\sqrt{145}$, and is not hard to realize that $\alpha + 2\beta < \pi$, for example because $\alpha < \pi/4$ and $\beta < \pi/2$. Hence we have a map $h : G_2 \to G_1$ such that $h(-1) = 1$ and $h'(-1) > 0$. Invert and obtain that the desired function is

$$f(z) = h^{-1}(z) = \sqrt{\frac{(2+i)w - (2\lambda+2)}{(-2-i)w + (3-3\lambda) + 2\lambda i}}$$

where we use the branch of the square root for which $\sqrt{1} = -1$, and λ is defined as above. The Riemann mapping theorem shows that this function is unique.

Chapter 7

Approximations of holomorphic functions

7.1 Runge's Theorem

The following proof I have found in the lecture notes of Ch.L. Epstein (at UPenn).

Theorem 52. (Runge's Approximation Theorem I) Let $K \subset \Omega \subset \mathbb{C}$, K compact, Ω open. The following conditions are equivalent:
(a) Every holomorphic function in a neighborhood of K can be approximated by holomorphic functions in Ω.
(b) The open set $\Omega \backslash K$ has no component whose closure is compact in Ω.

Proof. (a) implies (b). Assume by contrary this is not true, and let $L \subset \Omega\backslash K$ with $\bar{L}$ compact. Then the boundary of L lies in K. Let $w \in L$, and consider $h(z) = (z-w)^{-1}$ holomorphic in a neighborhood of K. Then we can find a sequence f_n of holomorphic functions in Ω such that $f_n \to h$ uniformly on K. But then $(z-w)f_n \to (z-w)h$ uniformly on K. Using the maximum modulus principle, we find that $(z-w)f_n$ converges to $(z-w)h$ uniformly on L, because the boundary of L lies in K. But $(z-w)f_n(z)|_{z=w} = 0$, but $(z-w)h(z)|_{z=w} = 1$, a contradiction.

(b) implies (a) Let H_K and H_Ω be the linear subspaces of the Banach space $C(K)$ of continuous functions on K with the sup norm that consist of the restrictions to K of the holomorphic functions in a neighborhood of K on the one hand, and of the holomorphic functions on Ω on the other hand. Consider the closures $\overline{H_K}$ and $\overline{H_\Omega}$. Assume that they do not coincide, and let $f \in \overline{H_K}\backslash\overline{H_\Omega}$. Then the Hahn-Banach theorem in functional analysis states that there is a continuous linear functional

$$\phi : C(K) \to \mathbb{C}, \text{ such that } \phi|_{\overline{H_\omega}} = 0, \quad \phi(f) = 1.$$

Another result in functional analysis, the Riesz representation theorem implies that ϕ can be represented by a finite measure on the Borel subsets of K, meaning that

$$\phi(g) = \int_K g d\mu.$$

So, to finish the proof, we have to show that if μ is a finite measure on K, then $\int_K f d\mu = 0$ for all $f \in H(\Omega, \mathbb{C})$ implies $\int_K f d\mu = 0$ for all f holomorphic in a neighborhood of K. Let μ be such a measure. The function

$$\phi(z) = \int_K \frac{d\mu(w)}{z-w}$$

vanishes for $z \in \mathbb{C}\backslash\Omega$. This function is holomorphic for $z \in \mathbb{C}\backslash K$. But since every component of $\mathbb{C}\backslash K$ meets a component of $\mathbb{C}\backslash\Omega$, we deduce that $\phi = 0$ on $\mathbb{C}\backslash K$.

Now let h be a holomorphic function in a neighborhood of K and let χ be a function that is supported in the domain of h and identically equal to 1 on K. Using the product rule and the Cauchy-Pompeiu formula, we obtain

$$\psi(z) = \frac{1}{2\pi i}\iint_{\mathbb{C}\backslash K} \frac{h(w)\frac{\partial\chi}{\partial\bar{w}}}{w-z} dw \wedge d\bar{w}.$$

We now compute

$$\begin{aligned}\int_K h(z)d\mu(z) &= \frac{1}{2\pi i}\int_K \iint_{\mathbb{C}\backslash K} \frac{h(w)\frac{\partial\chi}{\partial\bar{w}}}{w-z} dw \wedge d\bar{w}\, d\mu(z)\\ &= \frac{1}{2\pi i}\iint_{\mathbb{C}\backslash K}\int_K \frac{h(w)\frac{\partial\chi}{\partial\bar{w}}}{w-z} d\mu(z) dw \wedge d\bar{w} = 0,\end{aligned}$$

where in the second equality we changed the order of summation. The conclusion follows. □

Theorem 53. (Runge's Approximation Theorem II) Let $K \subset \mathbb{C}$ be compact and let $E \subset \mathbb{C}\backslash K$ be such that E intersects every component of $\mathbb{C}\backslash K$. If f is analytic in an open set containing K and $\epsilon > 0$, then there is a rational function $R(z) = P(z)/Q(z)$, with $P(z)$ and $Q(z)$ polynomials, such that all zeros of $Q(z)$ are in E and

$$|f(z) - R(z)| < \epsilon.$$

Proof. Let f be holomorphic on $U \supset K$, with U open and $\bar{U}$ compact. Choose one point in each component of $\mathbb{C}\backslash K$ that intersects $\mathbb{C}\backslash U$. Because K is compact, the distance from K to $\mathbb{C}\backslash U$ is positive, let it be δ. Also, because K is bounded, the number of connected components of $\mathbb{C}\backslash K$ that contain an open disk of radius $\delta/2$ is finite. But any component that does not contain such a disk lies inside U. So the number of connected components of $\mathbb{C}\backslash K$ that intersect $\mathbb{C}\backslash U$ is finite. Choose a (finite) point from E in each of these components, and let these points be $z_1, z_2, \ldots, z_k$. Let K_0 be the compact set obtained as the union of K and the connected components of $\mathbb{C}\backslash K$ that lie entirely in U. Then f is still holomorphic in a neighborhood of K_0, but now E is finite.

Let us apply the previous theorem to K_0 and $\Omega = \mathbb{C}\backslash\{z_1, z_2, \ldots, z_n\}$. Then there is $g \in H(\Omega, \mathbb{C})$ such that

$$|f(z) - g(z)| < \frac{\epsilon}{3}, \quad \text{for all } z \in K.$$

At every z_k, expand

$$g(z) = \sum_{j=-\infty}^{\infty} a_j^k (z - z_k)^j.$$

Because the distance from z_k to K is positive, for every k there is $r_k > 0$ such that $B(z_k, r_k) \subset \mathbb{C}\backslash K$. For every k, choose $N_k \in \mathbb{N}$ such that

$$\left|\sum_{j=-\infty}^{-N_k} a_j^k (z - z_k)^j\right| < \frac{\epsilon}{3n}, \text{ for } |z_k| \geq r_k.$$

This is possible because of the uniform convergence of the Laurent expansion on compact annuli (see the proof of the Laurent series expansion theorem for more details). Set

$$h(z) = g(z) - \sum_{k=1}^{n} \sum_{j=-\infty}^{-N_k} a_j^k (z - z_k)^j = \frac{\ell(z)}{(z - z_1)^{m_1} \cdots (z - z_n)^{m_n}} = \frac{\ell(z)}{Q(z)},$$

where $\ell(z)$ is integral (i.e. holomorphic in the entire plane. Next, choose a polynomial $P(z)$ such that

$$\sup\{|P(z) - \ell(z)| \,|\, z \in K\} < \frac{\epsilon}{3 \sup\{|Q(z)|} \,|\, z \in K\}.$$

Set $R(z) = P(z)/Q(z)$. Then $R(z)$ is rational, it only has poles or removable singularities at $z_1, z_2, \ldots, z_n$, and

$$\begin{aligned}
|f(z) - R(z)| &\leq |f(z) - g(z)| + |g(z) - h(z)| + |h(z) - R(z)| \\
&\leq |f(z) - g(z)| + \sum_{k=1}^{n} |\sum_{j=-\infty}^{-N_k} a_j^k (z - z_k)^j| + \frac{|\ell(z) - P(z)|}{|Q(z)|} \\
\frac{\epsilon}{3} + &\leq \sum_{k=1}^{n} \frac{\epsilon}{3n} + \frac{\epsilon}{3} = \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon,
\end{aligned}$$

as desired. □

Corollary 12. Let G be an open subset of the plane and let E be a subset of $(\mathbb{C} \cup \{\infty\}) \backslash G$ such that E meets every component of $(\mathbb{C} \cup \{\infty\}) \backslash G$. Let $R(G, E)$ be the set of rational functions with poles in E, viewed as a subspace of $H(G, \mathbb{C})$. Then $R(G, E)$ is dense in $H(G, \mathbb{C})$ in the compact-open topology.

Corollary 13. If G is an open subset of $\mathbb{C}$ such that $(\mathbb{C} \cup \{\infty\}) \backslash G$ is connected, then the set of polynomials is dense in $H(G, \mathbb{C})$.

Proof. Apply Runge's theorem with $E = \{\infty\}$ □

At this moment we have a nice characterization of simply connectedness.

Theorem 54. Let G be open and connected in $\mathbb{C}$. Then the following are equivalent

- G is simply connected;

- $n(\gamma, a) = 0$ for all loops γ in G and $a \in \mathbb{C} \backslash G$;

- $(\mathbb{C} \cup \{\infty\}) \backslash G$ is connected;

- any analytic function in G is the limit of a sequence of polynomials;;

- $\int_\gamma f = 0$ for all $f \in H(G, \mathbb{C})$ and γ a loop in G;

- every analytic function in G has an antiderivative;
- for every nonzero analytic function f in G, $\log f$ is well defined;
- every nonzero analytic function is G has a well defined square root;
- G is homeomorphic to the unit disk.

7.2 The Mittag-Leffler Theorem

We have seen that given finitely many points in an open set and fixing the singular parts of a holomorphic function at these points we can construct a function that is holomorphic off those points and has the prescribed singular parts. But what about infinitely many such points? We have seen before that we can construct a holomorphic function with prescribed zeros. Can we prescribe the poles? More than that, can we prescribe the singular parts of a meromorphic function?

Theorem 55. Let G be an open set, $(a_k)_k$ a sequence of distinct points in G without a limit point in G, and let

$$S_k(z) = \sum_{j=1}^{m_k} \frac{A_{j_k}}{(z-a_k)^j}, \quad k = 1, 2, ...$$

Then there is a meromorphic function f on G whose poles are exactly the points a_k, $k \geq 1$, and such that the singular part of f at a_k is $S_k(z)$.

Proof. The complete proof is at pages 205-206. Here are the main ideas of the proof.

First, if there are only finitely many a_k, then we can just take $f(z) = \sum_k S_k(z)$. The expression on the right is the partial fraction decomposition of the rational function f, and this partial fraction decomposition contains splits f into the sum of its singular parts at each of a_k. Indeed, at a_k all S_j, $j \neq k$, are holomorphic and so do not contribute to the singular part at a_k.

If there are infinitely many a_k we can still take $f(z) = \sum_k S_k(z)$, but the sum might not converge. But then we choose the compact sets K_n such that
(i) $K_n \subset \mathrm{int}K_{n+1} \subset G$,
(ii) $\cup_n K_n = G$,
(ii) every connected components of $\mathbb{C}\backslash K$ contains a point from $\mathbb{C}\backslash G$.

Divide the indices of the sequence (a_k) into the sets I_j, with I_1 containing the indices of a_k in K_1, and I_n containing the indices of a_k in $K_n \backslash K_{n-1}$, $n \geq 2$. All I_n are finite because of compactness and because a_k has no accumulation point in G.

Then use Runge's Theorem (Theorem 52) to approximate on K_{n-1} the function $f_n(z) = \sum_{k \in I_n} S_k(z)$ by a holomorphic function on G, $R_n(z)$. Do it such that

$$|f_n(z) - R_n(z)| \leq 1/2^n.$$

Set $f(z) = f_1(z) + \sum_{n>1}(f_n(z) - R_n(z))$.

Now on the open set $\mathrm{int}K_{n-1}$ we have

$$f(z) = \sum_{k \in I_1} S_k(z) + \sum_{j=2}^{n-1} \left(\sum_{k \in I_j} S_k(z) - R_j(z) \right) + \sum_{j \geq n} (f_j(z) - R_j(z)).$$

This sum converges by the Weirestrass M-test. The last term is holomorphic on $\text{int}K_{n-1}$, so it does not contribute to any singularities in $\text{int}K_{n-1}$. Nor does any of the $R_j(z)$ create or contribute to singularities. So on $\text{int}K_{n-1}$ we are in the situation of the finite sum discussed in the beginning, with the only singularities given by the a_k with $k \in I_1 \cup I_2 \cup ... \cup I_{n-1}$. These singularities have the singular parts $S_k(z)$.

Now as you vary n, you cover all G, as the union of the interior of K_n also cover G. $\square$

Chapter 8

Analytic continuation

8.1 Schwarz Reflection Principle

Theorem 56. (Schwarz Reflection Principle) Let G be a region that is symmetric with respect to the real axis. If $f : G \cap \{\mathrm{Im} z \geq 0\} \to \mathbb{C}$ is continuous and analytic on $G \cap \{\mathrm{Im} z > 0\}$ and $f(x) \in \mathbb{R}$ when $x \in G \cap \mathbb{R}$ then f can be extended to an analytic function on G.

Proof. pages 211-212 □

8.2 Analytic Continuation along a Path

8.3 The Monodromy Theorem

Theorem 57. Let $G \subset \mathbb{C}$ be open. Let (f, D) be so that $D \subset G$ and D open, and $f \in H(D, \mathbb{C})$ such that (f, D) can be analytically continued along any path in G that starts from a point in D. Let $a \in D$, $b \in G$, and γ_0 and γ_1 be two paths such that $\gamma_0(0) = \gamma_1(0) = a$, $\gamma_0(1) = \gamma_1(1) = b$, and γ_0 and γ_1 are homotopic with fixed endpoints. Consider the analytic continuations (f_t^0, D_t^0) and (f_t^1, D_t^1) of (f, D) along γ_0 and γ_1 (so $f_0^0 = f = f_0^1$ and $D_0^0 = D_0^1 = D$). Then $[f_1^0]_b = [f_1^1]_b$.

Chapter 9

Riemann surfaces

9.1 Riemann surfaces as domains of functions

This section uses some things from Donaldson's book on Riemann surfaces and from C. Teleman's lecture notes, as well as some ideas from my topology course.

The closed orientable surfaces are classified by their genus. More general surfaces can be quite complicated.

Recall the definition of Riemann surfaces from before. They were introduced by Riemann in the study of elliptic integrals, in relation to implicitly defined functions. The story was about functions $w(z)$ defined by an algebraic equation $P(z,w) = 0$, where P is a 2-variable polynomial, and the question was where does the function w naturally live. We have encountered this question before, for $P(z,w) = z - w^2$, when $w(z) = \sqrt{z}$. We have seen that the domain was a Riemann surface obtained by patching together two copies of the plane cut along a ray. This is one way to produce Riemann surfaces, but there are many other ways. I will show you below many, many examples.

9.1.1 The Riemann sphere revisited

We recall that the Riemann sphere is the one point compactification of the plane, and that it is itself a Riemann surface. We will now identify it with the projective line $\mathbb{C}P^1$. For this define the n-dimensional complex projective plane by:

$$\mathbb{C}P^n = \mathbb{C}^{n+1}\backslash\{0\}/\sim \quad \mathbf{z} \sim \mathbf{z}' \text{ if } \mathbf{z} = \lambda \mathbf{z}', \quad \lambda \in \mathbb{C}\backslash\{0\}.$$

We put on the quotient space the topology induced by $\mathbb{C}^{n+1}$. Using appropriate charts, teh result can be made into a complex manifold. For example when $n = 1$, then we let $U_0 = \{[Z_0, Z_1] \,|\, Z_0 \neq 0\}$ and $U_1 = \{[Z_0, Z_1] \,|\, Z_1 \neq 0\}$, where square brackets are used for equivalence classes. The maps $\phi_0 : U_0 \to \mathbb{C}, \phi_0([Z_1, Z_2]) = Z_1/Z_0$ and $\phi_1 : U_1 \to \mathbb{C}$, $\phi_\infty([Z_0, Z_1]) = Z_0/Z_1$ define charts that allow us to identify $\mathbb{C}P^1$ with the Riemann sphere $\mathbb{C} \cup \{\infty\}$.

In general we can turn $\mathbb{C}P^n$ into an n-dimensional complex manifold by using the charts $U_j = \{[Z_0, Z_1, ..., Z_{j-1}, 1, ..., Z_n] \,|\, Z_k \in \mathbb{C}\}$, and $\phi_j : U_j \to \mathbb{C}$, $\phi([Z_0, Z_1, ..., Z_{j-1}, 1, ..., Z_n]) = (Z_0, Z_1, ..., Z_n)$. In this picture, ϕ_0^{-1} embedds $\mathbb{C}^n$ in $\mathbb{C}P^n$, and the remaining part is referred to as the hyperplane at infinity. In particular $\mathbb{C}P^2\backslash U_0$ is the line at infinity L_∞. It is important to point out that the n-dimensional projective space is compact, and that it is an example of a compactification of the n-dimensional complex space, which when $n > 1$ is different from the one-point compactification.

9.1.2 Algebraic curves

Let us return to the equation $P(z,w)=0$ that defines w, and look at the so-called *affine curve* defined by it

$$X = \{(z,w) \in \mathbb{C}^2 \mid P(z,w) = 0\}.$$

It turns out that the curve is the right domain of w.

Compare to $x^2+y^2=1$. The function y is defined on $[-1,1]$ by either $y=\sqrt{1-x^2}$ or $y=-\sqrt{1-y^2}$, but the two overlap at endpoints. So it makes sense to glue the endpoints. But then you get a circle: the circle $x^2+y^2=1$! And indeed, the function y defined in the circle just projects on the first coordinate.

But the affine curve might not be a Riemann surface. Nevertheless we have the following result:

Theorem 58. If for each point on X either $\partial P/\partial z$ or $\partial P/\partial w$ is not zero, then X is a Riemann surface.

Proof. Assume that (z_0,w_0) is a point where $\partial P/\partial w$ does not vanish. Then using the implicit function theorem, there are disks $D_1, D_2 \subset \mathbb{C}$ centered at z_0, such that $X \cap (D_1 \times D_2)$ is the graph of a holomorphic function $\phi : D_1 \to D_2$. Then we can define a chart $\psi : X \cap (D_1 \times D_2)$ such that $\psi(z,\phi(z)) = z$. If (z_0,w_0) is a point where $\partial P/\partial z$ does not vanish, switch the roles of z and w.

We have to check that the changes of coordinates are holomorphic. Between two charts of the first kind, or two charts of the second kind, the changes of coordinates are just the identity maps. The change of coordinates from a chart of the first kind to a chart of the second kind is just $z \mapsto (z,\phi(z)) \mapsto \phi(z)$, and this is holomorphic. The theorem is proved. $\square$

Example 25. Let $\lambda \in \mathbb{C}$ and define the curve

$$X = \{(z,w) \mid w^2 = z(z-1)(z-\lambda)\}.$$

Then the conditions of Theorem 58 are satisfied, and consequently the curve is a Riemann surface. What surface? It is the Riemann surface of the function $w(z) = \sqrt{z(z-1)(z-\lambda)}$, which is a punctured torus. We will see below how to make it into a torus with no puncture.

This curve can be presented in a different form, by multiplying out and changing the z-variable:

$$X = \{(z,w) \mid w^2 = z^3 + az + b\}.$$

The condition that the cubic polynomial does not have a multiple root, namely that the cubic is nonsingular, is that the discriminant $\Delta = -16(4a^3 + 27b^2)$ be nonzero.

Example 26. Let

$$X = \{(w,z) \mid w^n = z\}.$$

This is a sphere with one puncture.

Example 27. The affine curve

$$X = \{(z,w) \mid w^2 = z^3\}$$

does not satisfy the conditions from the statement.

Now we turn to *projective curves*, which lie not in the 2-dimensional complex space, but in the the 2-dimensional complex projective space, which is obtained by adding a complex line at ∞ to $\mathbb{C}^2$. Consequently, we will add add points at infinity to the affine curves, turning the ones that are Riemann surfaces into closed Riemann surfaces (provided that they behave nicely at infinity).

For this let p be a homogeneous polynomial of degree n in 3 variables, meaning that

$$p(Z_0, Z_1, Z_2) = \sum_{i_0+i_1+i_2=n} c_{i_0,i_1,i_2} Z_0^{i_0} Z_1^{i_1} Z_2^{i_2}.$$

For technical reasons, let us further assume that p is nonconstant and that is not the product of two polynomials, in particular it is not divisible by any of the variables. Note that

$$p(\lambda Z_0, \lambda Z_1, \lambda Z_2) = \lambda^n p(Z_0, Z_1, Z_2)$$

meaning that it is n-homogeneous, so it satisfies the Euler identity

$$Z_0 \frac{\partial p}{\partial Z_0} + Z_1 \frac{\partial p}{\partial Z_1} + Z_2 \frac{\partial p}{\partial Z_2} = np.$$

Note that $p(Z_0, Z_1, Z_2) = 0$ is equivalent to $p(\lambda Z_0, \lambda Z_1, \lambda Z_2) = 0$ for all $\lambda \neq 0$. So the equation $p(Z_1, Z_2, Z_3) = 0$ defines a subset $\bar{X}$ of $\mathbb{C}P^2$ which we call the projective algebraic variety defined by p, which in this case is just the *projective curve.* Thus

$$\bar{X} = \{[Z_0, Z_1, Z_2] \in \mathbb{C}P^2 \,|\, p(Z_0, Z_1, Z_2) = 0\}.$$

Note that projective curves are compact. Note that

$$X = \bar{X} \cap U_0 = \bar{X} \cap \mathbb{C}^2 = \{(z, w) \,|\, P(z, w) = p(1, z, w) = 0\}.$$

Thus X is an affine curve, and $\bar{X}$ is obtained from it by adjoining the points at infinity

$$\bar{X} \cap L_\infty = \{[0, Z_1, Z_2] \,|\, p(0, Z_1, Z_2) = 0\}.$$

Note that in $\mathbb{C}^3$, $p(0, Z_1, Z_2) = 0$ is 1-dimensional, and so in $\mathbb{C}P^2$, this becomes 0-dimensional, hence a finite set of points on the line at infinity.

Theorem 59. Suppose $p(Z_0, Z_1, Z_2)$ is homogeneous of degree $d \geq 1$ and assume that all of its three partial derivatives are simmultaneously equal to zero only at the origin. Then

$$\bar{X} = \{[Z_0, Z_1, Z_2] \in \mathbb{C}P^2 \,|\, p(Z_0, Z_1, Z_2) = 0\}$$

is a compact Riemann surface.

Proof. Euler's identity implies right away that p is not divisible by any of the variables. Now let us work in a chart of $\mathbb{C}P^2$, say U_0. Thus $Z_0 \neq 0$. The intersection X of $\bar{X}$ with this chart is

$$X = \{(z, w) \,|\, P(z, w) = p(1, z, w) = 0\},$$

Note that $\partial P/\partial z = \partial p/\partial Z_1$ and $\partial P/\partial w = \partial p/\partial Z_2$, and the latter cannot both be zero on a point in the zero set of p, or else by Euler's identity $\partial p/\partial Z_0 = 0$ at such a point as well, which is ruled out by the hypothesis. So we are in the condition of Theorem 58 showing that via the Implicit Mapping Theorem we can put a Riemann surface structure on X. But on the part of $\bar{X}$ that lies in the intersection of two charts we can put via the same theorem the same Riemann surface structure. So $\bar{X}$ is a Riemann surface. $\square$

Example 28. Let us return to the Riemann surface of the nth root

$$X = \{(z, w) \,|\, w^n = z\}.$$

Its compactification in the projective space is

$$\bar{X} = \{[Z_0, Z_1, Z_2] \in \mathbb{C}P^2 \,|\, p(Z_0, Z_1, Z_2) = Z_2^n - Z_1 Z_0^{n-1} = 0\}.$$

We have

$$\frac{\partial p}{\partial Z_0} = (n-1)Z_0^{n-2}Z_1, \quad \frac{\partial p}{\partial Z_1} = Z_0^{n-1}, \quad \frac{\partial p}{\partial Z_2} = nZ_2^{n-1}.$$

On the curve also $Z_2^n - Z_1 Z_0^{n-1} = 0$. Note that if $n > 2$ at $(0, 1, 0)$ all of these are zero. So the conditions of Theorem 59 are satisfies exactly when $n = 2$. There is exactly one point at infinity. In that case $\bar{X}$ is a sphere. We will see later that it is conformally equivalent (i.e. biholomorphic) to the Riemann sphere.

Example 29. We now look at the famous example of *elliptic curves.* These are the compactifications of the curves of the form

$$X = \{(z, w) \,|\, w^2 = z^3 + az + b\} \quad -16(4a^3 - 27b^2) \neq 0.$$

We set

$$\bar{X} = \{[Z_1, Z_2, Z_3] \,|\, Z_2^2 Z_0 - Z_1^3 - 2aZ_1 - bZ_0^3 = 0\}.$$

If we set the partial derivatives equal to zero we obtain

$$\begin{aligned}
&Z_2^2 - 2aZ_1 - 3bZ_0^2 = 0\\
&3Z_1^3 - aZ_0^2 = 0\\
&2Z_2Z_0 = 0.
\end{aligned}$$

We only need to be concerned with what happens on the line at infinity, because we already know that the curve is smooth at finite points. But if $Z_0 = 0$ then the above conditions imply $Z_1 = Z_2 = 0$, so the conditions of Theorem 59 are satisfied.

How many points are there at infinity anyway? Well, $Z_0 = 0$ implies $Z_1 = 0$, so there is only one point at infinity, namely $[0, 0, 1]$. We add this point to the punctured torus X and obtain a torus with no puncture $\bar{X}$.

We will revisit elliptic curves in a moment.

9.1.3 One-dimensional abelian varieties

Let $\tau \in \mathbb{C}$ be such that $\operatorname{Im} \tau > 0$. Consider the following discrete subgroup of $(\mathbb{C}, +)$ (referred to as a lattice):

$$\Lambda = \{m + \tau n \,|\, m, n \in \mathbb{Z}\}.$$

The standard Riemann surface structure induces a Riemann surface structure on the quotient $\mathbb{C}/\Lambda$. We obtain what is called a *1-dimensional abelian variety.* The name variety comes from the fact that it is a projective variety with the same complex structure. It is worth mentioning the fact that these tori are projective varieties can be proved with the Lefschetz trick, which uses theta functions!

There is an apparently more general way to define these Riemann surfaces, using $\omega_1, \omega_2 \in \mathbb{C}\backslash\{0\}$ such that $\omega_2/\omega_1 \notin \mathbb{R}$. By switching ω_1 with ω_2 we can assume that Im $\omega_2/\omega_1 > 0$. Then set

$$L = \{n_1\omega_1 + n_2\omega_2 \mid n_1, n_2 \in \mathbb{Z}\}.$$

Then $\mathbb{C}/L$ is again a torus with a Riemann surface structure. As a homework exercise shows, this case is already covered by the previous situation.

Let us connect this to elliptic curves. We introduce the Weierstrass $\wp$-function (which makes the object of a homework exercise).

$$\wp(z) = \frac{1}{z^2} + \sum_{w \in L\backslash\{0\}} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2} \right).$$

This function is meromorphic in the plane, it is holomorphic in $\mathbb{C}\backslash L$.

Proposition 25. The series defining $\wp$ converges uniformly in $H(\mathbb{C}, \mathbb{C}P^1)$ (i.e. in the space of meromorphic functions on $\mathbb{C}$.

Proof. First observation is that , then

$$|n_1\omega_1 + n_2\omega_2| = |\omega_1||n_1 + n_2\omega_2/\omega_1| = |\omega_1|\sqrt{(n_1+n_2)^2 + n_2^2|\mathrm{Im}\omega_2/\omega_1|^2|}$$

So there is a constant $c > 0$ such that $|n_1\omega_1 + n_2\omega_2| \geq c\sqrt{n_1^2 + n_2^2}$. Consequently,

$$\sum_{w \in L\backslash\{0\}} 1/|w|^3 < \infty \text{ and } \sum_{w \in L\backslash\{0\}} 1/|w|^4 < \infty.$$

Let K be a compact set in $\mathbb{C}$. Note that $L \cap K$ is finite. Moreover, if we set $M = \sup_{z\in K}|z|$, there is N such that for $n_1^2 + n_2^2 > N$, $|n_1\omega_1 + n_2\omega_2| > 2M$. Set $L_N = \{n_1\omega_1 + n_2\omega_2 \mid n_1^2 + n_2^2 > N\}$. Then $\sum_{w\in L_N}\left(\frac{1}{(z-w)^2} - \frac{1}{w^2}\right)$ defines a holomorphic function on K. To prove this, for $w \in L_N$ note the estimate:

$$\left| \frac{1}{(z-w)^2} - \frac{1}{w^2} \right| = \left| \frac{2zw - z^2}{(z-w)^2w^2} \right| \leq 2\frac{|z|}{|w||z-w|^2} + \frac{|z|^2}{|w|^2|z-w|^2}$$

But

$$|z - w| \geq |w| - |z| \geq |w| - M > |w| - \frac{1}{2}|w| = \frac{1}{2}|w|.$$

Thus

$$\left| \frac{1}{(z-w)^2} - \frac{1}{w^2} \right| \leq 2\frac{|z|}{|w||z-w|^2} + \frac{|z|^2}{|w|^2|z-w|^2} \leq 8M\frac{1}{|w|^3} + 4M^2\frac{1}{|w|^4}.$$

Consequently $\sum_{w\in L_N}\left(\frac{1}{(z-w)^2} - \frac{1}{w^2}\right)$ converges uniformly and absolutely on K, thus defines a holomorphic function on K. Hence $\wp$ converges in the compact-open topology of $H(\mathbb{C}, \mathbb{C}P^1)$. □

Proposition 26.

$$\wp'(z) = -2\sum_{w\in L}\frac{1}{(z-w)^3}.$$

Proof. The series converges uniformly on compacts in $\mathbb{C}\backslash L$ in the topology of the space of holomorphic functions, so it can be differentiated term by term. □

Note that by construction $\wp$ is even and $\wp'$ is odd. And it is immediate that $\wp'$ is doubly periodic with periods ω_1 and ω_2. But we actually have the following result.

Proposition 27. The Weierstrass p-function $\wp$ is doubly periodic with periods ω_1 and ω_2.

Proof. We have

$$\frac{d}{dz}(\wp(z+\omega_i) - \wp(z)) = \wp'(z+\omega_i) - \wp'(z) = 0.$$

Thus there are constants C_i such that $\wp(z+\omega_i) = \wp(z) + C_i$, $i = 1, 2$. But for $z = -\omega_i/2$,

$$\wp(\omega_i/2) = \wp(-\omega_i/2) + C_i = \wp(\omega_i/2) + C_i,$$

Thus $C_1 = C_2 = 0$, and so $\wp$ is doubly periodic, as specified. □

Thus we can interpret $\wp$ as a function in $H(\mathbb{C}/L, \mathbb{C}P^1)$, namely as a meromorphic function on the torus. The functions in $H(\mathbb{C}/L, \mathbb{C}P^1)$ are called *elliptic functions.* They are 'inverses' of elliptic integrals, one of which computes the arc-length of the ellipse. The same is true for its derivative $\wp'$. These are examples of what are called elliptic functions. Note that by Liouville's Theorem, $H(\mathbb{C}/L, \mathbb{C}) = \mathbb{C}$, since any holomorphic function on $\mathbb{C}/L$ is bounded and can be lifted to a bounded holomorphic function on $\mathbb{C}$.

In some sense $\wp$ and $\wp'$ are *the* elliptic functions. Every other elliptic function can be written in the form $R_1(\wp) + \wp' R_2(\wp)$, where R_1, R_2 are rational functions. We will not prove this, nor will we prove a fact discovered by Jacobi, namely that every elliptic function can be expressed in terms of theta functions. But we will prove one thing, namely we will establish a conformal equivalence between elliptic curves and one-dimensional abelian varieties.

To this end, set

$$g_2 = 60\sum_{w\in L\backslash\{0\}}\frac{1}{|w|^4} \text{ and } g_3 = 140\sum_{w\in L\backslash\{0\}}\frac{1}{|w|^6}.$$

Theorem 60. The Weierstrass p-function satisfies the differential equation

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3.$$

Proof. This proof is sketchy, we ignore the convergence issues. The idea is from C. Teleman's lecture notes on Riemann surfaces. First, set $G_r = \sum_{w\in L\backslash\{0\})} w^{-r}$. We have the Taylor series expansion

$$(z-w)^{-k} = \frac{(-1)^k}{w^k}\left[1 + k\frac{z}{w} + \frac{k(k+1)}{2!}\frac{z^2}{w^2} + \frac{k(k+1)(k+2)}{3!}\frac{z^3}{w^3} + \cdots .\right]$$

Changing the order of summation and canceling the odd powers of z we have

$$\wp(z) = \frac{1}{z^2} + \sum_{m=1}^{\infty} \frac{2 \cdot 3 \cdots (2m+1)}{(2m)!} 2! G_{2m+2}(w) u^{2m}.$$

Do the same computation for the derivative, or simply differentiate this sum to obtain

$$\wp'(z) = -\frac{2}{z^2} + \sum_{m=1}^{\infty} \frac{2 \cdot 3 \cdots (2m+1)}{(2m-1)!} G_{2m+2}(w) u^{2m-1}.$$

Now compute $f(z) = \wp'(z)^2 - 4\wp(z)^3 - g_2\wp(z) - g_3$. Well, this is impossible! Instead, note that when computing the first few terms the singularity at the origin cancels out, and we get a function f that is elliptic and holomorphic, not meromorphic. But then it is constant, by Liouville's Theorem. The constant is zero, because $f(0) = 0$. Consequently $\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$, as desired. □

Proposition 28. In a fundamental domain given by the parallelogram with vertices $0, \omega_1, \omega_2, \omega_1 + \omega_2$ the function $\wp'(z)$ has exactly three zeros, namely $\omega_1/2, \omega_2/2, (\omega_1 + \omega_2)/2$. Moreover, if we set $e_1 = \wp(\omega_1/2)$, $e_2 = \wp(\omega_2/2)$, $e_3 = \wp((\omega_1 + \omega_2)/2)$, then e_1, e_2, e_3 are distinct and these numbers are the roots of the equation $4z^3 - g_2 z - g_3 = 0$.

Proof. Consider a complex number ζ whose real and imaginary part are positive, and sufficiently small when compared to $|\omega_1|, |\omega_2|, |\omega_1 + \omega_2|$ (say 1/10th of the smallest of these numbers). Consider the parallelogram D with vertices $\zeta, \omega_1 + \zeta, \omega_2 + \zeta, \omega_1 + \omega_2 + \zeta$. There is exactly one pole of order 3 inside D, and because of periodicity, if γ is the counterclockwise contour consisting of the boundary of this parallelogram, $\int_\gamma \wp''(z)/\wp'(z) dz = 0$. By the Argument Principle (Theorem 22):

$$\frac{1}{2\pi i} \int_\gamma \frac{\wp''(z)}{\wp'(z)} dz = \text{ number of zeros of } \wp' \text{ in } D - \text{ number of poles of } \wp' \text{ in } D,$$

$\wp'$ has exactly three zeros, multiplicities counted, inside D. By periodicity

$$\wp'(\omega_i/2) = -\wp'(-\omega_i/2) = \wp'(-\omega_i/2 + \omega_i) = \wp'(-\omega_i/2)$$
$$\wp'((\omega_1 + \omega_2)/2) = -\wp'(-(\omega_1 + \omega_2)/2) = \wp'(-(\omega_1 + \omega_2)/2 + \omega_1 + \omega_2) = \wp'(-(\omega_1 + \omega_2)/2)$$

so $\wp'(\omega_1/2) = \wp'(\omega_2/2) = \wp'((\omega_1 + \omega_2)/2) = 0$. As seen above, there are no other zeros in D, as there are only three zeros in any fundamental domain, and in particular in the parallelogram with vertices $0, \omega_1, \omega_2, \omega_1 + \omega_2$.

Let us now prove that e_1, e_2, e_3 are distinct. If say $e_1 = e_2$, then $f(z) = \wp(z) - e_1 = \wp(z) - e_2$ has zeros at $\omega_1/2$ and $\omega_2/2$, and since $f'(z) = \wp'(z)$ and $\wp'(\omega_1/2) = \wp'(\omega_2/2) = 0$. But then each of the zeros of $\wp$ is double. But the the Argument Principle:

$$0 = \frac{1}{2\pi i} \int_\gamma \frac{\wp'(z)}{\wp(z)} dz = \text{ number of zeros of } \wp \text{ in } D - \text{ number of poles of } \wp \text{ in } D$$

implies that $\wp$ has only 2 roots, multiplicities counted, in a fundamental domain. So e_1, e_2, e_3 are distinct.

Finally, because

$$4e_1^3 - g_2 e_1 - g_3 = 4\wp(\omega_1/2)^3 - g_2\wp(\omega_1/2) - g_3 = \wp'(\omega_1/2)^2 = 0,$$

and same for e_2, e_3, we deduce that e_1, e_2, e_3 are the three roots of the equation $4z^3 - g_2 z - g_3 = 0$. □

Note that as a biproduct of the proof, we deduce that every elliptic function, i.e. every map in $H(\mathbb{C}/L, \mathbb{C}P^1)$, has the same number of zeros and poles. In fact, by adding a constant to the function, we deduce that every value of the function is assumed the same number of times. This number of times is called the degree of the function. In our case $\wp$ has degree 2 and $\wp'$ has degree 3. There are no maps of degree 1 or else the one-dimensional abelian variety and the Riemann sphere would be conformally equivalent, and in particular would be homeomorphic. But one is simply connected and one is not, and to prove this you do not need a lot of toplogy, just notice that the integral of any holomorphic 1-form on a closed contour in the Riemann sphere is zero by Cauchy's theorem, while the integral of dz on the closed contour from 0 to ω_1 on the torus is $\omega_1 \neq 0$.

It is quite spectacular that from here we deduce that $g_2^3 \neq 27g^3$, so the discriminant of the cubic is nonzero! Now we have an abelian variety $\mathbb{C}/L$ *and* an elliptic curve $\bar{X}$ obtained by compactifying in the projective space the affine curve defined by $w^2 = 4z^3 - g_2 z - g_3$.

Proposition 29. The map $f : \mathbb{C}/L \to \bar{X}$ given by $f(z) = [1, \wp(z), \wp'(z)]$ if $z \neq 0$ and $f(0) = \infty$ is a conformal equivalence.

It can be shown conversely that every elliptic curve has an associated lattice, and hence an associated one-dimensional abelian variety.

9.2 Riemann surfaces through analytic continuation

9.2.1 The formulation of the problem

So we have addressed the problem of constructing the appropriate domain of a function w implicitly defined by a polynomial equation $P(z, w) = 0$, which, historically, was the starting point in the development of Riemann surfaces. Now we want to integrate such functions, or rather, functions of the form $R(z, w)$ where R is rational, or even better said, 1-forms $R(z, w)dz$ on the Riemann surface. When restricting ourselves to just a planar part of the Riemann surface, this leads naturally to the question of finding an antiderivative (also called primtive), via the Fundamental Theorem of Calculus. In other words, given $f(z)$ defined on an open set in the plane how to construct a function $F(z)$ such that

$$F'(z) = f(z).$$

(In general, for a 1-form $f(z)dz$ you want to find a function $F(z)$ such that $dF = fdz$.)

This problem can be solved easily if $f(z)$ is defined on an an open simply connected set D in the plane, by fixing $z_0 \in D$ and for each $z \in D$, a path γ_z from z_0 to z, and setting $F(z) = \int_{\gamma_z} f(w)dw$. This is a trick we have used extensively in the past, for example when we proved that differentiable implies analytic. The Cauchy Theorem implies that $F(z)$ is well defined. But if the domain D is not simply connected, then the primitive might be multivalued. So the topology of the domain plays an essential role.

Example 30. Let us visit an example that we understand very well by now. Let

$$f : \{z \,|\, 1 < |z - 3| < 3\} \to \mathbb{C}, \quad f(z) = \frac{1}{z}.$$

For everty z in the domain of f fix a path γ_z from 1 to z and define

$$F(z) = \int_1^z \frac{dw}{w} = \ln|z| + i\arg(z),$$

which is the principal branch of the logarithm. Then $F(z)$ is an antiderivative for $f(z)$.

But if we let

$$f : \{z \mid 1/2 < |z| < 3/2\} \to \mathbb{C}, \quad f(z) = \frac{1}{z},$$

then the integral formula, and any other attempt to define a primitive for f, leads to a multivalued function.

In both cases $f(z)$ is defined on an annulus by the same formula. But $f(z)$ lives naturally on $\mathbb{C}\backslash\{0\}$, and the first domain lies inside the simply connected domain $\{z \mid \text{Re } z > 0\} \subset \mathbb{C}\backslash\{0\}$.

But now where does $F(z)$ live? It can certainly be defined on a small disk $B(1, \epsilon)$ by $F(z) = \int_1^z f(w)dw$. For an arbitrary z in the plane, we can still fix a path γ_z from 1 to z and use the same formula. But the value of $F(z)$ depends on the path as well. At this point we would like to make a remark, that connects us to a more recent discussion. The value of $F(z)$ is the one obtained by analytic continuation of $(F, B(1, \epsilon)$ along γ_z, so once we have fixed an antiderivative in $B(1, \epsilon)$ we can forget about f altogether and the problem of finding the primitive becomes a problem of analytic continuation.

With this example we have shifted our attention to the following question. We are given an analytic function

$$f : U \subset \mathbb{C} \to \mathbb{C}$$

and we want to find the largest domain on which f can be extended. In more generality, let X, Y be Riemann surfaces and let U be an open subset of X. Assume that

$$f : U \to Y$$

is a holomorphic map. We want to find the largest domain of f. The construction will yield

- an open set V such that $U \subset V \subset X$,
- a Riemann surface Z and an onto holomorphic map $\pi : Z \to V$ such that every $z_0 \in V$ has a connected, simply connected open neighborhood W such π establishes a conformal equivalence between each of the connected components of $\pi^{-1}(W)$ and W,
- a holomorphic map $\tilde{f} : Z \to Y$ such that on one connected component of $\pi^{-1}(U)$, $f \circ \pi = \tilde{f}$.

It is important to point out that this procedure also covers the particular case that we have discussed before, that of finding the domain of an implicitly defined function w, given by $P(z, w) = 0$, P a polynomial. Both methods yield the same result, up to a conformal equivalence. The second method is less intuitive (it is pure abstract nonsense), but it works in more generality.

The map π is what is called a covering map, with Z a covering space of V.

Definition. A continuous map $\pi : E \to B$ is called a covering map if for every $b \in B$ there is an open set U containing it such that π establishes a homeomorphism between every connected component of $\pi^{-1}(U)$ and U.

9.2.2 The solution to the problem

We recall the notion of a germ of an analytic function at a point. This can be extended to the notion of a germ of an analytic map between Riemann surfaces. Here is how we define it. Assume that X, Y are Riemann surfaces, and that $z_0 \in X$. Consider all pairs (f, D), where $z_0 \in D \subset X$, D open, and $f : D \to Y$ is holomorphic. Two such pairs (f_1, D_1) and (f_2, D_2) are equivalent if $f_1 = f_2$ on $D_1 \cap D_2$. The equivalence class of (f, D) is called the germ of f at z_0 and is denoted by $[f]_{z_0}$.

It is important to point out that the Monodromy Theorem works for maps between Riemann surfaces the way it works for functions in the plane.

Problem. Given a germ of a holomorphic map between two Riemann surfaces, how do we construct the natural domain of the function whose germ this is?

Let $[f]_{z_0}$ be a germ, and let (f, D) be a representative for it, with D connected and simply connected. Let $V \subset X$ be the largest open subset on which we can perform analytic continuation along a path (which is the union of all open sets on which we can perform analytic continuation). We define Z to be the collection of all germs of these analytic continuations. Each point in Z is therefore of the form $[f_1]_{z_1}$, where $z_1 \in V$ and f_1 is an analytic function in some neighborhood of z_1 that is obtained by analytic continuation of f along some path from z_0 to z_1.

Now we put on Z the structure of a Riemann surface. For $[g]_z$, we want to construct a chart of Z containg this point. We know that $z \in V$, and we let $W \subset V$ be a simply connected open set containing z that lies entirely inside a chart $\phi_\alpha : U_\alpha \to \mathbb{C}$ of X. Consider all analytic continuations of $[g]_z$ to $[h_w]_w$ along a path from z to w that is contained entirely in W. Define the chart $\psi_W : W \to \mathbb{C}$, where W is the set of all such analytic continuations, and $\psi_W([h]_w) = \phi_\alpha(w)$.

Define the maps $\pi : Z \to V$, $\pi([g]_z) = z$ and $\tilde{f} : Z \to Y$, $\tilde{(f)}([g]_z) = g(z)$.

Theorem 61. *Z is a Riemann surface. The maps π and $\tilde{f}$ are holomorphic, π is a covering map and there is one connected component of $\pi^{-1}(D)$ on which $f \circ \pi = \tilde{f}$.*

Proof. The Monodromy Theorem implies that every ψ_W is one-to-one, so they can be used as charts. Let now $\psi_W, \psi_{W'}$ be two charts defined as above. If W lies inside the chart U_α and W' lies inside the chart U_β, then $\psi_W([h]_w) = \phi_\alpha(w)$ and $\psi_{W'}([h]_w) = \phi_\beta(w)$, thus $\psi_W \circ \psi_{W'}^{-1} = \phi_\alpha \circ \phi_\beta^{-1}$, and the latter is holomorphic. So Z is a Riemann surface.

Let W be an open set that defines a chart as above, and let ϕ_α the chart that defines ψ_W. Then $\phi_\alpha \circ \pi \circ \psi_W^{-1}$ is the identity map, which is holomorphic, so π is holomorphic. Let us show that π is a covering map. Choose $v \in V$, and choose a neighborhood V_v of v that is simply connected. Let us examine the set $\pi^{-1}(V_v)$. It contains every point in $\pi^{-1}(v)$. Let $[g]_v \in \pi^{-1}(v)$, and let W be the open set in $\pi^{-1}(V_v)$ defined by germs $[h_w]$ obtained by analytic continuation of $[g]_v$ along paths in V_v. Because every $[h]_w \in W$ is obtained by analytic continuation of $[g]_v$ along a path in V_v, by the Monodromy Theorem $[h]_w$ is determined by w, so π is one-to-one. Note also that different $[g]_v$ yield disjoint sets W because if W_1 and W_2 have $[h]_w$ in common, then by performing analytic continuation on a path from w to v we obtain the same germ at v. Moreover, the union of the sets W gives $\pi^{-1}(V_v)$. So π is a covering map.

Next, let $[g]_z \in Z$, and let $V_g \subset V$ be a simply connected open set on which the holomorphic map $g : V_g \to Y$ is defined, and let W be the open neighborhood of $[g]_z$ that is conformally equivalent to V_g via π. Then on W, $\tilde{f} = g \circ \pi$, which is a composition of holomorphic maps. So $\tilde{f}$ is holomorphic.

Finally, if we choose the connected component of $\pi^{-1}(D)$ that contains $[f]_{z_0}$, then on this component $\tilde{f} = f \circ \pi$. □

Example 31. The Weierstrass p-function $\wp(z)$ is a holomorphic map between a torus $\mathbb{C}/L$ and the Riemann sphere $\mathbb{C}P^1$:

$$\wp : \mathbb{C}/L \to \mathbb{C}P^1.$$

Using it we can define a 1-form $\wp(z)dz$, which we can integrate on every path on the torus. Fix some point $z_0 \neq 0$ and define $\zeta(z) = \int_{\gamma_z} \wp(w)dw$, where the integral is performed on some path that starts at some point $z_0 \neq 0$ on the torus and ends at z (set it equal to ∞ at the point that is the image on the torus of the nodes of the lattice). Because of Cauchy's Theorem this is well defined in some simply connected neighborhood D of z_0, so it defines the germ of a function $[\zeta]_{z_0}$ at z_0. Note that in our situation $X = V = \mathbb{C}/L$, $Y = \mathbb{C}P^1$. Now we apply the above algorithm. We obtain $Z = \mathbb{C}$, and $\zeta : \mathbb{C} \to \mathbb{C}P^1$,

$$\zeta(z) = \frac{1}{z} + \sum_{w \in L \setminus \{0\}} \left(\frac{1}{z-w} + \frac{1}{w} + \frac{z}{w^2} \right).$$

The function $\zeta(z)$, called the *Weierstrass zeta function*, is no longer periodic, it cannot be defined on the torus. It appears as the integral of an elliptic function. In the construction of $\zeta(z)$ we have produced a simply connected Riemann surface $\mathbb{C}$ and a holomorphic covering map $\pi : \mathbb{C} \to \mathbb{C}/L$.

9.2.3 Universal covering spaces

In the above example, the Weierstrass p-function and its integral, the Weierstrass zeta function, allowed us to find a covering space for one-dimensional abelian varieties (and in particular for elliptic curves) that is a simply connected Riemann surface, such that the covering map is holomorphic. We will discuss the case of hyperelliptic curves as a homework exercise, and explain that the situation repeats in that case. In fact, we will now construct, for every connected Riemann surface X, a simply connected Riemann surface $\tilde{X}$ and a covering map $\pi : \tilde{X} \to X$ that is holomorphic. Moreover, $\tilde{X}$ is unique up to a conformal equivalence, and is called the universal covering space of X. We will classify simply connected Riemann surfaces, which leads in some sense to a classification of all Riemann surfaces.

Theorem 62. Let X be a connected Riemann surface. Then there is a unique (up to conformal equivalence) connected, simply connected Riemann surface $\tilde{X}$ for which there is a holomorphic covering map $\pi : \tilde{X} \to X$.

Proof. Fix $z_0 \in X$ and consider the set of all paths in X starting at z_0. On the set of all paths in X beginning at z_0 we define an equivalence by

$$\alpha \sim \beta \text{ if and only if } \alpha(1) = \beta(1) \text{ and } \alpha \text{ homotopic to } \beta \text{ with fixed endpoints.}$$

We denote the equivalence class of α by $\hat{\alpha}$. We define $\tilde{X}$ to be the set of equivalence classes, and let e_0 the equivalence class of the constant path $e_{z_0}(t) = z_0$, $0 \leq t \leq 1$. The covering map $\pi : \tilde{X} \to X$ is defined by the equation

$$\pi(\hat{\alpha}) = \alpha(1).$$

Since X is connected, π is onto.

Let us put on $\tilde{X}$ a Riemann surface structure. For that we introduce the operation of composition of paths: if α ends where β starts, then $\alpha * \beta$ is $\alpha * \beta(t) = \alpha(2t)$ if $t \leq 1/2$ and $\beta(2t-1)$ if

$t \geq 1/2$. Note that path composition is compatible with homotopy (the compositions of homotopic paths are homotopic). Now choose a point $\hat{\alpha} \in \tilde{X}$ and let $v = \pi(\hat{\alpha}) = \alpha(1)$. Let U be a connected simply connected open neigborhood of v that is also a chart, with the complex coordinate defined by $\phi : U \to \mathbb{C}$ (just pick a chart around v and restrict it to a connected, simply connected open subset containing v). Let

$$W = \{\widehat{\alpha * \delta} \,|\, \delta \text{ is a path in } U \text{ beginning at } \alpha(1)\},$$
$$\psi : W \to \mathbb{C}, \quad \psi_W(\widehat{\alpha * \delta}) = \phi(\pi(\widehat{\alpha * \delta})) = \phi(\alpha * \delta(1)).$$

The proof that the pairs (W, ψ_W) defined this way for all points in $\tilde{X}$ define an atlas is similar to the proof given in Theorem 61. Also, by the same argument π is holomorphic.

Let us now show that $\tilde{X}$ is connected and simply connected. To show that $\tilde{X}$ is connected, let $\hat{\alpha}$ be a point in $\tilde{X}$. Define $f : [0,1] \to \tilde{X}$, $f(s) = \widehat{\alpha_s}$, where $\alpha_s(t) = \alpha(st)$. Then f is a path from $e_0 = \widehat{e_{z_0}}$ to $\hat{\alpha}$, which proves that $\tilde{X}$ is path connected.

To prove that $\tilde{X}$ is simply connected, we need the so called Path Lifting Lemma, which states that given a path $\gamma \in X$ starting at z and an element $\hat{\alpha} \in \pi^{-1}(z)$, there is a unique path $\tilde{\gamma}$ in $\tilde{X}$ starting at $\hat{\alpha}$ such that $\pi \circ \tilde{\gamma} = \gamma$. This is clearly true if γ lies in a set that is evenly covered, so it is true in general. Now let $\tilde{\gamma}$ be a loop in $\tilde{X}$ based at e_0. The notation is not accidental, as by the Path Lifting Lemma, it is the lifting of $\gamma = \pi \circ \tilde{\gamma}$. But now $s \mapsto \gamma_s(1)$ is another lifting of γ, and since the lifting is a loop, γ_0 and γ_1 are homotopic. But γ_0 is constant, so $\gamma_1 = \gamma$ is null-homotopic. Now there is another useful fact, the so called Homotopy Lifting Lemma, which states that homotopies can be lifted. So if γ is null homotopic, so is its lifting. Therefore any loop in $\tilde{X}$ is null homotopic; $\tilde{X}$ is simply connected.

To prove uniqueness, let $\tilde{X}'$ be another possible covering space of X that is simply connected, with covering map π'. For every $w_0 \in \tilde{X}$ and $w_0' \in \tilde{X}'$ such that $\pi(w_0) = \pi'(w')$ there is a unique map $h : \tilde{X} \to \tilde{X}'$ such that $\pi = \pi' \circ h$, and $h(w_0) = w_0'$:

$$\begin{array}{ccc} & & \tilde{X}' \\ & \overset{h}{\nearrow} & \downarrow \pi' \\ \tilde{X} & \xrightarrow{\pi} & X \end{array}$$

This map is defined as follows. Let $w \in \tilde{X}$ and let $\tilde{\gamma}$ be a path in $\tilde{X}$ from w_0 to w. Lift $\pi(\tilde{(\gamma)})$ to a path starting at w_0' and let w' be its endpoint. Then $h(w) = w'$.

If h' is the corresponding map when we switch $\tilde{X}$ and $\tilde{X}'$, then $h \circ h'$ and $h' \circ h$ are identity maps, making both h and h' conformal equivalences. The uniqueness is proved. □

Example 32. Let $g > 1$ be an integer number. Consider the unit disk $B(0,1)$ in the plane, and consider an arc of a circle that lies inside the unit disk and has enpoints on the unit circle, and that makes a right angle with the unit circle. Rotate this arc by $\frac{k\pi}{2g}$, $k = 0, 1, ..., 4g-1$. The arc of circle and its rotates forms a regular $4g$-gon with curved sides. Vary the original arc until the angles of the $4g$-gon are all equal to $\frac{\pi}{2g}$, so that the sum of the angles is 4π. When this happens, then by reflecting the regular polygon over the sides, and then reflecting again and again, using inversion (see the discussion on Möbius transformations), we obtain a tesselation of the unit disk by $4g$-gons. It is important to know that in the so called hyperbolic metric these polygons are all isometric.

Now let the oriented sides of the polygon at the center of the disk be

$$a_1 b_1, \alpha_1, \beta_1, a_2, b_2, \alpha_2, \beta_2, ..., a_g, b_g, \alpha_g, \beta_g$$

and consider the Möbius transformations ϕ_j mapping a_j to $-\alpha_j$, $j = 1, 2, ...g$ and ψ_j mapping b_j to $-\beta_j$, $j = 1, 2, ..g$, and let Γ be the group that they generate. It is not hard to see that the unit disk is invariant under Γ (the orthogonality of the arcs to the boundary plays a role). In fact the Möbius transformations that maps the unit disk to itself are of the form $\phi(z) = \frac{z-a}{1-\bar{a}z}$, $|a| < 1$. And Γ is a discrete subgroup of such transformations. We obtain the covering map

$$\pi : B(0,1) \to B(0,1)/\Gamma = \Sigma_g.$$

Here Σ_g is a closed genus g Riemann surface. In fact, as a consequence of the Uniformization Theorem, every closed genus g surface can be obtained as the quotient of the unit disk by a discrete subgroup of the group of automorphisms of the unit disk.

We conclude this example with one important remark. The universal covering space cannot be obtained by integrating a holomorphic 1-form, the way it was done for the torus where we integrated $\wp(z)dz$. In this setting, the correct construction is to integrate a complete system of linearly independent holomorphic 1-forms, and the result is a complex g-dimensional space. Factoring this space by the lattice of complete integrals (namely integrals along loops that define elements of the fundamental group) we obtain the so called Jacobian variety associated to the Riemann surface.

Theorem 63. (The Uniformization Theorem - Koebe, Poincaré, 1907) Every connected, simply connected Riemann surface is conformally equivalent to the unit disk, the complex plane, or the Riemann sphere.

Proof. Here is a sketch of the proof (following loosely some online notes by Kevin T. Chan). Let X be the Riemann surface.

Case 1. X is open.

We will need the following result due to Koebe:

Theorem 64. (Koebe Distortion Theorem) If $f : B(0,1) \to \mathbb{C}$ is univalent (i.e. injective) then

$$|f'(0)|\frac{|z|}{(1+|z|)^2} \leq |f(z) - f(0)| \leq |f'(0)|\frac{|z|}{(1-|z|)^2}.$$

Combining this with Montel's Theorem, we deduce that every family of functions that are holomorphic and univalent in a simply connected region of the plane, and so that at some point the set of values of the functions and of their derivatives is bounded, is normal.

We consider a countable triangulation of X. What this means is that we describe X as obtained by gluing countably many triangles, so that two triangles share one common side. Moreover, we do this so that the sides of each triangle lies in a coordinate chart, and in local coordinates the sides of the triangle are piecewise analytic arcs (i.e. images of a segment through a univalent analytic map). A result due to van der Waerden shows that for a triangulated simply connected open surface the triangles can be enumerated as $\Delta_1, \Delta_2, \ldots, \Delta_n, \ldots$ such that Δ_{n+1} has in common with $\Delta_1 \cup \Delta_2 \cup \cdots \cup \Delta_n$ either one side or two sides, but not a side and the opposite vertex.

For domains that have a piecewise analytic boundary the Riemann Mapping Theorem has a stronger version, namely that they can be mapped to the unit disk such that the map extends continuously to a bijection on the boundary.

We construct inductively a biholomorphic map ϕ_n from $\Delta_1 \cup \Delta_2 \cup \cdots \cup \Delta_n$ to a disk $B(z_0, R)$ in $\mathbb{C}$. For the base case, Δ_1 is (conformally equivalent to) a planar region with piecewise analytic boundary, so it can be mapped to the unit disk by a biholomorphic map that extends continuously to the boundary.

For the induction step, we add Δ_{n+1} to $E_n = \Delta_1 \cup \Delta_2 \cup \cdots \cup \Delta_n$. So we have $\phi_1 : E_n \to \Delta'$, where Δ' is a disk, that extends to the boundary. Also, Δ_{n+1} lies in a chart U, and let $\psi : U \to \mathbb{C}$ map it to a triangle Δ''.

Now part of E_n, call it V, lies inside the coordinate chart of Δ_{n+1}. In particular the common boundary lies inside the coordinate chart of Δ_{n+1}, which is an arc of the circle. Let this boundary be the arc a_1 that is mapped to arc a_1' in Δ' by ϕ_1 and to arc a_1'' in Δ'' by ψ. Choose a circular arc a_2' in Δ' that has the same endpoints as a_1' and such that the region in the shape of the bigon, B_1', that a_1', a_2' bound is inside $\phi_n(V)$. Set $B_1 = \phi_n^{-1}(B_1)$ and $B_1'' = \psi \circ \phi_n^{-1}(B)$.

Now we have a holomorphic function $f_1 : B_1' \to B_1''$, and more important a holomorphic map $g_1 : \Delta_{n+1} \cup B_1 \to \Delta' \cup B_1'$. Let B_2' be the reflection of B_1' over the arc a_2', and use the Schwarz reflection principle to extend f_1 to B_2', and let f_2 be the restriction of the extension to B_2'. This defines a univalent map from $\Delta_{n+1} \cup B_1 \cup B_2 \to \Delta'' \cup B_1'' \cup B_2''$, where $B_2 = \phi_n^{-1}(B_1)$ and $B_2'' = f_2(B_2)$, where on B_2' the map is given by $f_2 \circ \phi_n$. Now proceed inductively with reflections until the entire disk Δ' is covered, and hence we have a map on the entire E_n. Thus we have a conformal map from the interior of E_{n+1} to some bounded region in the plane that extends continuously to the boundary. Now map this region, using the Riemann mapping theorem, to the unit disk.

So for every n you have a conformal map

$$\phi_n : E_n = \Delta_1 \cup \Delta_2 \cup \cdots \cup \Delta_n \to B(0,1)$$

that extends to a bijection on the boundary.

Fix $z_0 \in \Delta_1$ and set $\phi_{1,n} = \phi_n(\phi_1^{-1})$. By composing with Möbius transformations normalize the ϕ_n such that $\phi_{1,n}(z_0) = 0$, $\phi_{1,n}'(z_0) = 1$, but now because of the second condition, the disk can be distorted, so we actually have maps from E_n to some disk R_n. As a consequence of the Koebe distortion theorem and Montel's theorem, the family $\phi_{1,n}$, $n \geq 1$, is normal and since the compact-open topology in $H(B(0,1), \mathbb{C})$ is metrizable, there is a convergent subsequence ϕ_{1,n_k} of $\phi_{1,n}$, $n \geq 1$. Then $\phi_{n_k} : E_{n_k} \to \mathbb{C}$ is a sequence of holomorphic functions that converges on E_1 to a univalent holomorphic function ϕ_0. Now do the same thing with E_2 but with the subsequence ϕ_{n_k} to extend ϕ_0 to E_2, and repeat for all n to obtain a function $\phi_0 : X \to \mathbb{C}$ that is both univalent and holomorphic. If the image is the whole plane, we are done. If the image is not the whole plane, map the image (which must be open and simply connected) using the Riemann mapping theorem to the unit disk.

Case 2. X **is closed.**

Take out one point $z_0 \in X$. The remaining part is still simply connected (because of the topological classification of surfaces your surface could only be a topological sphere). Now we are in the previous case, we have a surface that is either $\mathbb{C}$ or the disk. Let us prove that it must be $\mathbb{C}$. Let thus $\phi : X \backslash \{z_0\} \to B(0,1)$ be holomorphic. Let also $\psi : U \subset X \to B(0,1)$ be a local coordinate chart around z_0, say $\psi(z_0) = 0$. Then $\phi \circ \psi^{-1}$ is holomorphic and has a singularity at 0. But $\phi \circ \psi^{-1}$ is bounded, so z_0 is a removable singularity. Thus ϕ can be extended to z_0, and because of the Open Mapping Theorem, $\phi(z_0) \in B(0,1)$. But then we have a map $\phi : X \to B(0,1)$ that is onto, and X is compact while the disk is not. We have reached a contradiction. Thus $\phi(X \backslash \{z_0\}) = \mathbb{C}$. But then X is a one-point compactification of the complex plane, and so it must be the Riemann sphere. □

Remark 8. The universal covering space of the plane without a point, say $\mathbb{C} \backslash \{0\}$ is clearly the plane, and the exponential function is the covering map. A consequence of the Little Picard Theorem (page 297 in the book), which we do not have time to cover, is that the universal covering space of any other connected open set in the plane that is not the plane itself is the unit disk.