Complex Analysis

Final Exam Study Guide

1. Multivalued Functions and Branches

Many important functions in complex analysis are **multivalued**, meaning that they take more than one value at each point. For example, the logarithm is multivalued, with

$$\log z = \operatorname{Log} z + 2\pi n i \quad (n \in \mathbb{Z}).$$

Here $\operatorname{Log} z$ is the **principal branch** of the logarithm

$$\operatorname{Log} z = \ln|z| + i \operatorname{Arg} z$$

where $-\pi < \operatorname{Arg} z \leq \pi$. In general, a **branch of the logarithm** is any function $L(z)$ that satisfies

$$\exp\bigl(L(z)\bigr) = z$$

for all $z \in \mathbb{C}$. For example,

$$L(z) = \operatorname{Log}(e^{-i\phi} z) + i\phi$$

is a branch of the logarithm for any $\phi \in \mathbb{R}$. Every branch of the logarithm must have at least one curve on which it is discontinuous, known as a **branch cut**. For example, the branch cut for $\operatorname{Log} z$ is along the negative real axis.

Another multivalued function is the complex square root:

$$z^{1/2} = \pm\sqrt{z}.$$

Here $\sqrt{z} = \exp\bigl(\operatorname{Log}(z)/2\bigr)$ is the **principal square root**, which satisfies

$$-\frac{\pi}{2} < \operatorname{Arg}\sqrt{z} \leq \frac{\pi}{2}$$

for $z \neq 0$. In general a **branch of the square root** is any function $S(z)$ that satisfies $S(z)^2 = z$ for all $z \in \mathbb{C}$. For example,

$$S(z) = e^{i\phi}\sqrt{e^{-2i\phi} z}$$

is a branch of the square root for any $\phi \in \mathbb{R}$. Every branch of the square root has at least one branch cut, with the principal branch having a branch cut along the negative real axis.

The nth root function is also multivalued, with

$$z^{1/n} = \omega^k \sqrt[n]{z} \quad (k = 0, 1, \ldots, n-1)$$

where $1, \omega, \omega^2, \ldots, \omega^{n-1}$ are the nth roots of unity (with $\omega = e^{2\pi i/n}$) and $\sqrt[n]{z} = \exp\bigl(\operatorname{Log}(z)/n\bigr)$ is the principal nth root.

2. Integrals Involving Logarithms

Contour integrals of the form

$$\int_C \frac{1}{z-a}\,dz$$

can be tricky, since they involve the multivalued function

$$\log(z-a) \;=\; \ln|z-a| + i\arg(z-a).$$

If C is a contour that begins at $z = z_0$ and ends at $z = z_1$, then

$$\int_C \frac{1}{z-a}\,dz \;=\; \Big[\ln|z-a| + i\,\theta(z)\Big]_{z_0}^{z_1}$$

where $\theta(z)$ is a value for $\arg(z-a)$ that changes *continuously* as we move from the beginning to the end of C.

3. Power Series

A **power series** centered at $z = a$ is an infinite series of the form

$$\sum_{n=0}^{\infty} c_n(z-a)^n = c_0 + c_1(z-a) + c_2(z-a)^2 + c_3(z-a)^3 + \cdots$$

where the coefficients $c_0, c_1, c_2, \ldots$ are complex numbers.

Every power series centered at $z = a$ converges on the interior of a disk $|z-a| < R$, and diverges for $|z-a| > R$. The disk $|z-a| < R$ is known as the **disk of convergence**, and R is the **radius of convergence**. The sum of a power series is always a holomorphic function on its disk of convergence.

4. Finding the Radius of Convergence

The radius of convergence of a power series can be found using either the **ratio test** or the **root test**. Given a series $\sum a_n$, the ratio test involves the limit

$$r = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

and the root test involves the limit

$$r = \lim_{n\to\infty}\sqrt[n]{|a_n|}.$$

For either test, the series converges if $r < 1$ and diverges if $r > 1$. For example, consider the power series

$$\sum_{n=1}^{\infty}\frac{(-1)^n\,2^n}{n^2}(z-1)^n$$

centered at $z = 1$. Either the ratio test or the root test gives $r = 2|z-1|$, so the series converges for $2|z-1| < 1$, or equivalently $|z-1| < 1/2$. Thus the radius of convergence for this series is $R = 1/2$.

5. Taylor's Theorem

If $f(z)$ is a holomorphic function with no singularities on the disk $|z-a|<R$, then $f(z)$ is the sum of a power series on this disk:

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n \qquad \text{where } \; c_n = \frac{f^{(n)}(a)}{n!}.$$

This is called the **Taylor series** for $f(z)$ centered at $z=a$.

The radius of convergence R of the Taylor series is always the radius of the largest disk on which $f(z)$ can be made holomorphic. For a function with only isolated singularities, R is always the distance from a to the closest pole or essential singularity.

For a multivalued function, the radius of convergence of the Taylor series is the largest radius around a on which there exists a holomorphic branch of the function.

6. Working With Power Series

There are certain power series that you should be familiar with:

$$\begin{aligned}
\frac{1}{1-z} &= \sum_{n=0}^{\infty} z^n &&= 1 + z + z^2 + z^3 + \cdots &&\text{for } |z|<1. \\
\operatorname{Log}(1+z) &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} z^n}{n} &&= z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots &&\text{for } |z|<1. \\
e^z &= \sum_{n=0}^{\infty} \frac{z^n}{n!} &&= 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots &&\text{for all } z \in \mathbb{C}. \\
\cos z &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} &&= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots &&\text{for all } z \in \mathbb{C}. \\
\cosh z &= \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} &&= 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \cdots &&\text{for all } z \in \mathbb{C}. \\
\sin z &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} &&= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots &&\text{for all } z \in \mathbb{C}. \\
\sinh z &= \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} &&= z + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \cdots &&\text{for all } z \in \mathbb{C}.
\end{aligned}$$

There are also many functions whose series can be figured out by starting with one of these. For example,

$$\frac{1}{3-z} = \frac{1/3}{1-z/3} = \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n = \sum_{n=0}^{\infty} \frac{z^n}{3^{n+1}}.$$

You should be comfortable with such substitutions, as well as other operations such as addition and subtraction of power series, and derivatives and antiderivatives of power series.

You can multiply two power series using the distributive law, e.g.

$$\left(z + 3z^2 + 5z^3 + 7z^4 + \cdots\right)\left(z + z^2 + z^3 + z^4 + \cdots\right) \;=\; z^2 + 4z^3 + 9z^4 + 16z^5 + \cdots .$$

You can also divide power series using long division, e.g.

$$\begin{array}{r|r}
 & z + 3z^2 + 5z^3 + 7z^4 + \cdots \\ \hline
z + z^2 + z^3 + z^4 + \cdots & z^2 + 4z^3 + 9z^4 + 16z^5 + \cdots \\
 & -\left(z^2 + z^3 + z^4 + z^5 + \cdots\right) \\ \hline
 & 3z^3 + 8z^4 + 15z^5 + \cdots \\
 & -\left(3z^3 + 3z^4 + 3z^5 + \cdots\right) \\ \hline
 & 5z^4 + 12z^5 + \cdots \\
 & -\left(5z^4 + 5z^5 + \cdots\right) \\ \hline
 & 7z^5 + \cdots
\end{array}$$

7. Laurent Series

A **Laurent series** centered at $z = a$ is an infinite series of the form

$$\sum_{n=-\infty}^{\infty} c_n(z-a)^n \;=\; \cdots + \frac{c_{-2}}{(z-a)^2} + \frac{c_{-1}}{z-a} + c_0 + c_1(z-a) + c_2(z-a)^2 + \cdots$$

Such a series has an **annulus of convergence** $R_1 < |z-a| < R_2$. The series converges to a holomorphic function on this annulus, and diverges for $|z-a| < R_1$ or $|z-a| > R_2$.

Here R_2 is just the radius of convergence of the power series

$$c_0 + c_1(z-a) + c_2(z-a)^2 + \cdots$$

and R_1 can be found by applying the ratio or root test to the series

$$\frac{c_{-1}}{z-a} + \frac{c_{-2}}{(z-a)^2} + \frac{c_{-3}}{(z-a)^3} + \cdots .$$

Laurent's theorem states that if a function $f(z)$ is holomorphic on an annulus $R_1 < |z-a| < R_2$, then there exists a Laurent series for $f(z)$ that converges on this annulus. This includes the case where $R_1 = 0$, so a function with an isolated singularity has a Laurent series in a neighborhood of the singularity.

8. Finding Laurent Series

The methods for finding Laurent series are very similar to the methods for finding power series of complicated functions, such as substitution, addition, subtraction, multiplication, and division. For example,

$$\exp(1/z) \;=\; \sum_{n=0}^{\infty} \frac{1}{n!\,z^n} \;=\; \cdots + \frac{1}{3!\,z^3} + \frac{1}{2!\,z^2} + \frac{1}{z} + 1$$

and

$$\frac{e^z}{z^3} = \sum_{n=0}^{\infty} \frac{z^{n-3}}{n!} = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{2!\,z} + \frac{1}{3!} + \frac{z}{4!} + \cdots$$

Laurent series can often be obtained using the sum formula for a geometric series. For example,

$$\frac{1}{z-1} = \frac{1/z}{1-1/z} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n = \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} = \cdots + \frac{1}{z^4} + \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z}$$

for $|z| > 1$.

9. Residues

Suppose a holomorphic function $f(z)$ has an isolated singularity at $z = a$, and let

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$$

be the Laurent series for $f(z)$ centered at $z = a$. Then

$$\operatorname*{Res}_{z=a} f(z) = c_{-1}.$$

It follows that for any $n \in \mathbb{Z}$,

$$c_n = \operatorname*{Res}_{z=a} \frac{f(z)}{(z-a)^{n+1}} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}}\, dz.$$

10. Isolated Singularities

A singularity $z = a$ of a function $f(z)$ is called an **isolated singularity** if there exists a disk centered at a in which $z = a$ is the only singularity of $f(z)$. For such a singularity, there is always a Laurent series for $f(z)$ near $z = a$:

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n.$$

We classify isolated singularities into three types:

- A **removable singularity** is a singularity for which the Laurent series for z is actually a power series:

  $$f(z) = c_0 + c_1(z-a) + c_2(z-a)^2 + \cdots$$

  Such a singularity can be **removed** by defining $f(a)$ to be the value of c_0, which is the same as $\lim_{z\to a} f(z)$. For example, since

  $$\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots,$$

  the function $f(z) = \sin z / z$ has a removable singularity at $z = 0$.

- A **pole** is a singularity for which the Laurent series has finitely many terms with a negative power of $z - a$:

$$f(z) = \frac{c_{-n}}{(z-a)^n} + \frac{c_{-n+1}}{(z-a)^{n-1}} + \cdots + c_0 + c_1(z-a) + c_2(z-a)^2 + \cdots$$

  The largest power n that appears in the denominator of the Laurent series is called the **order** of the pole. If $f(z)$ has a pole of order n at $z = a$, then

$$f(z) = \frac{h(z)}{(z-a)^n}$$

  for some function $h(z)$ that is holomorphic in a neighborhood of $z = a$, and satisfies $h(a) \neq 0$.

- An **essential singularity** is a singularity for which the Laurent series has infinitely many terms with a negative power of $z - a$. For example, $\exp(1/z)$ has an essential singularity at $z = 0$.

We can determine what type of singularity a function has by examining the behavior of the limit as z approaches a:

- If $\lim_{z \to a} f(z)$, exists, then $f(z)$ has a removable singularity at $z = a$.
- If $\lim_{z \to a} f(z) = \infty$, then $f(z)$ has a pole at $z = a$.
- If $\lim_{z \to a} f(z)$ does not exist but is also not ∞, then $f(z)$ has an essential singularity at $z = a$.

There is a nice trick involving series that can be used to distinguish between poles and removable singularities. If

$$f(z) = \frac{g(z)}{h(z)}$$

and neither $g(z)$ nor $h(z)$ has an essential singularity at $z = a$, then the first term of the Laurent series for $f(z)$ is equal to the first term of the Laurent series for $g(z)$ divided by the first term of the Laurent series for $h(z)$. That is,

$$\frac{b_m(z-a)^m + b_{m+1}(z-a)^{m+1} + \cdots}{c_n(z-a)^n + c_{n+1}(z-a)^{n+1} + \cdots} = \frac{b_m}{c_n}(z-a)^{m-n} + \cdots .$$

In particular, if $m \geq n$ then the quotient has a removable singularity at $z = a$, and if $m < n$ then the quotient has a pole of order $n - m$ at $z = a$.

11. Evaluating Improper Integrals

Let $f(z)$ be a rational function satisfying the following conditions:

(a) $f(z)$ is an even function.

(b) The degree of the denominator of $f(z)$ is at least two higher than the degree of the numerator.

(c) $f(z)$ has no singularities on the real axis.

In this case, it follows that

$$\int_{-\infty}^{\infty} f(x)\,dx \;=\; 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=p_k} f(z)$$

where $p_1, \ldots, p_n$ are the singularities of $f(z)$ that lie in the half-plane $\mathrm{Im}(z) > 0$. For example,

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx \;=\; 2\pi i \left(\operatorname*{Res}_{z=i} \frac{1}{z^2+1} \right) \;=\; 2\pi i \left(\frac{1}{2i} \right) \;=\; \pi.$$