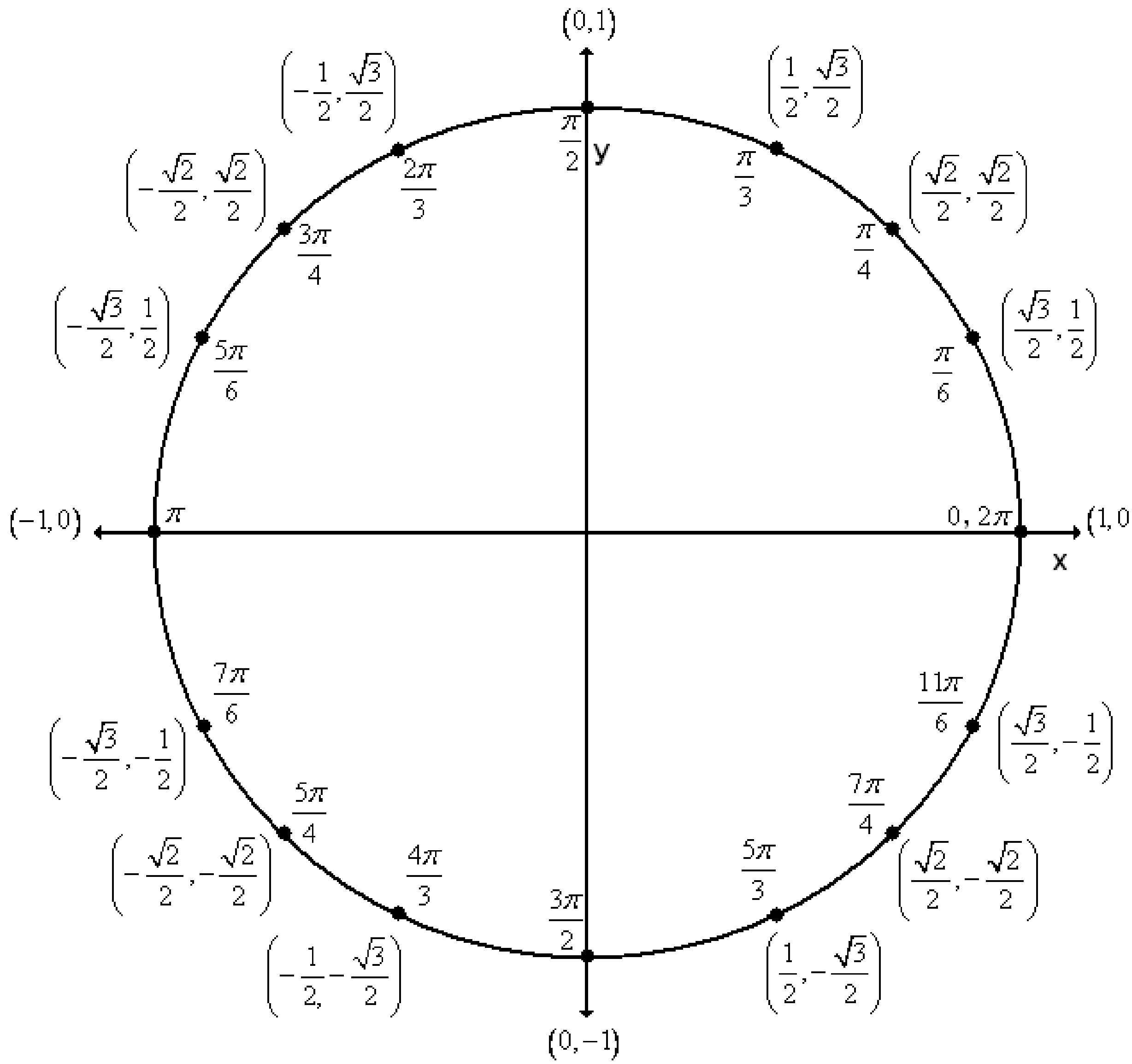
The Unit Circle

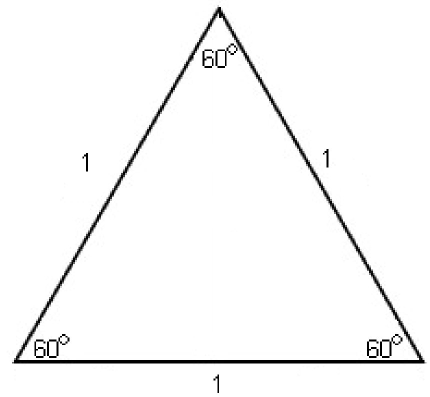
The unit circle is without a doubt the most critical topic a student must understand in trigonometry. The unit circle is the foundation on which trigonometry is based. If someone were to look at the unit circle and try to memorize it, they may find it difficult. In this section, we will discuss how to construct the unit circle, and exactly where those numbers on the unit circle come from.



This is called the unit circle, because the radius of the circle is exactly one unit. The numbers on the outside of the circle represent coordinates. These will be the x and y values with which various trigonometric functions can be evaluated. The numbers on the inside represent the radian measure of the angle. The construction of the unit circle entails the use of a conversion formula, and two different triangles. The two triangle used in the construction of a unit circle are a 30°-60°-90° right triangle, and a 45°-45°-90° right triangle. The lengths of the sides of the 30°-60°-90° triangle can be derived from a standard equilateral triangle.

The 30°-60°-90° triangle

To the right is a standard equilateral triangle. In this particular triangle, the length of each side is one unit.



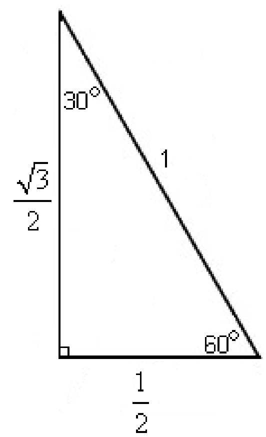
If the equilateral triangle is bisected, a 30°-60°-90° right triangle is formed. Since the triangle is bisected, the base is cut in half. The find the height of the triangle, use the Pythagorean Theorem.

$$x^2 + \left(\frac{1}{2}\right)^2 = 1^2$$

$$x^2 + \frac{1}{4} = 1$$

$$x^2 = \frac{3}{4}$$

$$x = \frac{\sqrt{3}}{2}$$



As a result, the side opposite the 60° angle has a length of $\frac{\sqrt{3}}{2}$ ***units, while the side opposite the 30° angle has a length of*** $\frac{1}{2}$ ***units. The hypotenuse was never touched, so the length of the hypotenuse remains 1 unit.***

The 45°-45°-90° triangle

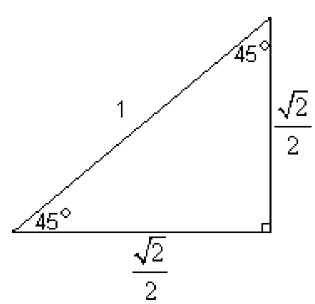
The lengths of the legs of a 45°-45°-90° triangle can be found using the Pythagorean Theorem. Since this is an isosceles triangle, the length of the two legs are equal to each other.

$$x^2 + x^2 = 1^2$$

$$2x^2 = 1$$

$$x^2 = \frac{1}{2}$$

$$x = \frac{\sqrt{2}}{2}$$



When dealing with a 45°-45°-90° triangle, the length of the sides opposite the 45° angles is $\frac{\sqrt{2}}{2}$

Building the Unit Circle

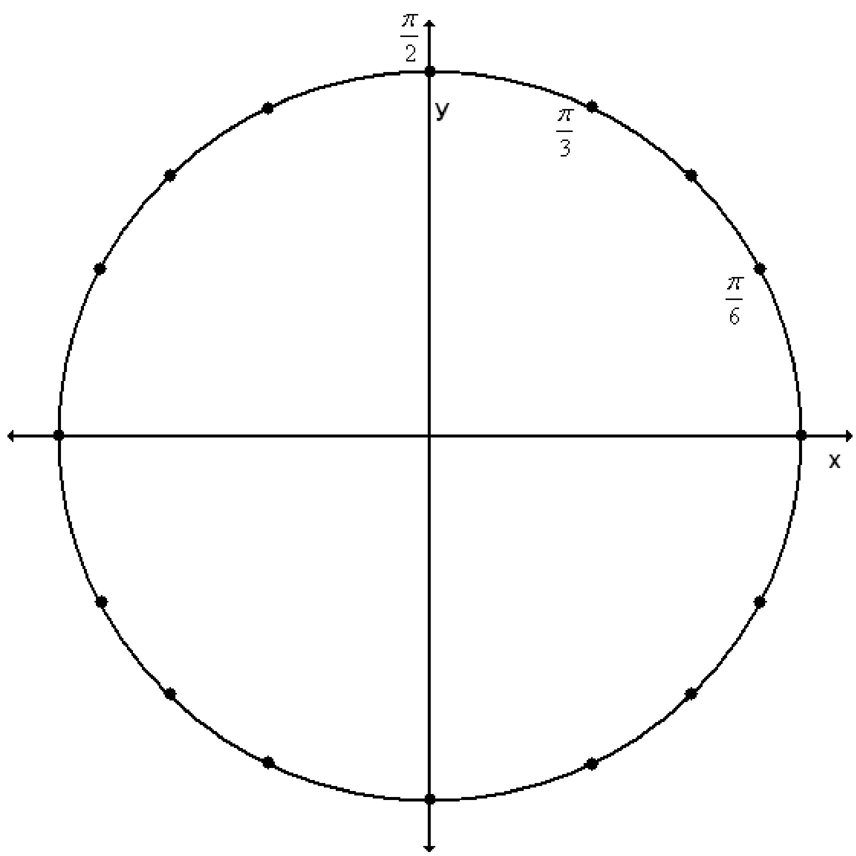
The first objective when building the unit circle is to use the conversion formula to find out the radian measures for a 30° angle, and a 45° angle. All of the angles used on the unit circle are multiples of the 30° angle and the 45° angle. Therefore, all that is needed it to add the required set. In other words, 120° is made up by 4 30° angles. A 30° angle is $\frac{\pi}{6}$ ***radians. Adding four of these together results in*** $\frac{4\pi}{6}$ ***radians which reduces to*** $\frac{2\pi}{3}$***.***

$$30° \cdot \frac{\pi}{180°} \qquad 45° \cdot \frac{\pi}{180°}$$

$$\frac{30\pi}{180} \qquad \frac{45\pi}{180}$$

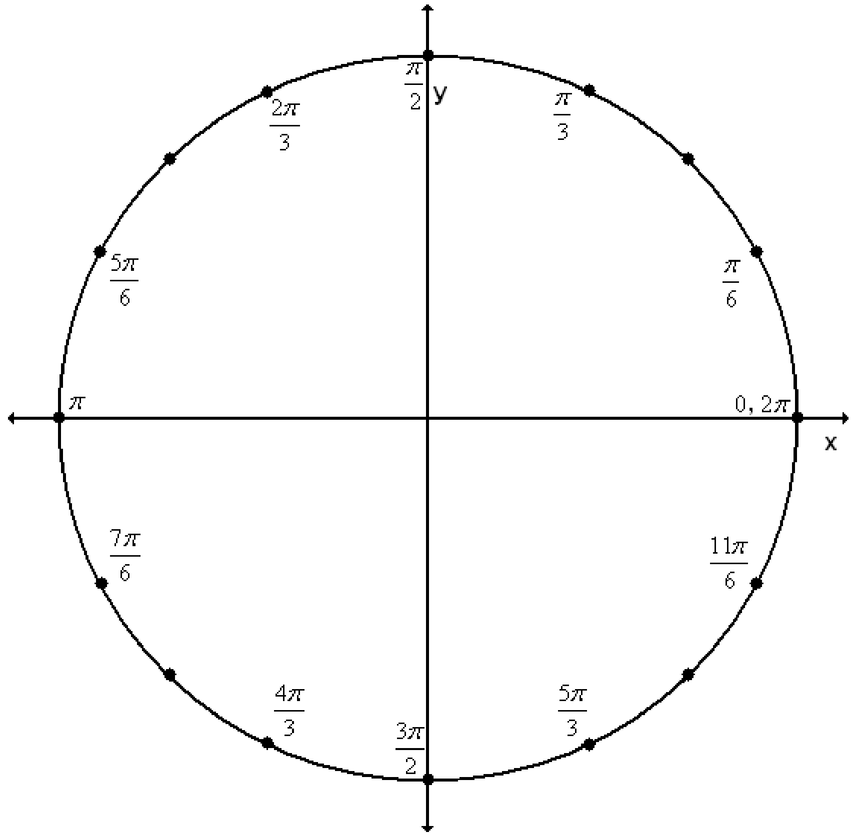
$$\frac{\pi}{6} \qquad \frac{\pi}{4}$$

Using the conversion formula, a 30° angle is $\frac{\pi}{6}$ ***radians, and a 45° angle is*** $\frac{\pi}{4}$ ***radians.***



Begin at the 30° angle. Place $\frac{\pi}{6}$ ***at that location and move around the circle in a counterclockwise direction adding by*** $\frac{\pi}{6}$ ***at every 30° increment. Make sure to reduce the totals when possible. For example, in the above diagram, to find the radian measure for 60°, add*** $\frac{\pi}{6}$ ***together twice. The results in*** $\frac{2\pi}{6}$ ***which is reduced to*** $\frac{\pi}{3}$***.***

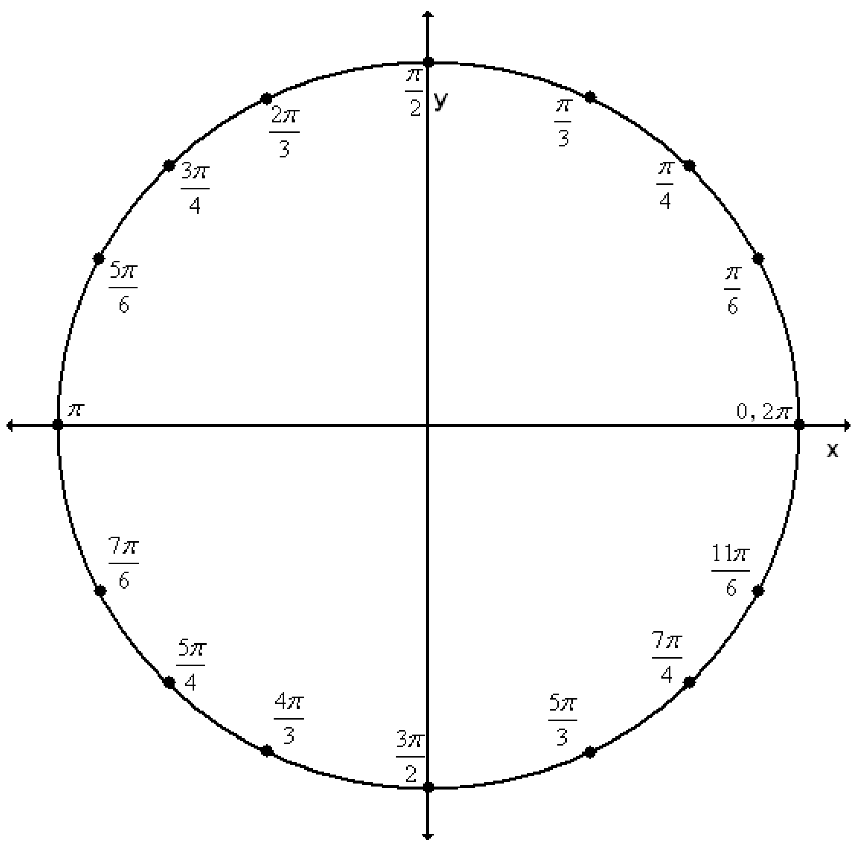
In the diagram on the right, all multiples of the 30° angles have been completed.



The next step is to continue moving counterclockwise, inputting all 45° angles.

A 45° angle is $\frac{\pi}{4}$ ***radians.***

Remember to reduce any fractions if possible.

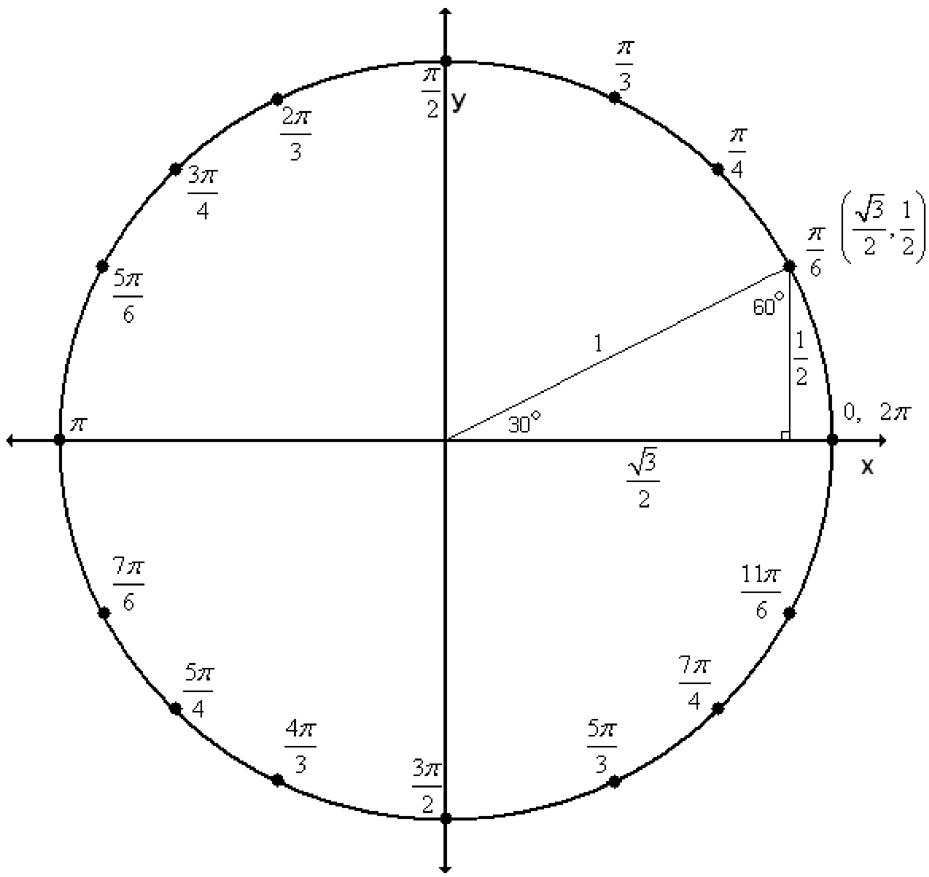


If a diagonal is drawn at the 30° angle, it will intercept the circle the first point. From there, a line is drawn to the x axis. This creates a 30°-60°-90° triangle. As demonstrated before:

The side opposite the 30° angle has a length of $\frac{1}{2}$.

The side opposite the 60° angle has a length of $\frac{\sqrt{3}}{2}$.

Since the length of the hypotenuse is 1 unit, we have found the x and y values at $\frac{\pi}{6}$. This yields to coordinates $\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$.

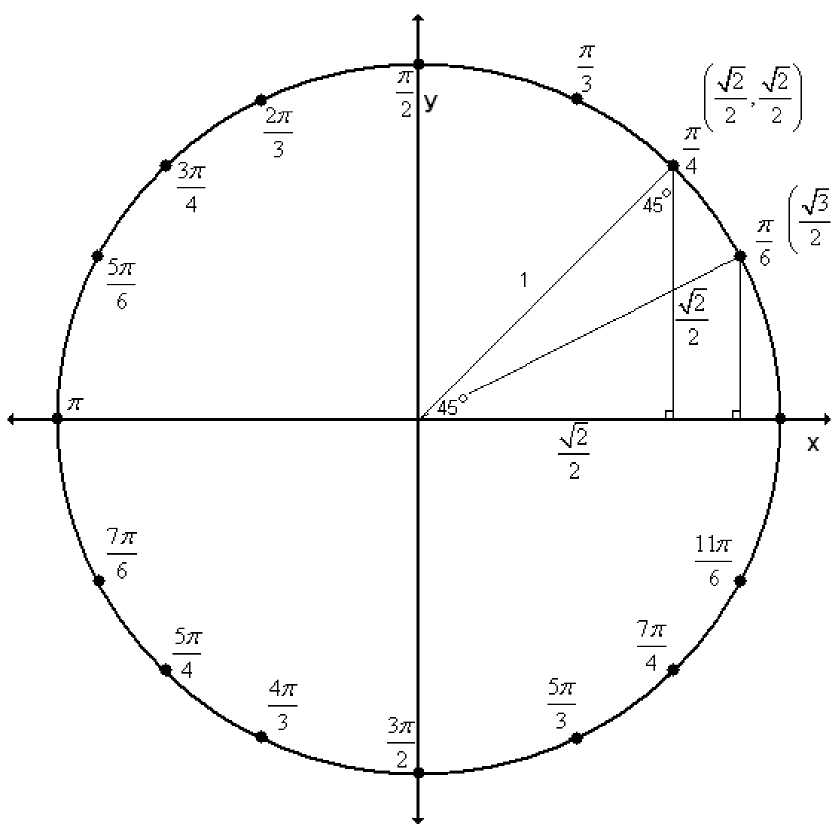


In this instance, a diagonal is drawn at the 45° angle to intercept the arc of the circle. The second line is then drawn to the x axis creating a 45°-45°-90° triangle.

The hypotenuse of the triangle has a length of 1 unit.

The length of the legs of the 45°-45°-90° triangle is $\frac{\sqrt{2}}{2}$ units.

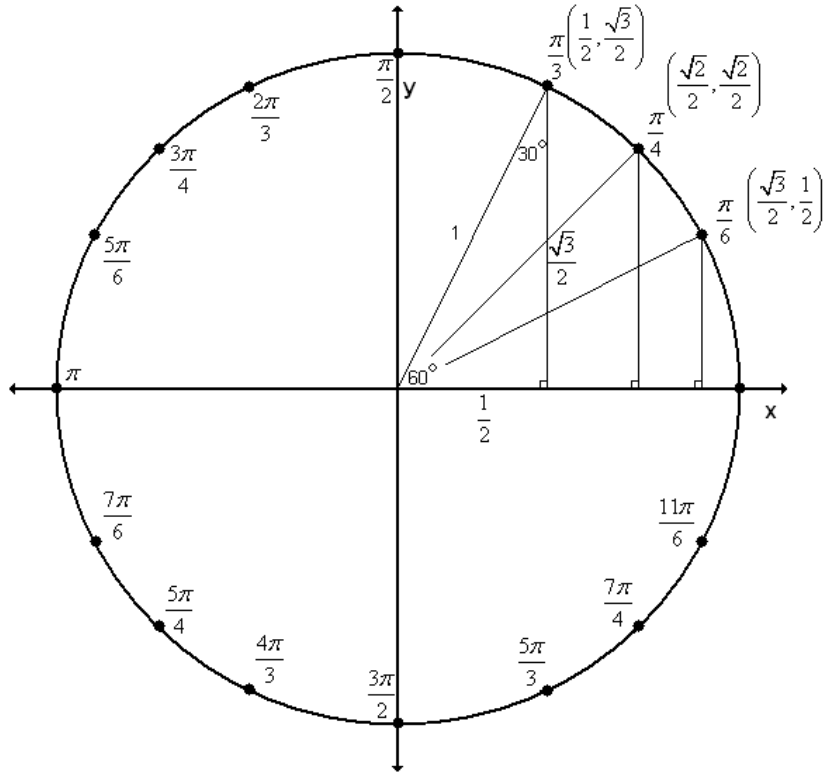
The x and y values here can be put together to find the coordinates of $\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$.



The same process is done for $\frac{\pi}{3}$.

Once again, a triangle is formed from which the lengths of the sides of the triangle are determined.

The coordinates here are $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.



To fill in the remaining coordinates use reflections of the triangle. As illustrated here, the lengths of the sides of the triangle formed at $\frac{11\pi}{6}$ are the same as those for $\frac{\pi}{6}$.

When labeling the coordinate here, however, the y value must be negative because the angle is in quadrant IV.

Once the coordinates are found in quadrant I, all others are reflections. Just take care with the sign being used.

Since the hypotenuse is always one, the coordinates on the axes are simple to find.