1. Introduction. When we meet a problem, which needs solving, what do we think of? Ordinarily, we may, first of all, wonder whether we have met this problem before. If so, we start to recall how we solved this problem before, what knowledge we used to solve it, what difficulty and setbacks we suffered, and how we succeeded finally, etc. All of these questions involve the previously encountered experience. When facing a similar problem, we will capitalize on previous experience and develop some ideas to design plans for solving it. In the course of carrying out the selected plan, we adjust and improve unceasingly until we reach the final solution of the problem. If we have never met such a kind of problem, and if no previous experience is available, we will have no ideas, no coping strategy and feel helpless . Just imagine we find

ourselves in a completely dark room, and we can see nothing, not even our hands; we couldn't find where the door is, not to mention escaping out of the room. But if we know how to think, we will find some ways to solve the problem. Let us read the following story, which is inspiring.

Joe is a boy of five years old. When he was eighteen months old, one day he was in the kitchen and saw some apples on the kitchen table. He asked his mother Laura: 'Mom, I wanted an apple, but I couldn't reach it. The table is too high and I am too short!' Laura didn't pass an apple to him, but pointed to a chair by the kitchen table and said, 'You can firstly climb on this chair and then pick up an apple'. Joe was happy to get what he wanted Laura's way. When he was two years old, it happened that one day the light bulb in the dining-room broke and a new light bulb needed to be replaced. Laura said: 'Joe, how can I change the light bulb? You see, the light is on the ceiling, it is too high for me.' She raised her arm and tried to reach for it. Joe looked at his mother's motion and the light on the ceiling, he estimated the distance between the location of the light and his mother, then he moved a stool under the light and said to his mother, 'Mom, this is enough for you to change the light bulb.' What a smart boy Joe was!

This story illustrates that Joe wanted an apple, but it was too high for him, he couldn't catch it and he didn't have any previous experience whereby to solve this problem by himself. He asked help from his mother. She showed to Joe the way to overcome the difficulty. Here, Laura of course had rich experience as to how to solve such kind of problems and she trained Joe in using her experience to solve the problem. We can say that Joe solved the problem by indirect experience. We can assume that Joe practiced many times later on and finally his mind reached a new level (let's call it methodical level or theoretical level). He was now smart enough to understand that when he meets a problem, the first thing to do was to find the difference between the goal (what he wanted to achieve; in this case the change of the light bulb) and the attending circumstances (what he knew; in this case his mother's height), then looking for ways to reduce the difference and reach the goal.

Examining our textbooks and our classroom teaching, we discover that most books and most teaching are 'experienced style', which means textbooks or teaching are designed in such an order: axioms-definitions-theorems-formulas-examples-exercises. In classroom, we state and explain definitions and basic concepts, have students learned the intension and extension of the concepts, then we show some basic properties represented by theorems or formulas. We then present some examples, which are meticulously designed, and tell students how they can use these theorems or formulas to solve problems. We usually suggest that students memorize all theorems, formulas and principles. We provide step-by-step explanation or instruction in solving each example and hope students will be able to solve the same type of problems over and over again until they feel confident and comfortable. It is hoped that students thereby accumulate experience in the process of solving problems and become smart and intelligent thanks to teachers' day-to-day nurturing.

In our teaching, we clearly know that the teaching of experienced style is not always effective. For example, the teaching of experienced style proves ineffectual when we teach how to verify trig identities in trigonometry, or how to make substitution in integration, etc. Most students, no matter how hard they try when confronted with a problem, do not know where to start, for instance, when we let them do a problem in verifying an identity in trigonometry or in finding an integral, despite the fact that we presented many examples before. But it is a different situation with solving a quadratic equation. They have only one choice: applying the quadratic formula. Most students can solve the quadratic equation by means of the formula. In trigonometry, students lose their minds when they are asked to verify an identity.

They do not know how to choose suitable formulas to perform the task, since there are too many formulas available (multiple choices!). The situation in which we present a large number of examples with a view to helping students to acquire the skills whereby to solve problems, with the result that students still feel lost compels us to seek a new way of teaching. We feel deeply that imitation and accumulating experience are important but not good enough for learning; students must learn how to think, not just how to do it. They need to learn to develop a correct idea and find out where a correct idea comes from.

There are currently so many exercise books in use; unfortunately, almost all books give solutions only, but fail to illustrate how to reach solutions and what methodology to apply. Can we say that almost all current books are experienced style, but not methodical style or thinking style? Mathematician George Polya wrote a famous book How to solve it, in which he tried to explain where a correct idea comes from or how a correct idea can be found in solving a problem. We think that the approach of this book belongs to methodical style. He expounded some basic problem solving principles in his book:

1. Understand the problem;

2. Think of a plan:

(1) Try to recognize something familiar,

(2) Try to recognize patterns,

(3) Use analogy, etc.

It is clear that the core of these principles is that the problem solver must have some experience. But it doesn't work for freshman or sophomore students when they learn trigonometry and verify trig identities. They have no such prior experience. Some books begin with logic and set theory to help students learn proofs, but our students do not have such a background. To verify a trig identity for the first time, students need the knowledge of logic and set theory. Even though students can learn something of logic and set theory, how many students can use the principles to verify a trig identity? Some philosophic books touch upon principles of ideology and methodology, but there is no suitable books that can make students apply them easily to solve mathematical problems, in particular, to verify a trig identity. We think the best way is for mathematics teachers to write special articles in an attempt to cultivate the thinking way for our students. This is the reason why I am writing this little article.

2. What is the essence of verifying a trig identity? Let us look at an example:

Example 1. Verify the following identity

$$\frac{\sin 2x+\sin 2y}{2\cos(x-y)}=\sin(x+y)$$

A good student starts the proof as follows

$$L=\frac{2\sin x\cos x+2\sin y\cos y}{2(\cos x\cos y+\sin x\sin y)}=\frac{\sin x\cos x+\sin y\cos y}{\cos x\cos y+\sin x\sin y}$$

Because she knows the double angle formula and the formulas involves the difference of two angle *s*. After she gets to the second step she does not know what the next step to help her reach the right side $\sin(x+y)$ of the equal sign is. We may deem that she does not know what the essence of verifying trig identities is.

Given an identity, if the expressions on each side of the equal sign are the same, we know we have nothing to prove. An identity only needs to be verified when the expressions on the two sides of the equal sign are different formally. What kinds of differences do we find in trig identities? Let us look at some examples.

Example 2. Verify the following identities:

(1) $1-\cos 2\theta+\cos 4\theta-\cos 6\theta=4\sin\theta\cos 2\theta\sin 3\theta$;

(2) $\csc\theta\tan\frac{\theta}{2}-\frac{\cos 2\theta}{1+\cos\theta}=4\sin^2\frac{\theta}{2}$;

(3) $\frac{\tan\theta+\sec\theta-1}{\tan\theta-\sec\theta+1}=\tan\theta+\sec\theta$.

Let us look at identity (1). The involving operations are additions and subtractions in the left side of the equal sign, but the involving operations are multiplications in the right side of the equal sign. We see the two sides have a difference of operations. The left side contains only cosine functions, but the right side contains sine functions and cosine function. We see that the two sides have a difference of functions. From the view of angle *s* we see that the left involves the angle *s* of $2\theta$, $4\theta$ and $6\theta$, but the right involves the angles $\theta$, $2\theta$ and $3\theta$. We see that the two sides have a difference of angle *s*.

Similarly, look at the identity (2) we see that the two sides of the equal sign have essentially three kinds of differences: angle*s*, functions and operations. In the identity (3), the two sides have only the difference of operations: the left contains division and the right side contains addition.

These three examples above show that we have an identity with one kind of difference of the two sides and two identities with three kinds of differences between the two sides of the equal signs. There is never going to be a trig identity with more than three differences between their two sides of the equal sign, and it is very important to keep this in mind. We can categorize the differences between the two sides of a trig identity in terms of three parameters: angle *s*, functions and operations.

Now we are ready to answer what is the essence of verifying a trig identity. Roughly it is identification of the differences between the two sides. To put it more precisely, it is the process of using some trig formulas to reduce the differences in the three viewpoints of angle *s*, functions and operations. When the differences are all gone, the verification is achieved.

3. What is the key to verifying a trig identity? We know that the process of verifying an identity is

also the process to using some basic formulas to transform one side of the identity to another side. The key to using all formulas is memorizing and understanding well all the formulas from the three perspectives of angle *s*, functions and operations.

You may raise the question: what does ‘good understanding’ mean?

In trigonometry, there are many basic formulas. For example, the square relationship of secant and tangent

$$1+\tan^2\theta=\sec^2\theta$$

And the double angle formula of cosine

$$\cos 2\theta=1-2\sin^2\theta$$

And so on. We need to understand all trig formulas from the new perspectives. The discussion above leads us to understand that using formulas is for the purpose of reducing the differences of angle*s*, functions and operations in two sides of the equal sign. Thus, when we look at a formula, we should look at the formula converting what angles to what angle *s*, what functions to what functions and what operations to what operations and finally memorizing them.

The name of the first formula above emphasizes the operation: square relation. Now looking at this formula, we should think of it from three viewpoints of angle*s*, functions and operations. We see that there is more than squaring involved when using the first formula from the left to the right or from the right to the left. In other words, the function of tangent is changed to the function of secant; the angle *s* however do not change; and the sum is changed to the product from left to right. On the other hand, the function of secant is changed to the function of tangent, and the product is changed to the sum from right to left. From the first formula we also have an equivalent formula

$$\sec^2\theta - \tan^2\theta = 1$$

which can help us to convert difference to product as well as deleting the functions tangent, secant and the angle $\theta$ from the left to the right.

The name of the second formula emphasizes the relation of double angle *s*. It converts double angle to single angle from the left to the right of the equal sign. At the same time it converts the function of cosine to the function of sine and the product (seeming single tern of $\cos 2\theta$ to be the product of 1 and $\cos 2\theta$) to the difference of 1 and $2\sin^2\theta$. From the right to the left, it converts single angle to double angle; function of sine to function of cosine and difference to product in operations. An equivalent form with the second formula is

$$1 - \cos 2\theta = 2\sin^2\theta$$

From left to right it converts double angle to single angle, function of cosine to function of sine and difference to product; from right to left, it converts single angle to double angle, the function of sine to the function of cosine and the product to the difference. Looking at and understanding each formula as used the way we explained above, then we reach the level of good understanding by classifying all formulas into three groups by the actions of converting angle $s$, functions and operations.

4. Every problem offers some clues which show us how to solve it. When we know the essence of verifying trig identities and completely understand all trig formulas from the three viewpoints of angle $s$, functions and operations, we will see that every problem offers some clues which show us how to solve it. Let us see the identity in Example 1

$$\frac{\sin 2x+\sin 2y}{2\cos(x-y)}=\sin(x+y)$$

We see there is the difference of angle $s$: the left contains angle $s$ of $2x$, $2y$ and $x-y$, the right side contains angle of $x+y$; there is the difference of operations: the left contains addition in numerator and division, the right side contains only multiplication; there is the difference of functions: the left contains cosine, but the right side only contains the function of sine. We have pointed out above that the process of verifying an identity is the process of using some trig formulas to reduce the differences in the three viewpoints of angle$s$, functions and operations. When the differences are all gone, the verification will be fulfilled. Hence, in judging whether each step in the process of verification is taken in the right direction or wrong direction, we just need normally to see whether the step is or isn't reducing the differences of angle*s*, functions and operations. The student we mentioned before used the double angle formulas and the formula involving difference of angle *s* formula, but did not reduce any difference of the two sides of the equal sign. That her two steps did not make any progress convinces us that the formulas she used are not suitable. Let us look for the clues which this identity hints. In fact, the differences of the three views, in particular the difference of operations, offers a clue to us: using the sum to product formula in the numerator. Looking at the numerator, we naturally choose the following formula

$$\sin\alpha+\sin\beta=2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

Thus

$$\sin 2x+\sin 2y=2\sin\frac{2x+2y}{2}\cos\frac{2x-2y}{2}=$$
$$2\sin(x+y)\cos(x-y)$$

Hence the left side of the identity in Example 1 should be

$$L=\frac{2\sin(x+y)\cos(x-y)}{2\cos(x-y)}$$

Now what is the difference of the left side with the right side? It is obvious that canceling the common factors of 2 $\cos(x-y)$, we will catch the right side. Taking just the two steps described above helps us finish the proof effortlessly, you see how powerful the approach is.

Next, let us examine how to prove the identity (1) in Example 2

$$1-\cos 2\theta+\cos 4\theta-\cos 6\theta=4\sin\theta\cos 2\theta\sin 3\theta$$

We see that the left contains four terms and the right side contains only one term. Do you feel that this identity strongly hints to us using the sum to product formulas? Looking at the left side only contains the function of cosine; we naturally choose the following formula

$$\cos\alpha-\cos\beta=2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

Thus

$$\cos 4\theta-\cos 6\theta=2\sin\frac{4\theta+6\theta}{2}\sin\frac{6\theta-4\theta}{2}=$$
$$2\sin 5\theta\sin\theta$$

From the discussion about double angle formula of cosine above, we know double angle formula can also help us convert sum to product: $1-\cos 2\theta=2\sin^2\theta$. Hence the left side of the identity (1) becomes

$$L=2\sin^2\theta+2\sin 5\theta\sin\theta$$

From four terms to two terms in the left, we have made progress; thus we know the direction of our proof effort is correct, but it has not reached the right side yet. Look at the right side of the identity, which contains the function of $2\sin\theta$ in product from.

Look at the left side now and we see the two terms both contain2 sin $\theta$. The common thing hints to us factoring the common factor in the left side. Thus we have made more progress

$$L=2\sin\theta(\sin\theta+\sin 5\theta)$$

Comparing the above with the right side of the i-dentity, we know by using just one more time the sum to product formula of

$$\sin\alpha+\sin\beta=2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

we will reach the right side of the identity (1). The proof goes as follows.

Proof of (1), Example 2

$$\begin{aligned}L=&2\sin^2\theta+2\sin 5\theta\sin\theta=\\&2\sin\theta(\sin\theta+\sin 5\theta)=\\&2\sin\theta(2\sin 3\theta\cos 2\theta)=\\&4\sin\theta\cos 2\theta\sin 3\theta=R\end{aligned}$$

Next, let us see the identity (2) of example 2

$$\csc\theta\tan\frac{\theta}{2}-\frac{\cos 2\theta}{1+\cos\theta}=4\sin^2\frac{\theta}{2}$$

We see the two sides have three differences of angle $s$, functions and operations. From the right we see the angle is $\frac{\theta}{2}$ and the function is sine, which hints to us converting functions of cosine, tangent and cosecant into function of sine; converting $2\theta$ to $\theta$ for reducing the differences. By the three formulas

$$\csc\theta=\frac{1}{\sin\theta},\tan\alpha=\frac{\sin\alpha}{\cos\alpha}\text{ and }\cos 2\theta=1-2\sin^2\theta$$

we get the first step from the left side

$$L=\frac{\sin\frac{\theta}{2}}{\sin\theta\cos\frac{\theta}{2}}-\frac{1-2\sin^2\theta}{2\cos^2\frac{\theta}{2}}$$

Now that the differences of the functions and the angle *s* are reduced, we know we have made further progress! The operation of the right side is product form, so we need to combine the two fractions in the left side. Firstly we need to find common denominator if we want to combine two fractions. Consider the right side, which contains only the angle of $\frac{\theta}{2}$, the best way for using the double angle formula of sine is

$$\sin 2\alpha=2\sin\alpha\cos\alpha$$

and then pushing the left side to

$$L=\frac{\sin\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos^2\frac{\theta}{2}}-\frac{1-2\sin^2\theta}{2\cos^2\frac{\theta}{2}}$$

In the first fraction we see $\sin\frac{\theta}{2}$ is the common factor for numerator and denominator. After canceling the common factor, two fractions have the common denominator of $2\cos^2\frac{\theta}{2}$; the condition of combining two fractions is ripe.

$$L=\frac{1}{2\cos^2\frac{\theta}{2}}-\frac{1-2\sin^2\theta}{2\cos^2\frac{\theta}{2}}=\frac{2\sin^2\theta}{2\cos^2\frac{\theta}{2}}$$

Using the double angle formula of sine again for reducing the difference of the angle *s* in the left side with the right side, we hit the goal.

Proof of (2), Example 2

$$L=\frac{\sin\frac{\theta}{2}}{\sin\theta\cos\frac{\theta}{2}}-\frac{1-2\sin^2\theta}{2\cos^2\frac{\theta}{2}}=$$

$$\frac{\sin\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos^2\frac{\theta}{2}}-\frac{1-2\sin^2\theta}{2\cos^2\frac{\theta}{2}}=$$

$$\frac{1}{2\cos^2\frac{\theta}{2}}-\frac{1-2\sin^2\theta}{2\cos^2\frac{\theta}{2}}=\frac{2\sin^2\theta}{2\cos^2\frac{\theta}{2}}=$$

$$\frac{4\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}}=$$

$$4\sin^2\frac{\theta}{2}=R$$

Finally let us look at the identity (3), Example 2

$$\frac{\tan\theta+\sec\theta-1}{\tan\theta-\sec\theta+1}=\tan\theta+\sec\theta$$

The solution one textbook offered goes as follows

$$\frac{\tan\theta+\sec\theta-1}{\tan\theta-\sec\theta+1}=\frac{\tan\theta+(\sec\theta-1)}{\tan\theta-(\sec\theta-1)}\cdot\frac{\tan\theta+(\sec\theta-1)}{\tan\theta+(\sec\theta-1)}=$$

$$\frac{\tan^2\theta+2\tan\theta(\sec\theta-1)+\sec^2\theta-2\sec\theta+1}{\tan^2\theta-(\sec^2\theta-2\sec\theta+1)}=$$

$$\frac{\sec^2\theta-1+2\tan\theta(\sec\theta-1)+\sec^2\theta-2\sec\theta+1}{\sec^2\theta-1-(\sec^2\theta-2\sec\theta+1)}=$$

$$\frac{2\sec^2\theta-2\sec\theta+2\tan\theta(\sec\theta-1)}{-2+2\sec\theta}=$$

$$\frac{2\sec\theta(\sec\theta-1)+2\tan\theta(\sec\theta-1)}{2(\sec\theta-1)}=$$

$$\frac{2(\sec\theta+\tan\theta)(\sec\theta-1)}{2(\sec\theta-1)}=$$

$$\sec\theta+\tan\theta$$

The natural question from students is why you both multiply $\tan\theta+(\sec\theta-1)$ on the numerator and denominator in the first step? It is not easy to explain it for students who didn't have any experience for proving problems. But using our thinking method it is not difficult to find a natural way to verify this identity.

First of all, let's see what the difference between the two sides of the equal sign in this identity is. We see that the angle *s* are the same and the functions are the same kind in the two sides. The only difference between the two sides of the equal sign is operations. The left side is the quotient of two expressions and right side is the sum of two expressions. We just need to convert the quotient to the sum, and the verification will be done. What is the clue which the identity offers to us? Look at the numerator in the left we see it contains the expression in the right: $\tan\theta+\sec\theta$. Can we factor $\tan\theta+\sec\theta$ from the numerator of the left? That means 1 should contain a factor of $\tan\theta+\sec\theta$. Does 1 have a factor of $\tan\theta+\sec\theta$ It reminds us of the formula

$$1=\sec^2\theta-\tan^2\theta=(\sec\theta+\tan\theta)(\sec\theta-\tan\theta)$$

Thus, we have the following proof.

$$L=\frac{(\tan\theta+\sec\theta)-(\sec^2\theta-\tan^2\theta)}{\tan\theta-\sec\theta+1}=$$

$$\frac{(\tan\theta+\sec\theta)-(\sec\theta+\tan\theta)(\sec\theta-\tan\theta)}{\tan\theta-\sec\theta+1}=$$

$$\frac{(\tan\theta+\sec\theta)[1-(\sec\theta-\tan\theta)]}{\tan\theta-\sec\theta+1}=$$

$$\frac{(\tan\theta+\sec\theta)(1-\sec\theta+\tan\theta)}{\tan\theta-\sec\theta+1}=$$

$$\tan\theta+\sec\theta=R$$

We introduce to you a think method with discussion above. See a method whether is or isn't powerful, the better way is applying it to solve some difficult problems, which normally couldn't be solved without a correct thinking way. In 1998, I coached three high school students and trained them to learn how to prove trig identities. Later they designed a web site. If you are interested in thinking methods, we welcome you and your students to visit our web site. The address is: http://thinkquest.outofcore.com in which you will see how powerful our method is.