Development of Trigonometric Formula for $\text{Sin}^4 n\theta$ and $\text{Cos}^4 n\theta$ Using Half Angle Identities as Mathematical Algorithm

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***Abstract*—In this paper, a fast and reliable formula for simplifying $\sin^4 n\theta$ and $\cos^4 n\theta$ using Half Angle Identities will be developed. This formula offers a simplified approach in simplifying sine and cosine function with exponent 4 to a function of cosine with exponent 1 with any value of n. From the algorithm of solving of half angle identities the new formula will emerge.**

***Index Terms*—Mathematical algorithm, half angle identity, multiple angle formula, half angle formula, trigonometric identities.**

I. Introduction

Half angle formulas are very useful. It is use to rewrite and evaluate multiple angle formulas. Usually it involves the other multiple angle formulas like the double angle formula to simplify the expression. Rational functions involving sine and cosine become easier to manipulate with the existence of half angle and the degree of difficulty to solve or to work on the problems involving sine and cosine becomes easier especially in the case of powers of sine and cosine. Half angle formulas and half angle identities all started from trigonometric identities. Simple ways on derivation of formula are shown in [1]-[3].

Half angle formula will direct you closer to the angle of the unit circle. It is the better option in order to find the trigonometric values of any angle that can be expressed as half of another angle on the unit circle.

Half-angle identities are derived from half angle formulas. Common application of these is to evaluate trigonometric function of an angle that isn't on the unit circle [4].

The purpose of this paper is to present a new formula to simplify the solution in expressing$\text{Sin}^4 n\theta$ and $\text{Cos}^4 n\theta$ in terms of cosine function with exponent 1. Some examples are given using the old method and the new one. The results of the new formula suggest that the new formula is easier to use than the old one.

Half angle identities for sine and cosine [5], [6].

$$\sin^2 u = \frac{1-\cos 2u}{2}$$

$$\cos^2 u = \frac{1+\cos 2u}{2}$$

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II. Procedure of the Old Method

The usual procedure in evaluating half angle identity is to apply double angle and half angle formulas. In expressing sine and cosine function in terms of cosine with exponent one is to factor the given to reduce the exponent. All functions with equivalent identity are substituted before manipulating and simplifying the equation.

If you are to express $sin^4 u$ in terms of the cosine function with exponent 1. The procedure will be as follows:

Factor $\sin^4 u$

$$\sin^4 u = (\sin^2 u)(\sin^2 u)$$

Substitute $\sin^2 u$ by $(\frac{1-\cos 2u}{2})$

$$\sin^4 u = (\frac{1-\cos 2u}{2})\ (\frac{1-\cos 2u}{2})$$

Multiply

$$\sin^4 u = \frac{1-2\cos 2u+\cos^2 2u}{4}$$

Simplify

$$\sin^4 u = \frac{1}{4}-\frac{1}{2}\cos 2u+\frac{1}{4}\cos^2 2u$$

Substitute $\cos^2 2u$ by $(\frac{1+\cos 4u}{2})$

$$\sin^4 u = \frac{1}{4}-\frac{1}{2}\cos 2u+\frac{1}{4}(\frac{1+\cos 4u}{2})$$

Simplify

$$\sin^4 u = \frac{1}{4}-\frac{1}{2}\cos 2u+\frac{1}{8}+\frac{1}{8}\cos 4u$$

Add $\frac{1}{4}+\frac{1}{8}$ then simplify

$$\sin^4 u = \frac{3}{8}-\frac{1}{2}\cos 2u+\frac{1}{8}\cos 4u$$

Similarly, if you are to express $sin^4 2u$ in terms of the cosine function with exponent 1. The procedure will be:

Factor $\sin^4 2u$

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$$\sin^4 2u = (\sin^2 2u)(\sin^2 2u)$$

Substitute $\sin^2 2u$ by $(\frac{1-\cos 4u}{2})$

$$\sin^4 2u = (\frac{1-\cos 4u}{2})\ (\frac{1-\cos 4u}{2})$$

Multiply

$$\sin^4 2u = \frac{1-2\cos 4u + \cos^2 4u}{4}$$

Simplify

$$\sin^4 2u = \frac{1}{4} - \frac{1}{2}\cos 4u + \frac{1}{4}\cos^2 4u$$

Substitute $\cos^2 4u$ by $(\frac{1+\cos 8u}{2})$

$$\sin^4 2u = \frac{1}{4} - \frac{1}{2}\cos 4u + \frac{1}{4}(\frac{1+\cos 8u}{2})$$

Simplify

$$\sin^4 2u = \frac{1}{4} - \frac{1}{2}\cos 4u + \frac{1}{8} + \frac{1}{8}\cos 8u$$

Add $\frac{1}{4} + \frac{1}{8}$ then simplify

$$\sin^4 2u = \frac{3}{8} - \frac{1}{2}\cos 4u + \frac{1}{8}\cos 8u$$

Another example is express $sin^4 3u$ in terms of cosine function with exponent 1. The same procedure will be followed:

Factor $\sin^4 3u$

$$\sin^4 3u = (\sin^2 3u)(\sin^2 3u)$$

Substitute $\sin^2 3u$ by $(\frac{1-\cos 6u}{2})$

$$\sin^4 3u = (\frac{1-\cos 6u}{2})\ (\frac{1-\cos 6u}{2})$$

Multiply

$$\sin^4 3u = \frac{1-2\cos 6u + \cos^2 6u}{4}$$

Simplify

$$\sin^4 3u = \frac{1}{4} - \frac{1}{2}\cos 6u + \frac{1}{4}\cos^2 6u$$

Substitute $\cos^2 6u$ by $(\frac{1+\cos 12u}{2})$

$$\sin^4 3u = \frac{1}{4} - \frac{1}{2}\cos 6u + \frac{1}{4}(\frac{1+\cos 12u}{2})$$

Simplify

$$\sin^4 3u = \frac{1}{4} - \frac{1}{2}\cos 6u + \frac{1}{8} + \frac{1}{8}\cos 12u$$

Add $\frac{1}{4} + \frac{1}{8}$ then simplify

$$\sin^4 3u = \frac{3}{8} - \frac{1}{2}\cos 6u + \frac{1}{8}\cos 12u$$

Generalizing the results for the three examples given

$$\sin^4 u = \frac{3}{8} - \frac{1}{2}\cos 2u + \frac{1}{8}\cos 4u$$

$$\sin^4 2u = \frac{3}{8} - \frac{1}{2}\cos 4u + \frac{1}{8}\cos 8u$$

$$\sin^4 3u = \frac{3}{8} - \frac{1}{2}\cos 6u + \frac{1}{8}\cos 12u$$

We can say that for every function $sin^4 nu$ the simplified expression in terms of cosine function with exponent 1 is:

$$\sin^4 nu = \frac{3}{8} - \frac{1}{2}\cos 2nu + \frac{1}{8}\cos 4nu$$

In the case of cosine function with exponent 4, the same method will be applied:

If you are to express $cos^4 u$ in terms of the cosine function with exponent 1. The procedure will be as followed:

Factor $\cos^4 u$

$$\cos^4 u = (\cos^2 u)(\cos^2 u)$$

Substitute $\cos^2 u$ by $(\frac{1+\cos 2u}{2})$

$$\cos^4 u = (\frac{1+\cos 2u}{2})\ (\frac{1+\cos 2u}{2})$$

Multiply

$$\cos^4 u = \frac{1+2\cos 2u + \cos^2 2u}{4}$$

Simplify

$$\cos^4 u = \frac{1}{4} + \frac{1}{2}\cos 2u + \frac{1}{4}\cos^2 2u$$

Substitute $\cos^2 2u$ by $(\frac{1+\cos 4u}{2})$

$$\cos^4 u = \frac{1}{4} + \frac{1}{2}\cos 2u + \frac{1}{4}(\frac{1+\cos 4u}{2})$$

Simplify

$$\cos^4 u = \frac{1}{4} + \frac{1}{2}\cos 2u + \frac{1}{8} + \frac{1}{8}\cos 4u$$

Add $\frac{1}{4} + \frac{1}{8}$ then simplify

$$\cos^4 u = \frac{3}{8} + \frac{1}{2}\cos 2u + \frac{1}{8}\cos 4u$$

Similarly if you are to express $cos^4 2u$ in terms of the cosine function with exponent 1. The procedure will be

Factor $\cos^4 2u$

$$\cos^4 2u = (\cos^2 2u)(\cos^2 2u)$$

Substitute $\cos^2 2u$ by $(\frac{1+\cos 4u}{2})$

$$\cos^4 2u = (\frac{1+\cos 4u}{2})\ (\frac{1+\cos 4u}{2})$$

Multiply

$$\cos^4 2u = \frac{1+2\cos 4u+\cos^2 4u}{4}$$

Simplify

$$\cos^4 2u = \frac{1}{4}+\frac{1}{2}\cos 4u+\frac{1}{4}\cos^2 4u$$

Substitute $\cos^2 4u$ by $(\frac{1+\cos 8u}{2})$

$$\cos^4 2u = \frac{1}{4}+\frac{1}{2}\cos 4u+\frac{1}{4}(\frac{1+\cos 8u}{2})$$

Simplify

$$\cos^4 2u = \frac{1}{4}+\frac{1}{2}\cos 4u+\frac{1}{8}+\frac{1}{8}\cos 8u$$

Add $\frac{1}{4}+\frac{1}{8}$ then simplify

$$\cos^4 2u = \frac{3}{8}+\frac{1}{2}\cos 4u+\frac{1}{8}\cos 8u$$

Another example is express $cos^4 3u$ in terms of cosine function with exponent 1. The same procedure will be followed

Factor $\cos^4 3u$

$$\cos^4 3u = (\cos^2 3u)(\cos^2 3u)$$

Substitute $\cos^2 3u$ by $(\frac{1+\cos 6u}{2})$

$$\cos^4 3u = (\frac{1+\cos 6u}{2})\ (\frac{1+\cos 6u}{2})$$

Multiply

$$\cos^4 3u = \frac{1+2\cos 6u+\cos^2 6u}{4}$$

Simplify

$$\cos^4 3u = \frac{1}{4}+\frac{1}{2}\cos 6u+\frac{1}{4}\cos^2 6u$$

Substitute $\cos^2 6u$ by $(\frac{1+\cos 12u}{2})$

$$\cos^4 3u = \frac{1}{4}+\frac{1}{2}\cos 6u+\frac{1}{4}(\frac{1+\cos 12u}{2})$$

Simplify

$$\cos^4 3u = \frac{1}{4}+\frac{1}{2}\cos 6u+\frac{1}{8}+\frac{1}{8}\cos 12u$$

Add $\frac{1}{4}+\frac{1}{8}$ then simplify

$$\cos^4 3u = \frac{3}{8}+\frac{1}{2}\cos 6u+\frac{1}{8}\cos 12u$$

Generalizing the result for the three examples given

$$\cos^4 u = \frac{3}{8}+\frac{1}{2}\cos 2u+\frac{1}{8}\cos 4u$$

$$\cos^4 2u = \frac{3}{8}+\frac{1}{2}\cos 4u+\frac{1}{8}\cos 8u$$

$$\cos^4 3u = \frac{3}{8}+\frac{1}{2}\cos 6u+\frac{1}{8}\cos 12u$$

We can say that for every function $cos^4 nu$ the simplified expression in terms of cosine function with exponent 1 is:

$$\cos^4 nu = \frac{3}{8}+\frac{1}{2}\cos 2nu+\frac{1}{8}\cos 4nu$$

Therefore, instead of simplifying the function $sin^4 nu$ and $cos^4 nu$ into a function with exponent 1 using the long method we can use the simplified formula which is:

$$\sin^4 nu = \frac{3}{8}-\frac{1}{2}\cos 2nu+\frac{1}{8}\cos 4nu$$

$$\cos^4 nu = \frac{3}{8}+\frac{1}{2}\cos 2nu+\frac{1}{8}\cos 4nu$$

III. Procedure of the New Method

With the new formula we can solve the same problem with different values of n. This makes the procedure easy and solution shorter.

Example:

Express the following example in terms of cosine function with exponent 1 using the formula written below

$$\sin^4 nu = \frac{3}{8}-\frac{1}{2}\cos 2nu+\frac{1}{8}\cos 4nu$$

$$\cos^4 nu = \frac{3}{8}+\frac{1}{2}\cos 2nu+\frac{1}{8}\cos 4nu$$

1) $\cos^4 5u = \frac{3}{8}+\frac{1}{2}\cos 2nu+\frac{1}{8}\cos 4nu$

$$= \frac{3}{8}+\frac{1}{2}\cos 2(5)u+\frac{1}{8}\cos 4(5)u$$

$$= \frac{3}{8}+\frac{1}{2}\cos 10)u+\frac{1}{8}\cos 20u$$

2) $\cos^4 8u = \frac{3}{8}+\frac{1}{2}\cos 2nu+\frac{1}{8}\cos 4nu$

$$= \frac{3}{8}+\frac{1}{2}\cos 2(8)u+\frac{1}{8}\cos 4(8)u$$

$$=\frac{3}{8}+\frac{1}{2}\cos 16u+\frac{1}{8}\cos 32u$$

3) $\cos^4 13u = \frac{3}{8}+\frac{1}{2}\cos 2nu+\frac{1}{8}\cos 4nu$

$$=\frac{3}{8}+\frac{1}{2}\cos 2(13)u+\frac{1}{8}\cos 4(13)u$$

$$=\frac{3}{8}+\frac{1}{2}\cos 26u+\frac{1}{8}\cos 52u$$

4) $\sin^4 7u = \frac{3}{8}-\frac{1}{2}\cos 2nu+\frac{1}{8}\cos 4nu$

$$=\frac{3}{8}-\frac{1}{2}\cos 2(7)u+\frac{1}{8}\cos 4(7)u$$

$$=\frac{3}{8}-\frac{1}{2}\cos 14u+\frac{1}{8}\cos 28u$$

5) $\sin^4 9u = \frac{3}{8}-\frac{1}{2}\cos 2nu+\frac{1}{8}\cos 4nu$

$$=\frac{3}{8}-\frac{1}{2}\cos 2(9)u+\frac{1}{8}\cos 4(9)u$$

$$=\frac{3}{8}-\frac{1}{2}\cos 18u+\frac{1}{8}\cos 36u$$

6) $\sin^4 18u = \frac{3}{8}-\frac{1}{2}\cos 2nu+\frac{1}{8}\cos 4nu$

$$=\frac{3}{8}-\frac{1}{2}\cos 2(18)u+\frac{1}{8}\cos 4(18)u$$

$$=\frac{3}{8}-\frac{1}{2}\cos 36u+\frac{1}{8}\cos 72u$$

IV. Conclusion

Simplification of $\text{Sin}^4 n\theta$ and $\text{Cos}^4 n\theta$ in terms of cosine function with exponent 1 can be solved using half angle identities as mathematical algorithm. The process to solve this kind of form is shorter and easier using the new formula presented in this paper. The formula presented is very easy to follow because it is the generalized formula for the said function. This confirms that simplifying trigonometric equations can be attained by sufficiently following this simple formula. This can be further used in higher mathematics courses such as Differential and Integral Calculus, Differential Equations, Complex Analysis and even in Physics and Mechanics. The most prominent application of this is in the field of engineering particularly in Electrical, Civil and Mechanical in which the alternating current (AC) and direct current (DC) are being analyzed. It can be widely used in Architecture especially in large scale infrastructures. In the field of music especially stringed instrument it can be used in calculating the frequency and the same with Physics which also requires the calculation of frequency in unit of Hertz.

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References

[1] E. Babolian and S. Javadi, "New method for calculating adomian polynomials," *Applied Mathematics and Computation*, vol. 153, no. 1, pp. 253–259, 2004.

[2] S. Abbasbandy, "Improving newton–raphson method for nonlinear equations by modified adomian decomposition method," *Applied Mathematics and Computation*, vol. 145, no. 2-3, pp. 887–893, 2003.

[3] A. Wazwaz, "A comparison between adomian decomposition method and Taylor series method in the series of solutions," *Applied Mathematics and Computation*, vol. 97, no. 1, pp. 37-44, 1998.

[4] Y. Kuang and E. Kase, *Pre-Calculus for Dummies*, 2nd ed, Wiley, June, 2012, ch. 8, pp. 101.

[5] E. Swokowski and J. Cole, *Algebra and Trigonometry with Analytic Geometry*, 13th ed, Cengage, 2012, ch. 7, pp. 574.

[6] O. Bible. Multiple Angle Formula. [Online]. Available: http://www2.volstate.edu/obible/Ppt/1720/Lsn30/index.html.

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