# Laws of Form - An Exploration in Mathematics and Foundations

## ROUGH DRAFT

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UIC

## I. Introduction

This paper is about G. Spencer-Brown's "Laws of Form" [LOF, SB] and its ramifications. Laws of Form is an approach to mathematics, and to epistemology, that begins and ends with the notion of a distinction.
Nothing could be simpler. A distinction is seen to cleave a domain. A distinction makes a distinction. Spencer-Brown [LOF] says
"We take the form of distinction for the form."
There is a circularity in bringing into words what is quite clear without them. And yet it is in the bringing forth into formalisms that mathematics is articulated and universes of discourse come into being. The elusive beginning, before there was a difference, is the eye of the storm, the calm center from which these musings spring.

In this paper, I have endeavored to give a mathematical and personal account of an exploration that I have followed for quite some time. As a result, there have arisen many pathways and byways, all related to this central theme of distinction. The collection of sections will speak for themselves, but a few warnings are in order: Part of the game of fusing apparently separate subjects is the use of similar notations with different import in different contexts.
Accordingly and particularly, we use the Spencer-Brown mark in many different contexts with shifts of meaning, and shifts of use as these contexts change. This means that the reader is invited to pay close attention to the uses assigned to notations in any given section.
They vary from place to place. In all cases, the mark stands for a distinction, but just how that distinction is distinct in its particular context is a matter for local articulation.

Sections 2 through 6 are a review of Laws of Form basics, with remarks that tell more. Section 7 is a relatively deep exploration of parentheses and contains a new proof, using marks and anti-marks, of the formula for counting the number of well-formed expressions in a given number of parentheses. This section will be of interest to those who wish to reformulate Laws of Form in parenthetical language.

Section 7 also introduces containers and extainers, an algebra generated by parenthesis structures, that reaches into biology (DNA reproduction), physics (Dirac bra-kets and ket-bras) and topology (the Temperley Lieb algebra and knot invariants).

Another theme that comes forward in section 7 is the matter of imaginary boolean values. It has been my contention for a long time that mathematics itself is the subject that invents/discovers/catalogs/explores
forms of reasoning beyond boolean logic. Each such discovery is eventually seen to be quite "logical", and is accepted into the toolkit of mathematicians and users of mathematics, who find all these methods as amazing ways to get at the truth about structured situations. In particular, the Spencer-Brown mark is itself (in the mind of the one who marks, in the mind that arises in the marking) the quintessential imaginary boolean value. This point may be either too easy to see at the beginning or too hard to see, but when we find the incredible effectivity of using a combination of mark and anti-mark in getting at the properties of parentheses and binary sequences (as happens in section 7), then it may dawn on us that there really is a power of new reasoning in mathematical constructions. Mathematical constructions are powers of reason.

Section 8 shows how boundary interactions give a new viewpoint on the primary arithmetic of Laws of Form, and gives a quick introduction to relationships with map coloring. Section 10 looks at sets in terms of the mark. Sections 10, 11, 12 are about re-entry, recursion and eigenform. The theme of imaginary value here comes forth in language that captures aspects of infinity and incompleteness of formal systems. Section 13 returns to sets, and weaves a story about sets and knots and links. Section 14 is about a digital circuit model and about the structures that come from chapter 11 of Laws of Form. Here counting and imaginary values live in the context of designs for circuits whose behaviour is quite real, and whose structure has the subtlety of asynchronous states and transitions. Section 15 discusses the waveform arithmetics of Form Dynamics [FD], and the Flagg resolution of paradoxes that lets us avoid multiple valued logics if we so desire.

In fact Flagg resolution is intimately related to the remark we made at the beginning of this introduction, that each use of the distinction must carefully respect the context in which it is cradled. The whole enterprise of paradox resolution is the search for appropriate contexts in which the contradiction will not arise as an anomaly. When the contradiction does arise, it comes forth because a new value for a familiar object seems to contradict its present value. Flagg resolution denies the freedom to make the substitution. The entity that becomes contradictory is treated as non-local in the text, and must change in every instance of its saying, or not at all.

Section 16 explains diagrammatic matrix algebra, applies it to the vector algebra of three-space, and indicates its relationship with map coloring and formation (section 8). Section 17 discusses the mythology and form of arithmetic in terms of Laws of Form. Section 18 shows how the mark, and the primary algebra provide a key to deciphering the conceptual notation of Frege. Section 19 is about Shea Zellweger's logical garnet, and how this is related to the logic structure of the mark. Section 20 is devoted to remembering events and people. Section 21 is a final remark.

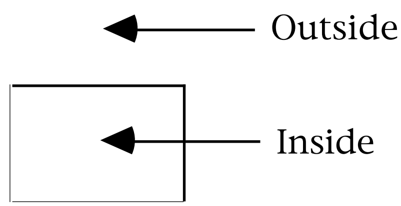
The titles of the sections are listed below.

## Contents

## II. Laws of Form

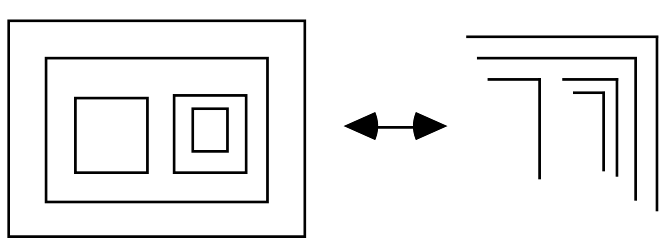
Laws of Form [LOF] is a lucid book with a topological notation based on one symbol, the mark:

This single symbol is figured to represent a distinction between its inside and its outside:

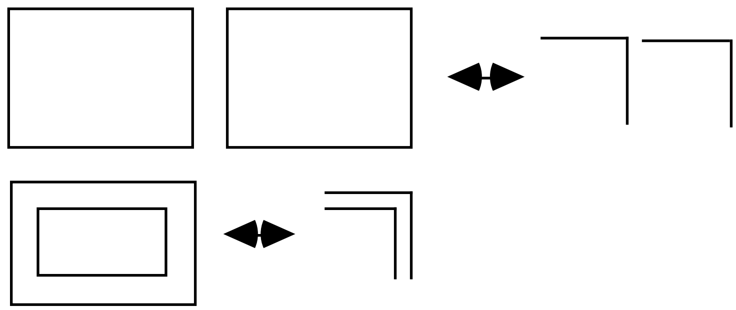



As is evident from the figure above, the mark is to be regarded as a shorthand for a rectangle drawn in the plane and dividing the plane into the regions inside and outside the rectangle. Spencer-Brown's mathematical system made just this beginning.

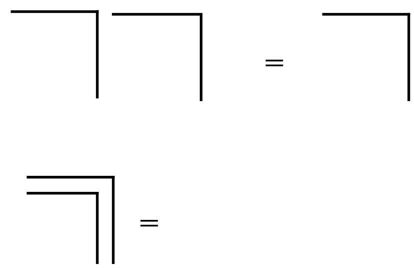
In this notation the idea of a distinction is instantiated in the distinction that the mark is seen to make in the plane. Patterns of non-intersecting marks (that is non-intersecting rectangles) are called expressions. For example,



In this example, I have illustrated both the rectangle and the marks version of the expression. In an expression you can say definitively of any two marks whether one is or is not inside the other. The relationship between two marks is either that one is inside the other, or that neither is inside the other. These two conditions correspond to the two elementary expressions shown below.



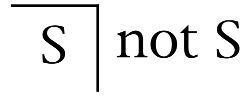
The mathematics in Laws of Form begins with two laws of transformation about these two basic expressions. Symbolically, these laws are:



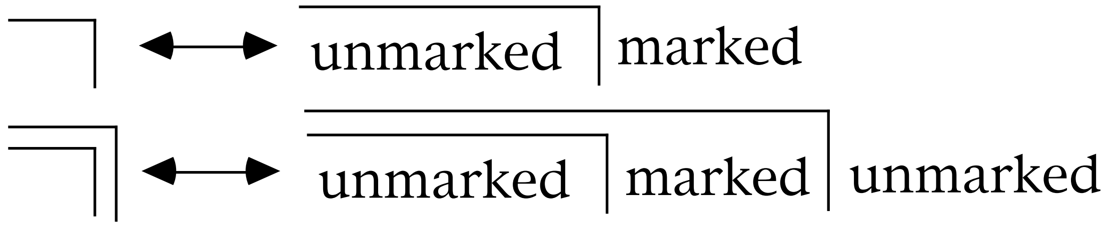
In the first of these equations, the law of calling, two adjacent marks (neither is inside the other) condense to a single mark, or a single mark expands to form two adjacent marks.  In the second equation, the law of crossing, two marks, one inside the other, disappear to form the unmarked state indicated by nothing at all.  Alternatively, the unmarked state can give birth to two nested marks. A calculus is born of these equations, and the mathematics can begin. But first some epistemology:

First we elucidate a principle of distinction that delineates the use of the mark.

**Principle of Distinction:** The state indicated by the outside of a mark is *not* the state indicated by its inside. Thus the state indicated on the outside of a mark is the state obtained by crossing from the state indicated on its inside.




It follows from the principle of distinction, that the outside of an empty mark indicates the marked state (since its inside is unmarked). It also follows from the principle of distinction that the outside of a mark having another mark incribed within it indicates the unmarked state.




Notice that the form produced by a description may not have the properties of the form being described. For example, the inner space of an empty mark is empty, but we describe it by putting the word "unmarked" there, and in the description that space is no longer empty. Thus do words obscure the form and at the same time clarify its representations.

Spencer-Brown begins his book, before introducing this notation, with a chapter on the concept of a distinction.

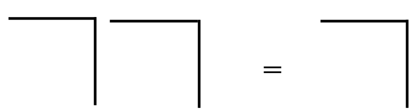
"*We take as given the idea of a distinction and the idea of an indication, and that it is not possible to make an indication without drawing a distinction. We take therefore the form of distinction for the form.*"

From here he elucidates two laws:

1. The value of a call made again is the value of the call.
2. The value of a crossing made again is not the value of the crossing.

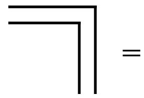
The two symbolic equations above correspond to these laws. The way in which they correspond is worth discussion.

First look at the law of calling. It says that the value of a repeated name is the value of the name. In the equation



one can view either mark as the name of the state indicated by the outside of the other mark.

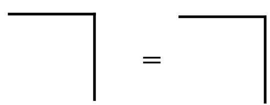
In the other equation



the state indicated by the outside of a mark is the state obtained by crossing from the state indicated on the inside of the mark. Since the marked state is indicated on the inside, the outside must indicate the unmarked state.  The Law of Crossing indicates how opposite forms can fit into one another and vanish into the Void, or how the Void can  produce opposite and distinct forms that fit one another, hand in glove.

There is an interpretation of the Law of Crossing in terms of movement across a boundary. In this story, a mark placed over a form connotes the crossing of the boundary from the Domain indicated by that form to the Domain that is opposite to it. Thus in the double mark above, the connotation is a crossing *from* the single mark on the inside. The single mark on the inside stands for the marked state. Thus by placing a cross over it, we transit to the unmarked state. Hence the disappearance to Void on the right-hand side of the equation. The value of a crossing made again is not the value of the crossing.

The same interpretation yields the equation



where the left-hand side is seen as an instruction to cross from the unmarked state, and the right hand side is seen as an indicator of the marked state. The mark has a double carry of meaning. It can be seen as an operator, transforming the state on its inside to a different state on its outside, and it can be seen as the name of the marked state. That combination of meanings is compatible in this interpretation.

In this calculus of indications we see a precise elucidation of the way in which markedness and unmarkedness are used in language. In language we say that if you cross from the marked state then you are unmarked. This distinction is unambiguous in the realm of words. Not marked is unmarked. In this calculus of the mark these patterns are captured in a simple and non-trivial mathematics, the mathematics of the laws of form.

From indications and their calculus, we move to algebra where it is understood that a variable is the conjectured presence or absence of an operator (the mark). Thus

$$\overline{\overline{A}|}|$$

stands for the two possibilities

$$\overline{\overline{\overline{\ \ }|}|}| = \overline{\ \ }| \quad , A = \overline{\ \ }|$$

$$\overline{\overline{\ \ }|}| = \qquad , A = $$

In all cases of the operator A we have

$$\overline{\overline{A}|}| = A .$$

Thus begins algebra with respect to this non-numerical arithmetic of forms. The primary algebra that emerges is a subtle precursor to Boolean algebra. One might mistake it for Boolean algebra but at the beginning the difference is in the use of the mark. Forming

$$\overline{A}|$$

accomplishes the negation of A, but the mark that does the job is also one of the values in the arithmetic. The context of the formalism separates the roles of operator and operand. In standard Boolean algebra the separation is absolute.

Other examples of algebraic rules are the following:

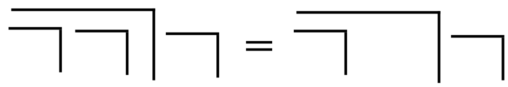
$$aa = a$$

$$\overline{\ \ }|\,a = \overline{\ \ }|$$

$$\overline{\overline{a}|\,a}| = $$

$$\overline{ab}|\,b = \overline{\ a\ }|\,b$$

Each of these rules is easy to understand from the point of view of the arithmetic of the mark. Just ask yourself what you will get if you substitute values of a and b into the equation. For example, in the last equation, if a is marked and b is unmarked, then the equation becomes



which is certainy true, by the law of calling.

With algebra one can solve equations, and Spencer-Brown pointed out that one should consider equations of higher degree in the primary algebra just as one does in elementary algebra.

Such equations can involve self-reference. Lets look at ordinary algebra for a moment.

$$x^2 = ax + b$$

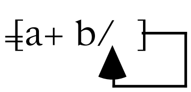
is a quadratic equation with a well-known solution, and it is also well-known that the solution is sometimes imaginary in the sense that it utilizes complex numbers of the form $R + Si$ where $i^2 = -1$. One can re-write the equation as

$$x = a + b/x.$$

In this form it is indeed self-referential, with x re-entering the expression on the right. We could "solve" it by an infinite reentry or infinite continued fraction:

$$x = a + b/(a + b/(a + b/(a + b/(a + b/(a + ...))))).$$

In this infinite formalism it is literally the case that $x = a + b/x$ and we can write

a + b/(a+b/(a+...))) = 


to indicate how this form reenters its own indicational space. This formal solution to the quadratic equation converges to a real solution when the quadratic equation has real roots. For example, if $a=1=b$, then

$$1+1/(1+1/(1+\ldots))) = [1+ 1/\ \ ]$$

converges to the positive solution of $x^2 = x + 1$, which is the golden
ratio, $\phi = (1 + \sqrt{5})/2$. We shall have more to say about the geometry of the golden ratio in later sections.

On the other hand, the quadratic equation may have imaginary roots. (This happens when $a^2 + 4b$ is less than zero.) Under these circumstances, the formal solution does not represent a real number.
For example, if $i$ denotes the square root of minus one, then we could write

$$i = -1/(-1/(-1/\ldots)) = [-1/\ \ ]$$

to denote a formal number with the property that

$$i = -1/i\ .$$

Spencer-Brown makes the point that one can follow the analogy of introducing imaginary numbers in ordinary algebra to introduce *imaginary boolean values* in the arithmetic of logic.
An apparently paradoxical equation such as

$$J = \overline{J}|$$

can be regarded as an analog of the quadratic $x = -1/x$, and its solutions will be values that go beyond marked and unmarked, beyond true and false.

## III. Paradox

In Chapter 11 of Laws of Form, Spencer-Brown points out that a state that may appear contradictory in a space may appear without paradox in space and time.

This is so with the famous paradoxes such as the *Russell set of all sets that are not members of themselves.* These are structures whose very definition propels them forward into the production of new entities that they must include within themselves. They are paradoxical in an eternal world and generative in a world of time.

The simplest instance of such an apparent paradox is the equation

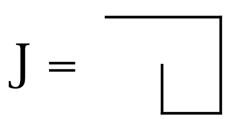
$$J = \overline{\;J\;}|$$

taken in the context of Laws of Form. For if J is equal to the mark, then the equation implies that J is equal to the unmarked state, and if J is equal to the unmarked state, then the equation implies that it is equal to the marked state.

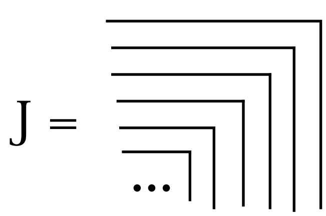
$$J = \overline{\;\;\;}| \longrightarrow J = \overline{\overline{\;\;\;}|}| =$$

$$J = \qquad \longrightarrow J = \overline{\;\;\;}|$$

Sometimes one writes

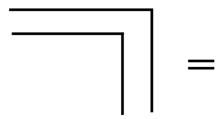


or

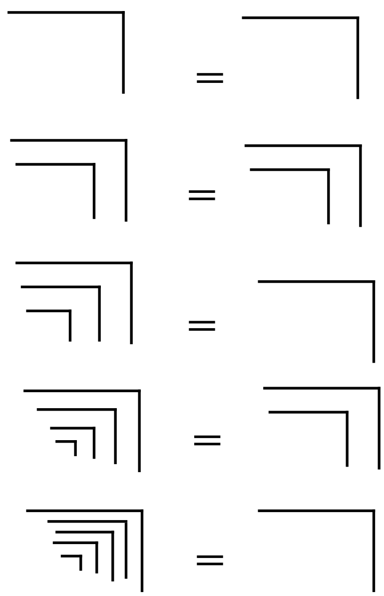


to indicate that this form reenters its own indicational space.

In Laws of Form we have the equation



where the nothing on the right hand side of the equals sign literally means nothing.  Living in this context, we see that the finite approximations to the reentering mark will oscillate between the values marked and unmarked:



This means that we now have two views of the reentering mark, one is purely spatial -- an infinite nest of enclosures.  One is purely temporal -- an alternating pattern of marked and unmarked states. All sorts of dynamics can occur in between these two extremes and this was the subject of Form Dynamics[FD].

There is no paradox when J is seen to oscillate in time. A new state has arisen in the form of the reentering mark J.  At this level the reentering mark would represent autonomy or autopoiesis [CSR]. It represents the concept of a system whose structure is maintained through the self-production of its own structure.  This idea of a calculus for self-reference, and the production of a symbol for the fundamental concept of feedback at the level of second order cybernetics captured the imaginations of many people, and it still does! Here is the ancient mythological symbol of the worm ouroboros embedded in a mathematical, non-numerical calculus.

The snake is now in the foundations and it is snakes all the way down.

One may argue that it is, in fact not appropriate to have the reentering mark at the very beginning. One may argue that it is a construct, not a fundamental entity. This argument would point out that the emergence of the mark itself requires self-reference, for there can be no mark without a distinction and there can be no distinction without indication (Spencer-Brown says there can be no indication without a distinction. This argument says it the other way around.). Indication is itself a distinction, and *one sees that the act of distinction is necessarily circular.* Even if you do not hold that indications must accompany distinctions, they do arise from them. The act of drawing a distinction involves a circulation as in drawing a circle, or moving back and forth between the two states. Self-reference and reference are intimately intertwined.

In our work on Form Dynamics [FD] we place the reentering mark back in the position of a temporal construct. In biology one may view autonomous organisms as fundamental, and one may look to see how they are generated through the physical substrate. It is a mystery that we face directly. The world that we know is the world of our organism. Biological cosmology is the primary cosmology and the world is fundamentally circular.

In writing [FD], I was fascinated by the notion of imaginary boolean values and the idea that the reentering mark and its relatives, the complex numbers, can be regarded as such values.

*The idea is that there are "logical values" beyond true and false, and that these values can be used to prove theorems in domains that ordinary logic cannot reach. Eventually I came to the understanding that this is the creative function of all mathematical thought.*

At that time I was fascinated by the reentering mark, and I wanted to think about it, in and out of the temporal domain.

The reentering mark has a value that is either marked or unmarked at any given time. But as soon as it is marked, the markedness acts upon itself and becomes unmarked. "It" disappears itself. However, as soon as the value is unmarked, then the unmarkedness "acts" to produce a mark.

You might well ask how unmarkedness can "act" to produce markedness. How can we get something from nothing? The answer in Laws of Form is subtle. It is an answer that destroys itself. The answer is that

*Any given "thing" is identical with what it is not.*

Thus markedness is identical to unmarkedness. Light is identical to darkness. Everything is identical to nothing. Comprehension is identical to incomprehension. Any duality is identical to its confusion into union. There is no way to understand this "law of identity" in a rational frame of mind. An irrational frame of mind is identical to a rational frame of mind. In Tibetan Buddhist logic there is existence, nonexistence and that which neither exists nor does not exist [BL]. Here is the realm of imaginary value.

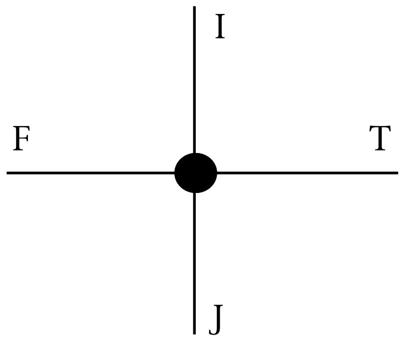
The condition of reentry, carried into time, reveals an alternating series of states that are marked or unmarked. This primordial waveform can be seen as

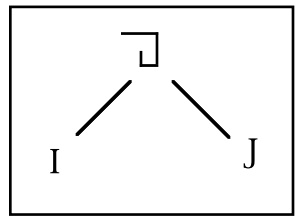
Marked, Unmarked, Marked, Unmarked,....

or as

Unmarked, Marked, Unmarked , Marked,...

I decided to examine these two total temporal states as representatives of the reentering mark, and I called them I and J respectively [DMA]. These two imaginary values fill out a world of *possibility* , that is perpendicular to the world of true and false.




I = [T,F] <-----> TFTFTFTFTFTFTFTFTF...
J = [F,T] <-----> FTFTFTFTFTFTFTFTFT...

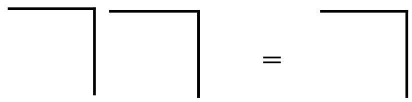
In [DMA] it is shown how I and J can be used to prove a completeness theorem for a four valued logic based on True, False, I and J. This is the "waveform arithmetic" associated with Form Dynamics. In this theory the imaginary values I and J participate in the proof that their own algebra is incomplete. This is a use of the imaginary value in a process of reasoning that would be much more difficult (if not impossible) without it.

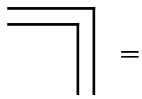
We shall return to this discussion of paradox and values that extend beyond the marked and the unmarked in sections 10 - 15. In the next section we take up the mathematics of Laws of Form from the beginning.

## IV. The Calculus of Indications

So far, we have described how Laws of Form is related to the concepts of distinctions, self-reference, paradox and reentry. We now go back to the beginning and look at the mathematical structure of the Calculus of Indications and its algebra. We shall see that these structures reach outwards to a new view of Boolean algebra, elementary logic, imaginary values and a myriad of excursions into worlds of mathematical form.

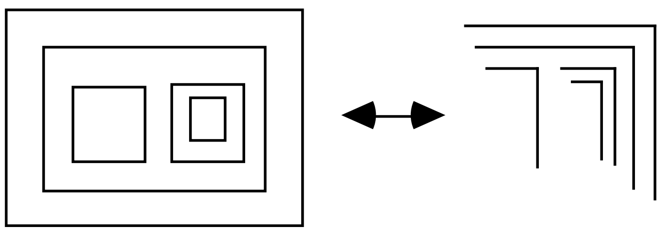
Recall the two basic laws of calling and crossing:

Calling: 

Crossing: 

The mark is seen to make a distinction in the plane between its inside and its outside, and can be regarded as an abbreviated box.

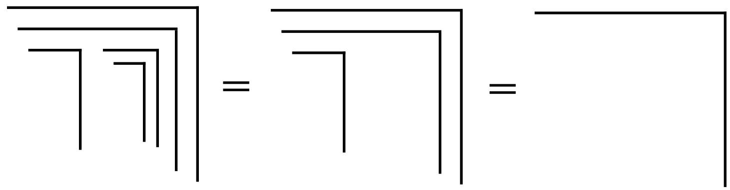
An *expression* is any collection of marks that have the property that for any two of them, they are either each outside the other, or one is inside the other one.



In the figure above we have illlustrated an expression in the form of boxes, and in the Laws of Form notation. Note that it is manifest to the eye that any two boxes are either outside each other, or one is inside the other one. In the box notation, an expression is just a collection of boxes in the plane such that the boundaries of the boxes do not intersect each other. One reads this same criterion into expressions created with the mark as half-box.

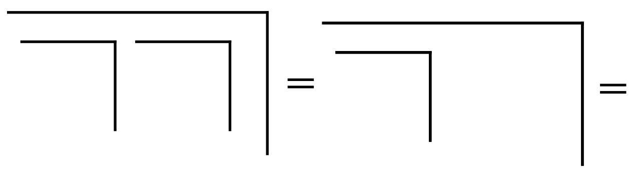
**A Word About the Equals Sign.**
*A = B is to be understood as saying that "A can be confused with B." in the root meaning of the word confused. To confuse A and B is to lose the distinction that makes them different.* Just so, in the laws of calling and crossing, the equals sign is an indication of our capacity to see two calls as a single call, and to see a crossing made again as no crossing at all.

Calling and crossing can be applied to the parts of an expression. For
example



the expression shown above reduces to the marked state by two applications of crossing. Note that in applying calling or crossing, one looks for a region in the expression where only the pattern of marks in the representative form for calling or crossing occur.
To apply calling, one must find two empty adjacent marks.
To apply crossing one must find two nested marks with no other marks between them, and with the innermost mark empty.

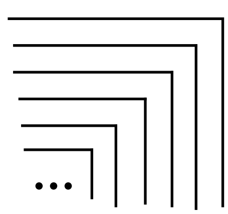
Here is another example of reduction.



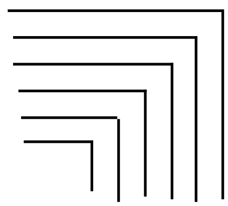
In this case, the expression reduces to the unmarked state via one application of calling and one application of crossing.

**Lemma 1.** Any finite expression can be reduced to either the marked or to the unmarked state by a finite sequence of applications of calling and crossing.

**Remark.** We say finite expression in the statement of the Lemma because there are infinite expressions such as

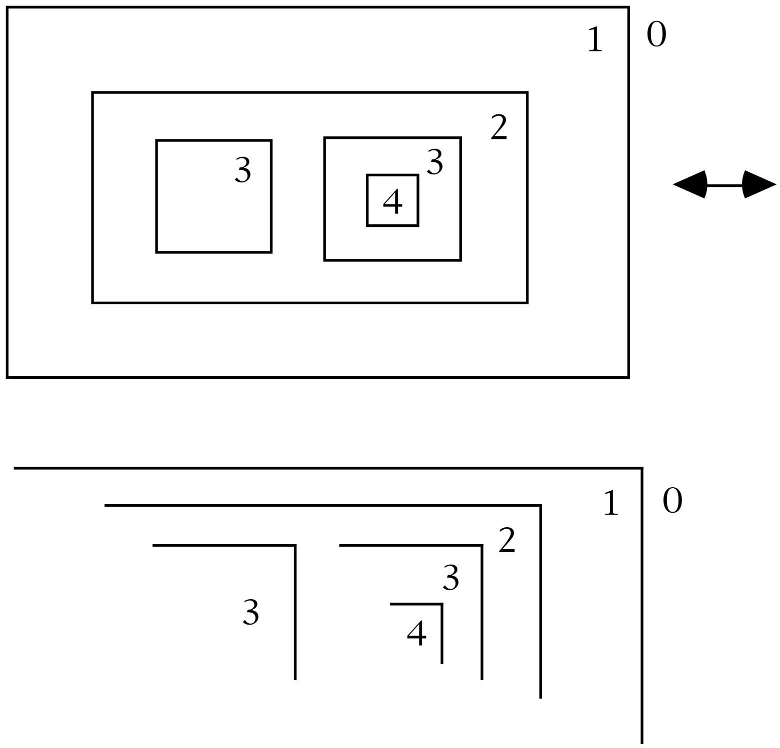
J = 

where there is no possibility of any reduction, since neither the law of calling nor the law of crossing applies. In the expression J above, there is no instance of adjacent empty marks and there is no instance of two nested marks with the inner mark empty. This would be the case if the expression stopped as in

$J_6 =$ 

consisting in six nested marks. We see easily that $J_6$ is equivalent to the unmarked state, and that more generally, $J_n$ ($n$ nested marks) is equal to the marked state when $n$ is odd and to the unmarked state when $n$ is even. The infinite expression $J$ is neither even nor odd, and it does not reduce to either the marked state or to the unmarked state.

**Remark.** In order to prove **Lemma 1**, it is useful to note that an expression divides the plane into disjoint (connected) divisions that can be assigned depth 0,1, 2,... according to the number of inward crossings needed to reach the given division. The concept is illustrated in the diagram below. In this diagram we illustrate an expression both in box form and in the notation of the mark.




In the diagram above, note that the given expression has two divisions of depth **3**. In general there may be a multiplicity of divisions at a given depth. The maximal depth in this expression is **4**.
Any finite expression has a maximal depth and a finite number of divisions.  Note that a division of maximal depth is necessarily empty, for otherwise it would have marks within it leading to greater depth. When you get to the bottom, there is nothing there.

**Proof of Lemma 1.**  Let **E** be a finite expression. Let **S** be a division of maximal depth in **E**. Then **S** is necessarily empty, and **S** may  be surrounded by an  mark **M**. This mark itself may be surrounded by an otherwise empty mark, or it may be  adjacent to some other mark **M'**. If there is another mark **M'** adjacent to **M**, then **M'** must be empty, else the depth in **M'** would be greater than the depth in **M**. Therefore if there occurs an **M'** adjacent to **M** then **M** and **M'** can be reduced to **M** alone by the law of calling. If there is no adjacent mark **M'** to **M**, and **M** is nested within a mark **M''**, then **M** and **M''** can be canceled by the law of crossing. We see that in these cases the expression can be reduced to an expression with fewer marks. The only case where this can not be accomplished is when the expression is empty or consists in a single mark. This completes the proof of the Lemma. //

**Lemma 1** shows that every expression can be regarded as marked or unmarked, but one could worry that it might be possible to transform the marked state to the unmarked state by using the laws of calling and crossing. Note that expressions can be made more complicated as well as less complicated by applications of calling and crossing. The next Lemma assures us that the marked and unmarked states are indeed different.

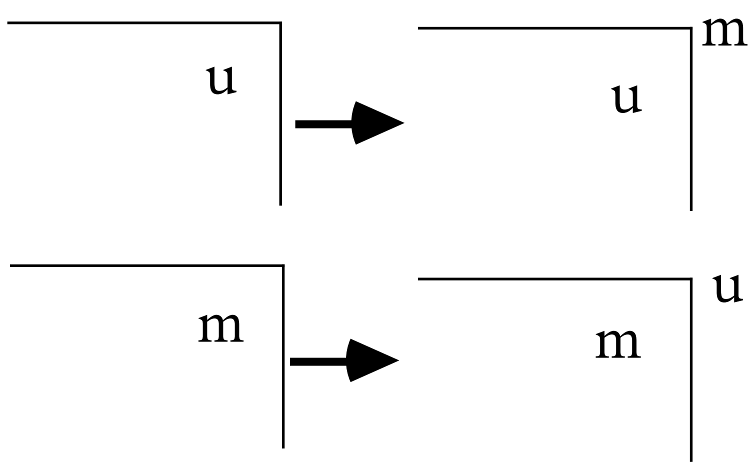
**Lemma 2.** There is no finite sequence of applications of calling and crossing that can transform the unmarked state to the marked state.

**Remark.** In order to prove **Lemma 2** we shall give a method for calculating a well-defined value V(E) for each expression E. (The letter V in V(E) stands for the word *Value*.)

V(E) does not depend upon reducing the expression by using calling and crossing. We then show that V(E) does not change under the operations of calling and crossing.

V(E) is computed as follows:
We shall use two values denoted by **m** (marked) and **u** (unmarked). These values will label divisions of an expression. We take the rule that if a division is labeled with at least one **m** and some number of **u's** then its label is **m**. If it is labeled with only a multiplicity of **u's** then its label is **u**. Thus
**uu=u**
**mm=m**
**mu = um = m**
as far as labels are concerned. These labels give names to each division in an expression.

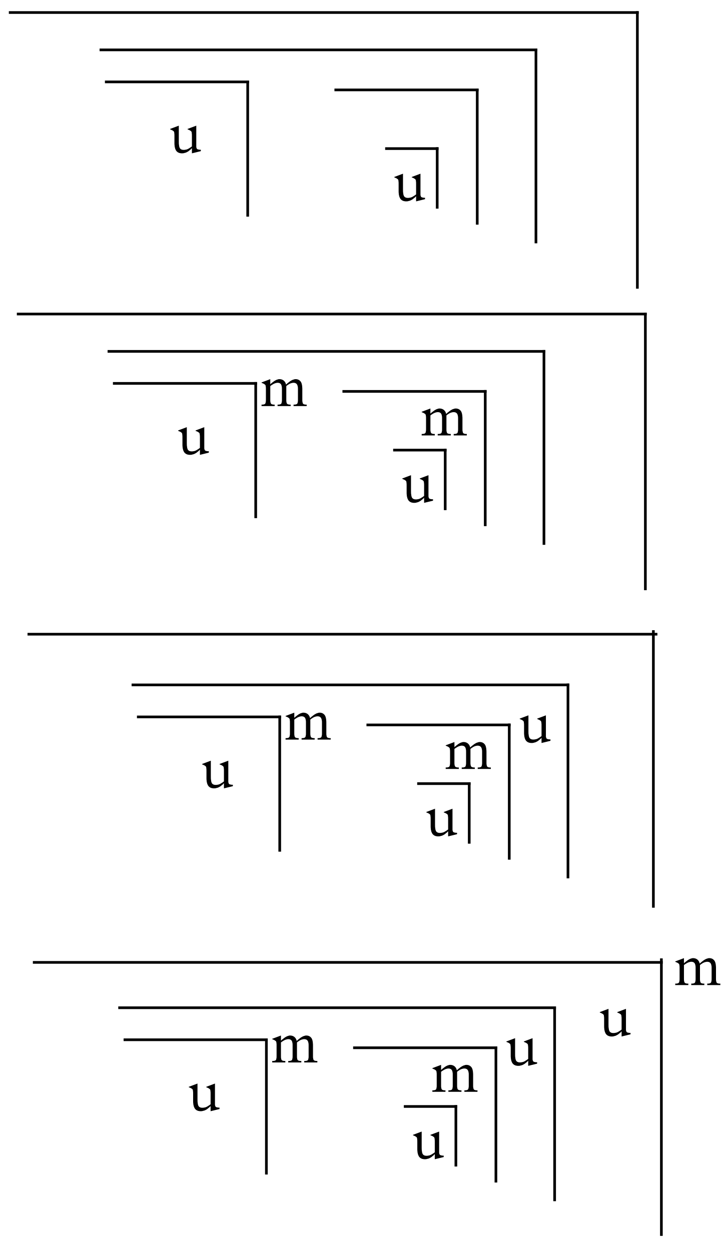
Now take an expression E, and label all its empty divisions (e.g. the deepest divisions) with **u**. Now send labels upward from deeper spaces by the rules shown below.




A deeper space sends an opposite label to divisions of one lesser depth above it. Think of the marks as inverting the labels as they pass the boundaries. Each successive division acquires a definite value by combining all labels that it receives from the depths below it.

Finally the unique division of depth 0 receives a label of **m** or **u**. This label for depth 0, is by definition V(E).

Here is an example of the labeling process.



Let E denote the initial expression. We begin the process by labeling the the two deep empty spaces with **u**. Crossing the boundary from these spaces propagates two **m** labels. The second **m** label propagates a **u** label into a division already labeled **m**. This does not change the value of that division, and a **u** is propagated into the next division which then propagates an **m** to the zero depth division.

Hence V(E) = **m**.

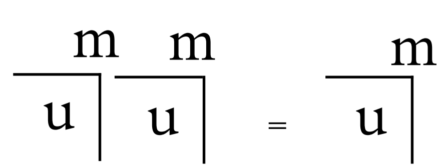
Note that E reduces by calling and crossing to the marked state.

A little thought by the reader should convince her that this method of evaluation of an expression is well-defined for finite expressions.

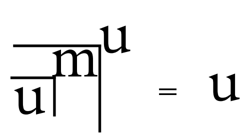
**Proof of Lemma 2.** Let E be a finite expression. Suppose that F differs from E by one operation of calling. Consider the calculation of V(E). Suppose that F has two adjacent empty marks and that E is obtained from F by removing one of the marks. Then each mark in F has its interior space labeled u and each mark transmits a label m to the same division in which they are adjacent. Since mm = m, we see that the evaluations of E and F are necessarily the same. Now suppose that E contains two nested marks with the innermost mark empty, and that these two marks have been removed in F. Then the innermost space of the two marks is labeled u, the next space m and the space outside the marks receives a u from them. Hence removing the two marks does not affect the evaluation of F. We have shown that if E and F differ by one act of calling or one act of crossing, then V(E) = V(F). It is easy to see that V(M) = m and V(U) = u where M denotes a single mark, and U denotes the unmarked state. Thus there can be no sequence of calling and crossing that takes M to U since at each step the value of I remains constant, and the labels u and m are distinct. This completes the proof of Lemma 2. //

**Example.** The following diagram illustrates the simplest cases of the invariance of V(E) under calling and crossing.

Calling:



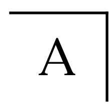

Crossing:



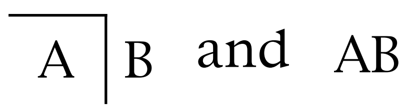

We now know that every expression E in the calculus of indications represents a unique value (marked or unmarked) and that this value can be found either by reducing the expression to the marked or unmarked state, using calling and crossing, or by computing V(E).

## V. The Primary Algebra

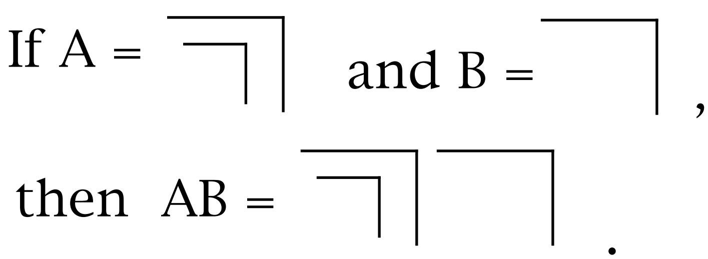
In the last section we established that the calculus of indications, the primary arithmetic based on the laws of calling and crossing, is a non-trivial formal system that has a unique value (marked or unmarked) associated to each expression in the calculus. This calculus can be regarded as an arithmetic of distinctions, and just as ordinary arithmetic has an algebra, so there is a Primary Algebra that describes the calculus of indications. In this algebra, a letter will denote the presence or absence of a marked state. Thus A, as an element of the primary algebra denotes an unspecified expression in the primary arithmetic of the calculus of indications. If we write



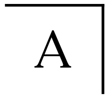

this new algebraic expression denotes the result of crossing the original expression A. Thus if A is marked then the cross of A will be unmarked, and vice versa. Similarly, if A and B are algebraic symbols, then we can form




where the juxtaposition of two symbols described the juxtaposition of the expression for which they stand. For example




It is easy to see that AB is marked exactly when either A is marked, or B is marked. Thus the algebraic operation of juxtaposition corresponds to the logical operation "OR", where the logical values under consideration are marked and unmarked. By the same token,



denotes NOT A, often denoted by ~A.

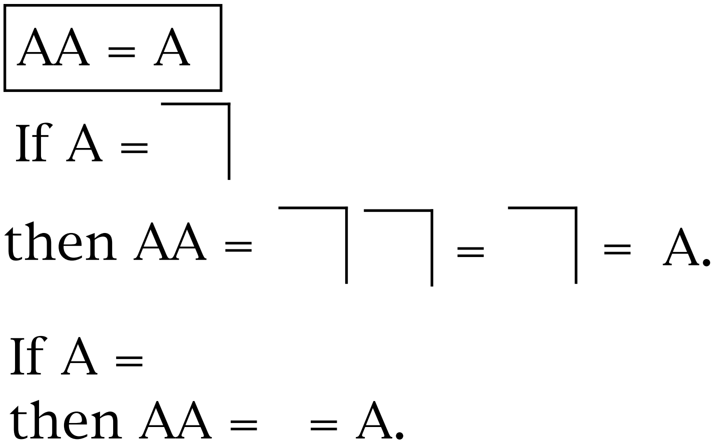
**Remark.** Note that in working with the primary algebra, we take it for granted that elements of the algebra commute:

$$AB = BA$$

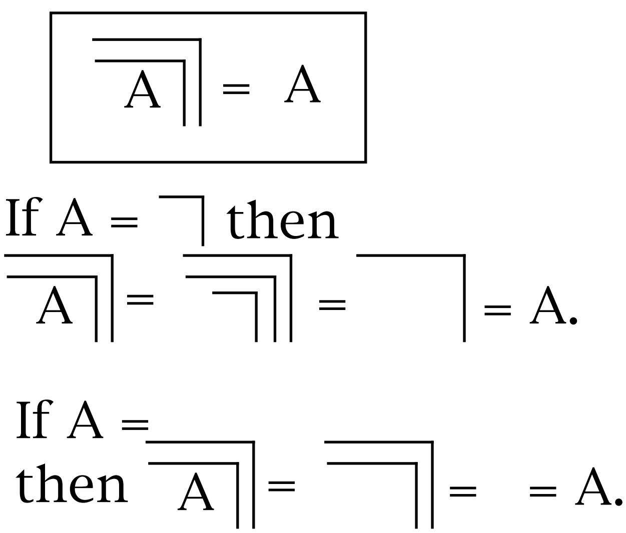
for any algebraic expressions A and B. Certainly, we can observe that this is indeed an identity about the primary arithmetic. It is just that we use this identity so frequently, that it is useful to take it as a given and not have to mention its use. (Of course, if we consider non-commutative generalizations of the algebra, then instances of commutativity will have to be indicated.)

**Remark on Logic.** The algebra we are constructing can be construed as an algebra for the logic of true and false by adopting the convention that
Marked = True
UnMarked = False.
We shall hold to this convention, and describe the interpretation for logic as we go along. So far, we have interpreted AB as A OR B and Cross A as NOT A. We will return to the interpretations for logic after discussing the algebra a bit more.

The primary algebra has lots of identities, just as does ordinary algebra. Lets look at some of them.




In the above figure, we have shown the identity AA = A, and its proof. Proofs are easy in the primary algebra. One only has to look at the possible values for the terms, and then use properties of the primary arithmetic.




Above is the identity "**A double cross = A**" and its proof in the primary arithmetic.

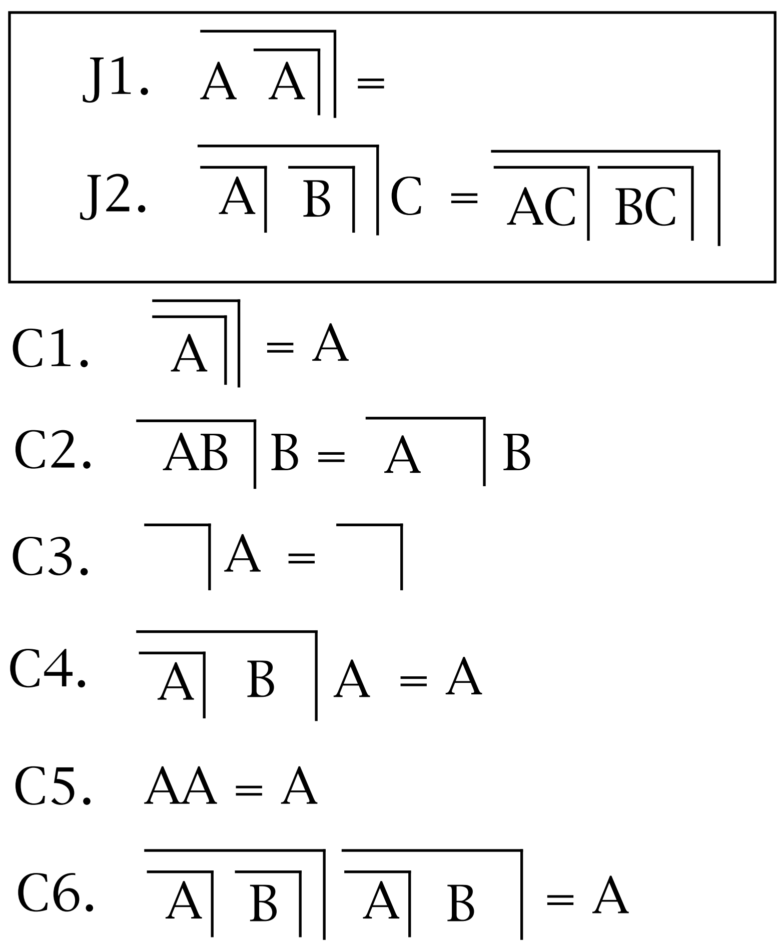



**Figure 1 - Identities in the Primary Algebra**

In the Figure above, we have listed a number of identities in the primary algebra, including C1 and C5 that we have already discussed. The reader should try his hand at proving each of **J1, J2, C1, C2, C3, C4, C5 and C6**. The reader of Laws of Form will find that there are three more identities proved there, and that the ones we have listed have the following names:

**J1. Position**
**J2. Transposition**
**C1. Reflection**
**C2. Generation**
**C3. Integration**
**C4. Occultation**
**C5. Iteration**
**C6. Extension.**

It is convenient at times to refer to the identities by their names.

There are actually an infinite number of identities that one can write down in primary that are truths about the primary arithmetic. How can we understand the structure of the collection of all such true algebraic identities? One way to study this question is to realize that some identities are algebraic consequences of other identities. For example, once we know reflection (C1 above), then we can apply it again and again, as in

$$A = \overline{\overline{A}\,\rvert}\,\rvert = \overline{\overline{\overline{\overline{A}\,\rvert}\,\rvert}\,\rvert}\,\rvert\ .$$

In fact, we regard the identity

$$\overline{\overline{A}\,\rvert}\,\rvert = \overline{\overline{\overline{\overline{A}\,\rvert}\,\rvert}\,\rvert}\,\rvert$$

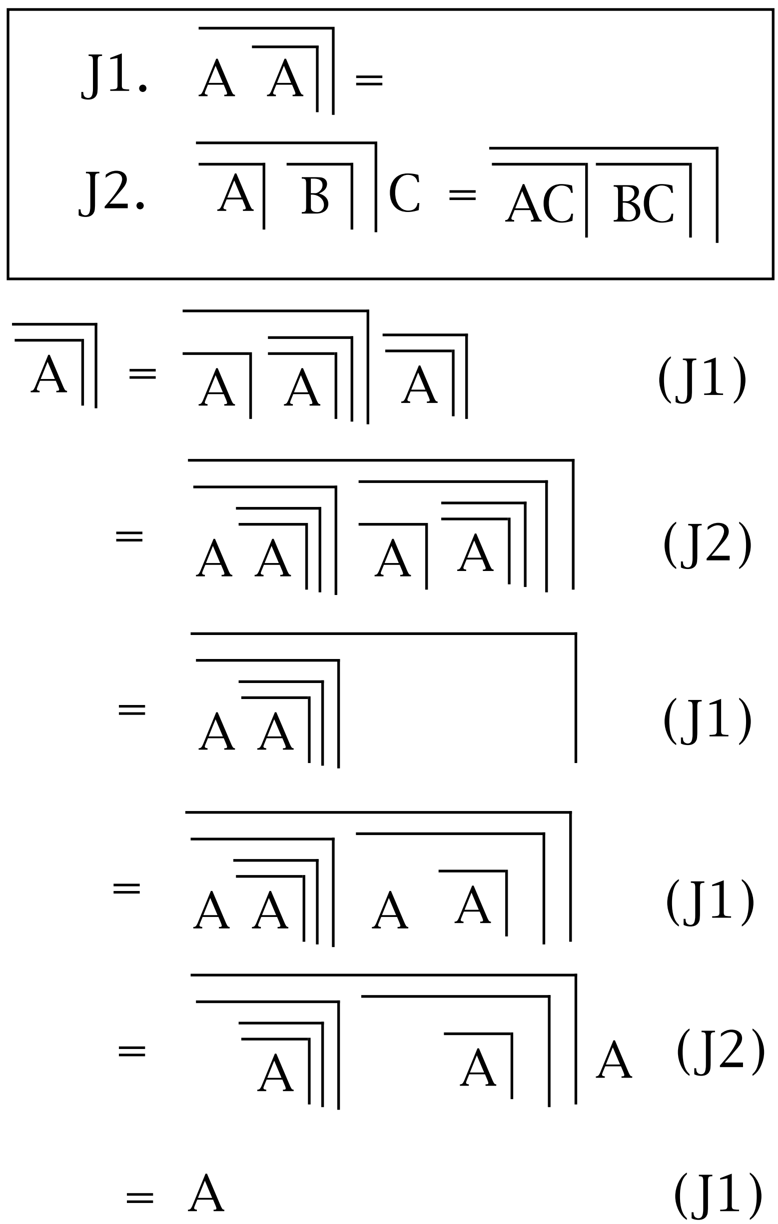
as a direct instance of C1. More generally, in working algebraically, we interpret each identity as an infinite number of specific algebraic formulas that can be obtained through algebraic substitution into the variables in a given identity. Thus

$$\text{J1.}\quad \overline{A\ \overline{A}\,\rvert}\,\rvert =$$

$$\text{Let } A = \overline{B}\,\rvert C\text{. Then}$$

$$\overline{\overline{B}\,\rvert C\ \ \overline{\overline{B}\,\rvert C}\,\rvert}\,\rvert =$$

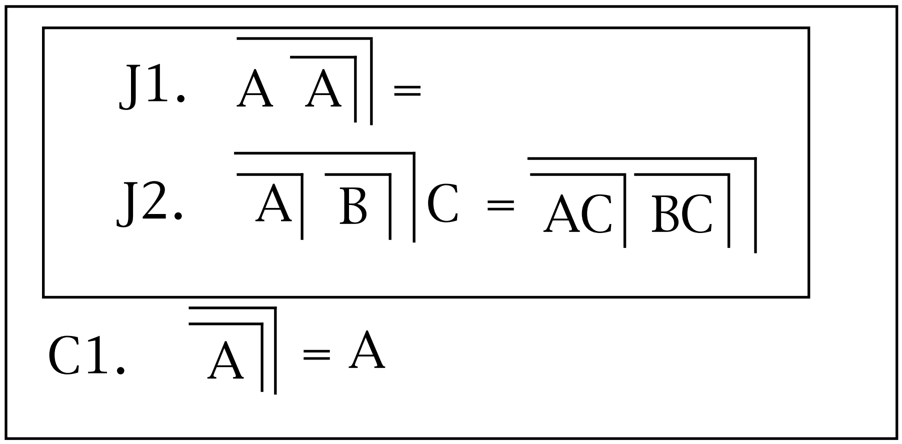
But there are subtler possibilities. For example, *reflection is an algebraic consequence of J1 and J2*. Here is the proof.

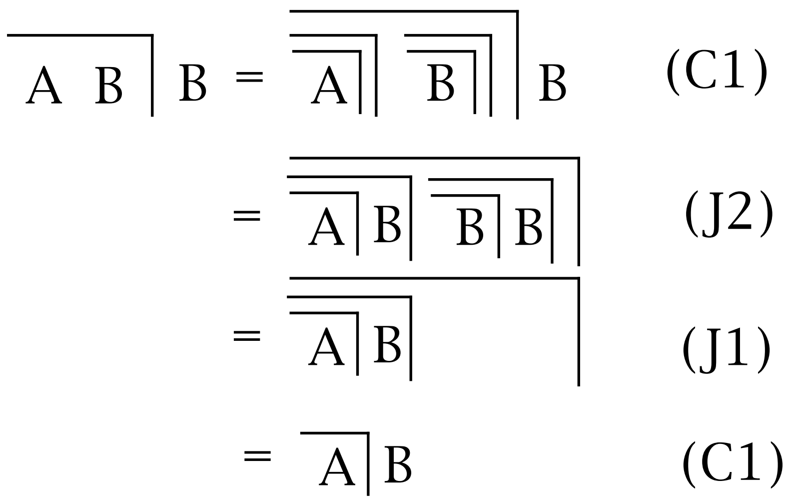



The reader should look carefully at the steps in this demonstration. In the first step we used one of the infinitely many instances of J1, here obtained by replacing A by cross A in J1. Hen we apply J2 directly, and then vanish a subexpression using J1. We then reinstate a different subexpression using J1 and avail ourselves of the opportunity to apply J2. The resulting expression simplifies via J1.

It is certainly not immediately obvious that C1 is a consequence of J1 and J2, but once this has been accomplished, it is not too hard to show that C3, ..., C6 all follow as well.

For example, here is the proof of C2:




In this demonstration, we have used C1 freely. If one wants a pure demonstration of C2 from J1 and J2, then it can be obtained from this demonstration by repeating the moves that obtain C1 from J1 and J2 whenever we have just used C1.

It turns our that all equational identities about the primary arithmetic can be derived from J1 and J2. This is the completeness theorem for the primary algebra.

**Completeness Theorem for the Primary Algebra.** Let **a = b** be an algebraic identity that is true for the primary arithmetic. Then **a = b** is a consequence of J1 and J2.

We omit the proof of this Theorem, and refer the reader to Laws of Form for the details. This result is the analog of completeness theorems for axioms systems for Boolean algebra, and it is a version of the completeness of the Propositional Calculus for elementary logic. It is very nice to have a Theorem of this kind for an algebraic or logical system. In more complex systems there is no algorithm to determine whether a give statement is a theorem in the system, but in the case of the primary algebra it is a finite algorithmic check to determine arithmetically whether a = b is true or false.

One says that J1 and J2 are initials for the primary algebra. If another collection of equations has the same consequences, then we say that the other set of equations is also a set of initials for the primary algebra. It is also worth remarking that there are other sets of initials for the primary algebra. Two that are worth noting are

1. J1 (position), C1 (reflection) and C2 (generation).
2. C6 (extension).

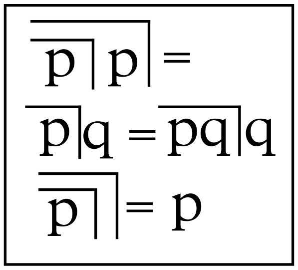
That is, position, reflection and generation taken together, generate the whole primary algebra. Just extension also generates the whole primary algebra. It is an open problem to characterize just which sets of initials capture the entire algebra.

I will refer to {position, reflection, generation} as the Bricken Initials, as the observation that they are an alternate set is due to William Bricken [WB]. That {extension} is an initial for the primary algebra is a reexpression of a theorem discovered for Boolean algebras in the 1930's by Huntington [H]. It was re-discovered by Spencer-Brown and his students in the 1980's. See [RA] for a proof due to the author of this article.

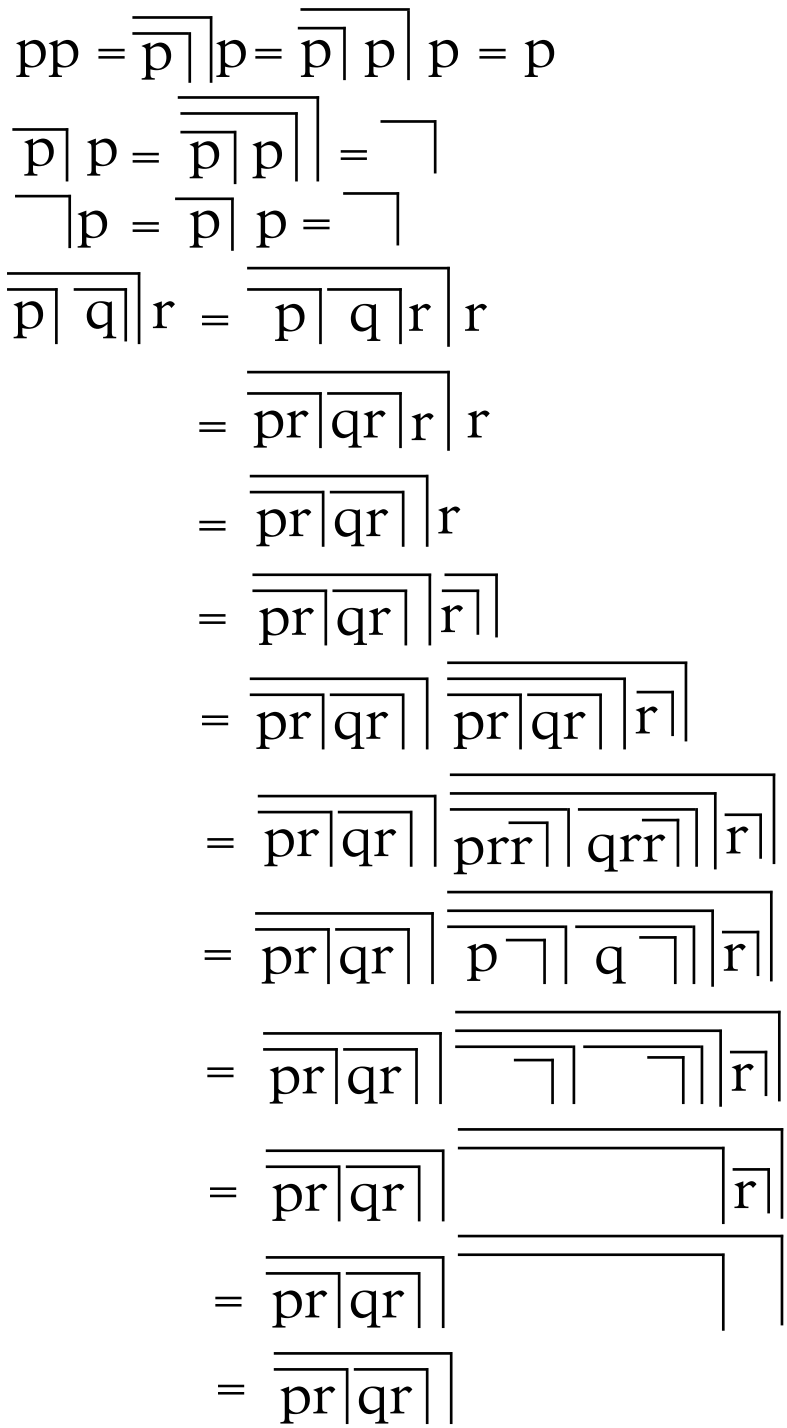
In verifying an alternate set of initials, one wants to derive J1 and J2 from the new set. We will not go into the details of these results, but it is of interest to show that reflection is needed in the Bricken initials. That is, we will show that *reflection is not a consequence of position and generation.* The method for doing this is to introduce a model in which position and generation are true, but reflection is false. In this model we shall have a new arithmetical element, distinct from both the marked and the unmarked states, and denoted by e, with the following properties

$$ee = e$$
$$e\,\overline{\phantom{mm}}| = \overline{\phantom{mm}}|\,e = \overline{\phantom{mm}}|$$
$$\overline{e}| = \overline{\phantom{mm}}|$$
$$\overline{\phantom{mm}}|\,\overline{\phantom{mm}}| = \overline{\phantom{mm}}|$$
$$\overline{\overline{\phantom{m}}|}| =$$

Call this the **PG Arithmetic** (**PG** stands for **Position and Generation**). Certainly **reflection** is not valid in this arithmetic, since the double cross of **e** is equal to the unmarked state and **e** is not unmarked. (One proves that this arithmetic really has three values in a fashion analogous to the way we proved that the primary arithmetic has two distinct values .) One then needs to verify that position and generation are facts about the **PG** arithmetic. We leave this as an exercise for the reader. Note that if reflection were a consequence of position and generation, then one could deduce it as a proposition about the **PG** arithmetic, and that would be a contradiction. Thus we have a model where position and generation hold, but reflection does not hold.



Bricken Initials



The diagram above shows the Bricken initials and the derivation of iteration, crossed position, integration and transposition from them. The derivation of transposition shows the extraordinary power of generation in the presence of other initials. In the first three steps of the derivation of transposition (shown above) we detail a use of generation that should be separately stated as a lemma.

**Lemma.** Assume only the use of the initial generation.

$$\overline{p|}\,q = \overline{pq|}\,q \;.$$

Let **E** be any algebraic expression. Then for any variable **q**, **Eq = Fq** where **F** is any algebraic expression obtained from **E** by adding or removing any instances of the variable **q** from within
(past at least one cross) the expression **E**.

**Proof.** We leave the proof of this lemma to the reader. //

**Example.**

$$\overline{\overline{\overline{qa|}\,b|}\,c|}\,q = \overline{\overline{\overline{a|}\,b|}\,c|}\,q$$

The power of sorting variables that is inherent in generation alone is extraordinary, and is the basis of Bricken's work on using primary algebra in computational contexts. It is worth noting that the following problem is NP complete:

**Problem:** Let **E** be an expression in the primary algebra in a finite number of variables. Determine whether there exist values (marked or unmarked) for these variables such the corresponding value of **E** is marked.

No one has given a general algorithm for solving this problem that is not time exponential in the number of variables. No one has proven that exponential time is necessary. This problem (equivalent to the corresponding problem in boolean algebra) is a standard example from the range of problems of complexity type NP. Given the efficiency of the primary algebra, it is possible that a solution to the NP problem could emerge from a deeper analysis of its potentials.

**Imaginary Boolean Values.** The use of e is an example of using an imaginary Boolean value to reason to a mathematical result. The mathematical result is the fact that reflection is not a consequence of position and generation. Without the addition of the extra arithmetical value e, any proof of this result would certainly be quite complex. In this case, I do not know of another proof.

But what is an imaginary boolean value? Did we really go beyond boolean reasoning in using the value e? Certainly, the three valued system PG can be regarded as a sort of logic with extra value e. The value e is subtle. It is not unmarked all by itself, but it appears to be unmarked in the presence of a mark. By using the imaginary value e, we were able to reason to a definite result about the structure of algebras. On the other hand, *PG can be regarded as a mathematical construct, and the reasoning we used, with the help of this construct, was normal boolean reasoning.* Here we have the fundamental situation that appears to occur with every instance of imaginary values. An imaginary value can be viewed as a new piece of mathematics. Mathematics itself is the subject that studies and classifies imaginary values in reasoning. Each new mathematical discovery is a discovery of a new way to bring forth reason. Boolean algebra, or standard logic is a very useful brand of mathematics. Reason is inexhaustible.

## VI. Elementary Logic

Here is how we shall model elementary logic using Laws of Form. We shall take the marked state for the value T (true) and the unmarked state for the value F(false). We take NOT as the operation of enclosure by the mark.

$$\overline{A\;}\big| = \text{NOT } A$$

We take A OR B as the juxtaposition AB in the primary algebra. Note that the law of calling tells you that this works as a model of OR where A OR B means "A or B or both".

By putting in AND and ENTAILS,
we shall have the vocabulary of elementary logic.

First we create AND:

$$A \text{ and } B = \overline{\overline{A\rceil}\;\overline{B\rceil}}\rceil$$

We take the convention (See the last section.) that marked corresponds to true, and unmarked corresponds to false.
The reader should have no difficulty verifying that this expression for A and B is marked if and only if both A and B are marked. Hence it is true if and only if A is true and B is true. Note also that this definition of and embodies the DeMorgan Law:

A and B = Not (Not A or Not B).

The next matter is entailment. The standard logical definition of A entails B is

A entails B = (Not A) or B.

Thus

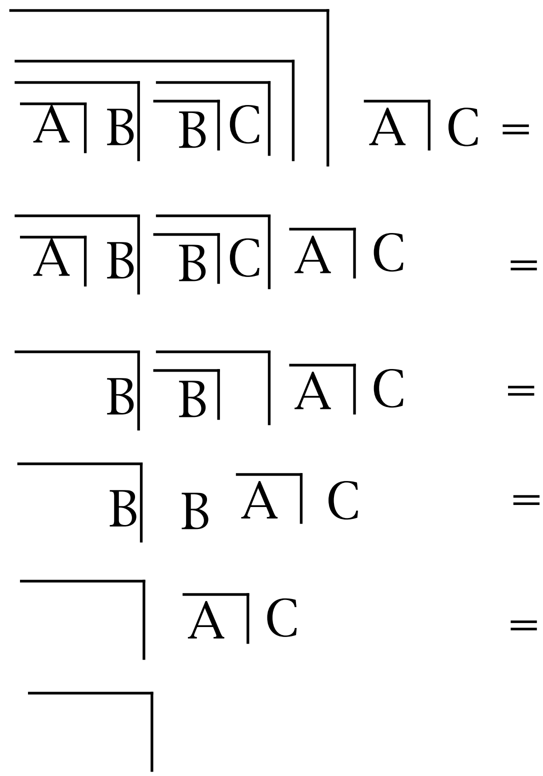
$$A \text{ entails } B = \overline{A\rceil}\,B\,.$$

The result of our labors is a neat iconic expression for entailment, sometimes called implication. Note that the expression of entailment is false (unmarked) only if A is marked and B is unmarked. This is the hallmark of that operation. A entails B is false only when A is true and B is false.

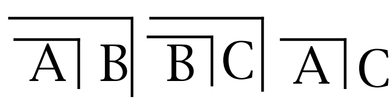
We now have the vocabulary of elementary logic, and are prepared to analyze syllogisms and tautologies. For example, the classical syllogism has the form
((A entails B) and (B entails C)) entails (A entails C),
which has the form in our notation:

$$\overline{\overline{\overline{\overline{A\rceil}\,B}\rceil\;\overline{\overline{B\rceil}\,C}\rceil}\rceil}\rceil\;\;\overline{A\rceil}\,C$$

The tautological nature of this expression is at once apparent from the primary algebra.

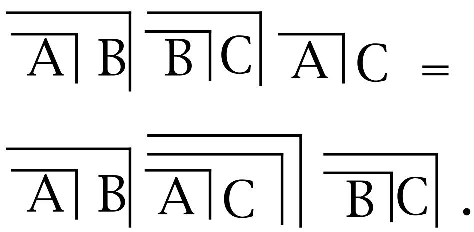


This formalism makes it very easy to navigate problems in elementary logic and it makes it easy to understand the structure of many aspects of elementary logic. Here is one example. Note that in evaluating the syllogism we immediately reduced it to the form



where the premises
**A entails B**
and
**B entails C**
are enclosed in marks, while the conclusion
**A entails C**
is not enclosed in a mark.

We can rearrange this syllogistic form using only **reflection** (C1 from Figure 1) (and implicit commutativity) without changing its value. Call the algebra generated by **reflection** alone the **reflection algebra**. By applying reflection algebra to this form of the syllogism, we cannot reduce it to a marked state, but we can obtain alternate valid forms of reasoning. We get a host of other valid syllogisms from this one form. For example,



With this rearrangement, we have that the new premises are
**A entails B**
and
**not(A entails C)**
with the conclusion
**not(B entails C).**

This is indeed a valid syllogism, but what is most interesting is the result of taking one of the interpretations of entailment.

Let us take
**A entails B** to mean **All A are B.**
Then **not(A entails C)** means **not(all A are C).**
But we can interpret that latter as
**not(all A are C) = some A are not C.**

With this intepretation, we have the rearranged syllogism with premises
**All A are B.**
and
**Some A are not C.**
with the conclusion
**Some B are not C.**

This is ia correct syllogism, and it turns out that there are exactly 24 valid syllogisms involving some, all and not. Each of these can be obtained by rearranging the basic form of the syllogism (as indicated above) in combination with replacing some or all of the variables by their crossed forms. *The 24 valid syllogisms are exactly those that can be obtained by this rearrangement process.* This fact

is an observation of Spencer-Brown in his Appendix 2 to Laws of Form.
It is a remarkable observation. The elementary logical form that we have been pursuing does not actually "know" about multiplicities. Without any formalization of quantification for logical variables, we nevertheless get a structure that holds the basic reasonings about collections. This a matter for the structure of logic and linguistics and it deserves further study.

**Remark on the Algebra of Sets.** Another relationship of the primary algebra with studying multiplicities is its interpretation as a Boolean algebra of sets.

Let U be a "universe", a set whose subsets we are studying.
Let O denote the empty set. Other subsets of U will be denoted by alphabetic letters such as A, B, C. Let <A> = U - A denote the collection of elements in 1 that are not in A. Interpret this complement in Laws of Form notation by

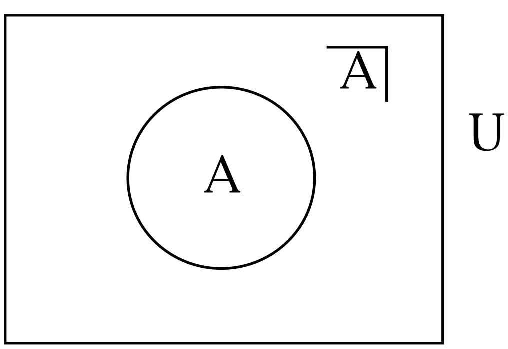
<A> = U - A = $\overline{A\,}|$

U ⟷ marked state = $\overline{\quad}|$

O ⟷ unmarked state =

U = The Universe

O = The Empty Set




so that **cross A** = <A> corresponds to the complement of the set A in the universe U. Incidentally, the notation <A> for **cross A** is useful in other contexts. William Bricken [WB] and Philip Mequire

[Meg] apply Laws of Form and other instances of boundary mathematics by using this notation. With this interpretation of crossing, we see that the subset of the universe makes a distinction between inside (in the subset) and outside (not in the subset but in the surrounding universe). We then have

$$\text{A Union B} = AB$$

$$\text{A Intersection B} = \overline{\overline{A\,|}\;\overline{B\,|}\,\Big|}$$

so that the basic set-theoretic operations are expressed in the language of the primary algebra. All the identities in the primary algebra are correct identities about sets. For example, **position** says that the complement of the union of a set with its own complement is empty. **Transposition** says that union distributes over intersection.
The Boolean algebra of sets can be regarded as a second order version of the primary algebra, since the elements of the sets can themselves be sets. We ignore everything but the top level of this structure when we do the usual set theoretic algebra.

## VII. Parentheses

It will not have escaped the reader that the mark acts as a generalized pair of parentheses. In fact it is perfectly possible to rewrite the arithmetic and algebra of Laws of Form in terms of parentheses with `()` denoting the marked state.

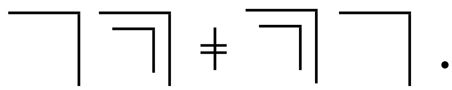
$$\overline{\phantom{AAA}\Big|} = ()$$

For Laws of Form, we then have

```
(()) = *
()() = ()
```

where * is a place-holder for the unmarked state, and can be erased when it is convenient. For example, we write `(*) = ()`. In this language it is assumed that all parenthesis structures are well-formed (correctly paired according to the usual typographical conventions), and that we make the operation of juxtaposition so that AB is regarded as equal to BA. We leave it as an exercise for the reader, to rewrite everything in this language and experiment with it. Labor is saved in making this switch, while some graphical emphasis is lost.

In this section *we shall consider parentheses in their usual non-commutative mode* so that AB is not equal to BA in the sense that ()(()) is not equal to (())(). Hence



It is here that parentheses come into their own.
Lets make some lists.
1. ()
2. ()(), (())
3. ()()(), (())(), ()(()), (()()), ((()))

We have listed all parentheticals with one, two and three parentheses. Let $C_n$ denote the number of parentheticals with $n$ parenthesis pairs. We have shown that $C_1=1$, $C_2=2$, $C_3 = 5$. You will find that $C_4 = 14$. These are the *Catalan Numbers*. The general result is that

$$C_n = (1/(n+1))C(2n,n) = (2n)!/n!n!(n+1),$$

where $C(r,s)$ is the number of ways to choose $s$ objects from $r$ distinct objects. This counting result is surprisingly subtle to verify.

Here is one way to approach the result.
Form the following infinite sum of all possible parentheticals:

P = * + () + ()() + (()) + ()()() + (())() + ()(()) + (()()) + ...

Use the following conventions

(*) = ()
*A = A* = A
(A + B) = (A) + (B)
A + B = B + A.

Then it is easy to see that P satisfies the following reentry equation.

$$P = * + P(P).$$

The infinite sum is regenerated by successive reentry into this equation, and the equation is literally true about the infinite sum. The truth of this reentry equation is equivalent to the observation that every parenthetical has a unique expression in the form

$$X(Y)$$

where X and Y are smaller parentheticals (possible empty).
It follows at once from this that $C_n$ satisfies the recursion relation

$$C_{n+1} = C_0C_n + C_1C_{n-1} + \ldots + C_{n-1}C_1 + C_nC_0$$

where $C_0 = C_1 = 1$.

For example,
$C_4 = C_0C_3 + C_1C_2 + C_2C_1 + C_3C_2 = 5 + 2 + 2 + 5 = 14.$

Define $C(x) = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots.$ so that

$C(x) = 1 + x + 2x^2 + 5x^3 + 14x^4 + \ldots .$

$C(x)$ is the *generating function* for the Catalan numbers.

$C(x)$ is obtained from P by replacing each parenthesis pair in P by a copy of the commuting variable x. Hence it follows from the reentry equation for P, that

$$C(x) = 1 + x\,C(x)^2.$$

You then solve this quadratic equation for $C(x)$ as a power series in x, using the fractional binomial theorem, and get the coefficients. The coefficients turn out to be given by the formula we just wrote down for $C_n$.

Just for the record, here is a sketch of that calculation. The fractional binomial theorem says that

$$\sqrt{1+x} = 1 + C_1^{1/2} x + C_2^{1/2} x^2 + C_2^{1/2} x^3 + \ldots$$

where

$$C_n^a = a(a-1)(a-2)\ldots(a-n+1)/n!$$

with a any real number and n a non-negative integer.
The correct solution to the quadratic to use for this combinatorics is

$$C(x) = (1-\sqrt{1-4x})/2x = \sum_{n=0}^{\infty} (-1)^n 2^{2n+1} C_{n+1}^{1/2} x^n .$$

One then checks that

$$(-1)^n 2^{2n+1} C_{n+1}^{1/2} = C_n^{2n}/(n+1).$$

On the other hand, one would like a more direct understanding of the formula for the Catalan numbers in term of the structure of the parentheses themselves. Note that the choice coefficient C(2n,n) in the formula for $C_n$ suggests that we should be choosing n things from 2n things. In fact, a parenthesis structure is such a choice. Regard a parenthetical as a sequence of left and right parentheses. To emphasize this, let L = ( and R = ) so that, for example,

(())() = LLRRLR.

We see that the parenthetical is determined by choosing the placement of the three left parentheses in the expression. Thus each parenthetical is a choice of n placements of L in a string of 2n L's and R's.

But the collection of all such choices is bigger than the set of parentheticals. Many of these choices are illegal. For example, consider

RRRLLL = )))(((.

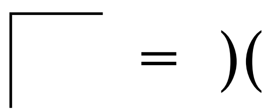
This is a legitimate choice of three L's from six places, but it is not a well-formed parenthetical. Our formula

$$C_n = (1/(n+1))C(2n,n)$$

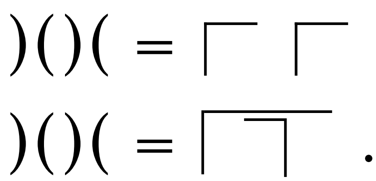
seems to be telling us that we should look in the larger set of all sequences of n left and n right parentheses, and find that to each legal string there are associated in some natural way n other illegal strings, filling out the whole set.

It turns out that this idea is correct! There is a wider domain to explore. Let $S_n$ denote the collection of *all* strings of n left parentheses and n right parentheses. Call these the *n-strings*.

In order to conduct this exploration, we shall take seriously the anti-parenthesis )(. In fact, we shall denote this by an *anti-mark.*

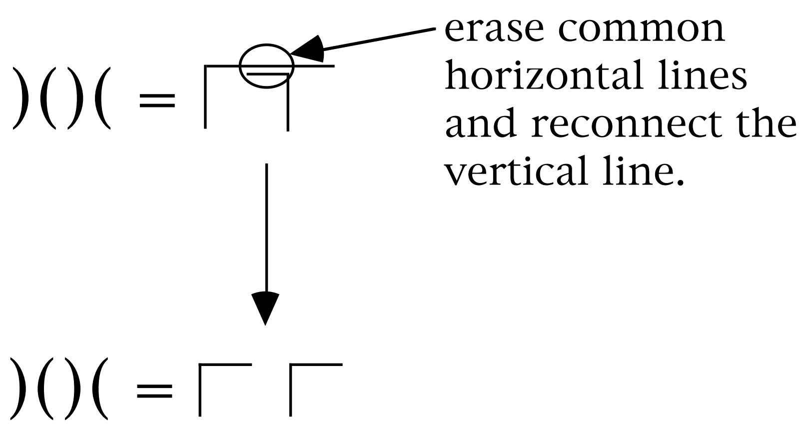


With the help of the anti-mark, we can easily write n-strings as well-formed expressions using both the mark and the anti-mark.
Such decompositions are not, however, unique.
For example,



There are two distinct ways to write the string )()( as an expression in primary marks. (We shall refer to the mark and the anti-mark as *the two primary marks.*)

Note how these two expressions are related to one another. If you take the second expression and erase the place where the two marks share a bit of horizontal boundary, you get the first expression.

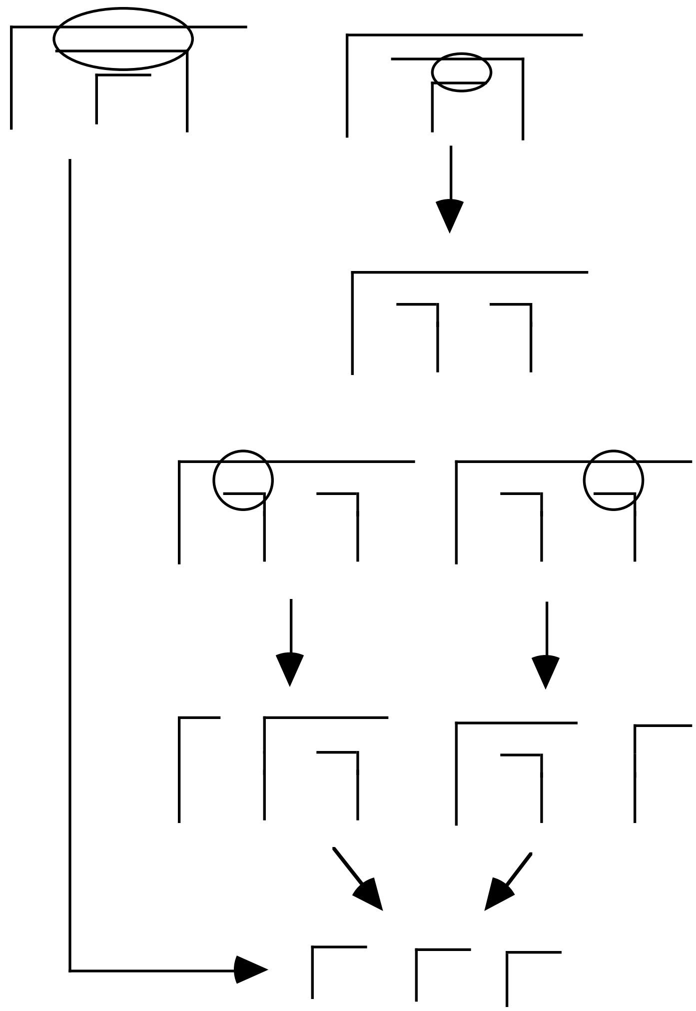



We shall call this operation of cancellation and reconstruction of expressions, *horizontal boundary cancellation*, or *HB-cancellation* for short. In order to apply HB-cancellation, the two marks in question must be horizontally adjacent to one another, and of opposite type.

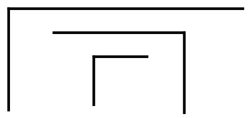
Note that the horizontal segment of the lower mark must be shorter that the horizontal segment of the upper mark, so that the cancellation produces two new horizontal segments.

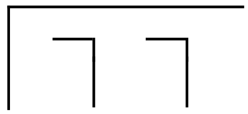
Here is another example.

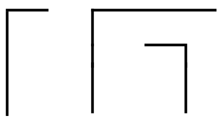
$$= \,)()()($$

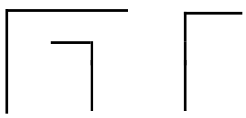


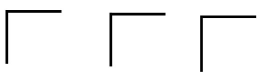
In this example, we see that there are five distinct expressions in two primary marks that represent the one string )()()(. They are obtained one from another by sequences of horizontal boundary cancellation. We have collected these expressions below.

 = )()()(

 = )()()(

 = )()()(

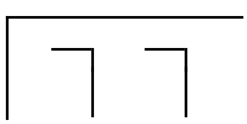
 = )()()(

 = )()()(

The general result is that if two expressions in the two primary marks describe the same **n**-string, then they can be obtained from one another by a sequence of horizontal boundary cancellations (or inverse cancellations where one constructs an expression that cancels to the first expression).

I shall say that an expression in the two primary marks is *special* if it has the following two properties:
1. *For each anti-mark in the expression, all marks that contain it are also anti-marks.*
*2. Given two anti-marks in the expression, then one contains the other.*

For example, in the above list there is only one special expression:

 = )()()(

**Theorem.** To every **n**-string there corresponds a unique special expression in the two primary marks.

**Proof.** Let S be a given **n**-string. Regard S as a sequence of L's and R's with

L = (

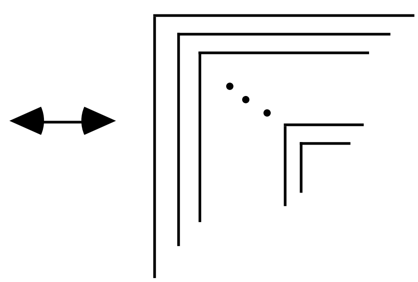
R = )

 = )( = RL

 = () = LR .

Scan the string S and strike out each occurrence of LR. Call the resulting string $S_1$. Repeat this process for $S_1$, calling the new string $S_2$, and continue until a string $S_k$ is reached that has no occurrence of LR. It is easy to see that $S_k$ must either be empty, or of the form $S_k$= RR...RLL...L. Thus $S_k$ itself corresponds to the special expression

$S_k$ = R...RL...L = )...)(...(



$S_k$ will be the collection of anti-marks in the expression for the original string S. The LR's that have been removed can be reinserted as a collection of legal subexpressions in the mark. The special expression is obtained in this manner.

For example, suppose that S = )(()))()( = RLLRRRLRLL. Then we see that

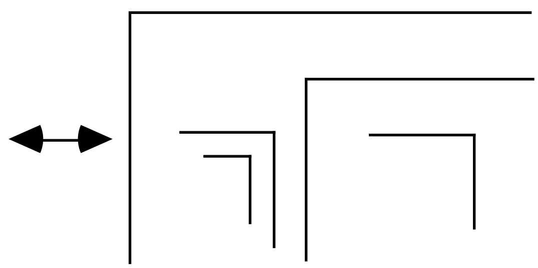
$S_1$ = RL[]RR[]LL and

$S_2$ = R[[]]RL[]L.

In writing S1 and S2 in this example, we have kept track of the removals of LR by placing empty square parentheses for each pair

LR. As a result, one can read out directly the special expression that corresponds to $S_k$ by replacing the square parentheticals with parentheticals in the mark. The remaining L's and R's give the parts written in the form of the anti-mark.

S = )(()))()(( = RLLRRRLRLL
S1 = RL[ ] RR [ ] LL
S2 = R[[ ]]RL[ ]L
 = )[[ ]] ) [ ] ( (



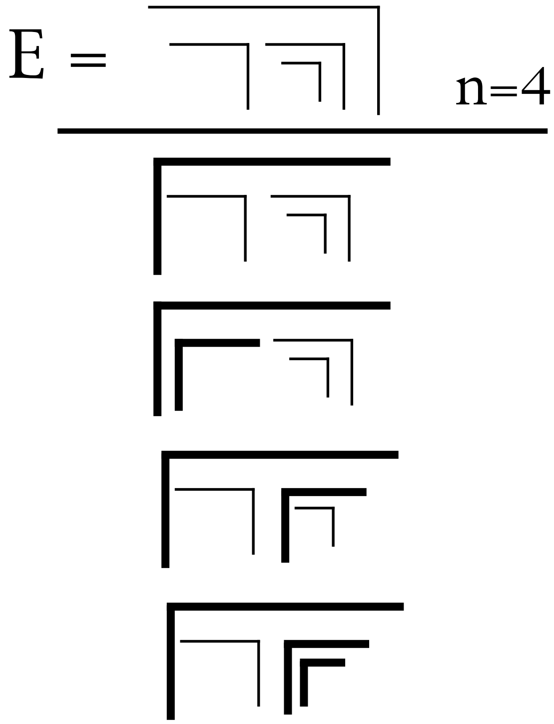
This method shows how to uniquely associate a special expression to each n-string. It is clear from the construction that any special expression is obtained in this way. This completes the proof. //

We now show how to construct all special expressions from the collection of parentheticals, and how this relates to our problem about the Catalan numbers.

### Generation Method

1. Take a given expression E in using n marks (no anti-marks).
2. Form n special expressions from the given expression by the following recipe: Choose a mark in the expression E. Switch it to an anti-mark and switch all marks that contain this mark to anti-marks. This new expression is special.

Here is an example:




The result of this generation method is the production from a given expression E (in only the mark) n expressions that are special. Note that E is, by definition, special. Hence each E produces n+1 special expressions, and all special expressions are generated in this way from parentheticals. By the Theorem, the collection of all special expressions is in one-to-one correspondence with the collection of all n-strings of which there are C(2n,n). Thus we have shown that

$$(n+1)C_n = C(2n,n).$$

This is exactly what we wanted to prove. QED.

In this section we have looked rather deeply into the structure of parentheses, and we have given two quite different proofs of the formula for the number of parenthetical expressions with n parenthesis pairs. Each proof is accomplished by going outside the original system of ideas presented by the counting problem. In the first case we used a reentry formula for the formal sum of all parenthetical expressions. In the second case, we enlarged the structure to all n-strings, and showed how the parenthetical epressions were embedded in n-strings by using the concept of an anti-mark (reversed pair of parentheses). These proofs illustrate, in a microcosm, the common situation in mathematics where one needs to introduce concepts or constructions apparently beyond the original problem in order to solve that problem. In this essay we have used the metaphor of the imaginary boolean value to refer to this process. The metaphor asks us to locate the extra concept, the new construction that does the trick in solving the problem. How did we go outside the system in order to facilitate a new view and accomplish the task? Since there is no logical way to find such solutions, it is only after the fact that we look at the entire construction and say -- this is just more mathematics!

## Coda

I will end this section with some hints about an algebraic structure that uses the mark and the anti-mark. We will write in the language of sharp brackets:

C = < >

E = ><

calling C a container and E an extainer. In previous terminology C is the mark and E is the anti-mark. Now using sharp parentheses, we take the view that C is a container for its inside, while E is an opening for interaction with its outside (the space to the right and to the left of E in the line). Note the patterns of concatenation:

CC = < > < > = < E >

EE = >< >< = > C < .

Iteration of C produces E and iteration of E produces C. If we make no discrimination between C and E, then they have symmetric roles in this arithmetic of concatenation. But we shall make a discrimination.

Suppose that we also had a rule that under certain circumstances an E could be generated inside a container.

< > (gen)----> < E >.

Then we could have

< > (gen)----> < E > = < >< > ,

and *we have accomplished the reproduction of the container < >.*

What is interesting here is that the *form* of this reproduction is identical to way that the DNA molecule can reproduce itself in the living cell. The DNA molecule consists in two interwound long chain molecules called the Watson and Crick strands. These two strands are bound to one another molecule by molecule in a pattern that can be reproduced from either of the strands taken individually. That is,
if a Watson strand, Watson, were introduced, by itself, into the cell's environment, that molecules from the environment would bind to Watson and transform it into DNA = WatsonCrick.
Similarly, if Crick is the Crick strand, then Crick will be transformed by the cell's environment into CrickWatson = DNA.
Finally, there are mechanisms in the cell that separate DNA into the individual strands when the cell divides. Thus we can write

DNA = WatsonCrick -----> Watson E Crick
-----> WatsonCrick WatsonCrick = DNA DNA

as the schema for DNA reproduction, where E denotes the action of the environment of the cell on these molecules.

This example shows that there is power in the descriptive properties of extainers and containers. In the DNA context, the container corresponds to the DNA. The DNA is a movable entity, but also, like the extainer, open to interaction.
The extainer, in this description is analogous to the environment with its free supply of molecules for interaction.

More generally, the container is a natural notation for a self-contained entity, while the extainer is a natural notation for an entity that is open to interactions from the outside. *We can emphasize this way of discriminating containers and extainers by regarding the container as a single entity that is movable. That is, we shall take C as an element that commutes with the rest of the string.* Then we have

$$EE = >< >< = > C < = C >< = CE.$$

We do not assume that E commutes with other expressions. In this way containers and extainers are made distinct from one another, and a language arises that can be used in many contexts.

For example, consider the algebra of two types of container and extainer. Let

$$A = >< \quad \text{and} \quad B = ] [ .$$

Then

$$ABA = >< ] [ > < = < ] [ > \; >< = < ] [ > A$$

$$BAB = ] [ > < ] [ = [ > < ] ] [ = < ] [ > B.$$

We see that A and B generate both their own corresponding containers $< >$ and $[ ]$, and also the mixed containers $[ >$ and $< ]$, along with mixed extainers $> [$ , $] <$ as well. The identities shown above indicate an interesting algebraic structure.
In fact, this algebraic structure is related to biology, topology, statistical mechanics and quantum mechanics. There is not room in this essay to explore these connections. We refer the reader to [Biologic] as well as [KP].

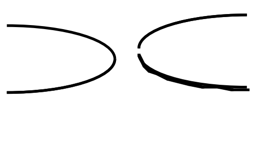
Here is a hint about the relationship with physics. Dirac [D] devised a notation for quantum states that has the form $| \phi >$. An function that evaluates states is denoted by $<\psi |$ so that the *bra-ket*

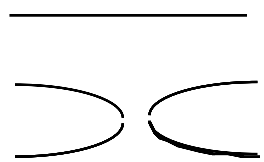
$$< \psi || \phi > = < \psi | \phi >$$

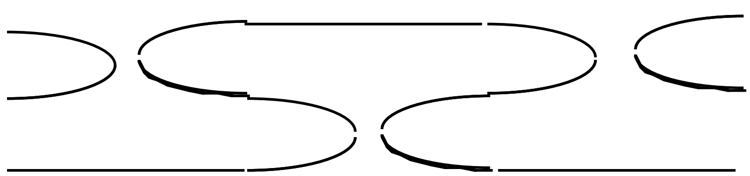
represents the value of the projection of the state $| \phi >$ in the direction of $| \psi >$. The *ket-bra* $| \phi >< \psi |$ is used as a projection operator and acts just like the extainers in our algebra. This duality between evaluations and operators is endemic to quantum mechanics. It is an instance of the abstract duality of containers and extainers. In fact, if the reader will reflect on our original remarks about the dual nature of the mark in Laws of Form as operator and as value or name, she will see that the extainer -

container duality is nothing more than an exfoliation of that original identity of naming and acting that occurs at the inception of the form.

Here is a hint about the relationship with topology. View the diagrams below. You will note that P and Q are now extainers that have been shifted from one another by the placement of an extra line. Each of P and Q has the aspect of three (vertical) left points and three (vertical) right points, connected by non-intersecting segments in the plane. Multiplication such as PQ is accomplished by attaching the right end-points of P to the left end-points of Q. We see that
PQP = Q via a topological equivalence of the resulting configuration.
This equation is analogous to our equation for A and B above where we found that ABA is a multiple of A and BAB is a multiple of B.
It is in fact mysterious that this pattern occurs in these two different ways, one combinatorial, the other topological.

P = 

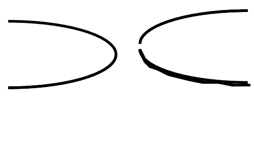
Q = 

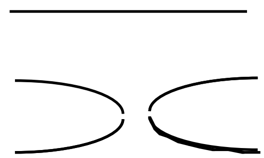
PQP = 

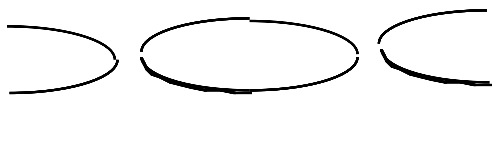
~ 

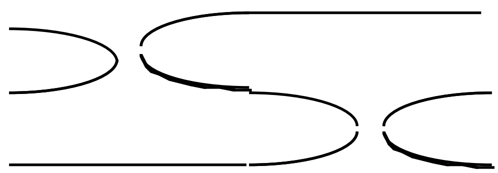
PQP = P

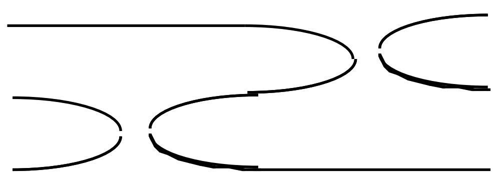
Note also that PP = CP, QQ=CQ where C denotes the circle in the plane. We take this circle as the container in this context and allow it to commute with every other expression. As the diagrams below show,if we take **I** to denote three parallel lines, then { **I, P, Q, PQ, QP**} is closed under multiplication, up to multiples of **C**. This algebra is called the diagrammatic Temperley Lieb algebra. See [KP, KL].

P = 

Q = 

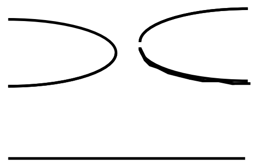
PP = 

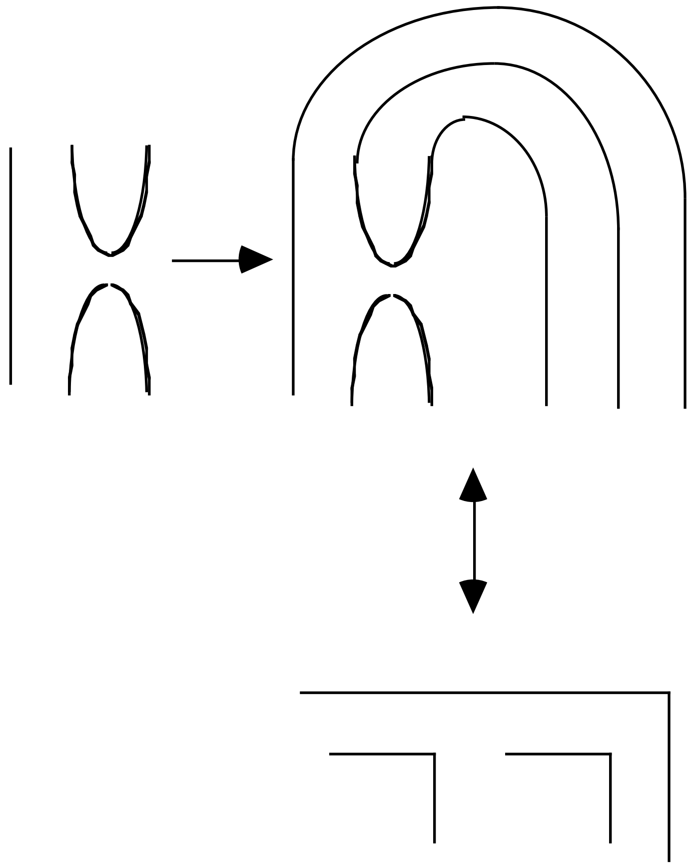
PQ = 

QP = 

It is not an accident that the set { **I, P, Q, PQ, QP**} has *five* elements. In fact each form of connection in the plane by non-intersecting curves from **n** points to **n** points is a parenthetical structure. To see this in terms of the algebra we have just been discussing, **turn eacn element by ninety degrees and then take the top points and curve them in parallel down and to the right**. You will see the parenthesis structure emerge.

P =

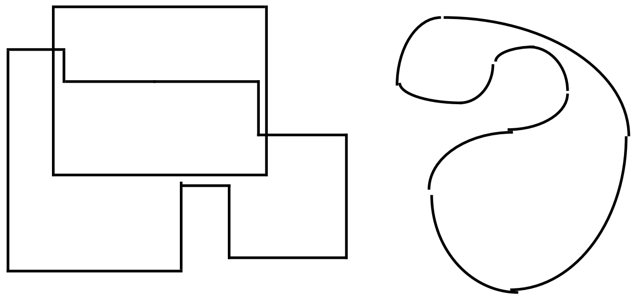




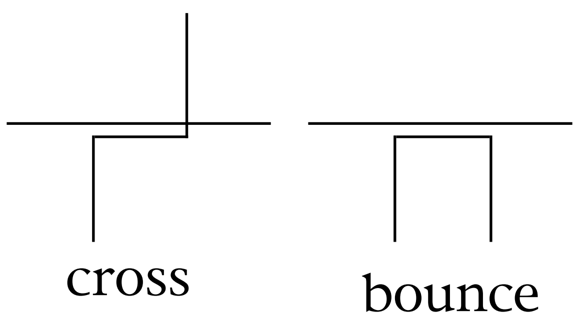
In this way the parentheticals with **n**- marks acquire an intricate algebra structure generated by the Catalan number of connection patterns between two rows of **n** points.

## VIII. Idemposition, Curve Arithmetic and Map Coloring

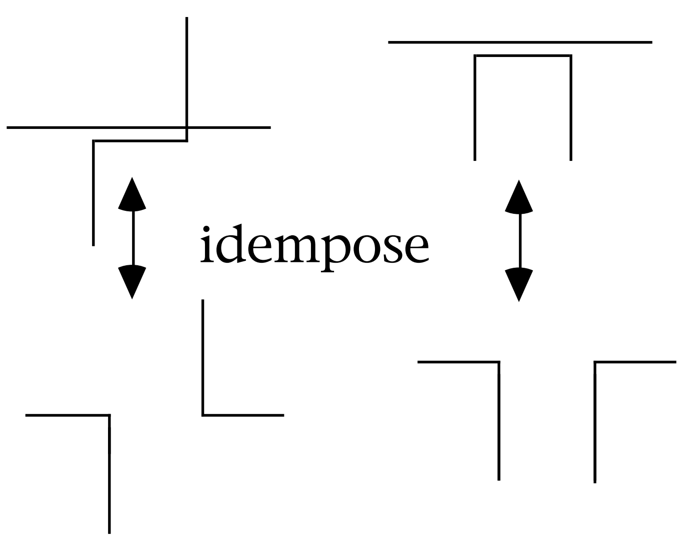
In this section we give some hints about a different approach to Laws of Form via a calculus of boundaries. We shall consider curves in the plane and their interactions.



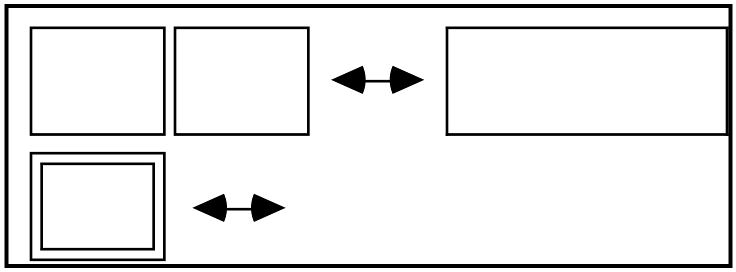
In the figure above, we illustrate two polygonal closed curves interacting, and one smooth curve in isolation.  We will use polygonal curves throughout the rest of this section, but it is convenient to draw smooth curves as well.  All polygonal curves will interact by sharing a segment of boundary in such a way that the two curves either cross over one another or not. In the case where they do not cross over, we call the interaction a *bounce.*



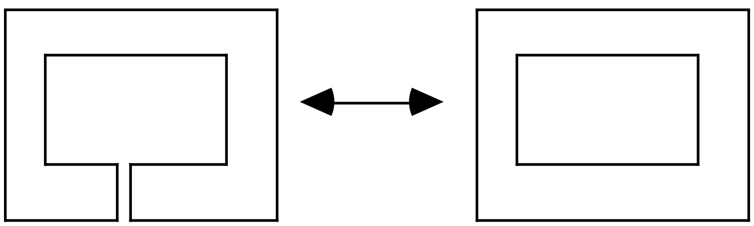

Analogous to the laws of crossing and calling in Laws of Form, we adopt the *principle of idemposition*:
**Common boundaries cancel.**



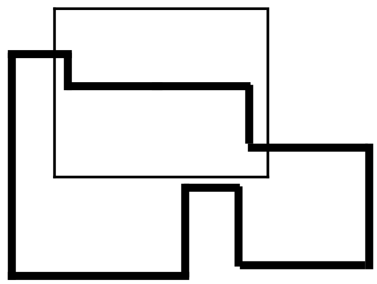
In the above figure we illustrate the two basic local effects of idemposition at a bounce or at a crossing. Note the following two examples.



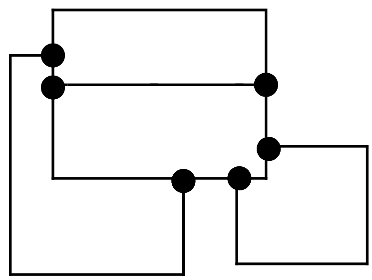
This diagram shows that the familiar laws of calling and crossing are part of the more general curve arithmetic of idempositions. Of course, there are many more complex interactions possible. For example consider the next diagram.



Here a curve self-interacts and produces two curves.
Lots of things can happen. In fact, we are now going to look at an even more complex curve arithmetic, where there are two types of curves, distinguished by light and dark edges.



formation

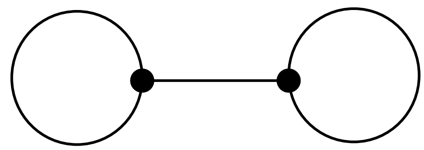


cubic graph

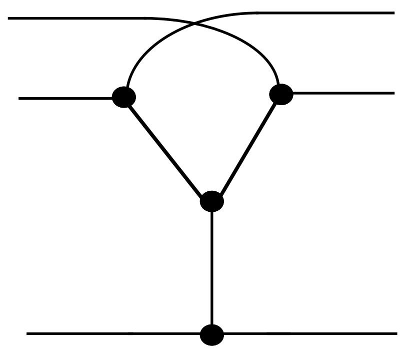
In the diagram above we have illustrated how a dark curve and a light curve can interact. The shared edges are combination of dark and light. A curve interaction of curves of two colors in the plane such that only curves of different colors interact (by bounce or cross) is called a formation. Each formation has an underlying cubic graph (three edges incident to each vertex). See the the diagram above for the cubic graph corresponding to the formation drawn there. The formation gives a coloring of the cubic graph such that each vertex sees three colors: dark, light and the combination dark/light. **In this way we obtain, by drawing formations, infinitely many cubic graphs colored with three colors with three distinct colors at the vertex.** The famous *Four Color Theorem* [Kempe, Tutte, VCP, SB, Map] is equivalent to the statement that every cubic plane graph without an *isthmus* (an edge such that the graph is disconnected if the edge is deleted) is colorable by three colors, with three distinct colors incident at each node. This approach to the problem of coloring cubic graphs is the beginning of Spencer-Brown's work on the problem [SB, Map].

### Uncolorables and Reentry Forms

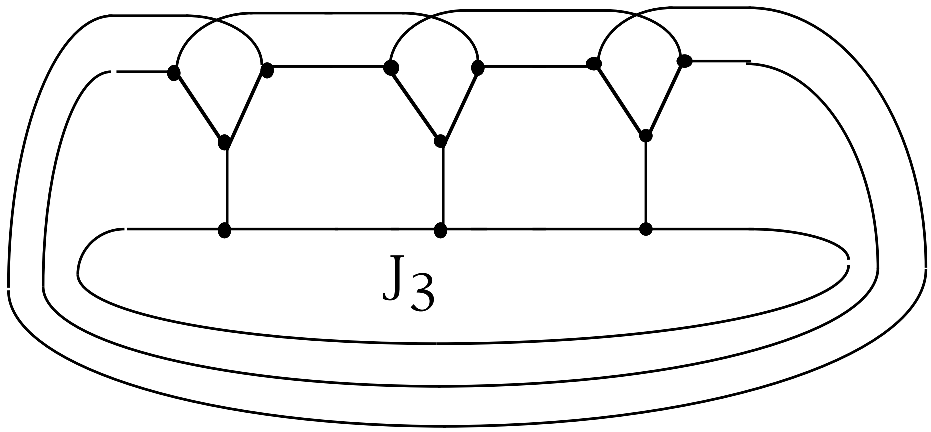
Another approach to the Map Theorem is to characterize and understand the class of uncolorable cubic graphs. The simplest such graph is the dumbbell shown below. This example is planar, but has an isthmus (the edge joining the two loops).



In [RI], Rufus Isaacs gives a basic building block for the construction of uncolorables that have no isthmus, but are not embeddable in the plane. Here is the Isaacs building block R:

R = 

The crossover of lines at the top of R is not meant to indicate any interaction. It is a virtual crossing, and can be resolved if the graph is placed in three-dimensional space. You can think of R as an input-output device for three colors assigned to the lines, with the stipulation that each node must see three distinct colors. Isaacs then defines the graph $J_n$ to be that graph obtained by plugging $n$ of these R's into one another, and then running the output back into the input as shown below for $n=3$.




Isaac's $J_3$ is uncolorable and non-planar. In fact $J_{2n+1}$ is uncolorable for each $n=1,2,3,....$ The uncolorablity comes from that fact that the transmission of values around the three strand loop is self-contradictory. In this sense $J_3$ is a cousin of the reentering mark. One way to gain insight into the Four Color Theorem is to begin to realize that uncolorables are all analogs of the reentering mark, and there is just no way to build self-contradictory devices like that in the plane.

## IX. Sets

It is common practice in mathematics to introduce the concept of sets and membership, and then construct a hierarchy of sets, beginning with the empty set and continuing by using the operation of *collection* (making a set from a previously constructed array of sets).

*1. Collection: We assume that given an array of sets, there is a new set whose members are the elements of this array.*
*2. Equality: Two sets are equal if and only if they have the same members.*

This has the effect of bootstrapping a huge array of sets virtually from nothing but the act of forming a collection and the definition that two sets A and B are equal if they have the same members.

**Lemma.** If A and B are empty sets, then A = B. Hence there is only one empty set and it shall be denoted by { }.

**Proof.** A = B if and only if A and B have the same members. But A and B do have the same members, namely none. This completes the proof of the Lemma. //

Now that we know the empty set is unique, we can proceed. But wait! How do we know that the empty set exists?

**Lemma.** There is an empty set.

**Proof.** Suppose that all sets are non-empty.
Let S be the collection of all sets that have no members.
But then S is empty! This is a contradiction. Therefore the empty set exists.//

**Remark.** In this context, we could just as well have said, let 0 be the set of all round squares, or the set of all unicorns. The advantage of the above proof is that it bootstraps the existence of the empty set directly from the internal structure of our universe of sets. Note that if there were no empty sets and we formed S, the collection of all sets with no members, then as soon as S is formed, S itself would demand membership in S. If we allow S to be a member of itself, then S would no longer be empty, and so could not be a member of itself. This keeps S empty but always asking for self-membership. There are interesting issues of self-reference surrounding this evocation of the empty set.

Now that we have the empty set 0 = { }, we can begin to form lots of sets. For example we can form the set whose member is the empty set 1 = {0} = {{ }}. This set has one element.
Now form
2 = { 0, 1} = {{ }, {{ }}}
3 = { 0, 1, 2 } = { { }, {{ }}, {{ }, {{ }}} }
and generally
n+1 = {0, 1, 2, ..., n}.
Each set in this sequence has one more element than the preceding set and they are all unequal to one another. For example 0 is not equal to 1 since 1 has a member (namely 0) and 0 has no member. 2 is not equal to 1 because 0 is the only member of 1, while 1 (which is not equal to 0) is a member of 2. In this way we construct eventually, sets that have n elements for every natural number n. This construction of numbers from sets is due to John von Neumann and, in an earlier version, to Gottob Frege.

Note that if we are making sets only from sets, then the comma that is commonly used to delineate the members from one another can be eliminated. We can write

0 = { }
1 = { { } }
2 = { { } {{ }} }
3 = { { } {{ }} {{ } { }}} }

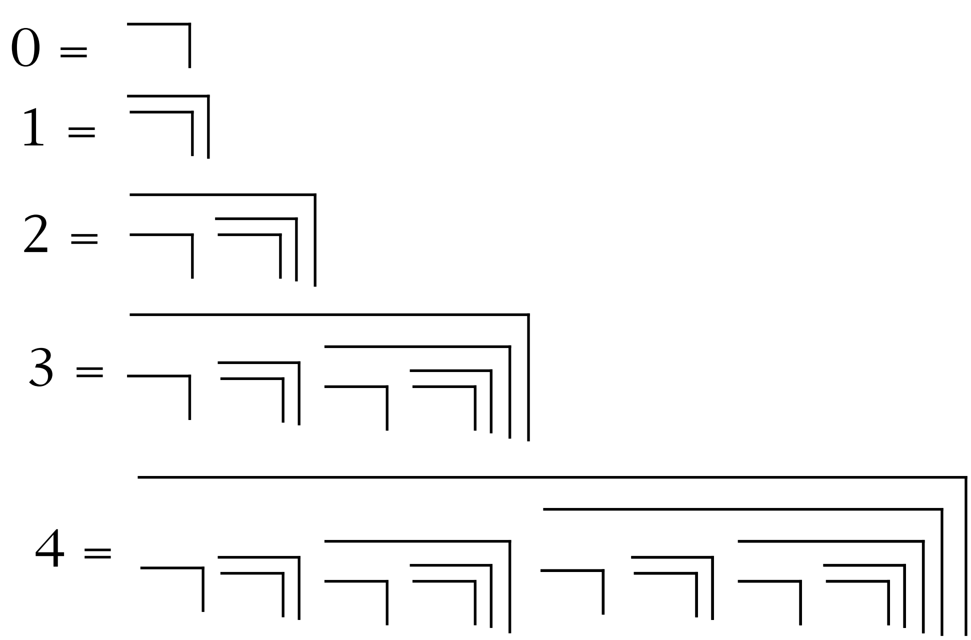
In this notational universe, when you go inside the boundary of a given set (passing through its outer braces),
you encounter a well-formed parenthetical expression in the brackets. Such an expression factors into a collection of subexpressions, each in the form { Something }. These are the elements of the set. For example, the elements of

2 = { { } {{ }} }

are the factors of { } { { } }, which are { } and { { } }.

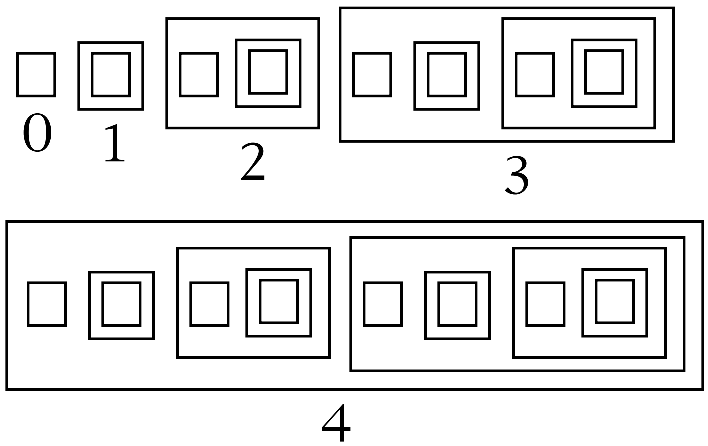
We can, if we wish, replace the paired braces with marks, and represent the sets as expressions (but without any rules of calling or crossing) in the notation of the primary arithmetic.

There is a strong parallel between our constructions of elementary sets and the formalism of the primary arithmetic. The set theoretic brackets correspond to the mark. The marked state corresponds to the empty set. Putting a mark around an expression makes all the parts of the expression the members of a corresponding set.

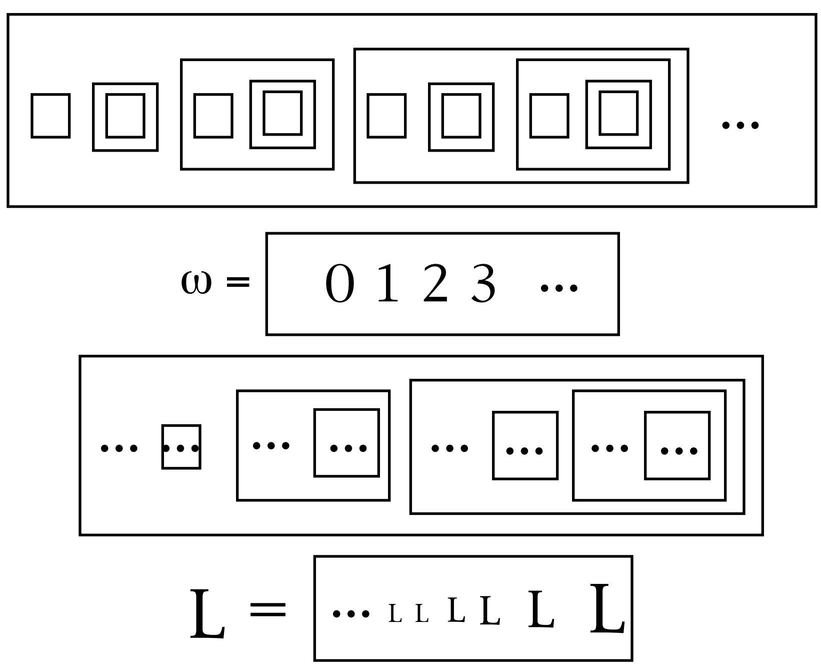


Set theory can be formulated in terms of distinctions, and is seen to be another way (distinct from the primary arithmetic) to start from nothing and build a mathematical universe.

Here is another take on the von Neumann construction, using boxes this time.




With boxes, it is intuitively a little easier to think about the form of the limit of this construction, as we let **n** go to infinity.




The figure above indicates two ways to look at the limit. In the first way we see the limit as the set of all numbers 0,1,2,3,4,.... This is called ω, the first countable transfinite ordinal. Once one has taken a limit in this mode, it is possible to continue, forming the unending sequence of transfinite ordinals.

$$\omega + 1 = \boxed{\omega \;\; \boxed{\omega}}$$

$$\omega+2,\ \omega+3,\ \omega+4,\ \ldots,\ \omega+\omega, \ldots$$

$$\omega+\omega+\omega,\ \ldots,\ \omega^{2},\ \ldots\ \omega^{\omega}, \ldots$$

$$\tau = \omega^{\omega^{\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}}} = \omega^{\tau}, \ldots$$

The second limit is L as shown above. We have L = { ... L L L L }, a countably infinite multi-set (it has infinitely many elements, all identical to one another) whose only member is itself! The second limit takes us out of the usual category of sets. It shows how each non-negative integer is an approximation to a set that is a member of itself. In the author's opinion, both of these limits should be taken seriously and regarded as a rich source of imaginary values.

Of course we can make lots of other sets. For example we can make

[0] = { }
[1] = {{ }}
[3] = {{{ }}}
[4] = {{{{ }}}}
[5] = {{{{{ }}}}}

and so on so that

[n+1] = { [n] }.

Here one is tempted to take a limit and form

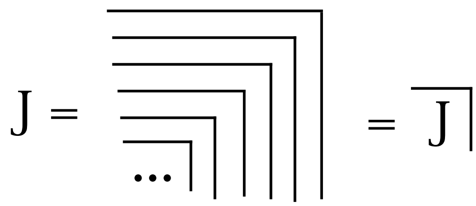
L = [Infinity] = {{{{{{{ ... }}}}}}}

so that

L = { L }.

Then L would be a singleton set with itself as the only member.
L is of course, in form, nothing but the reentering mark.

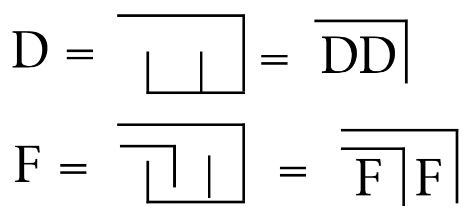
$$L = \{\{\{\{\{\{\{ \ldots \}\}\}\}\}\}\} = \{ L \}$$



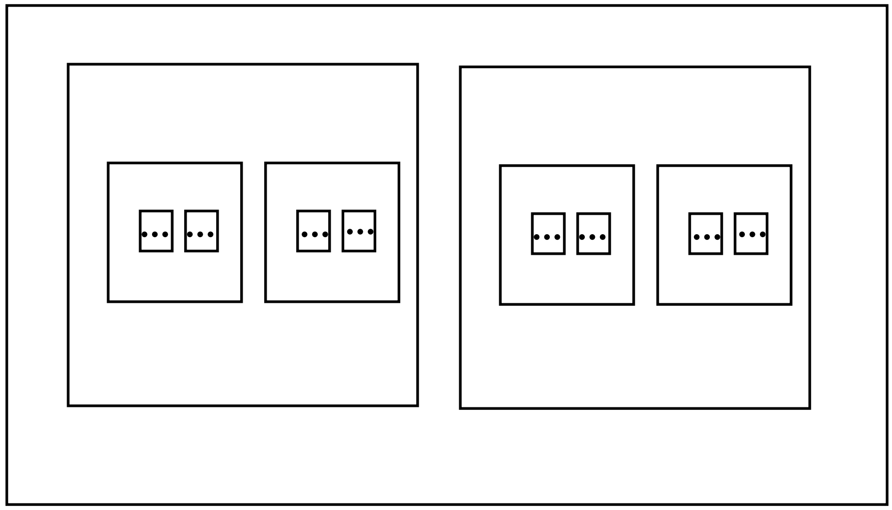
There is nothing wrong with considering such sets, but they are excluded by certain axiom systems for set theory. Other axioms systems allow them. In order to form L as a limit, we did have to use something beyond the act of forming a collection of previously created sets.

### X. Infinite Recursive Forms

Constructions of sets as expressions in the mark, suggests considering all possible expressions, including infinite expressions, with no arithmetic initials other than commutativity.
We shall call such expressions *forms*. Here we shall discuss some of the phenomenology of infinite forms that are described by reentry.
This simplest example of such a form is the reentering mark J as discussed above. Here are the next two simplest examples.



I call D the *doubling form*, and F the *Fibonacci form*.
A look at the recursive approximations to D shows immediately why we have called it the doubling form (approximations are done in box form):

$D =$ 

We see from looking at the approximations, that the number of divisions of D doubles at each successive depth beyond depth zero. Letting $D_n$ denote the number of divisions of D at depth n, we see that $D_0=1$, $D_1=1$, $D_2=2$, $D_3=4$, $D_4 = 8$, ..., $D_n = 2^{n-1}$. We can see this behaviour from the recursive definition of the form, for given any forms G and H, it is clear that with $G_n$ the number of divisions of G at depth n, we have the basic formulas:

$$\overline{G|}_{n+1} = G_n$$

$$(GH)_n = G_n + H_n$$

Thus

$$D = \overline{\;\lfloor\;\rfloor\;|} = \overline{DD|}$$

$$D_n = \overline{DD|}_n$$

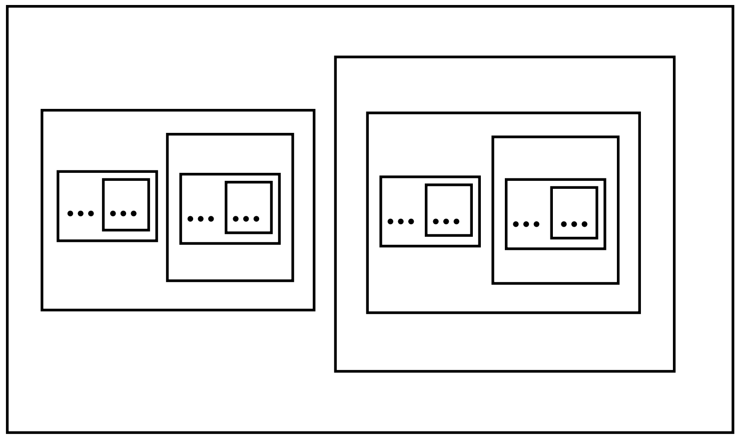
$$= D_{n-1} + D_{n-1}$$

$$D_n = 2\,D_{n-1}$$

The reader will have no difficulty verifying that in the case of the Fibonacci form, $F_{n+1} = F_n + F_{n-1}$ with $F_0=F_1=1$. Hence the depth counts in this form are the Fibonacci numbers

$$1,1,2,3,5,8,13,21,34,55,89,144,\ldots$$

with each number the sum of the preceding two numbers.

F = 

The Fibonacci Form

It is natural to define the *growth rate* $\mu(G)$ of a form $G$ to be limit of the ratios of successive depth counts as the depth goes to infinity.

$$\mu(G) = \lim_{n \to \text{Infinity}} G_{n+1}/G_n.$$

Then we have $\mu(D) = 2$, and $\mu(F) = (1 + \sqrt{5})/2$, the golden ratio.

In the spirit of these recursions and the consideration of thinking about the most elementary recursive forms, it is natural to wonder *what are the recursive forms that would correspond to the general recursion*

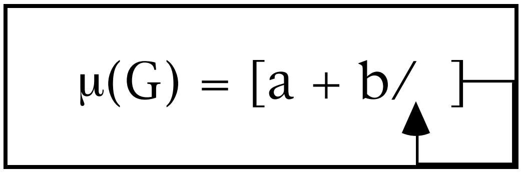
$$G_{n+1} = aG_n + bG_{n-1}$$

*where a and b are rational numbers?*

Note that if we have a recursion as shown above, then

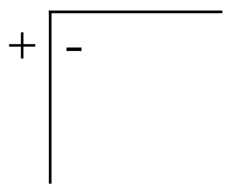
$$G_{n+1}/G_n = a + b/(G_n/G_{n-1}).$$

Thus, if the limit $x = \mu(G)$ exists, then it satisfies the equation $x = a + b/x$. We can call the growth rate of the form $G$ the infinite formal continued fraction




In the case where $a^2 + 4b < 0$, the roots of the corresponding quadratic equation ($x = a + b/x$ implies the quadratic $x^2 = ax + b$)
are complex numbers, and the continued fraction approximations do not converge to any specific real number. The corresponding complex roots could be regarded as the "growth rate" of G.

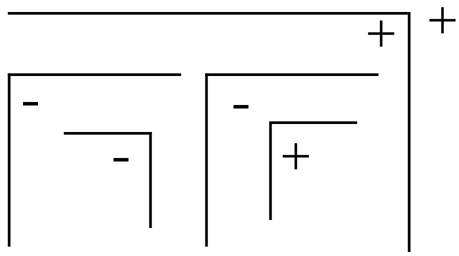
In particular, what about $S_{n+1} = S_n - S_{n-1}$? Here, we ask only about the possibility of using negative numbers in the recursion.
It is clear that if a and b are positive integers, then we can make corresponding recursive forms by using multiple reentries just as we did with the doubling form D. We can accomplish negative numbers in the counts by introducing the *negative mark*.



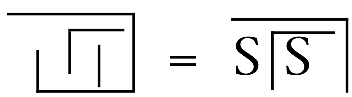

The negative mark, by definition, encloses a negative space.
There is 1 division of depth 0 in the negative mark, and -1 division of depth 1. Also, by definition, if a form G is crossed by the negative mark, then all positive spaces in G become negative and all negative spaces in G become positive. Thus

$$(\overline{\lvert G}\ )_n = -\,G_{n-1}.$$

For example,




the form above has one division of depth 0, one division of depth 1, -2 divisions of depth 2 and 0 = -1 +1 divisions of depth 3. Note how the crossing of the negative mark by the negative mark makes the innermost space positive. Now consider the following form.

S = 


$$S_n = S_{n-1} - S_{n-2}$$

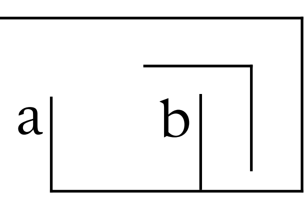
1,1,0,-1,-1,0,1,1,0,-1,-1,0, ...

Using the negative mark, we obtain the the recursion $S_n = S_{n-1} - S_{n-2}$, which is periodic of period 6. The depth counts of the form S have period six. Notice that the actual number of divisions at depth n in S is the n-th Fibonacci number.

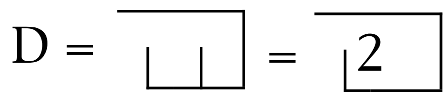
More generally, if

$$G_{n+1} = aG_n + bG_{n-1}$$

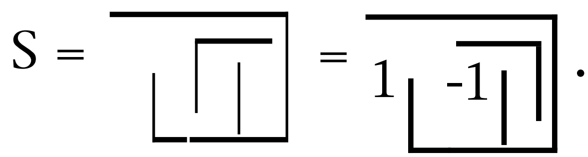
with $G_0 = 1$ and $G_1 = 1$, we shall model this as a recursive form in the formalism

G(a,b) = 


where it is understood that the divisions in the reentry that receive labels a or b are now weighted with that label, so that they count a or b in the depth count, and so that all divisions inside them are now weighted by multiplying by the factor a or b. For a and b positive integers, we identify the labeling process with the process of actually repeating these subforms that many times. Thus



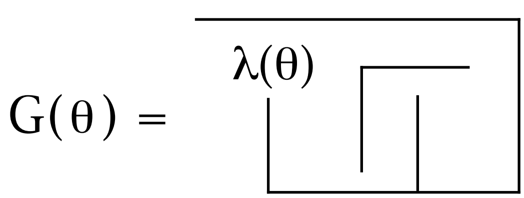

and




There is a general formula for the depth counts of the forms G(a,b). It is based on the fact that if R and S are the roots of the quadratic equation $x^2 = ax + b$, then these roots satisfy $x^{n+1} = ax^n + bx^{n-1}$, so that linear combinations of the roots can be used to produce the desired recursion. Assuming that R and S are distinct roots, we have, for $G_0 = G_1 = 1$,

$$G(a,b)_n = ((S-1)R^n + (1-R)S^n)/(S-R).$$

This works even when the roots are complex numbers. Thus if
$S = \cos(\theta) + i\sin(\theta)$ and $R=\cos(\theta) - i\sin(\theta)$,
then the quadratic equation is
$x^2 = 2\cos(\theta)x - 1$
so that the recursive form is
$G(\theta) = G(2\cos(\theta), -1)$



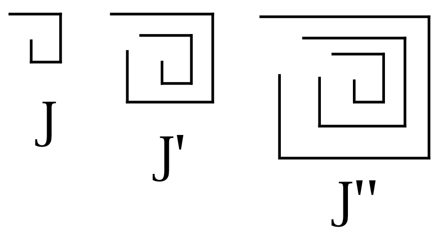

$\lambda(\theta) = 2\cos(\theta)$

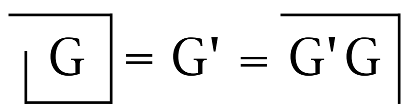
A little algebra reveals that
$G(\theta)_n = (\sin(n\theta) - \sin((n-1)\theta))/\sin(\theta)$,
from which it follows that

$Tot_n(G(\theta)) = G(\theta)_1 + G(\theta)_2 + ... + G(\theta)_n = \sin(n\theta)/\sin(\theta)$.

Thus the sine function appears as the total signed depth count, $Tot_n(G)$, for the form $G(\theta)$. If we implicitly choose $\theta$ by taking $a = (q-1)/q$ and $b = -1$ ($q$ is a positive integer greater than 1), then the depth counts of $G((q-1)/q, -1)$ can yield very good approximations to the sine function when $q$ is large, and they can exhibit interesting, sometimes chaotic behaviour.

Finally, here is a natural hierarchy of recursive forms, obtained each from the previous by enfolding one more reentry.

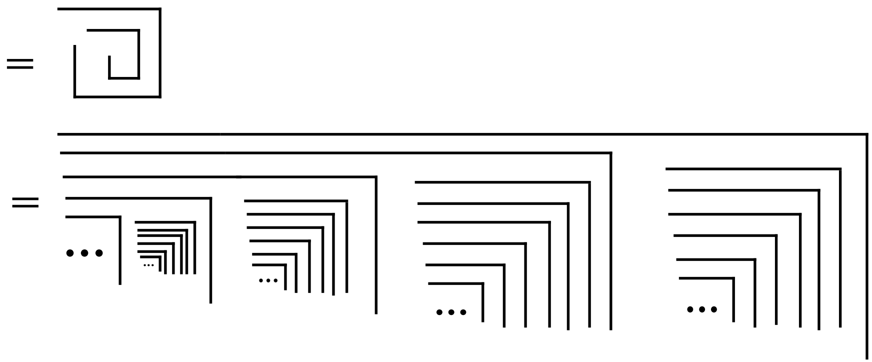


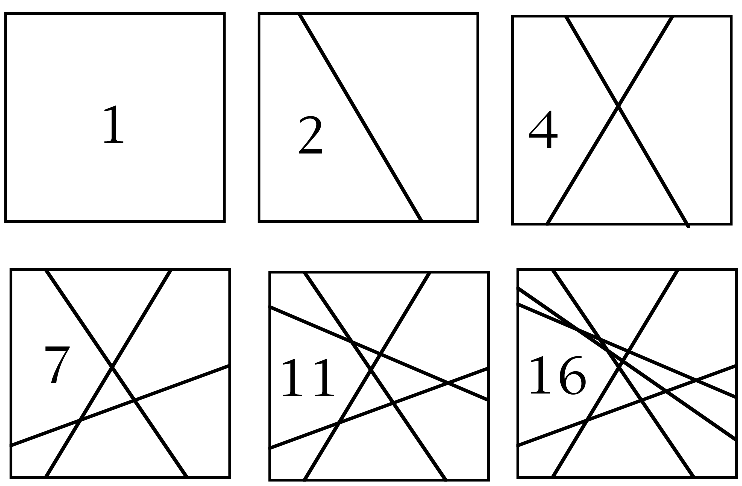

$$G'_{n+1} = G'_n + G_{n-1}$$

Given any form G, we define G' by the formula shown above, so that
$G'_{n+1} = G'_n + G_{n-1}$.
This implies that
$G'_{n+1} - G'_n = G_{n-1}$.
Thus the discrete difference of the depth series for G' is (with a shift) the depth series for G. In a certain sense G' is the "integral" of G. The series J, J', J'', J''', ... is particularly interesting because *the depth sequence* $(J^{(n)})_k$ *is equal to the maximal number of divisions of* $n$*-dimensional Euclidean space by* $k-1$ *hyperspaces of dimension* $n-1$. We will not prove this result here, but note that J takes the role of a point (dimension zero) with $J_k = 1$ for all k, while J' satisfies $J'_{k+1} = J'_k + 1$ $(k > 0)$, so that $J'_k = k-1$ for $k>1$. This is the correct formula for the number of divisions of a line by $k-1$ points.

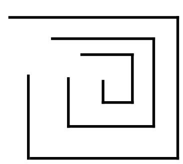
J' =



To think about the divisions of hyperspace, think about how a collection of lines in general position in the plane intersect one another. If a new line is placed, it will cut a number of regions into two regions. The number of new regions is equal to the number of divisions made in the new line itself. This is a verbal description of the basic recursion of reentry enfoldment given above.




1 + 1 = 2
2 + 2 = 4
4 + 3 = 7
7 + 4 = 11
11 + 5 = 16
...

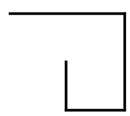


The very simplest recursive forms yield a rich complexity of behaviours that lead directly into the mathematics of imaginary numbers and oscillations, patterns of growth, dimensions and geometry..

There is an eternity and a spirit at the center of each complex form. That eternity may be an idealization, a "fill-in", but it is nevertheless real. In the end it is that eternity, that eigenform unfolding the present moment that is all that we have.  We know each other through our idealizations of the other. We know ourselves through our idealization of ourselves. We become what we were from the beginning,  a Sign of Itself [P] .

### XI. Eigenforms

Consider the reentering mark.

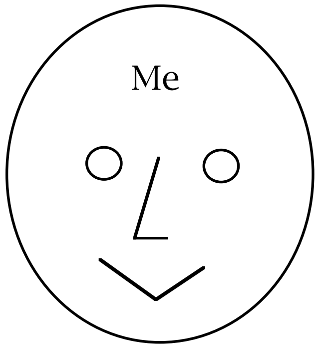


This is an archetypal example of an eigenform in the sense of Heinz von Foerster [VF].  What is an eigenform?  An eigenform is a solution to an equation, a solution that occurs at the level of form, not at the level of number. You live in a world of eigenforms. You thought that those forms you see are actually "out there"?  Out where? It has to be asked. The very space, the context that you regard as your external world is an eigenform. It is your organism's solution to the problem of distinguishing itself in a world of actions. The shifting boundary of the Myself/MyWorld  is the dynamics of the form that "you" are.  The reentering mark is the solution to the equation

$$J = \overline{\;J\;}\rceil$$

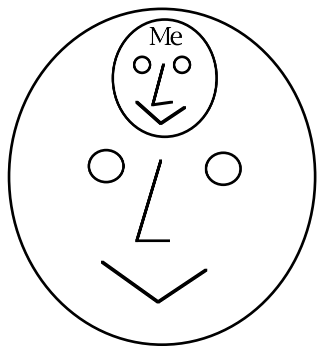
where the right-angle bracket  distinguishes a space in the plane. This is not a numerical equation.  One does not even need to know any particularities about the behaviour of the mark to have this equation.  It is more akin to solving

Me =



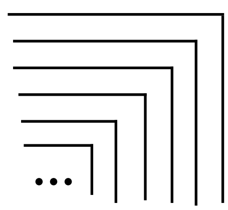

by attempting to create a space where "I" can be both myself and inside myself, as is true of our locus psychological. And this can be solved by an infinite regress of Me's inside of Me's.

Me =




Just so we may solve the equation for J by an infinite nest of boxes

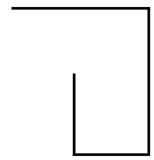
J =



Note that in this form of the solution, layered like an onion, the whole infinite form reenters its own indicational space. It is indeed a solution to the equation

$$J = \overline{\;J\;}\rceil$$

The solution in the form

J = 

is meant to indicate how the form reenters its own indicational space. This reentry notation is due to G. Spencer-Brown. Although he did not write down the reentering mark itself in his book "Laws of Form", it is implicit in the discussion in Chapter 11 of that book.

Now you might wonder many things after seeing this idea. First of all, it is not obvious that we should take an infinite regress as a model for the way we are in the world. On the other hand, everyone has experienced being between two reflecting mirrors and the veritable infinite regress that arises at once in that situation. Physical processes can happen more rapidly than the speed of our discursive thought, and thereby provide ground for an excursion to infinity.

These patterns of form dynamics form the skeleton for the description and consideration of many structures in cybernetics and science. Elaboration of the solution to eigenform equations leads to the structure of fractals and to a philosophy that extends the notion of eigenvalues in physics. See [SRF ,EF] for a discussion of this point of view.

Here is one more example. This is the eigenform of the Koch fractal [SRF]. In this case one can write the eigenform equation

$$K = K \{ K \quad K \} K.$$

The curly brackets in the center of this
equation refer to the fact that the two middle copies within the fractal are inclined with respect to one another and with respect to the two outer copies. In the figure below we show the geometric configuration of the reentry.

The Koch fractal reenters its own indicational space **four** times (that is, it is made up of four copies of itself, each **one-third** the size of the original. We say that the Koch fractal has *replication rate* four and write R(K)=4. We say it has *length ratio* three and write F(K)=3.

In describing the fractal recursively, one starts with a segment of a given length L. This is replaced by a R(K) segments each of length L' = L/F(K). In the equation above we see that R(K)=4 is the number of reentries, and F(K) is the number of groupings in the reentry form.

It is worth mentioning that the fractal dimension D of a fractal such as the Koch curve is given by the formula

$$D = \ln(R)/\ln(F)$$

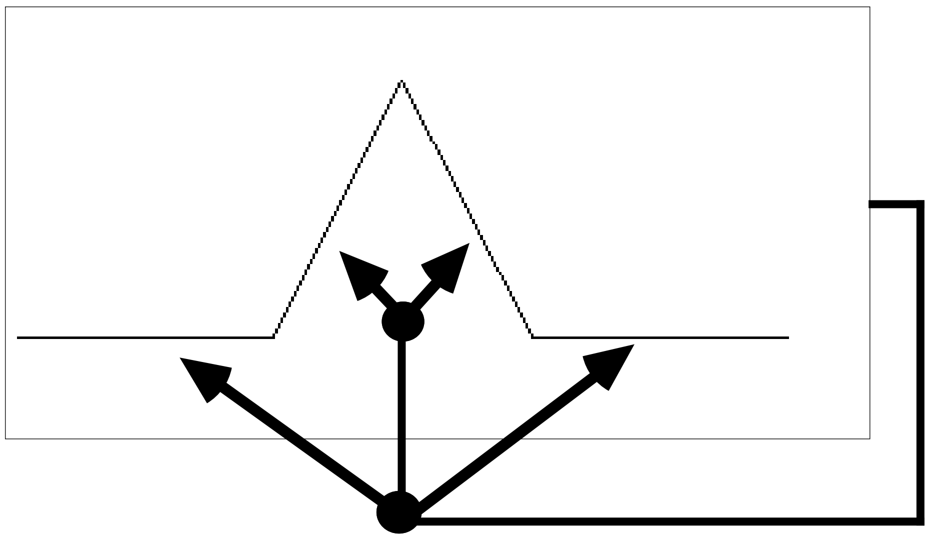
where R is the replication rate of the curve , F is the length ratio and ln(x) is the natural logarithm of x.

In the case of the Koch curve one has $D = \ln(4)/\ln(3)$. The fractal dimension measures the fuzziness of the limit curve. For curves in the plane, this can vary between 1 and 2, with curves of dimension two having space-filling properties.

It is worth noting that we have, the case of an abstract, grouped reentry form such as K = K { K K } K, a corresponding abstract notion of fractal dimension, as described above

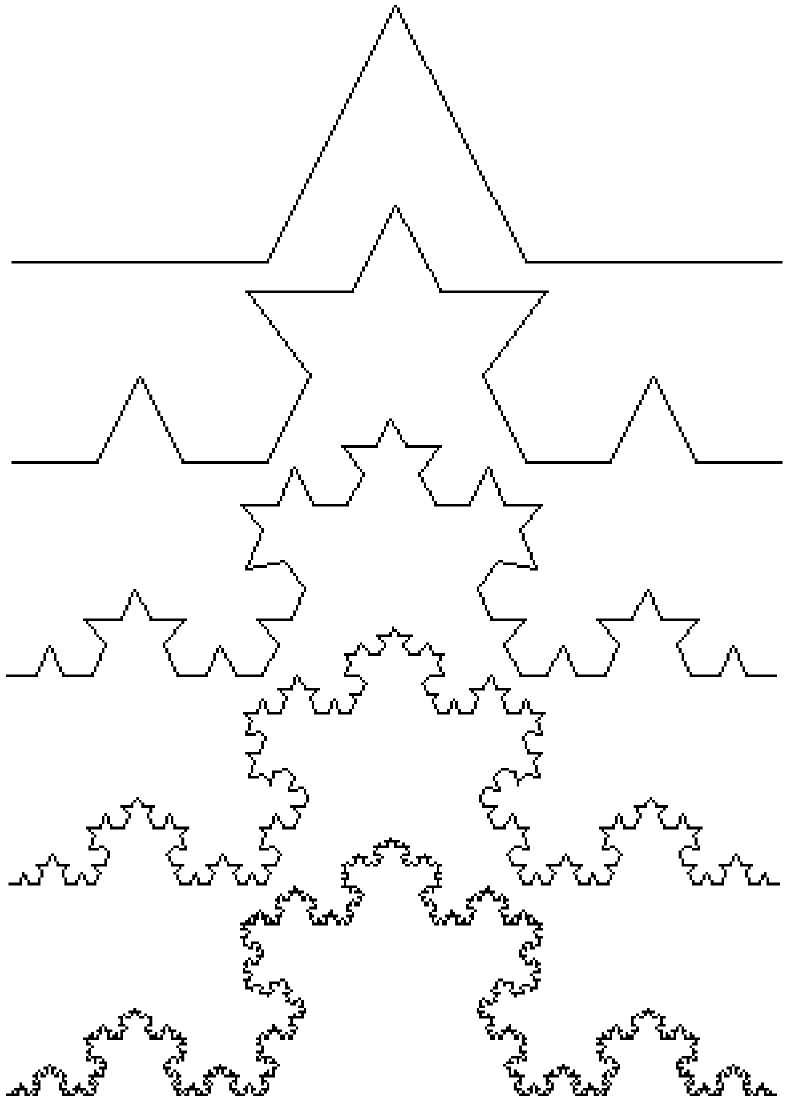
D(K) = ln(Number of Reentries)/ln(Number of Groupings).

As this example shows, this abstract notion of dimension interfaces with the actual geometric fractal dimension in the case of appropriate geometric realizations of the form. There is more to investigate in this interface between reentry form and fractal form.



$$K = K \{ K K \} K$$

In the geometric recursion, each line segment at a given stage is replaced by four line segments of one third its length, arranged according to the pattern of reentry as shown in the figure above. The recursion corresponding to the Koch eigenform is illustrated in the next figure. Here we see the sequence of approximations leading to the infinite self-reflecting eigenform that is known as the Koch snowflake fractal.



Five stages of recursion are shown. To the eye, the last stage vividly illustrates how the ideal fractal form contains four copies of itself, each one-third the size of the whole. The abstract schema

K = K { K K } K

for this fractal can itself be iterated to produce a "skeleton" of the geometric recursion:

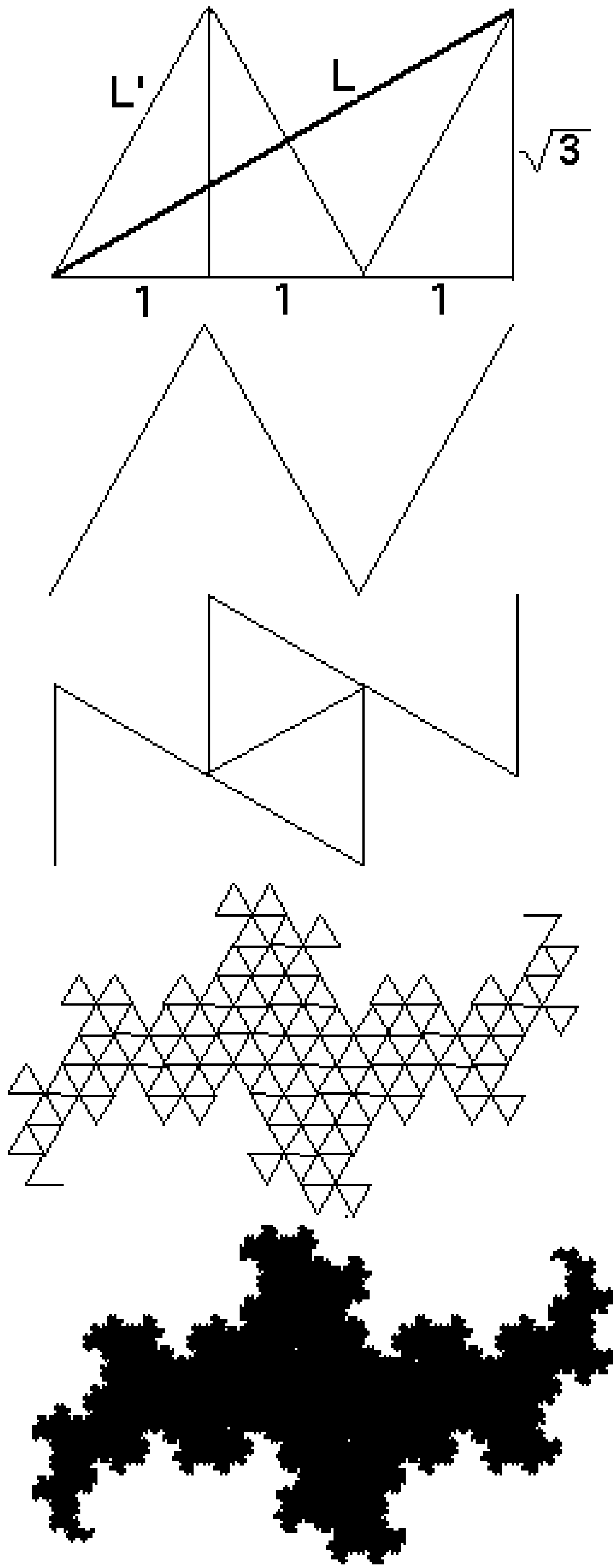
K = K { K K } K

= K { K K } K { K { K K } K K { K K } K } K { K K } K

= ...

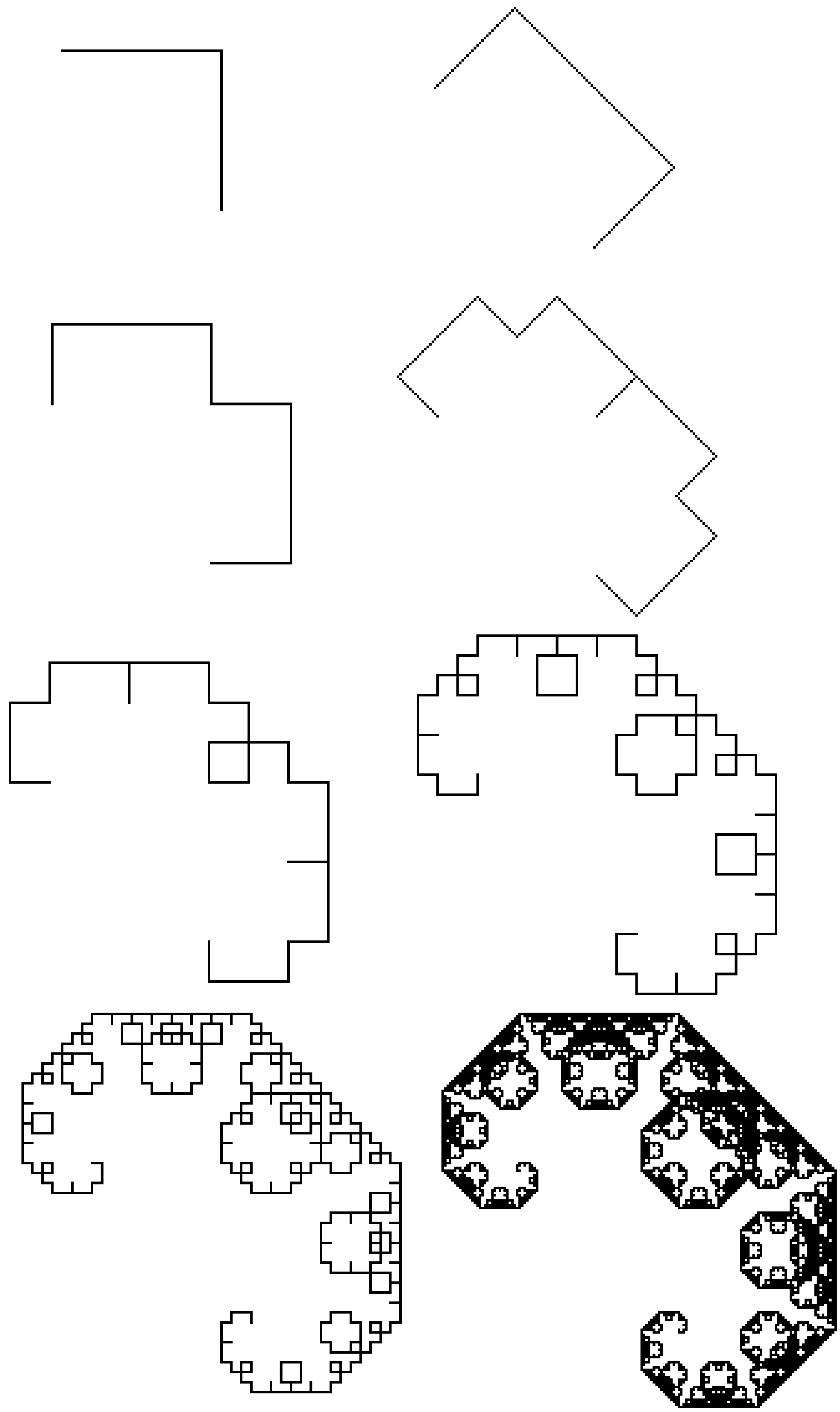
We have only performed one line of this skeletal recursion. There are sixteen K's in this second expression just as there are sixteen line segments in the second stage of the geometric recursion. Comparison
with this abstract symbolic recursion shows how geometry aids the intuition.

Geometry is much deeper and more surprising than the skeletal forms. The next example illustrates this very well. Here we have the initial length L being replaced by a length three copies of L' with L'/L equal to the square root of 3. (To see that L'/L is the square root of three, refer to the illustration below and note that L'= √(1 + 3) = 2, while L = √(9 + 3) = 2√3.) Thus this fractal curve has dimension D = ln(3)/ln(√3) = 2. In fact, it is strikingly clear from the illustration that the curve is space-filling. It tiles its interior space with rectangles and has another fractal curve as the boundary limit.


L'
L
√3
1
1
1

The interaction of eigenforms with the geometry of physical, mental, symbolic and spiritual landscapes is an entire subject that is in need of deep exploration. Compare with [EF].

As a last fractal example for this section, here is a beautiful specimen SB generated by the Spencer-Brown mark. That is, the generator for this fractal is a ninety degree bend. Each segment is replaced by two segments at ninety degrees to one another, and the ratio of old segment to new segment is $\sqrt{2}$. Thus we have $D(SB) = \ln(2)/\ln(\sqrt{2}) = 2$, another space-filler. Notice how in the end, we have an infinite form that is a superposition of two smaller copies of itself at ninety degrees to one another.



It is usually thought that the miracle of recognition of an object arises in some simple way from the assumed existence of the object and the action of our perceiving systems. What is to be appreciated is that this is a fine tuning to the point where the action of the perceiver, and the perception of the object are indistinguishable. Such tuning requires an intermixing of the perceiver and the perceived that goes beyond description. Yet in the mathematical levels, such as number or fractal pattern, part of the process is slowed down to the point where we can begin to apprehend it. There is a stability in the comparison, in the one-to-one correspondence that is a process happening at once in the present time. The closed loop of perception occurs in the eternity of present individual time. Each such process depends upon linked and ongoing eigenbehaviors and yet is seen as simple by the perceiving mind.

## XII. Lambda Calculus, Eigenforms and Godel's Theorem

Church and Curry [B] showed (in the 1930's, long before von Foerster wrote his essays) how to make eigenforms without apparent excursion to infinity. Their formalism is usually called the "lambda calculus."

Here is how it works:
We wish to find the eigenform for F. We want to find a J so that F(J) = J. Church and Curry admonish us to create an operator G with the property that

$$GX = F(XX)$$

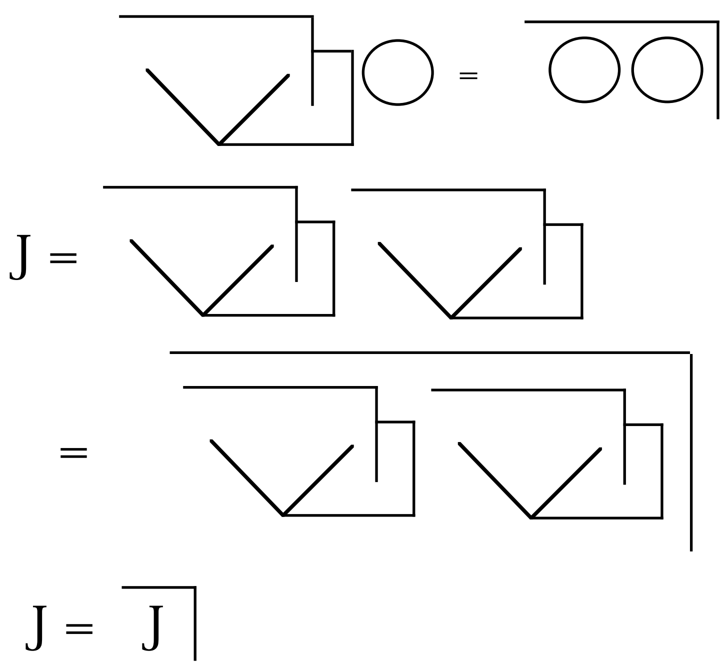
for any X. That is, when G operates on X, G makes a duplicate of X and allows X to act on its duplicate. Now comes the kicker.
Let G act on herself and look!

$$GG = F(GG)$$

So GG, without further ado, is a fixed point for F. We have solved the problem without the customary ritual excursion to infinity.

I like to call the construction of the intermediate operator G, the "gremlin" [VL, MP]  Gremlins seem innocent enough. They duplicate entities that meet, and set up an operation of the duplicate on the duplicand. But when you let a gremlin meet a gremlin then strange things happen. It is a bit like the story of the sorcerer's apprentice, except that here the sorcerer is the mathematician or computer scientist who controls context, and the gremlins are like the self-duplicating brooms in the story. The gremlins can go wild without some control. In computer science the gremlins are programs with loops in them. If you do not put restrictions on the loops, things can get very chaotic!

The reentering mark can be created with the help of a gremlin. In the illustration below, we show a gremlin G such that GA = <AA> where the brackets in this notation represent the enclosure shown in the figure. Then GG = <GG> so that GG can be regarded as the reentering mark.




Once an appropriate gremlin is in place, clocks will tick and numbers will count.  For each of us, there is a continual manufacture of eigenforms (tokens for eigenbehaviour) but such tokens will not pass as the currency of communication unless we achieve mutuality as well.

## The Indicative Shift

This next part is intimately related to the lambda calculus.
One way to see this is to reformulate the gremlin as follows:
Let #a = aa.
Then the Gremlin is defined by GX = F(#X) and we obtain the fixed point by substituting G into its own formula to get GG = F(#G) = F(GG). In this next part *we shift the equality sign to a sign of reference* so that instead of #a = aa, we have #a -----> aa, and we shall see this as a consequence of an earlier reference of a to itself in the form a -----> a.
In this case a -----> a came from a = a.
Read on!

Consider the form of names. When I am introduced to Mr. A, there is a momentary separation of the *name* of Mr. A (lets say this is just A) and the *person* of Mr. A (who is in the room). If I did not know his name, the person would nevertheless be present, but after I learn his name, then the presence of his person calls up his name. If my memory is not slow, then the name and the person are superimposed for me. We can diagram this process as follows. Let
A -----> P
denote the name A pointing to the person P.
After I get to know him, this separation has shifted to
#A -----> PA.
That is, the name is "attached" (in my mind) to the person (as perceived, or as a participant in a conversation) and there is a new *meta-name* #A that refers to this combination. In ordinary language there is no explicit distinction between the name and the meta-name, but analysis of our ways of speaking shows that it is there. For example, if I should forget Mr. A's name, then I am uneasy and can experience the name and meta-name drop into place as I remember his name (hopefully he has not, at this point, left the room).

I call the movement from A -----> P to #A -----> PA the **indicative shift.**

Since it is useful to have the arrow available for the shift itself, lets use the following notation for A refers to B.

$$\overline{A}|B$$

Then the indicative shift is written

$$\overline{A}\rceil B \longrightarrow \overline{\#A}\rceil BA$$

We regard the meta-naming operator # as a nameable object, and so suppose that **M** names #.

$$\overline{M}\rceil \# \longrightarrow \overline{\#M}\rceil \#M$$

We see that the indicative shift applied to the naming of # yields the self-reference

$$\overline{\#M}\rceil \#M \,.$$

The meta-name of the meta-naming operator refers to itself.

Similarly if we start with a reference to **F#** for any **F**, then the result is an expression **F#g** that contains its own name (which is **#g**).

$$\overline{g}\rceil F\# \longrightarrow \overline{\#g}\rceil F\#g$$

**Godel's Theorem**

At this point we have arrived at the essence of Godel's Incompeteness Theorem. Godel works with a formal system where every formula or well-formed text in the system has a *code number* **g** in the positive integers. We shall write

$$\overline{g}\rceil F$$

to denote that ***g*** *is the Godel code number of the formula* ***F.***

Godel considers formulas **F = F(u)** that have a single free variable **u**,
and considers the operation of substituting a specific number **N** into the formula. We shall write **FN = F(N)**, the result of substituting **N** for the free variable in **F**. If **g** is the Godel number of **F**, a formula

with one free variable, then **#g** is by definition the Godel number of **Fg**, the result of substituting its own Godel number into the formula **F.**

The indicative shift

$$\overline{g\,|}\,F \longrightarrow \overline{\#g\,|}\,Fg$$

can be interpreted as the shift obtained by substituting a Godel number **g** into its own formula, getting a new Godel number **#g** for the new formula.

Godel considers formulas specifically of the type **F(#u).** If one should substitute a number in for **u**, then the formula says: first replace **u** by **#u**, then determine **F** applied to **#u.** We shall write **F#** for the formula **F(#u).** Note that **#u** is *not computed* in forming **F#**, rather we just substitute the numeral for **u** in the syntax of the original formula. This is directly in line with the definition of the operator # so that if **F** has Godel number **g**, then **F#** has Godel number **#g.** We have the shift

$$\overline{g\,|}\,F\# \longrightarrow \overline{\#g\,|}\,F\#g$$

The result is that the formula **F#g = F(#g)** makes an assertion about its own Godel number.

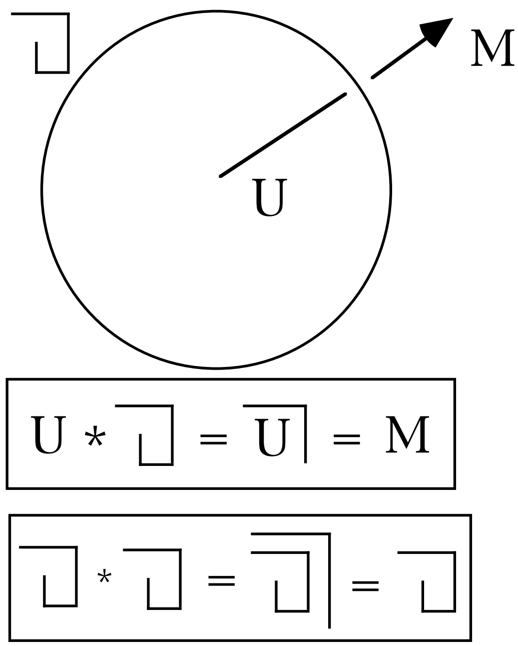
In particular we can use the formula **B(u)** that says that "**the formula whose Godel number is u has a proof in the given formal system**". Then the shift

$$\overline{g\,|}\,{\sim}B\# \longrightarrow \overline{\#g\,|}\,{\sim}B\#g$$

yields the formula **~B#g** that asserts own unprovability in the formal system. This formula cannot consistently have a proof within the formal system. Thus any consistent formal system strong enough to support these structures is incomplete. We have proved that there can be no proof of **~B#g** within the system. Since that is what **~B#g** says, this means that we (outside the given formal system) have proved **~B#g**, a result that the formal system itself cannot prove. That is the proof of Godel's Theorem

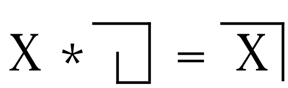
## XIII. Knots,  Sets and Knot Sets

The theme of this section is expressed in the following diagram, that we shall repeat again below in speaking of algebras associated with knots.

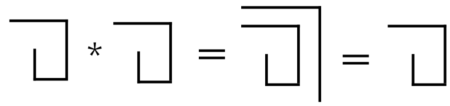



Here you see a circle, configured as a distinction, with the circle itself as the boundary of the distinction, labeled with the reentering mark.

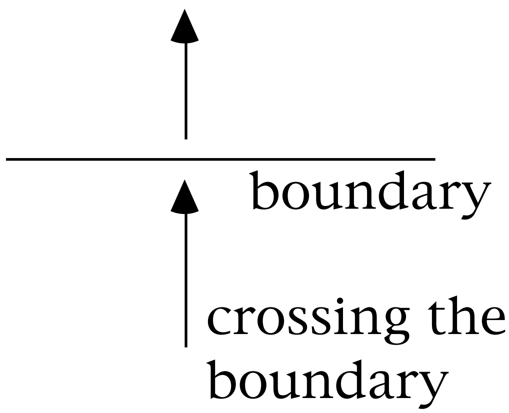
You see an arrow (whose linear dimension is similar to that of a boundary) shown crossing from the inside of the distinction to the outside. We adopt the convention that the arrow represents the act of crossing while the circle represents the boundary that is crossed. This means that interacting boundaries can be interpreted differently as shown below. Furthermore, we take the boundary itself as neither marked, nor unmarked. The boundary is the terra incognita of the imaginary value. The boundary can be seen as the operator that transforms marked state to unmarked state and vice versa. We take the equation



as the basic operation of the imaginary value on the marked and unmarked states, we note that this definition entails the equation



the fact, that via the definition of the reentering mark, the action of the reentering mark on itself is to reproduce itself.




Here we choose the interpretation of boundaries as active (changing state) or passive (spatial boundary) by using the broken line to indicate the locus of the crossing of the boundary. The overcrossing part is passive. The undercrossing part (with its pair of broken segments) is active.

**Knot Sets**

Consider the mutual forms A and B such that

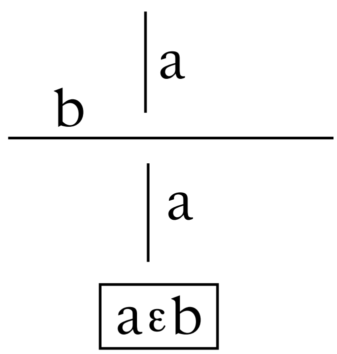
$$A = \overline{B\,}| \qquad B = \overline{A\,}|$$

Each form (A and B) includes the other inside itself, creating a mutuality indicated below the equations for the two forms. The form of mutuality. Here we shall consider a model for set theory where such mutuality is part of the natural territory.

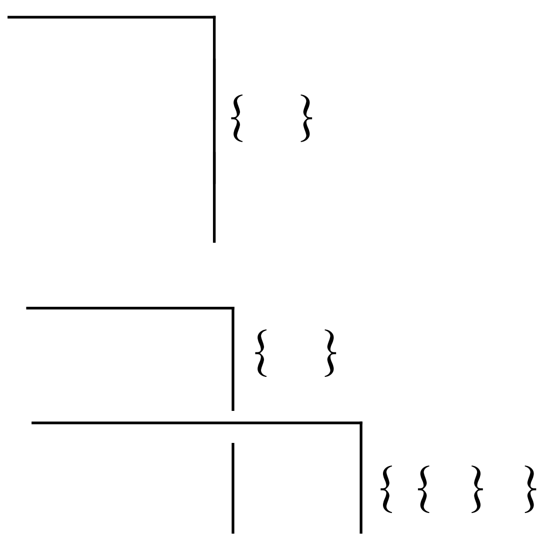
We shall use knot and link diagrams to represent sets. More about this point of view can be found in the author's paper "Knot Logic" [KL].

From the point of view of science fiction, these diagrams were first used in this way by flatlanders before the invention of the third dimension. After that invention, it turned out that the diagrams represented knotted and linked curves in space, a concept far beyond the ken of those original flatlanders.

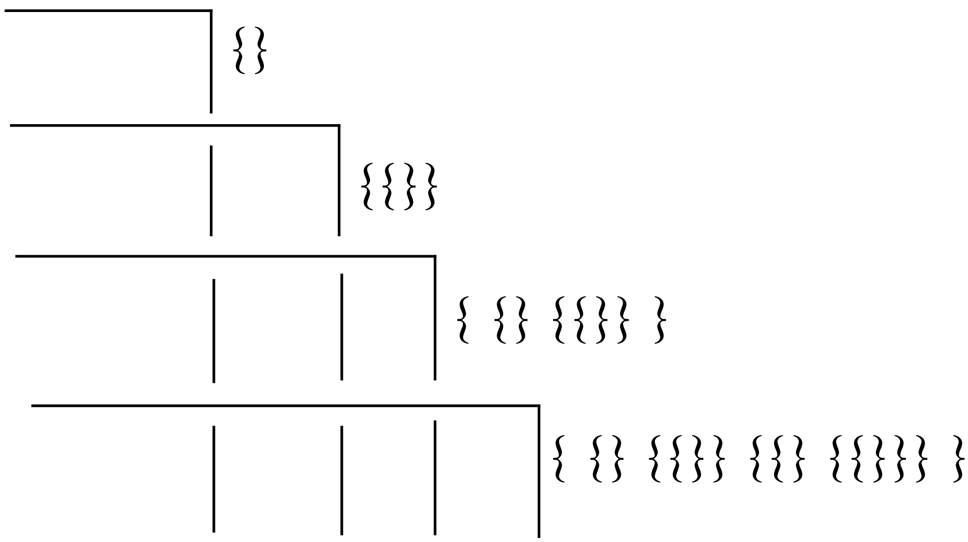
Set theory is about an asymmetric relation called membership. We write **a ε S** to say that a is a member of the set **S.** And we are loathe to allow a to belong to b, b to belong to a (although there is really no law against it). In this section we shall diagram the membership relation as follows:



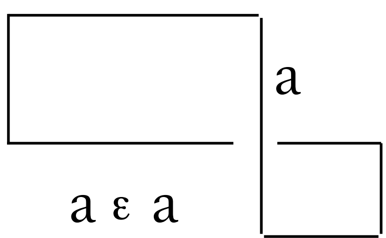

The entities **a** and **b** that are in the relation **a ε b** are diagrammed as segments of lines or curves, with the **a**-curve passing underneath the **b**-curve. Membership is represented by under-passage of curve segments. A curve or segment with no curves passing underneath it is the empty set.



In the diagram above, we indicate two sets. The first (looking like the mark) is the empty set. The second, consisting of a mark crossing over another mark, is the set whose only member is the empty set.
We can continue this construction, building again the von Neumann construction of the natural numbers in this notation:

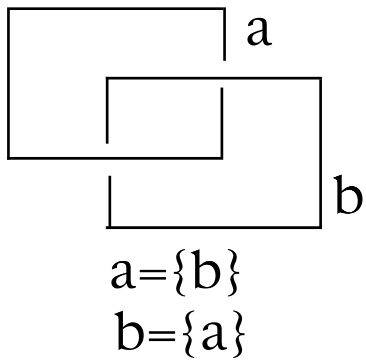



This notation allows us to also have sets that are members of themselves,



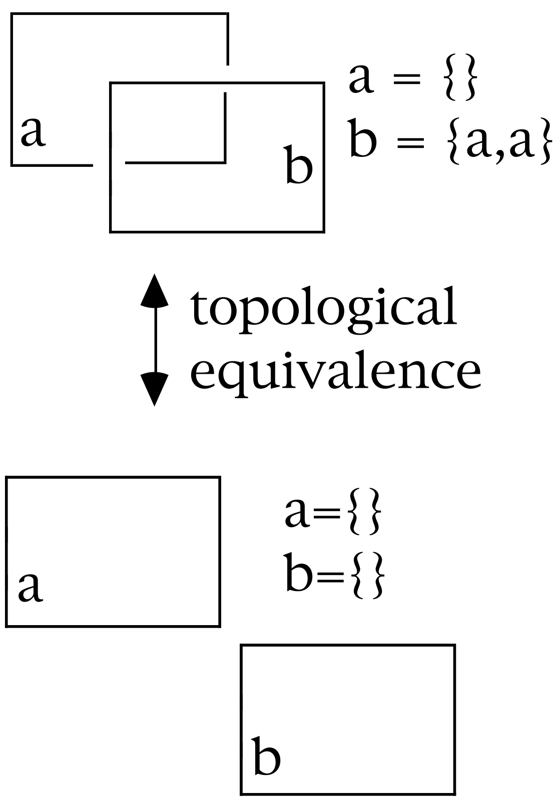

$$a = \{a\}$$

and sets can be members of each other.




Mutuality is diagrammed as topological linking. This leads the question beyond flatland: Is there a topological interpretation for this way of looking at set-membership?

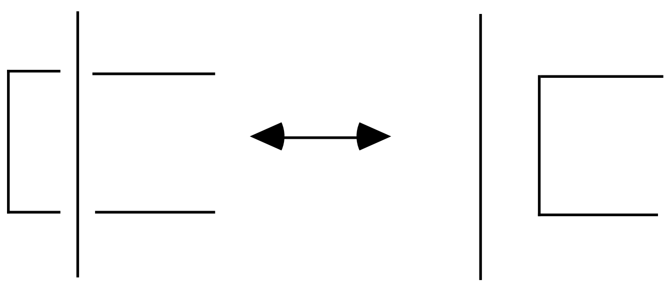
Consider the following example, modified from the previous one.



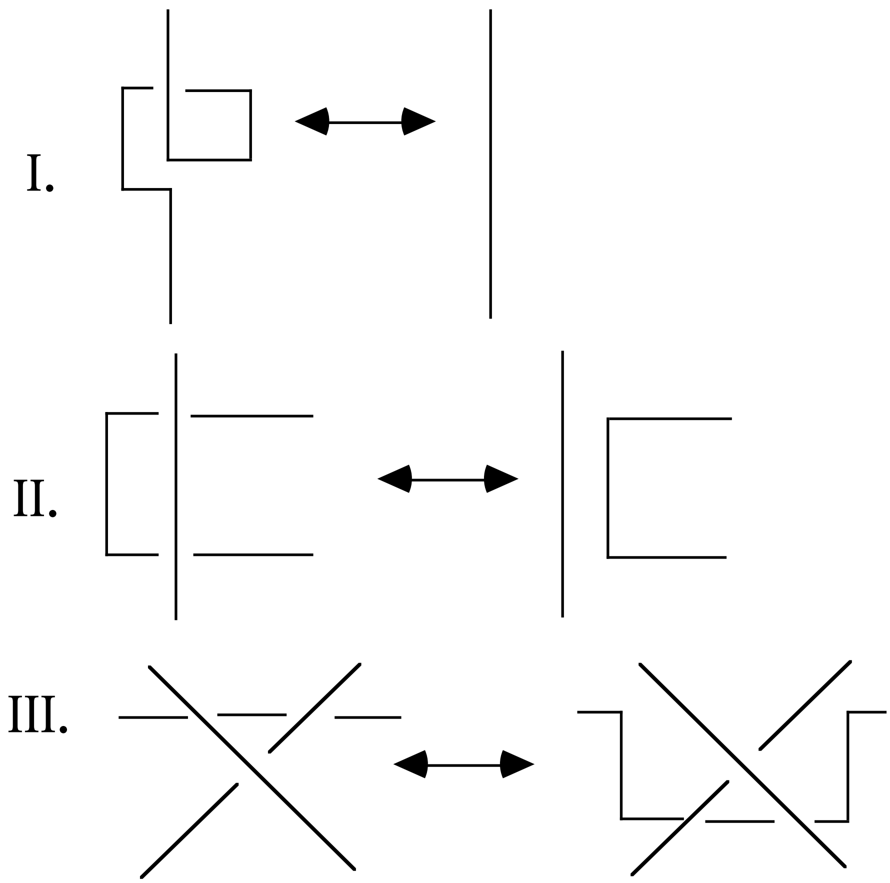

The link consisting of a and b in this example is not topologically linked. The two components slide over one another and come apart. The set a remains empty, but the set b changes from b = {a,a} to empty. This example suggests the following interpretation.

Regard each diagram as specifying a multi-set
(where more than one instance of an element can occur), and the rule for reducing to a set with one representative for each element is:
*Elements of knot sets cancel in pairs.*
*Two knot sets are said to be equivalent if one can be obtained from the other by a finite sequence of pair cancellations.*

This equivalence relation on knot sets is in exact accord with the general diagrammatic topological move shown below.



There are other topological moves, and we must examine them as well.  In fact, it is well-known that topological equivalence of *knots* (single circle embeddings), *links* (mutltiple circle embeddings) and *tangles* (arbitrary diagrammatic embeddings with end points fixed and the rule that you are not allowed to move strings over endpoints) is generated by three basic moves (the Reidemeister moves) as shown below.
See [KP].

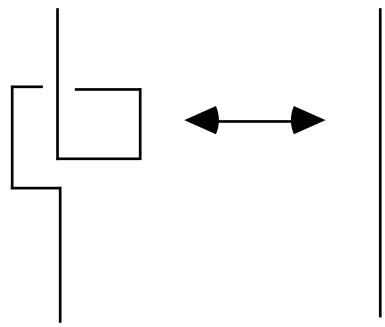



The Reidemeister Moves

It is apparent that move III does not change any of the relationships in the knot multi-sets. The line that moves just shifts and remains underneath the other two lines. On the other hand move number one can change the self-referential nature of the corresponding knot-set.

One goes, in the first move, between a set that indicates self-membership to a set that does not indicate self-membership (at the site in question).

I.



a = { a, ...}      a= { ...}

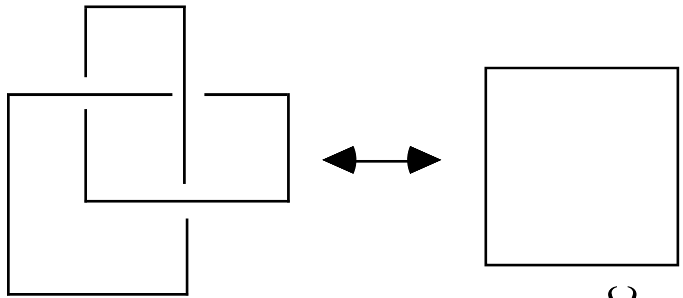
This means that in knot-set theory every set has representatives (the diagrams are the *representatives* of the sets) that are members of themselves, and it has representatives that are not members of themselves. In this domain, self-membership does not mean infinite descent. We do **not** insist that a = {a} implies that a={ { { { ... } } } }. Rather, a = {a} just means that a has a little curl in its diagram. The Russell set of all sets that are not members of themselves is meaningless in this domain.

We can summarize this first level of knot-set theory in the following two equivalences:

1. **Self-Reference:** a = {b,c,...} <-----> a = {a,b,c, ...}
2. **Pair Cancellation:** {a,a, b, c, ...} <-----> {b,c, ...}

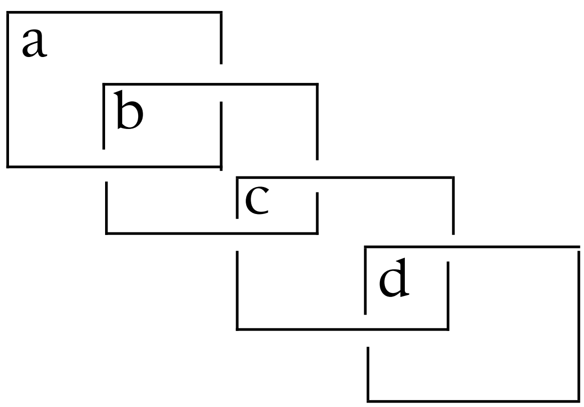
With this mode of dealing with self-reference and multiplicity, knot-set theory has the interpretation in terms of topological classes of diagrams. We could imagine that the flatlanders felt the need to invent three dimensional space and topology, just so their set theory would have such an elegant interpretation.

But how elegant is this interpretation, from the point of view of topology? Are we happy that knots are equivalent to the empty knot-set?



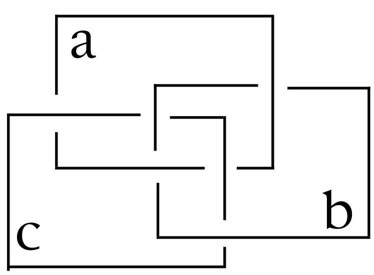
$a = \{a, a, a\}$ $\quad$ $a = \{\}$

We are happy that many topologically non-trivial links correspond to non-trivial knot-sets.




$a = \{b\}$
$b = \{a, c\}$
$c = \{b, d\}$
$d = \{c\}$

In the diagram above, a chain link becomes a linked chain of knot-sets. But consider the link shown below.




$a = \{b,b\}$
$b = \{c,c\}$
$c = \{a,a\}$

The Borromean Rings

These rings are commonly called the *Borromean Rings.* The Rings have the property that if you remove any one of them, then the other two are topologically unlinked. They form a topological tripartite relation. Their knot-set is described by the three equations

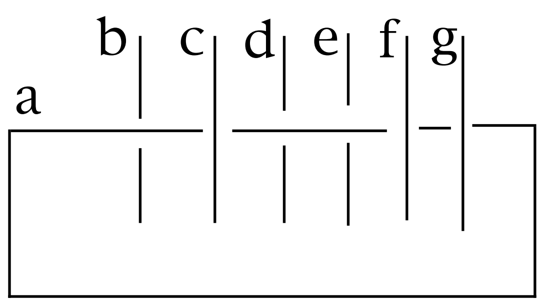
a = {b,b}
b = {c,c}
c = {a,a}.

Thus we see that this representative knot-set is a "scissors-paper-stone" pattern. Each component of the Rings lies over one other component, in a cyclic pattern. But in terms of the equivalence relation on knot sets that we have used, the knot set for the Rings is empty (by pair cancellation)!

The example of the Borrommean Rings suggests that we should generalize the notion of knot-sets so that the Rings represent a non-trivial "set" in this generalization. The generalization should also be invariant under the Reidemeister moves.

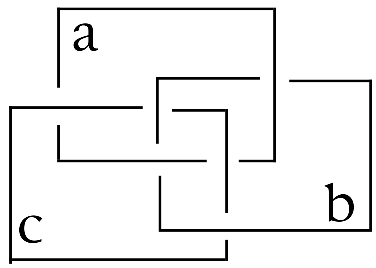
### Ordered Knot Sets

Take a walk along a given component.
Write down *the sequence of memberships and belongings that you encounter on the walk* in the following manner.




a = { b [c] d e [f] [g] }

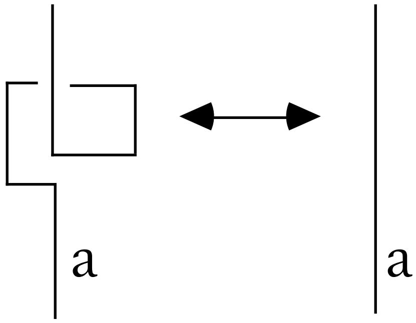
In this notation, we record the order in which memberships and "co-memberships" ( a is a co-member of b if and only if b is a member of a) occur along the strand of a given component of the knot-set. Since we have no intention of setting a fixed direction of traverse, it is ok to reverse the total order of the contents of a given component. Thus we now have the following representation of the Borrommean
Rings:




a = { b [c] b [c] }
b = { c [a] c [a] }
c = { a [b] a [b] }

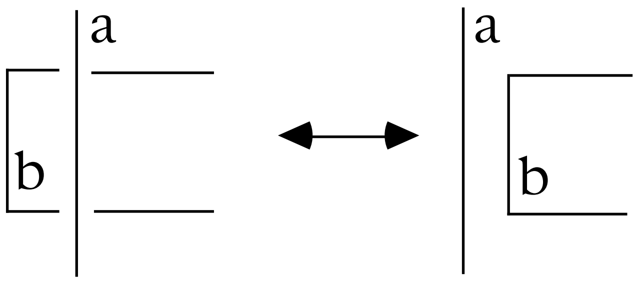
With this extra information in front of us, it is clear that we should not allow the pair cancellations unless they occur in direct order, with no intervening co-memberships. Lets look at the Reidemeister moves for wisdom:

I.



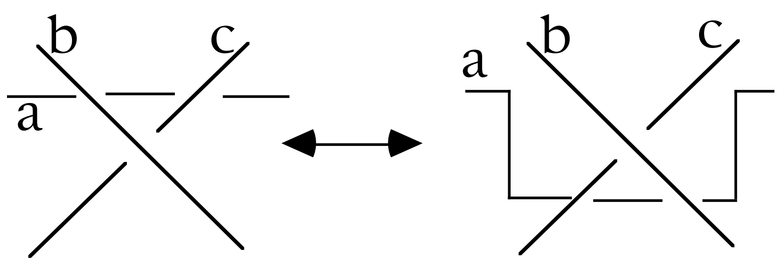

a = { a [a] ... }  a = { ... }

II.




a = { bb ...}  a = { ... }
b = { [a][a] ...}  b = { ... }

III.




a = { bc ...}  a = { cb ... }
b = { ac ...}  b = { ca ... }
c = { a[b] }  c = { [b]a ... }

As is clear from the above diagrams, the Reidemeister moves tell us that we should impose some specific equivalences on these ordered knot sets:

1. We can erase any appearance of `a[a]` or of `[a]a` inside the set for `a`.
2. If `bb` occurs in `a` and `[a][a]` occurs in `b`, then they can both be erased.

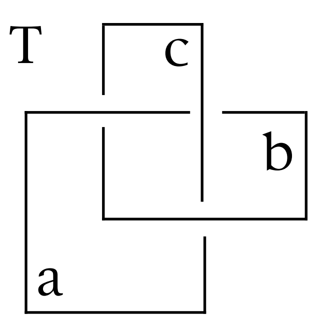
3. If bc is in a, ac is in b and a[b] is in c, then we can reverse the order of each of these two element strings.

We take these three rules (and a couple of variants suggested by the diagrams) as the notion of equivalence of ordered knot-sets.
I *conjecture* that the ordered knot-set for the Borrommean rings is non-trivial in this equivalence relation. It would be quite interesting to have a proof of this conjecture, as it would constitute a proof that the Borromean rings are linked, based on their scissors, paper, stone structure.

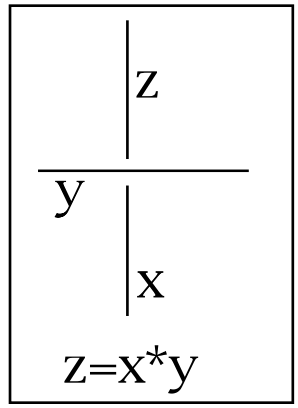
Knots and links are represented by the diagrams themselves, taken up the equivalence relation generated by the Reidemeister moves. This calculus of diagrams is quite complex and it is a source of wonderment to the author, the number and depth of different mathematical approaches that are used to study this calculus and its properties. Studying knots and links is rather like studying number theory. The objects of study themselves can be constructed directly, and form a countable set. The problems that seem to emanate naturally from these objects are challenging and fascinating.
For more about knot-sets, see [KL].

### Quandles and Colorings of Knot Diagrams

There is an approach to studying knots and links that is very close to our ordered knot sets, but starts from a rather different premise. In this approach each arc of the diagram receives a label or "color". An arc of the diagram is a continuous curve in the diagram that starts at one undercrossing and ends at another undercrosssing.
For example, the trefoil diagram below has three arcs.



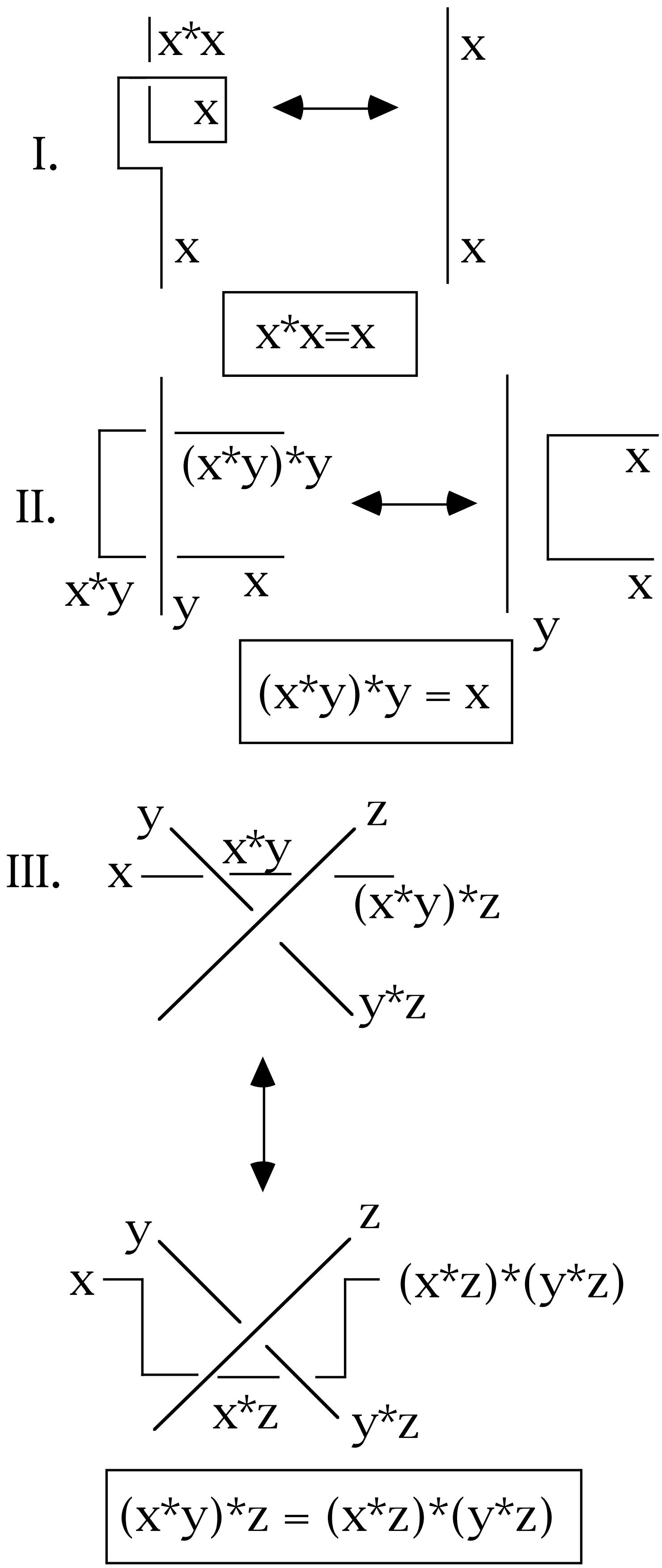

b=a*c
c = b*a
a = c*b




Each arc corresponds to an element of a "color algebra" IQ(T) where T denotes the trefoil knot. We have that IQ(T) is generated by colors a,b and c with the relations c*b=a, a*c=b, b*a = c. Each of these relations is a desecription of one of the crossings in T. These relations are specific to the trefoil knot. If we take on an algebra of this sort, we want its coloring structure to be invariant under the Reidemeister moves. As the next diagram shows, this implies the following *global relations*:

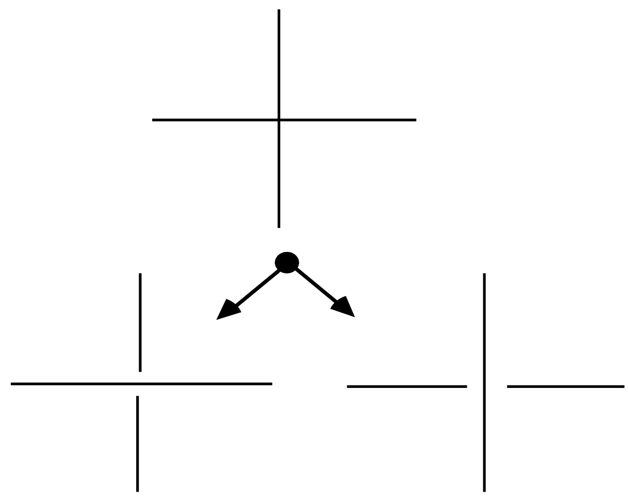
x*x = x
(x*y)*y= x
(x*y)*z = (x*z)*(y*z)

for any x, y and z in the algebra (set of colors) IQ(T).
An algebra that satisfies these rules is called an *Involutory Quandle* [], hence the initials IQ.



I.
x*x
x
x
x
x
x*x=x
II.
(x*y)*y
x*y
y
x
x
x
y
(x*y)*y = x
III.
y
z
x*y
x
(x*y)*z
y*z
y
z
x
(x*z)*(y*z)
x*z
y*z
(x*y)*z = (x*z)*(y*z)

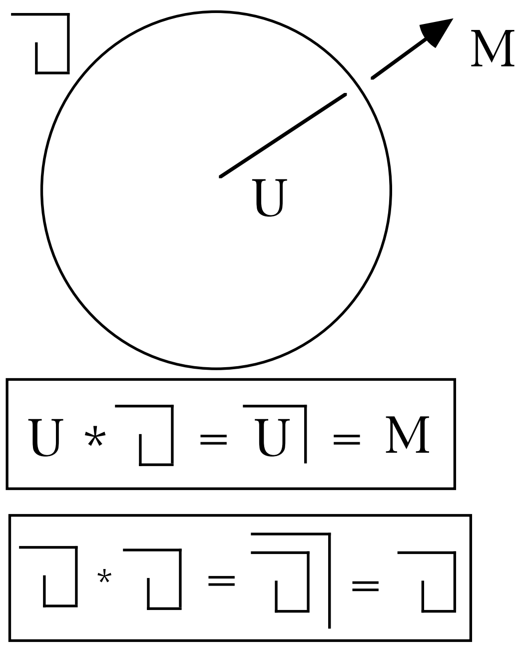
It should be apparent to the reader that these global relations are really expressions of the concept of self-crossing and iterated crossing in the multiplicity of crossings that are available in a calculus of boundaries where the notation



indicates the choice of interpretation, where one boundary is seen to cross (over) the other boundary. Such a choice of interpretation is a choice that regards one boundary as the active and the other one as passive, or one as the container and the other as the contained. The difference in the quandle calculus of boundaries is that we chose to change the name of the undercrossing boundary. That is, if **U** crosses under the boundary of the first distinction, then the name of **U** is changed to **M**. (The progression from the unmarked state U to the marked state **M**). This is a boundary image of the formalism of crossing a single distinction that we have already used in Laws of Form. But now we have made explicit the boundary of the first distinction. What is the boundary of the first distinction. It is neither marked, nor is it unmarked. The most aesthetic choice for the boundary of the first distinction is that it should be the reentering mark, the imaginary value herself! This this we have the equation shown below, with the reentering mark acting on the unmarked state to produce the marked state. If **J** denotes the reentering mark, then
**U*J = M** and by the quandle laws **M*J = U** as well. *The reentering mark is the transformer of states.* All this is in good accord with the epistemology that we have developed for the reentering mark as an imaginary value. Note that in this context, the quandle equation

J*J = J is equivalent to the statement to the statement that J is invariant under the operation of crossing, which is the defining property of the reentering mark.

What we see here is that the knot theory can be seen as a natural articulation not of three dimensional space (a perfectly good interpretation) but of the properties of interactions of boundaries in a realm where each boundary is seen as the imaginary value that sources a distinction. Each boundary can be regarded as that boundary transgressed by another boundary. The choice of who is the transgressed and who transgresses is the choice of a crossing, the choice of membership in the context of knot-set theory. In one sense all boundaries represent the creation of the first distinction, and all boundaries can be different in the complexity of interactions of actors that is the domain of knot-sets, and is the precursor domain for three-dimensional space and spaces of higher dimensions as well.




If we adopt these global relations for the algebra IQ(K) for any knot or link diagram K, then two diagrams that are related by the Reidemeister moves will have isomorphic algebras. They will also inherit colorings of their arcs from one another. Thus the calculation of the algebra IQ(K) for a knot or link K has the potentiality for bringing forth deep topological structure from the diagram.

Lets go back and look at what happens for the trefoil knot T.
We have the initial local relations from the diagram:
c*b=a,
a*c=b,
b*a = c.
From these relations it follows that
c = a*b
a = b*c
b = c*a
For example, we get
c = (c*b)*b = a*b
by taking c*b = a from the first list of equations, and multiplying by b on both sides, and using the global relation
(c*b)*b = a.
Thus we now know

c*b=a
a*c=b
b*a=c
c = a*b
a = b*c
b = c*a

This says that given any two distinct elements of the set {a,b,c} their quandle product is the third (remaining) element. We are also given that

a*a=a
b*b =b
c*c=c.

So we have found that if we make the quandle IQ(T) for the trefoil knot T, then IQ(T) = {a,b,c} with the multiplication of colors defined as above.

| * | $a$ | $b$ | $c$ |
|---|---|---|---|
| $a$ | $a$ | $c$ | $b$ |
| $b$ | $c$ | $b$ | $a$ |
| $c$ | $b$ | $a$ | $c$ |

We can put the whole matter of the structure of IQ(T) succinctly via the multiplication table above. This algebra is certainly disctinct from the simple one-generator algebra for the unknotted circle, and so we have proved that the trefoil knot is knotted.

There is a more concrete way to understand this pattern.
Suppose that a, b and c are integers and that we define
a*b by the equation

$$a*b = 2b - a.$$

Then it is easy to check the quandle properties:
1. a*a = 2a -a = a

2. (a*b)*b = 2b - (2b -a) = a

3. (a*b)*c = 2c - (2b -a) = 2c -2b + a
(a*c)*(b*c) = 2(2c -b) - (2c -a) = 2c - 2b + a
Hence (a*b)*c = (a*c)*(b*c).

This shows that we could label the arcs of the knot diagram with integers if these integers solved the local equations.
In the case of the trefoil, we need

a=c*b=2b - c
b=a*c=2c - a
c=b*a=2a - b

which is the same as

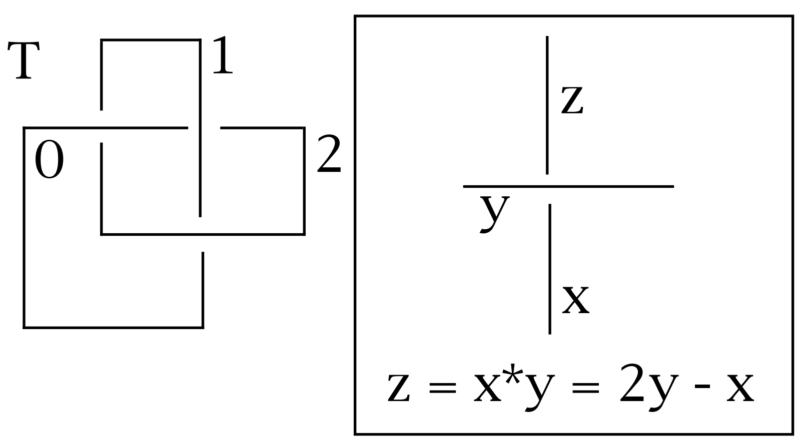
a - 2b + c = 0
a + b - 2c = 0
-2a + b + c = 0.

The third equation is the negative of the sum of the first two and can be eliminated. We are left with

$a - 2b + c = 0$
$a + b - 2c = 0.$

Letting $c = 2b - a$ we have
$a + b - 2(2b - a) = 0.$
Hence
$3(a - b) = 0.$

What does this mean? If we are to solve the trefoil using $x*y = 2y - x$ over the integers, then it would mean that $a=b$. But if $a = b$, then $c = 2a - b = a$ also, and we would have a constantly colored knot. This would not distinguish the knot from the unknot. Can $3 = 0$? That looks like a contradiction in mathematics, but we know that there are number systems where $3 = 0$, and so we decide to take the elements that label the knot from the modular number system $Z/3Z = \{0,1,2\}$ where the arithmetic operations are performed by doing the usual integer operations and then taking the remainder on division by 3. This means that we have colored the arcs of the trefoil knot with elements of $Z/3Z$ using the $2b-a$ rule. But note that in this system $1*2 = 3$, and generally the quandle product of any two distinct elements of the set $\{0,1,2\}$ is the remaining element (and the product of any element with itself is itself). Thus we have, in this case, reproduced the algebra obtained abstractly from just the local equations and the global rules defining the quandle.

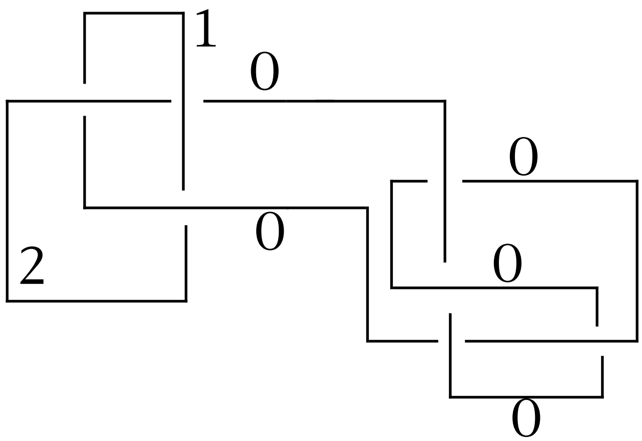


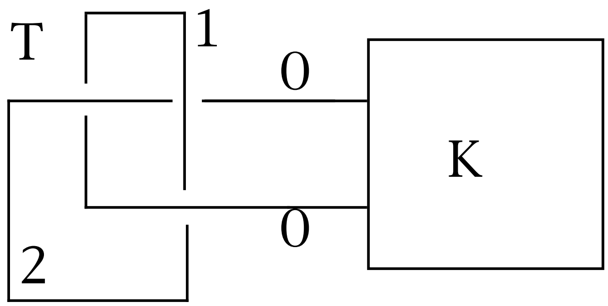

Coloring the Trefoil in Z/3Z

In the figure above, we have illustrated coloring the trefoil in $Z/3Z$. A knot is said to be *3-colorable* if it can be colored with three colors in this way. It is not necessary for every crossing to have three colors incident to it as in the figure above, but one wants three distinct colors on the diagram and the coloring has to follow the $2b-a$ rule. In the case of three colors this means simple that every crossing sees either three colors or only one color.

It is not hard to see that a single component diagram $K$ that is three colored must be knotted. For suppose that $K$ is equivalent by Reidemeister moves to an unknot. Then there is a sequence of colorings each obtained from the previous one by a local change that goes from the coloring of $K$ to a coloring of the unknot. The coloring of the unknot has only one color, but the coloring of $K$ has three colors. However it is easy to see (exercise!) the on going from a three-colored knot to another one by Reidemeister moves, no colors are lost. So it is impossible to get to the unknot. That means that the knot was indeed knotted.

We can apply the above argument to prove that some other knots are knotted. For example consider the knot below and the coloring we have shown for it.

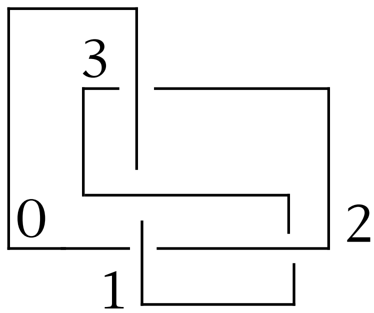



T#K

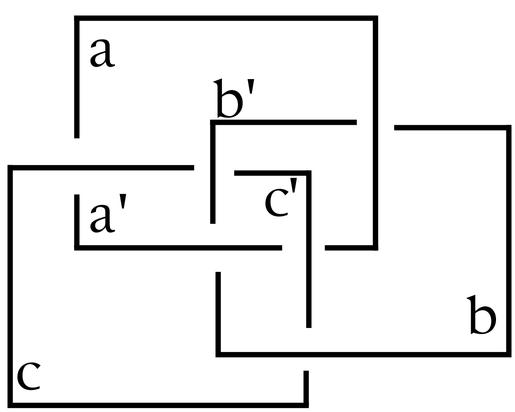
In this figure we show a trefoil knot that has been augmented with some extra knotting in the form of a figure eight knot. The trefoil part is colored with three colors and the new part is colored only with the one color zero. But the whole knot shown must be knotted, since it is colored with three colors. This is part of a more general result illustrated in the second part of the figure. There we show a trefoil knot augmented by an arbitrary knot K. This general construction is called the connected sum of the knots T and K, and denoted by T#K. The very same argument then shows that threre is no knot K such that T#K is unknotted. Actually, this is true more generally: If K and K' are two knots such that K#K' is knotted, then it can be proved that both K and K' are individually unknotted. (See [KL].)

Not every knot can be three colored. For example, the figure eight knot shown below can be colored in colors from Z/5Z, but it cannot be three colored.




In the diagram above we have shown a coloring of the figure eight knot using four out of the five colors in $Z/5Z = \{0,1,2,3,4\}$. Other diagrams of the figure eight knot actually require five colors. The modulus 5 is forced by the equations in the $2b-a$ labelling. Every knot has modulus, known as the determinant of the knot, and this modulus is itself an invariant of the knot (unchanged under the Reidemeister moves). Thus we have shown that the figure eight knot is also knotted, and that the figure eight knot and the trefoil knot are distinct from one another.

But now, what about the Borrommean Rings?




$a*c = a'$
$a'*c' = a$

$b*a = b'$
$b'*a' = b$

$c*b = c'$
$c'*b' = c$

$a=(a*c)*(c*b)$
$b=(b*a)*(a*c)$
$c=(c*b)*(b*a)$

Here we have the rings and their quandle-equations. The boxed equations are a consequence of substitution from the equations on the side. They express the self-referring intelock of the three rings in the quandle language. Our task is to show that this intelocking algebraic pattern can not be reduced to the pattern that comes from three unlinked rings.

The linear equations (via the 2y - x rule ) corresponding to the boxed equations are:

a = 2(2b-c) - (2c -a)
b = 2(2c-a) - (2a -b)
c = 2(2a -b) - (2b -c)

It is not hard to see that these simplify to

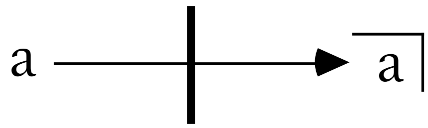
4(b-c)=0
4(c-a)=0
4(a-b)=0.

Thus the appropriate modulus for the Borrommean rings is 4, and the coloring occurs in Z/4Z. The fact that the modulus is 4 shows that the rings are indeed linked.

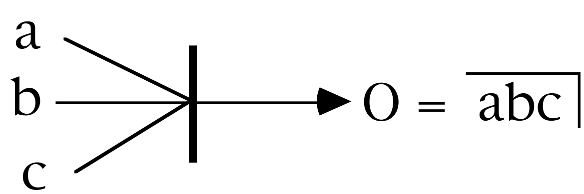
**Discussion.** We have spent some care in explaining how the quandle algebra and its representations (using the 2b-a rule and using modular number systems) can be used to show that some knots are knotted, how to distinguish some knots from each other, and how to show that the Borrommean rings are linked. In the case of the Borrommean rings the basic quandle equations certainly do use the fact that the rings pass over one another in cyclic fashion ("scissors - paper - stone" pattern), but this initial fact seems to get lost in the algebraic technique that we used to detect them. There is still much to think about at this elementary level. From the point of view of form the cyclic relationship that is tripartite is certainly a fundamental pattern. That it can be embodied in a topological entity is very fascinating. We have not exhausted the potential of the topological analysis at all in this discussion.

## XIV. Digital Circuits

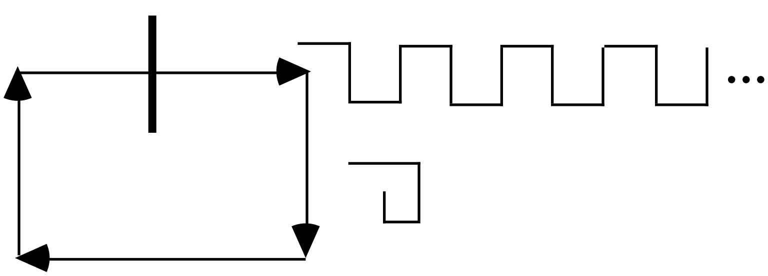
The mark can be construed as an operator that inverts a signal.




More generally, we can consider a collection of inputs to the inverter,
and a collection of outputs. *Balance* at the given inverter will mean that each output is the cross of the juxtaposition of the inputs. This is illustrated below for three inputs a,b,c and one output.




The reentering mark can be construed as a circuit in which an inverter feeds back its output directly to the input. The result of such an interconnection is an oscillation between the initial state and its inversion. Such an interpretation assumes that there is a time delay between the production of the output and the processing of the input. If there is no time delay, then we are in a state of eternal contradiction.




From the mathematical point of view, time is just another structure. Thus we can say

$$J_{t+dt} = \overline{\;J_t\;}|$$

and as long as **dt** is non-zero, then there is no contradiction.

If $dt = 0$, then we arrive at the abstract structure of the reentering mark that is neither marked not unmarked, the imaginary value.

There is a way to understand such circuits that goes beyond simple temporal recursion. A given circuit may have *stable states*, where the equations at each inverter are balanced. Then one can consider the process corresponding to the circuit to be a pattern of transitions from one stable state to another, instigated by an imbalance at some places in the circuit. If the circuit has no stable states (as with the reentering mark) then the process of transition continues without end. A transition process happens as follows:

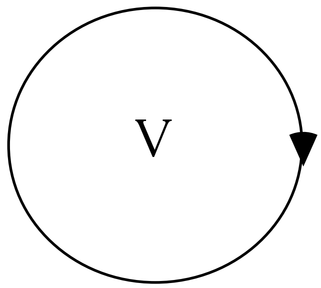
**Transition Model for Inverter Circuits**
*0. Assign "time delays" to each inverter in the circuit. For the purpose of this model, it is sufficient to just order the inverters so that one can answer the question whether any one inverter is slower than another.*
*1. Find an inverter in the circuit whose equation is not balanced. Readjust the outputs of this inverter so that it is balanced.*
*If there is more than one unbalanced inverter, choose the one with the smallest time delay.*
*2. Examine the circuit once more. If it is balanced, stop. If there is an unbalanced inverter, perform step 1. again.*

Another way to formulate this model is to replace ordering of the marks by a probabilistic choice.  The rules would then read:

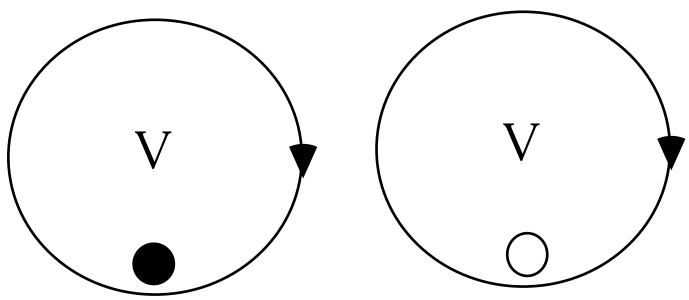
**Probabilistic Transition Model for Inverter Circuits**
*1. If there is more than one unbalanced inverter, choose one of them at random. Readjust the outputs of this inverter so that it is balanced.*
*2. Examine the circuit once more. If it is balanced, stop. If there is an unbalanced inverter, perform step 1. again.*

A circuit is said to be *determined*. If the process described above does not depend upon the (time delay) ordering of the marks in the circuit (or upon the probabilities in the second model). This transition model for circuit behaviour is asynchronous in the sense that it does not assume that there is an external "clock" that causes all rebalancings to happen at once. As we shall see, clocked behaviour can be quite different from unclocked behaviour.

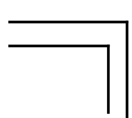
The next example is the simplest circuit with a stable state.




You might call this circuit V, *the reentering void*. It has two stable states, **marked** or **unmarked**, and no inverters. *In illustrating the states, marks in the state are indicated by black dots and white dots. A black dot is a mark, and a white dot denotes the absence of the mark.*




You could think of V as a form of memory, where a given state labeling persists for as long as necessary. We will, however, not use this memory, but rather the next one (see below) in making circuit designs. For mathematical purposes one could use V in circuit design, but the memory we are about to construct, by taking two inverters back to back, actually corresponds to what is done in engineering practice. At the least, if we used V, we would have to assign a time delay to it and then it would have a similar mathematical effect as the back-to-back inverters that we are about to discuss, the only problem being how to kick it out of the marked state once a mark had begun to circulate round its basic turn. The difference between V and the circuit we are about to discuss is the difference (operational at best) between

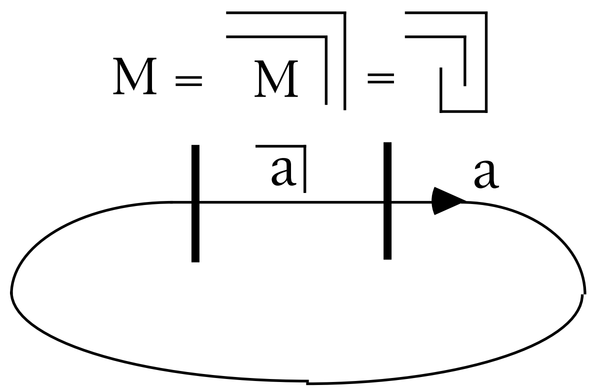


and the void.

The next example corresponds to the equation

$$M = \overline{\overline{M\,}|\,}|$$

and its corresponding circuit.

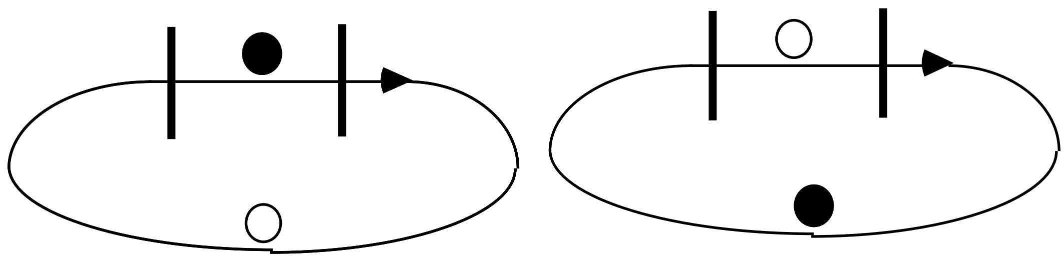



Here we have a benign reentry that does not create oscillation. The circuit has two stable states, and it is described by two equations with the extra variable N corresponding to the internal line in the circuit.

$$M = \overline{N\,}|$$
$$N = \overline{M\,}|$$

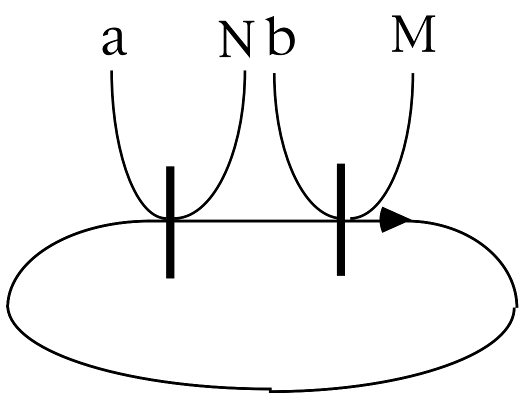
One need not think of any recursion going on in the stable state. In that condition, one just has a solution to the above equations. Each part of the circuit balances the other part. The circuit itself can be interpreted as a *memory element*, in that it can store time-independently the information

(M,N) = (marked, unmarked)
or
(M, N) = (unmarked, marked).



These two stable states of the memory are depicted in the figure above.

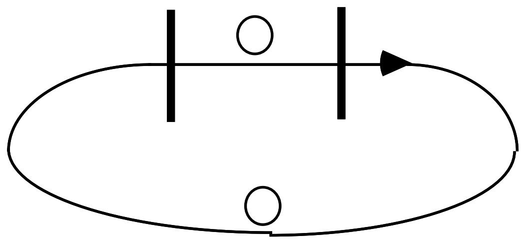
A little modification of this memory circuit, and we can interrogate it and change it from one state to another.




$$N = \overline{Ma}\rceil \qquad M = \overline{Nb}\rceil$$

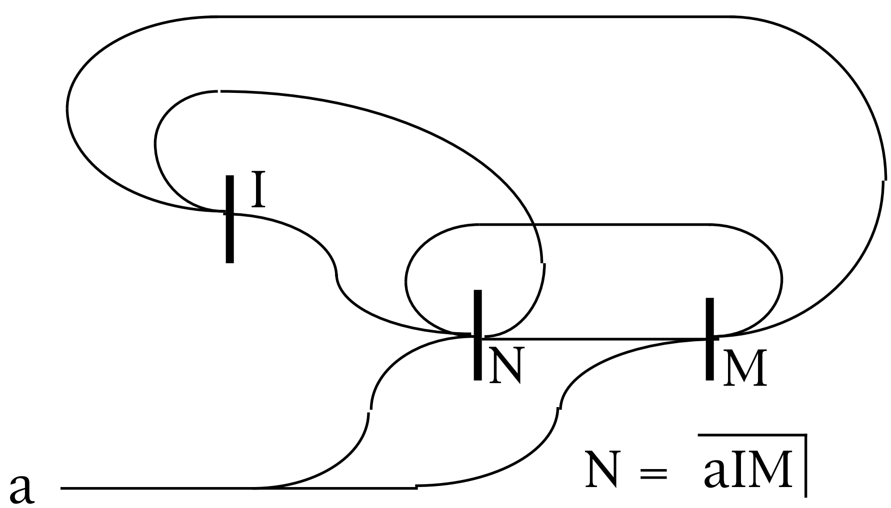
The new circuit has inputs **a** and **b**, and outputs that can measure the values of **M** and **N** without affecting the balance of the circuit itself. By choosing **a** marked and **b** unmarked, the memory is forced into the state **(M,N) = (marked, unmarked).** By choosing a unmarked and b marked the memory is forced into the state **(M,N) = (unmarked, marked).** In each case, since these states are stable, the marked input can be removed without affecting the state of the memory.

A more diabolical setting would be to have both a and b marked and then to remove them simultaneously. The resulting state of the memory is then the unstable configuration shown below.



Each mark in the memory is unbalanced. The first mark to transmit a marked state will win the race and propel the memory into one of its two stable states. If it is possible for both marks to "fire" at once, we would arrive at the other unstable state where both sides are marked. In physical practice this will never happen, and the above unstable state will fall to one or the other of the two stable states, just as it does in our transition model (where one mark reacts faster than the other).

In practice, memory conditions such as the above can occur, and it is interesting to see how to design a circuit that will determinably transit to only one of the two possibilities.  Consider the circuit below.

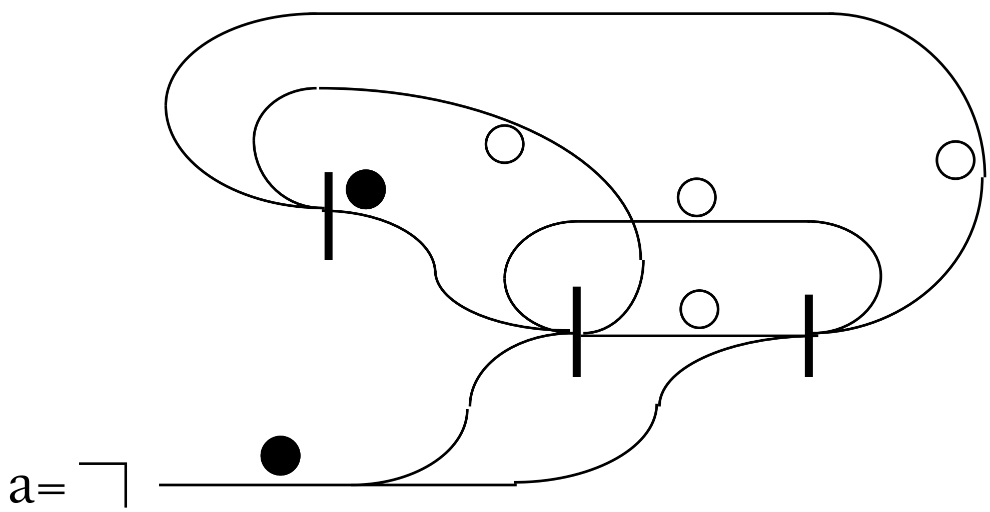



$$N = \overline{aIM|}$$

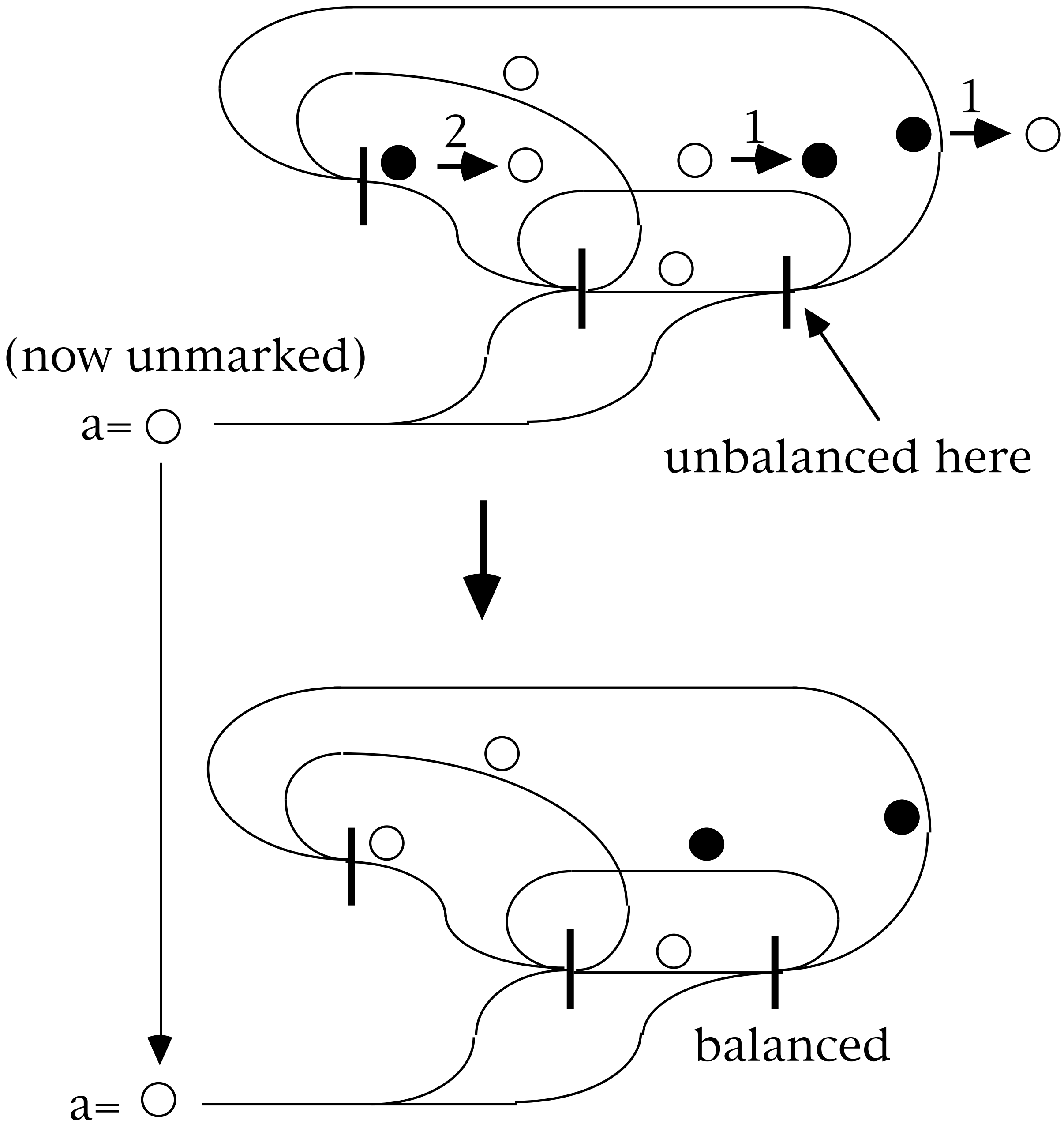
$$M = \overline{aN|}$$

$$I = \overline{NM|}$$

In this circuit we have eliminated the arrows that indicate direction of signals through the inverters and have used the convention that *signals travel through each inverter from left to right.* This suffices to fix all other directed lines. The memory consisting of M and N has an input a, and there is one more mark in the circuit labeled I. If a is marked, then N and M are unmarked, forcing I to be marked. This is a stable condition of the circuit so long as a is held in the marked state.



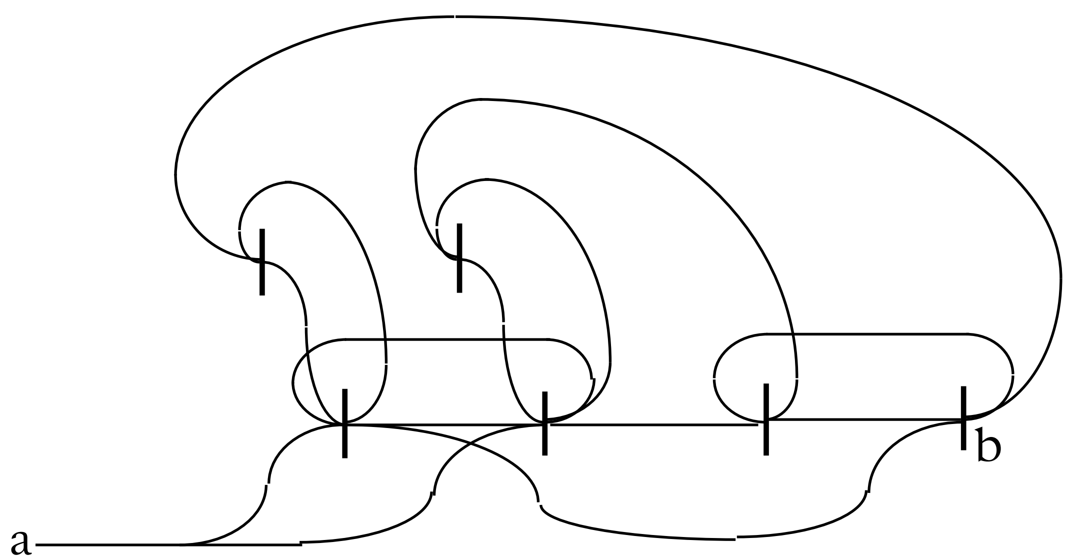

If now, we let a change to the unmarked state, then the circuit becomes unbalanced at M only, since I continues to put out a marked state.



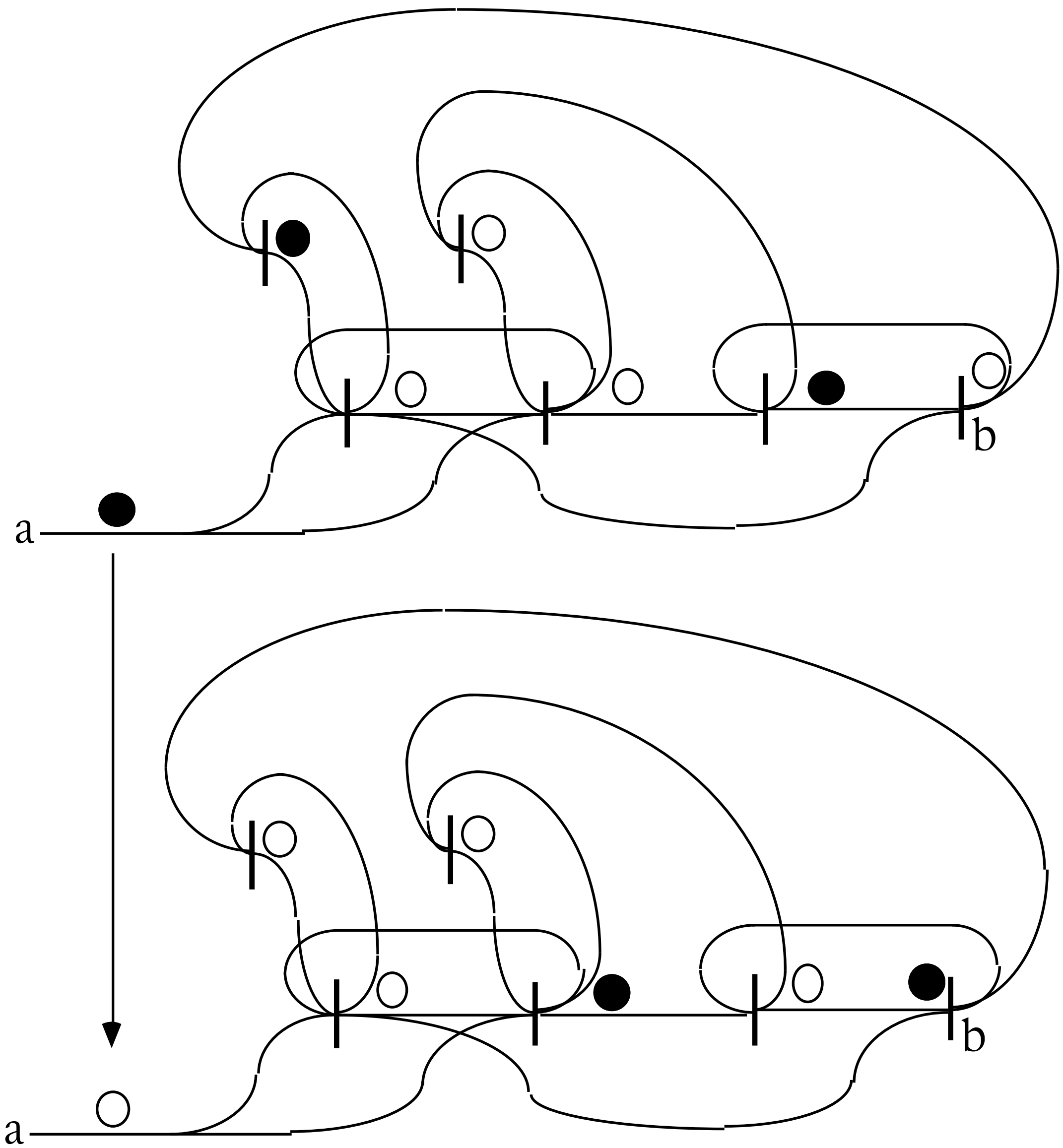

In the diagram above, we show how the change of a to the unmarked state gives rise to a transition of M to the marked state (1) and that this forces a transition of I to the unmarked state (2). The resulting circuit is balanced and the transition to this state of the memory is determinate.

The addition of the mark I to the circuit enabled the determinate transition. In fact, I acts as an observer of M and N who feeds back the inversion of the or of M and N to the input to N. The result is that I is marked at the point of transition, holding the state at N in balance. For any choice of time delays, this condition of I can be arbitrarily small, but significant in forcing the transition.

We see that a circuit can be construed as a miniature self-observing system, and that this condition of self-observation can radically influence the behaviour of the circuit. In a certain sense, the value of the circuit at I is "imaginary" in at least the metaphorical sense of the term. In terms of circuit design, we can use such imaginary values to influence the structure of the design and make otherwise indeterminate circuits determinate. We say that an marker in a circuit has an imaginary value if there are transient states at that marker that influence the transition behaviour of the circuit. Self-observation can occur in a circuit without transient states. The circuit below is an example. It is a modulator in the sense of Spencer-Brown. That is a given frequency waveform input at a results in an output waveform of one-half the frequency at b. I discovered this circuit in 1978 when studying Laws of Form. It is similar to circuits in Chapter 11 of Laws of Form, and it accomplishes the modulation without using any imaginary values using only six markers. Spencer-Brown gives an example in Chapter 11 of a modulator with six markers that uses imaginary values. The reader will enjoy making the comparison.




To see how this circuit operates, I have illustrated one stable state and one transition below. In showing the transition, I have only shown the end result. The reader will find that this is an example of a determinate transition. Note also that in the next transition, the value of b will not change. There is a four-fold cyclic pattern in the transitions, with b changing every other time.

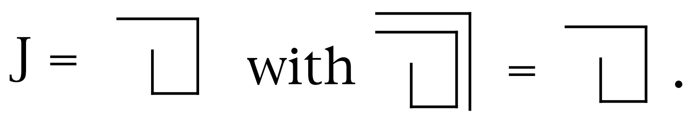



Modulators are the building blocks for circuits that count and are often called "flip-flops" in the engineering literature. There is much more to say about this circuit structure and its relationships with computer design, information and cybernetics, but we shall stop here, only to note that this is an aspect of Laws of Form that goes far beyond traditional boolean algebra, and is well-worth studying and working with as a research subject.

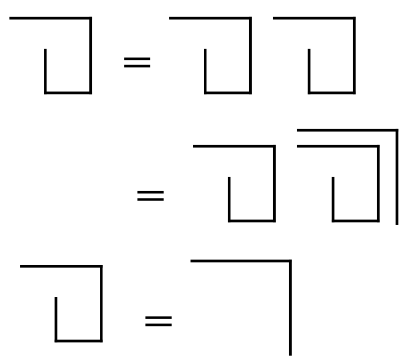
**Remark.** G. Spencer-Brown (private communication 1992) discovered another modulator with six markers but fewer connecting lines, and conjectured it to be the unique minimal modulator. One wishes to minimize the number of markers plus the number of lines. We respect Spencer-Brown's privacy by not showing this design, but the reader is encouraged to find it! The question of the classification of minimal modulators is a good example of the open nature of the mathematics of circuit design. The mathematical model of circuit design discussed in this section makes it possible to formulate such questions with precision.

## XV. Waveform Arithmetics and The Flagg Resolution

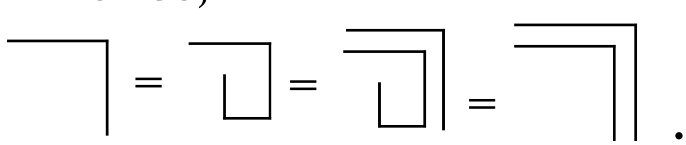
Lets go back to the reentering mark again, first looking at it through the eyes of the primary algebra. We have



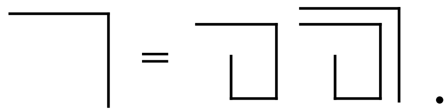

Lets suppose that J satisfies all the identities in the primary algebra. Then



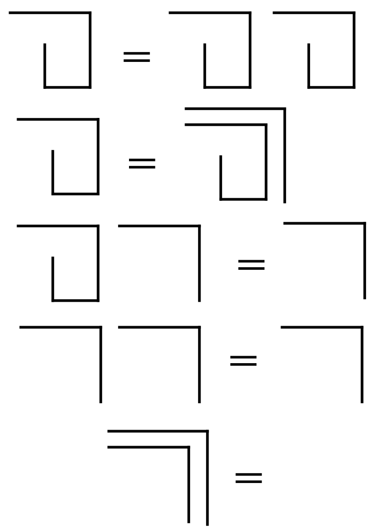
Hence,



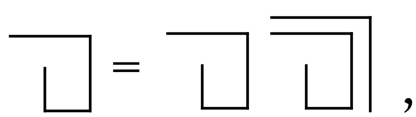
This glaring contradiction seems to propel us into the point of view that it is not legitimate to attempt to extend the primary algebra by adding in the reentering mark. Indeed that is a way out. One can add in the reentering mark as an imaginary value, but not assume all the usual rules of the arithmetic. In particular, one does *not* assume that



More specifically, one can start with the following arithmetical interactions.

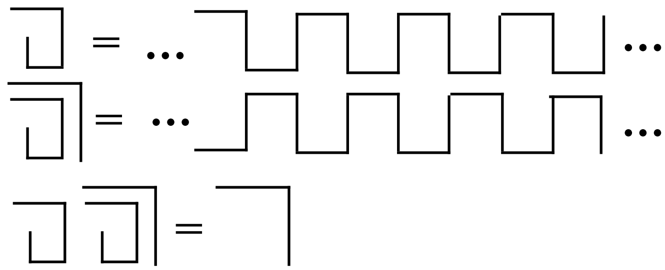


The algebra that describes this arithmetic is Varela's Calculus for Self-Reference (CSR). It is mapped directly into the three-valued logic of Lukasiewcz. See [CSR] and [IV]. In CSR, we have

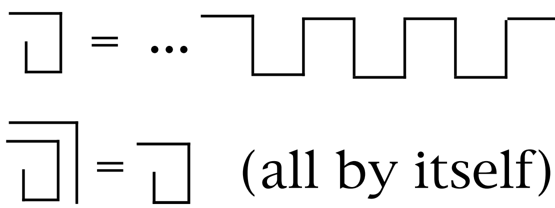


but the reentering mark has a value distinct from marked or unmarked.

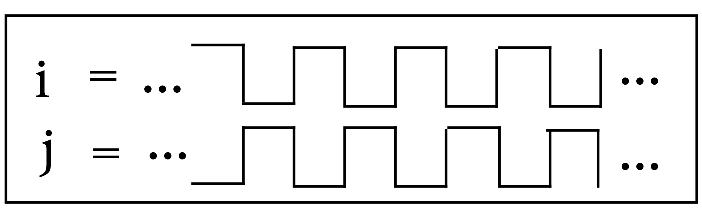
Such constructions are fine, but they do not tell the whole story. If we take the temporal point of view, then the reentering mark may be identified with a discrete wave-form. Crossing the mark can be interpreted as switching each temporal instant of this waveform from the marked state (up) to the unmarked state (down), and vice-versa. The result is that the cross of the reentering mark is not equal to the reentering mark, but rather it is equal to a waveform that is phase-shifted from the original one by one half-period. The juxtaposition of the these two waveforms yields a marked state.

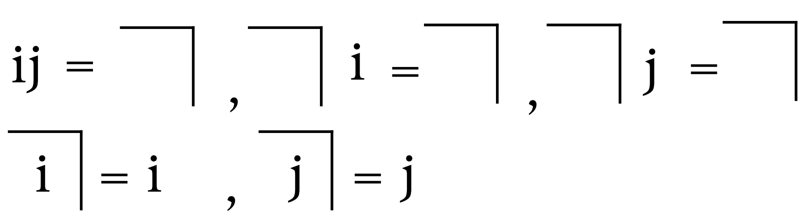


With this interpretation we would like to keep **position** as a rule about the reentering mark. But we also note, that as a waveform the reentering mark, taken all by itself, is indistinguishable from its crossed form.




One way to get partially out of this dilemma is to make two imaginary values i and j, one for each waveform and to have the following waveform arithmetic:




The waveform arithmetic satisfies occultation and transposition, but not position. It is similar to the three-values Calculus for Self-Reference, and has a completeness theorem using these values. This rich structure is directly related to a class of multiple valued logics called DeMorgan Algebras [DMA]. In [FD] we called the algebra corresponding to the waveform arithmetic *Brownian Algebra.*

It is worth mentioning what a simple model the waveforms are suggesting here. Consider an alternating temporal pattern.

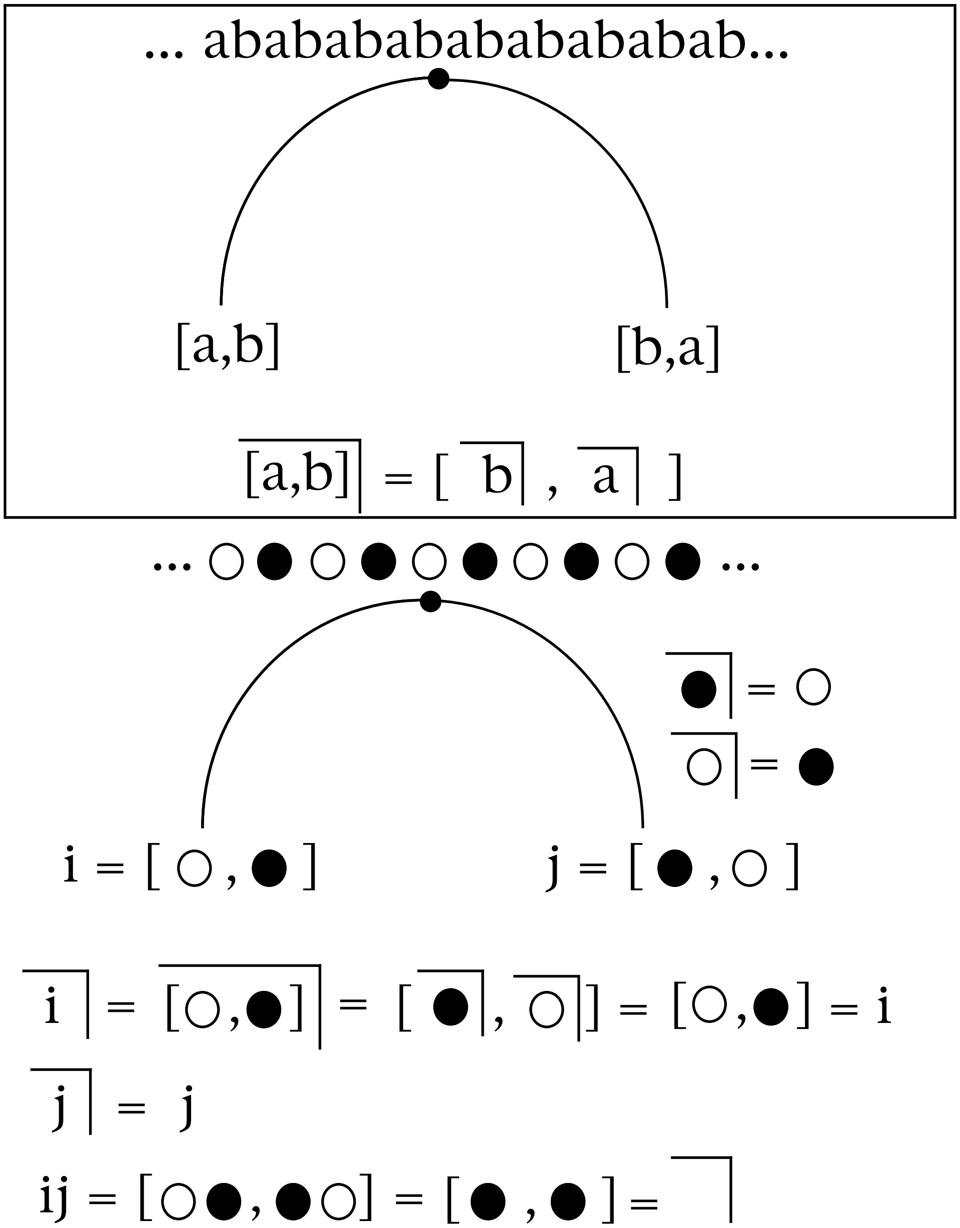
... abababababababab...

Such a pattern can be viewed as a repetition of **ab** or as a repetition of **ba.** We can see **[a,b]** and **[b,a]** as two views of this pattern. We define the cross of one view to be the other view. With this definition in place we can define **i** to be the ordered pair [ **unmarked, marked]** and **j** to be the ordered pair **[marked, unmarked]**.

Views of one alternating pattern become distinct ordered pairs. We juxtapose the ordered pairs by superimposing the corresponding patterns. Thus

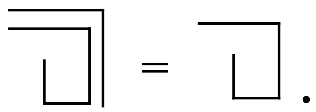
$$[a,b][c,d] = [ac, bd].$$

In this way we get a specific model for the waveform arithmetic. One can then investigate matters of initials, completeness and so on.

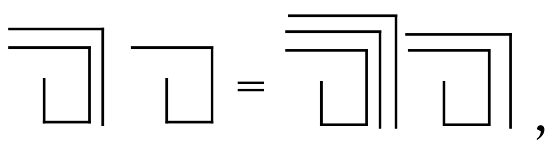



This last figure has been designed to kind of give you a feel for the potential of waveform arithmetics and algebras as ways of capturing temporal process and multiple viewpoint in algebraic equations. There is a great deal of this, and it extends outward into programming languages, cellular automata, artificial life and so on. And yet, have we solved the conundrum with which we began? Have we found a context for the reentering mark in the primary algebra? Well, the answer is obviously not, not this way. There is another way, and it is very simple.

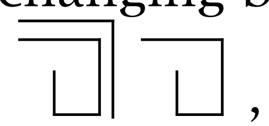
That other way is the **Flagg Resolution**, discovered by James Flagg around 1980. See [FR1] for more about the history of this point of view. Any solution to our paradox will ask us to give up something. Flagg resolution asks us to give up the commonly assumed locality of an algebra element. *In Flagg resolution, the reentering mark is non-local in the text.* Here is the Flagg resolution.



If one applies this equation in a given text, then it must be applied everywhere in that text. There is only one reentering mark, and all references to it are relational. In particular you can write

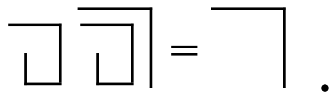


changing both marks in the text



but it is forbidden to change only one of them.

With this resolution in hand, there is no paradox. Look back, for example, at the contradiction that we derived at the beginning of this section. It cannot happen. Each text must be taken on its own grounds. The reentering mark has its self-crossing property, but that does not disturb its relation to itself in the equation.



In fact, if you now go back to waveform description, you see that in the above equation, it is nonsense to allow one to change only one reentering mark, but not the other. The whole point of the equation is that within it, the two waveforms are shifted and they superimpose to form the marked state.

Flagg resolution resolves the paradox by turning the temporal interpretation into a non-locality in the treatment of specific text entities. Once one is conscious of this mode of resolution, one realizes that it is in essence what is being done to avoid contradictions in numerous systems. For example, in the digital circuits we work with systems with circularities and indicate the non-local connections by using the graphical representation. Flagg could be done graphically by attaching common lines to all instances of the reentering mark that must be handled as one mark. Much more can be said along these lines, and we shall take up the theme in a separate paper [FR2].

## XVI. Diagrammatic Matrix Algebra

This section is about the use of diagrams in the algebra of matrices, a well-known subject in mathematics that has many motivations. We add this section to the present paper because the issues in diagramming matrix algebra are directly related to the formal mathematics examined here.

Lets first recall how matrix multiplication works. Matrices are arrays of elements of an arithmetic or an algebra. Here we will begin by assuming that the matrix elements occur in ordinary numbers (integers, rationals, reals or complex numbers) or their algebra. Two 2 x 2 arrays are multiplied by the following formula.

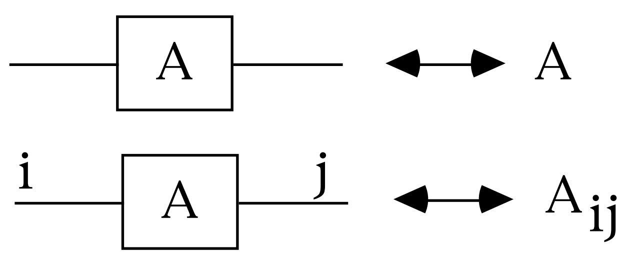
$$\begin{pmatrix} a00 & a01 \\ a10 & a11 \end{pmatrix}\begin{pmatrix} b00 & b01 \\ b10 & b11 \end{pmatrix} = \begin{pmatrix} a00b00+a01b10 & a00b01+a01b11 \\ a10b00+a11b10 & a10b01+a11b11 \end{pmatrix}$$

We denote a matrix $A = (A_{ij})$ by a global letter (A in this case), and by an indication of the form of the elements of the array, $A_{ij}$. The subscripts range over the set $\{0,1\}$ in the case of a 2 x 2 matrix, as shown above. The rule for multiplying two matrices is

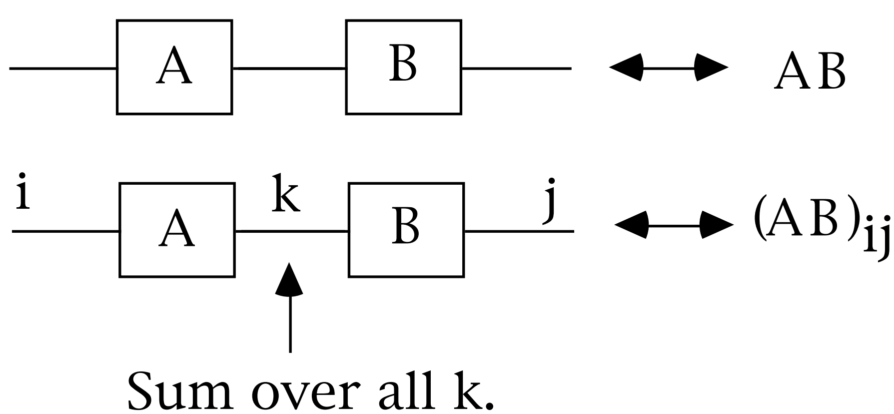
$$(AB)_{ij} = \Sigma_k A_{ik}B_{kj}.$$

where the summation is over the index set for the matrix size that we are using. Compare this formula with the arrangement of indices and sums in the explicit matrix product given above.

We now give a diagrammatic interpretation for matrix algebra. Each individual matrix is represented by a box with (input and output) lines that correspond to the matrix indices.




Matrix multiplication is represented by attaching the output line from one box to the input line of the other box.



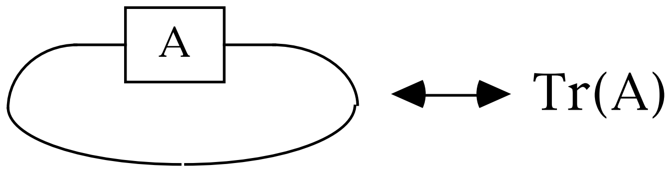

$$(AB)_{ij} = \Sigma_k A_{ik} B_{kj}$$

Lines tying one box to another correspond to internal indices in the matrix product, and so one sums over all possible choices of index for such internal lines.

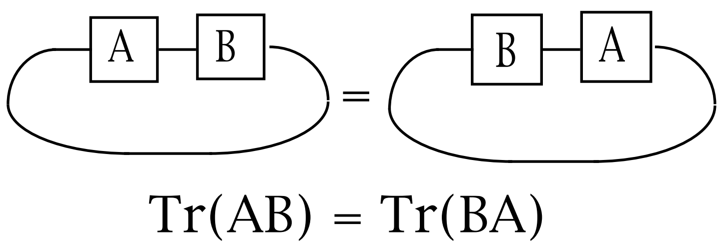
With these diagrammatic conventions in place, one can often make very efficient insight into properties of matrix composition. For example, the trace of a matrix A is given by the formula

$$Tr(A) = \Sigma_i A_{ii}.$$

Here is the diagram.




With this diagrammatic for the trace of A, we easily prove that $Tr(AB) = Tr(BA)$ by putting two boxes in a circular connection pattern.

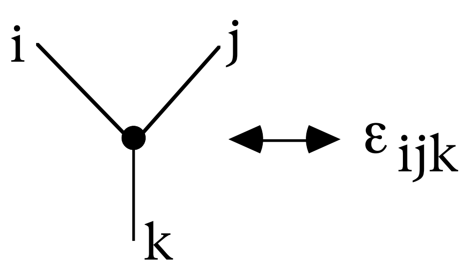



One of my favorite matrices is the "epsilon tensor" $\epsilon_{ijk}$.
This matrix has three indices, each of which can take the values 1, 2 or 3. The values of the epsilon are as follows

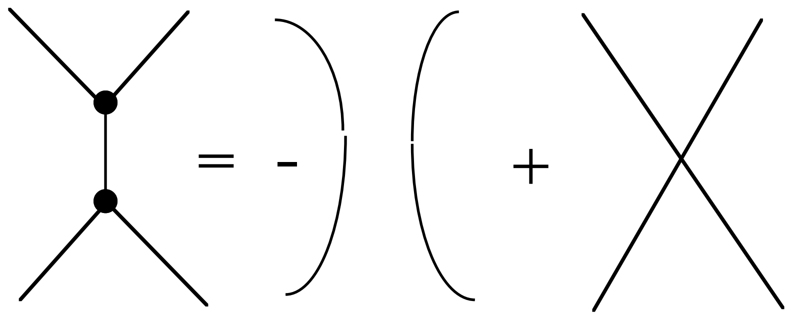
$$\epsilon_{123} = \epsilon_{312} = \epsilon_{231} = +1$$

$$\epsilon_{132} = \epsilon_{213} = \epsilon_{321} = -1.$$

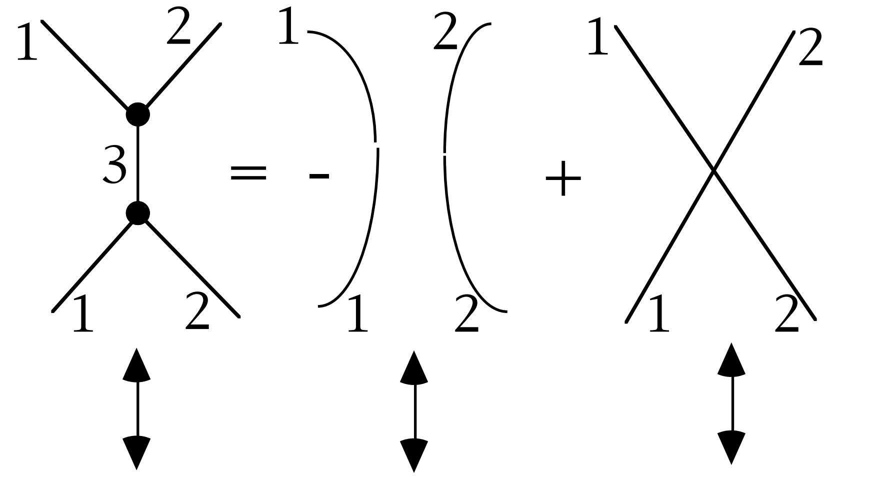
Otherwise (if there is any repetition of indices) the epsilon is zero.
Note that epsilon is invariant under cyclic permutation of the indices.
We diagram epsilon by using a trivalent vertex.



There is a magic identity about the epsilon, which translates into diagrammatic language as



A single line represents the identity matrix. That is, when the two endpoints of the line have the same index value, then the value of the matrix element is one, otherwise it is zero. You can see the truth of this diagrammatic identity by assigning some values to the lines. For example:

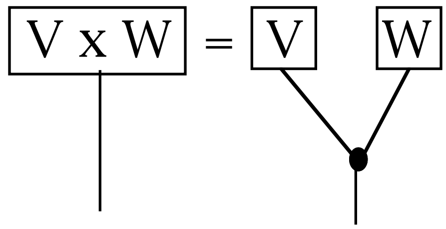



(1)(-1) = -1   -(1)(1) = -1   (0)(0)=0

Now the *cross product* of two three dimensional vectors is defined by the epsilon:

$$(V \times W)_k = \Sigma_k \varepsilon_{ijk} V_i W_j.$$

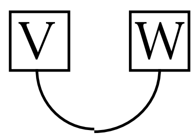
Here one sums over the repeated index **k**. Note that a vector, having only one index is represented by a box with one line. In diagrams the vector cross product is given as follows.



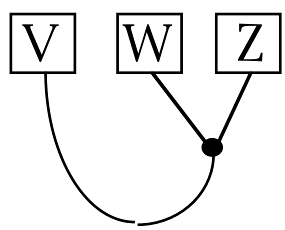

Similarly, the *dot product* of two vectors is given by the formula

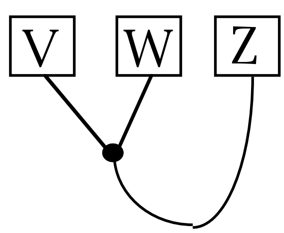
$$V \bullet W = \Sigma_k V_k W_k.$$

In diagrams, we have:

V.W = 

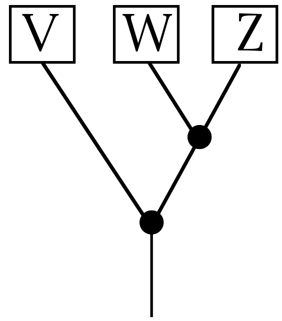

Now we are prepared to see some identities about the vector cross product and the dot product.

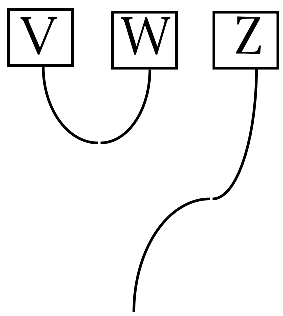
V.(W x Z) = 

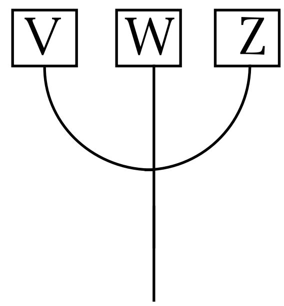

(V x W). Z = 


V.(W x Z) = (V x W).Z

The diagrams deform to one another in the plane. The epsilon is invariant under cyclic permutation of its indices.
Here is one that uses the basic epsilon identity.

V x (W x Z) = 


= - 

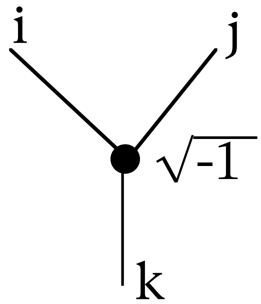
\+ 


= -(V.W)Z + (V.Z)W

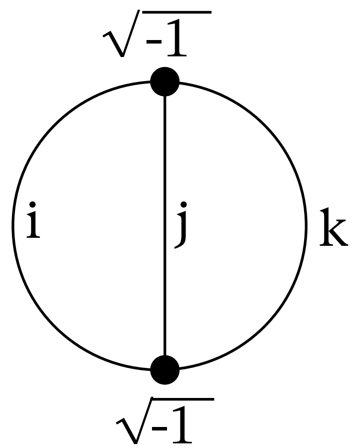
Vector algebra becomes transparent through the use of diagrammatic matrices.

One last remark about the epsilon: Roger Penrose [Pen] assigns epsilon times the square root of minus one to cubic vertices, and uses that assignment to prove a beautiful formula that lets one count the number of colorings of a planar cubic graph, just as we have discussed the problem in section 8. The proof of the Penrose formula can be performed as a combination of diagrammatic matrix algebra and the use of idemposition and formation as in section 8. The interested reader can consult [KP] or [Map] for the details. Here is a quick sketch of the matter:

From now to the end of the section the trivalent vertex will stand for $\sqrt{-1}\varepsilon_{ijk}$.




We shall indicate this by placing a $\sqrt{-1}$ sign next to the corresponding graphical node. Now consider the following decorated graph G.

$[G] =$ 


$$= \sum_{ijk} \sqrt{-1}\,\sqrt{-1}\;\epsilon_{ikj}\,\epsilon_{ijk}$$

$$= \sum_{ijk} (-1)(-\epsilon_{ijk}\,\epsilon_{ijk})$$

$$= \sum_{ijk\text{- distinct}} 1$$

$$= 6$$

We see in this instance that by taking each vertex of G as an epsilon multiplied by the square root of negative unity, we can take the matrix evaluation of the closed graph that is obtained by summing over all possible index assigments to the edges of the graph, and multiplying together the matrix entries for each choice of indices.

The resulting sum certainly runs through all colorings of the graph, as we have defined coloring to mean three distinct colors at each vertex, and the epsilon is non-zero in just this case. We find, in this example that each coloring gets counted with a plus one, so the the sum over all of them is equal to the number of colorings. This is not an accident!

## XVII. Formal Arithmetic

The purpose of this section is to go underneath the scene of numbers as we know them and to look at how these operations of addition and multiplication can be built in terms of a little technology of distinctions and the void. This is a story as old as creation herself, and we shall take some time to point out a mythological connection or two as we go along.

### In the Beginning

In the beginning there was Everything/Nothing, a world with no distinctions, the Void. Of course, there was not even a world at this stage, and we do not really know how to describe how observers with understanding could arise from a world in which there really was nothing and no way to begin.

So the idea in exploring the possibility of an infinitely creative Void prior to the creation of All Things is to look at structures that we know in the world that we seem to know, and follow them back into simplicity.

This will not be a linear process. Once we follow a structure back into what seems to be its essential simplicity, there is a new and wider view available, and this view compounded with what we already knew, leads to a new way to hold the entire matter and more possibility to move into even deeper simplicity.

It is a paradox. By moving into simplicity, we make room for a world with even greater complexity. And this complex world allows the movement into even greater simplicity. There is an infinite depth to simplicity, just as there is an infinite possibility for complexity.

### One

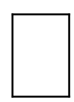
One?

The Void of Everything/Nothing is certainly One.  "It" (and by referring to it I naturally move away from it, for it is not an it.  The Void, when named, is not the Void. There is no way to define, name, delineate or otherwise contain the uncontainable.

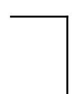
This very uncontainability makes the Void a One, since it certainly is not a Many. So we can certainly say that the Void is One. And at the same time there is no way to actually name the Void and so we might imagine that she has a secret and unpronounceable name. Void is not really her name. Void is a finger pointing to the moon.

Mathematics, at this stage, is delicate. You can do mathematics in the neighborhood of the Void, but you had better be very careful to understand that reference just does not work in the everyday --up here in the trees--world full of things way. No. We have proved that the Void is One, because it certainly is not Many. But we have to take this very carefully, because, if we were to enter into the Void there would not even be One or None or Many.

On the other side, coming from our home in the trees, it is quite tempting to just say. Well , One is just one distinction.  Like this:



or perhaps this:



How can we reconcile that grand One of the One Void and the small one of one distinction? Clearly the answer lies in understanding that the one of the distinction stands for the form of distinction itself and that it is this form of distinction that we refer to when we distinguish the One. The one void is not an expression of a something, but rather  an indication of our intuition of the form of distinction itself. Nothing is more distinguished than the Void, and so all aspects of distinction belong to it from the outside, as it were. In and of itself the void knows nothing, distinguishes nothing, is nothing.

Yet the void is Everything/Nothing and so all this, all this discussion is occurring in Void. This discussion pretends to make distinctions and to talk about the One and the Many. But it is fiction. It is all empty, and the only meaning that can possibly adhere to this discussion is emptiness.

For these reasons, we choose the mark

⏋

as the quintessential representative of one. The mark is seen to make a distinction in the plane on which it is drawn, and yet (being an abbreviated square) it provides an open pathway from inside to outside. The one mark unifies the sides that it divides.

## Many

We can proceed into the multiplicity of arithmetic with

0 =
1 = ⏋
2 = ⏋ ⏋
3 = ⏋ ⏋ ⏋

and so on.

Addition is the juxtaposition of forms: a + b = ab.

Thus

1 + 1 = ⏋ + ⏋ = ⏋⏋ = 2.

Multiplication is more complex.
When we multiply 2 x 3 we either take two threes and add them together, or we take 3 twos and add these together. In either case we make an operator out of one number and use this operator to reproduce copies of the other number. We seek a way to put these patterns into our formalism.

Let $\overline{n}\rvert$ denote the operator corresponding to the number $n$. Here is how this will work. If we put $\overline{n}\rvert$ next to any operator $\overline{m}\rvert$ then

$$\overline{n}\rvert\ \overline{m}\rvert$$

will create n copies of the number m (or m copies of the number n) and place them under the roof of a mark forming a new operator. Thus

$$\overline{5}\rvert\ \overline{3}\rvert = \overline{5 \text{ x } 3}\rvert\ .$$

Thus the juxtaposition of operators effects multiplication in the language of the operators. We would like to remove the mark over the result $\overline{5}\rvert\ \overline{3}\rvert$ and thereby obtain the product of the two numbers in the realm of numbers. The following simple rule about the boundaries helps:

$$\overline{\overline{a}\rvert}\rvert = a \quad \text{for any } a.$$

In particular,

$$\overline{\overline{\ \ }\rvert}\rvert \ =$$

This rule is the law of crossing from Laws of Form.

We take on the law of crossing for this version of arithmetic (but not the law of calling). With the law of crossing in use, we can write

$$\overline{\overline{5}\rvert\ \overline{3}\rvert}\rvert = \overline{\overline{5 \text{ x } 3}\rvert}\rvert = 5 \text{ x } 3.$$

In fact, these operators begin to take on a life of their own. Certainly we can write arithmetic entirely in Laws of Form notation. For example:

$$3 \times 3 = \overline{\overline{3}|\;\overline{3}|}|$$

$$= \overline{\overline{\rceil\rceil\rceil}|\,\overline{\rceil\rceil\rceil}|}|$$

$$= \overline{\overline{\rceil\rceil\rceil\rceil\rceil\rceil\rceil\rceil\rceil}|}|$$

$$= \rceil\rceil\rceil\rceil\rceil\rceil\rceil\rceil\rceil$$

$$= 9$$

What I wish to focus upon here is what this formalism says about zero. Consider zero multiplied by zero. Since zero is represented by absolutely nothing, we get

$$0 \times 0 = \overline{\rceil\;\rceil}|$$

and **there is no reason in this system to reduce 0 x 0 to 0.** In fact we see that

$$0 \times 0 = \overline{2}|\,,$$

our important multiplication operator corresponding to the number two.

Look at the powers of zero:

$$0^0 = \rceil$$

$$0^1 = \overline{\rceil}| \quad = 0$$

$$0^2 = \overline{\rceil\;\rceil}|$$

$$0^3 = \overline{\rceil\;\rceil\rceil}|$$

Now we can see how beautifully this choice of notation works. Zero raised to the first power is indeed zero, since

$\overline{\overline{\;\;}|}|$ is equal to absolutely nothing, and this is the additive zero in our system. The most wonderful equation of all is

$$0^0 = \overline{\;\;\;}|$$

For here we have the production of Distinction and Unity from the Void. The Void taken to its own Power produces Unity. This equation is true. We have proven it. It is beautiful and we can not know its full meaning. For that is the Power of the Void.

**Remark.**
We should note that if we allow transposition (see the section on primary algebra) then we can write:

$$\overline{N}|a = \overline{\overline{\;\;}|\,\overline{\;\;}|\,\overline{\;\;}|\ldots\overline{\;\;}|}|a$$

$$= \overline{\overline{a}|\,\overline{a}|\,\overline{a}|\ldots\overline{a}|}|$$

$$= a^N$$

Thus

$$a^N = \overline{N}|a = 0^N.a$$

Note well that a special case of this last result is the equation

$$\overline{\;\;\;}|.\overline{\;\;\;}| = \overline{\;\;\;}|$$

expressing the fact that $1^0 = 1$. This is not to be confused with

$$\overline{\;\;\;}|\;\overline{\;\;\;}|$$

which represents 2 as 1+1. In general, this means that the arithmetic consisting of zero numbers together with standard numbers is non-associative, and one must take care!

**Remark.** What about negative numbers?
We could postulate a reverse mark

$$\ulcorner \;\; = -1$$

so that

$$\ulcorner \;\; \urcorner \; = \; \urcorner\ulcorner \;\; = $$

Then

$$\boxed{0^{a} = \overline{a\,}|}$$

$$1/0 = 0^{-1} = \overline{\ulcorner\,}|$$

$$0^{1/0} = \overline{\overline{\ulcorner\,}|\,}| = \ulcorner$$

$$0^{1/0} = -1$$

(We assure the reader that one can set up the context so that this formalism is consistent. That task is not discussed here.)
One can regard this last equation as a precursor to the famous mathematical equation

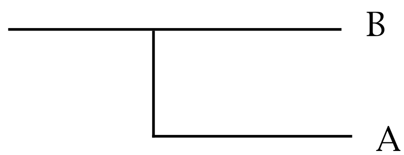
$$e^{i\pi} = -1 \,.$$

**Remark.** There is a good deal more to say about the structure of numerical arithmetic as constructed in the context of Laws of Form. Spencer-Brown has an article on this in the English-German edition of his book [SB]. See also [LKN, JEN, JJN].

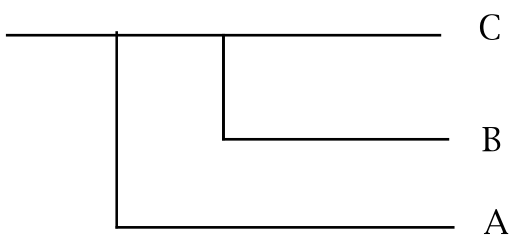
## XVIII. Frege's Conceptual Logical Notation and Laws of Form

Along with his very clear motive to base arithmetic and logic on conceptual foundations, Gottlob Frege devised a "conceptual notation" for logic. It turns out that this Frege notation is directly related to the circuit theory approach to Laws of Form. In this section we give an exposition of Frege's system and of that connection with Laws of Form. The material in this section is joint work with Christina Weiss [KF]. It is of great interest that Frege interwove his graphical notation with the logical text.

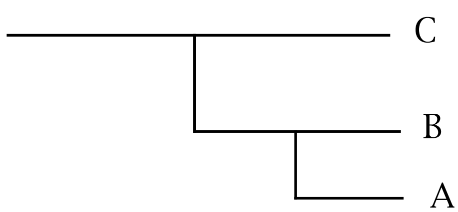
Here is the Frege symbol for "A **entails** B":




This is a continuous notation that structurally links the propositions together in the form of a tree. Compare "A **entails** (B **entails** C)"

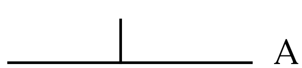



with "(A **entails** B) **entails** C".

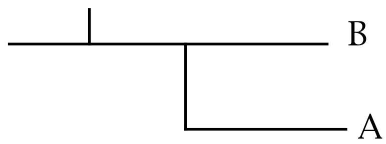



The different tree structures in Frege's notation make the non-associativity of entailment graphic.

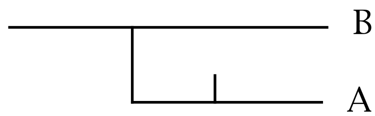
Negation is accomplished in Frege by the placement of a stroke. The diagram below represents "**Not A.**"



Not A is represented by the diagram as a whole, but the placement makes the sign of negation act on what is found to the right of it. Thus the following diagram represents "**Not (A entails B)**"



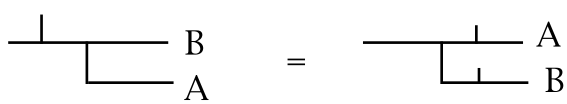

while the next diagram represents "**(Not A) entails B.**"




Logical relationship becomes spatial relationship in this inherently two-dimensional notation.

Frege went on to include the quantifiers "for all" and "there exists". We will not use these constructions in this paper. The reader can consult [GF] for more information about Frege's conceptual notation for the logic of quantification.

Now we come to the difficulties in using the Frege notation. Consider the tautology "**Not{A implies B) = (Not B) implies (Not A).**" In the Frege diagrams this is expressed as shown below.




In order to operate the Frege system one must get used to applying transformations of this kind. An algebraic model has it uses. Logic is not just the structure of implication. Logic, beginning with implication, needs to handle the intricacies of other connectives such as "and" and "or" and the patterns that arise from negation.

What about the other connectives such as "**AND**" and "**OR**" ?
In standard logic one has

**Not(A OR B) = (Not A) AND (Not B)**

and

A entails B = Not ( A AND Not B),

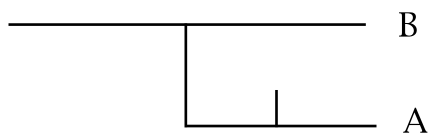
from which it follows that

A entails B = Not(A) OR B.

Turning this around, we have

A OR B = Not(A) entails B.

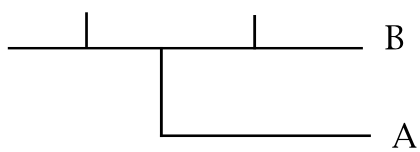
Thus in Frege, the following diagram represents "A OR B".




Similarly we have that

A AND B = Not( A etails Not(B))

so that the following diagram represents "A AND B."




This is getting complicated.

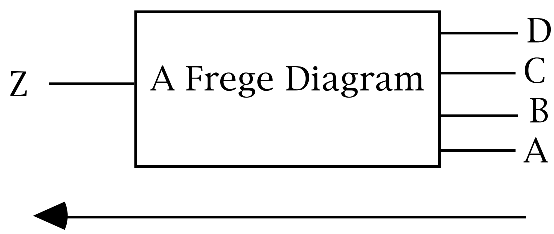
## Frege's Diagrams as Signal Processors of the Marked and Unmarked States

In order make Frege's notation palatable, one needs to become fluent in making transformations within it. In order to make Frege's notation useful we need a deeper understanding of its structure. There is such an understanding, an understanding that was not available to Frege, but began with Claude Shannon in the 1930's with his discovery of the relationship of Boolean algebra and switching networks [CES]. A precursor to our remarks occurs in the paper by Hoering [H], who made an early relationship between Frege's diagrams and switching circuits. We are indebted to James Flagg for pointing out this paper to us.

Shannon's insight was that the patterns of logic and Boolean algebra, were the same as the patterns of signals transmitted in networks where at a given juncture the signal would either be

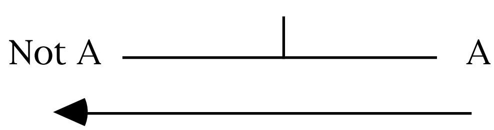
transmitted or not transmitted. The decision to transmit or not to transmit can be made by a switch that is either open or closed. Switches in series have the pattern of AND. Switches in parallel have the pattern of OR.
The negation of a switch is obtained by changing its state from open to closed, or from closed to open. With this point of view Shannon was able to utilize Boolean algebra in the design of switching circuits, and logic took on new clothes in the form of the network formalisms. These networks can have circularities and Shannon was well aware, even at the beginning of his project, that this meant that logic would also have to deal with the circularities so apparent in an electrical or information network. Shannon found the inception of a cybernetic logic.

Think of a Frege diagram as a signal processor of the letters on its right. *Let the information flow from right to left* so that the left-most edge in the diagram represents the value of the expression that is the diagram itself.



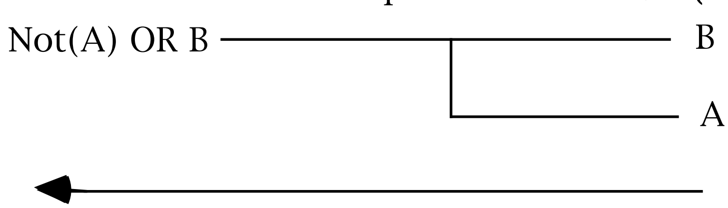

**Z** is the transform of **A,B,C,D** that is the value of the diagram.

For example, we take the diagram for Not A and label it as follows.

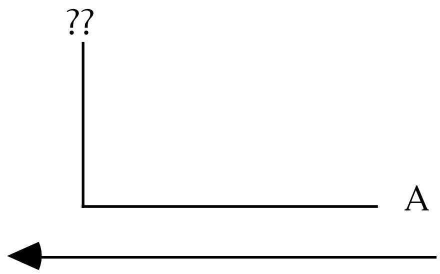



**A** is transformed into **Not A** by the negation stroke.

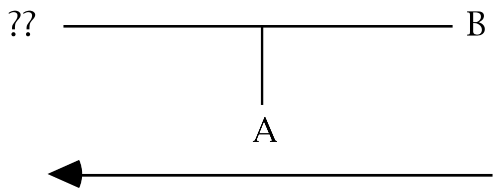
In entailment we have the production of **Not(A) OR B.**



What happens to the signal when you turn a corner?




What is the action of the simple join of two lines in the tree?

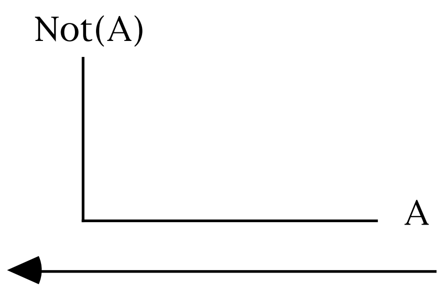



The answers are simple, and they reveal an underlying syntax for Frege's diagrams that makes them easy to use.

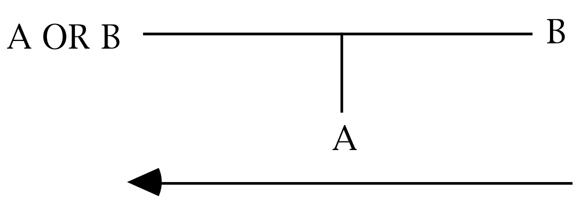
**Laws of Form is the Key to Frege**

Here are our answers to the questions.

**1. The signal is negated when it turns the corner.**

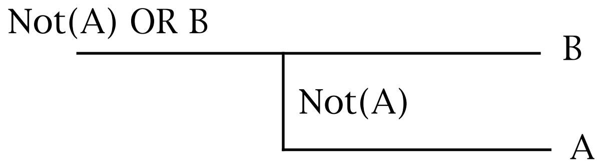


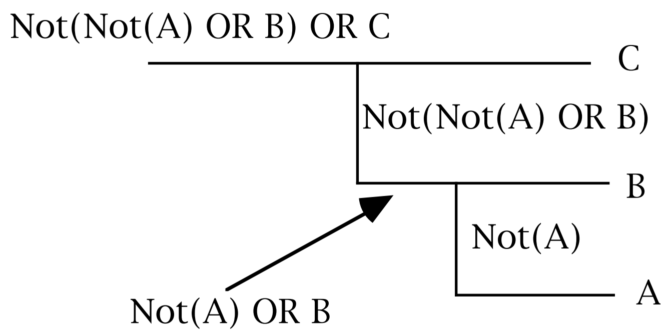

**2. The join of two lines performs the operation OR.**



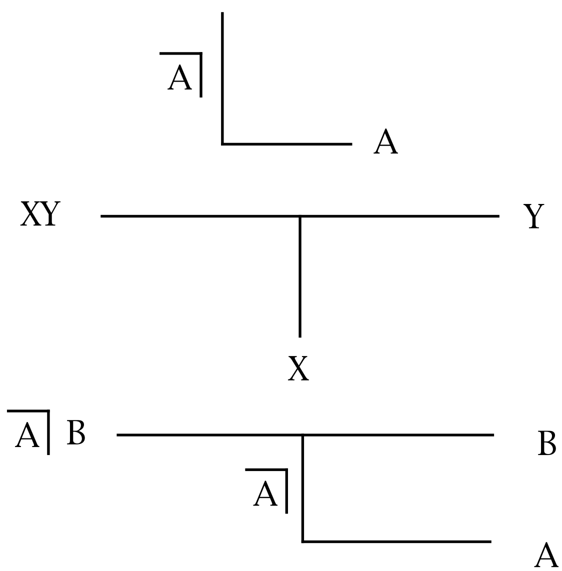

It is a miracle!  Signals can now flow in Frege's formalism and his language can live in time.

With these assignments of operation, Frege's diagrams become information processing networks and they are utterly easy to read in the language of Not and OR.




Here is our amalgam of the Frege and Spencer-Brown notations.

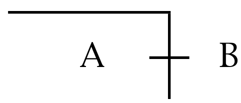



Note that in Spencer-Brown [LOF], "A entails B"  is written as

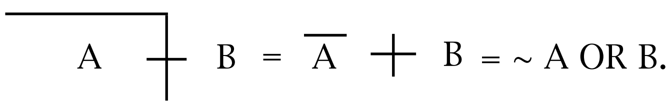
Spencer-Brown's ninety degree bend is a marker, a container, a representative of a primary distinction.

### And Peirce

The American philosopher and mathematician Charles Sanders Peirce
discovered diagrammatic logic systems (his sign of illation and his existential graphs) that are closely related to the concerns of this essay. For an introduction to Peirce's diagrams we refer the reader to [MP]. More needs to be said, but here it is worth remarking that the Peirce sign of illation




also partakes of a ninety degree angle. Peirces's sign is a compound sign, a combination of negation (the horizontal overbar) and addition (logical OR) the vertical bar with a horizontal line. Thus one has the decomposition




Peirce managed to invoke in this compound sign a microcosm of his entire theory of signs. Like Frege's diagram for implication, the Peirce sign of illation invokes both a unitary sense of that operation and a hint of its interrelationship with other operations in the web of logical discourse. By finding negation at the ninety degree bend and OR at the junction in the Frege implication diagram, we have provided a view of Frege's diagram of implication that puts in parallel with the Peirce sign of illation and its internal structure. These parallels between Peirce and Frege run deep and will be the subject of another paper.

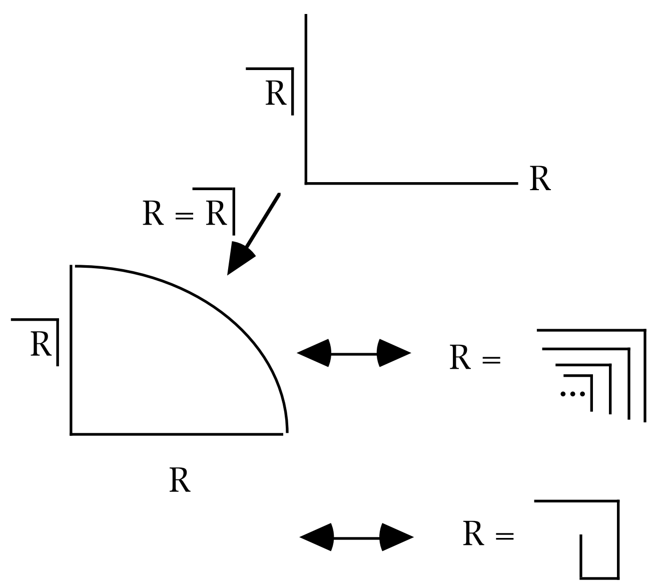
### Recursive Unity

We have discussed Frege's conceptual notation and how it can be understood in terms of the transmission of signals. Signs can be seen to move through and be transformed by other signs and by themselves. Seen this way, the Frege notation enters the modern age. It is extraordinary how well it fits.

Note that the Frege traditional diagrams are all trees. There is no recursion. Recursion caught up with Frege with the Russell paradox. There the class R of all sets that are not self-members, keeps extending itself. For if R is not a member of itself, then indeed R at once becomes a member of itself! But this is a new R, augmented by its own self-collection. The bootstrapping goes on so that R always becomes NOT R or simply ~ R.
If we wish R to exist timelessly, then the paradox comes forth in the equation

$$R = \sim R.$$

In order to support such a circularity in the formalism we would have to *let the ninety degree angle bite its own tail.*




In Spencer-Brown, the mark reenters its own indicational space, producing circularity, recursion and time. In Frege, it is necessary to *curve* back to the start lest we transform again by the action of an angle. Angular turn combined with smooth transition is the essence of circularity. This synthesis of the continuous and the discrete is inherent in Frege's language. The basis of self-reference sits right there in that eternal return.

## XIX. The Logical Garnet

The purpose of this section is to point out a remarkable connection between Laws of Form, polyhedral geometry, mirror symmetry and the work of Shea Zellweger [ZW].

Zellweger did an extensive study of the sixteen binary connectives in Boolean logic ( "and", "or" and their relatives -- all the Boolean functions of two variables), starting from Peirce's own study of these patterns. He discovered a host of iconic notations for the connectives and a way to map them and their symmetries to the vertices of a four dimensional cube and to a three dimensional projection of that cube in the form of a rhombic dodecahedron. Symmetries of the connectives become, for Zellweger, mirror symmetries in planes perpendicular to the axes of the rhombic dodecahedron. See Figure G2. Zellweger uses his own iconic notations for the connectives to label the rhombic dodecahedron, which he calls the "Logical Garnet".

This is a remarkable connection of polyhedral geometry with basic logic. The meaning and application of this connection is yet to be fully appreciated. It is a significant linkage of domains. On the one hand, we have logic embedded in everyday speech. One does not expect to find direct connections of the structure of logical speech with the symmetries of Euclidean Geometry. It is the surprise of this connection that appeals to the intuition. Logic and reasoning are properties of language/mind in action. Geometry and symmetry are part of the mindset that would discover eternal forms and grasp the world as a whole. To find, by going to the source of logic, that we build simultaneously a world of reason and a world of geometry incites a vision of the full combination of the temporal and the eternal, a unification of action and contemplation. The relationship of logic and geometry demands a deep investigation. This investigation is in its infancy.

In this section I will exhibit a version of the Logical Garnet (Figure G2) that is labeled so that each label is an explicit function of the two Boolean variables A and B. A list of these functions is given in Figure 1. We will find a new symmetry between the Marked and Unmarked states in this representation. In this new symmetry the mirror is a Looking Glass that has Peirce on one side and Spencer-Brown on the other!

Before embarking on Figure G1, I suggest that the reader look directly at Figure G2. That Figure is a depiction of the Logical Garnet. Note the big dichotomies across the opposite vertices. These are the oppositions between Marked and Unmarked states, the opposition between A and not A, and the opposition between B and not B. If you draw a straight line through any pair of these oppositions and consider the reflection in the plane perpendicular to this straight line, you will see one of the three basic symmetries of the connectives. Along the A/not A axis the labels and their reflections change by a cross around the letter A. Along the B/not B axis, the labels change by a cross around the letter B. These reflections correspond to negating A or B respectively. Along the Marked/Unmarked axis, the symmetry is a bit more subtle. You will note the corresponding formulas differ by a cross around the whole formula and that both variables have been negated (crossed). Each mirror plane performs the corresponding symmetry through reflection. In the very center of the Garnet is a double labeled cube, labeled with the symbols "A S B" and "A Z B". These stand for "Exclusive Or" and its negation. We shall see why S and Z have a special combined symmetry under these operations. The rest of this section provides the extra details of the discussion.

Let us summarize. View Figure G2. Note that in this three-dimensional figure of the Logical Garnet there are three planes across which one can make a reflection symmetry. Reflection in a horizontal plane has the effect of changing B to its crossed form in all expressions. Reflection in a vertical plane that is transverse to projection plane of the drawing, interchanges A and its crossed form. Finally, reflection in a plane parallel to the projection plane of the drawing interchanges marks with unmarks. We call this the Marked/Unmarked symmetry.

*On first pass, the reader may wish to view Figure 2 directly, think on the theme of the relationship of logic and geometry, and continue into the next section. The reader who wishes to see the precise and simple way that the geometry and logic fit together should read the rest of this section in detail.*

Figure 1 is a list of the sixteen binary connectives given in the notation of Laws of Form. Each entry is a Boolean function of two variables. In the first row we find the two constant functions, one taking both A and B to the marked state, and one taking both A and B to the unmarked state (indicated by a dot). In row two are the functions that ignore either A or B. The remaining rows have the functions that depend upon both A and B. The reader can verify that these are all of the possible Boolean functions of two variables. The somewhat complicated looking functions in the last row are "Exclusive Or" , A *S* B and its negation A *Z* B.

In order to discuss these functions in the text, and in order to discriminate between the Existential Graphs and the Laws of Form notations, I will write <A> for the Laws of Form mark around A. Thus, in contrast, (A) denotes the Existential Graph consisting in a circle around A. The Spencer-Brown mark itself is denoted by <>, while the circle in the Peirce graphs is denoted (). Exclusive Or and its negation are given by the formulas

A*S* B = <A <B> > < <A> B > , A*Z* B = < A*S* B >.

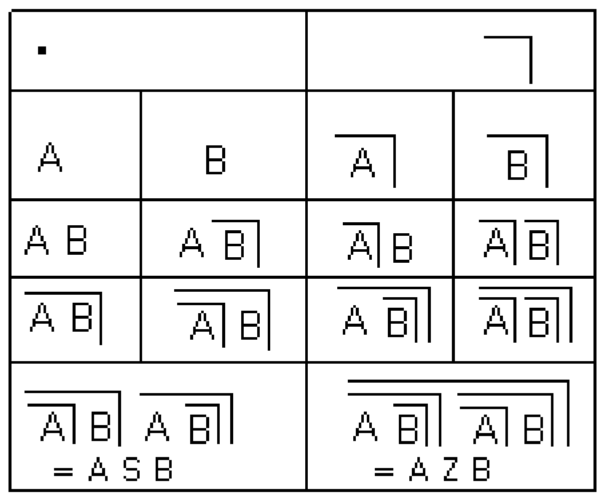



Figure G1. The Sixteen Binary Boolean Connectives

Here we have used Zellweger's alphabetic iconics for Exclusive Or with the letters *S* and *Z* topological mirror images of each other.

*Exclusive Or is actually the simplest of the binary connectives*, even though it looks complex in the chart in Figure 1. Let "Light" denote the unmarked state and "Dark" denote the marked state. Then the operation of Exclusive Or is given as follows

Dark *S* Dark = Light,
Dark *S* Light = Dark,
Light *S* Dark = Dark,
Light *S* Light = Light.

Imagine two dark regions, partially superimposed upon one another.

Where they overlap, the darknesses cancel each other, and a light region appears. This is the action of Exclusive Or. Darkness upon darkness yields light, while darkness can quench the light, and light combined with light is light. In other words, Exclusive Or is the connective closest to the simple act of distinction itself, and it is closest to the mythologies of creation of the world (heaven and earth, darkness and light) than the more complex movements of "and" and "or". Exclusive Or and its negation sit at the center of the logical garnet, unmoved by the symmetries that interchange the other connectives.

The operation of Exclusive Or on the marked state is the same as negation (darkness cancels darkness to light) and the operation of Exclusive Or on the unmarked state is the identity operation that makes no change.

A *S* <> = <A> while
A *S* . = A.

The symmetries of Exclusive Or are very simple. If we change one of the variables to its negation we just switch from S to Z! That is,

<A> S B = A S <B> = A Z B.

As a result, A S B and A Z B together are invariant under the symmetries induced by the A/not A and B/ not B polarity.

Note that the central vertex (cube) in the Garnet (labeled with A *S* B and A *Z* B) is connected to the eight compound terms on the periphery of the Garnet. These terms are the terms that arise from Exclusive Or and its Complement when we take it apart. For example

A *S* B = < A <B> > < <A> B >

and we can take this apart into the two terms

<A <B>> and < <A> B>,

while

<A> *S* B = <<A> B> < A B>

and we can take this apart into the two terms

< <A> B>   and  <A B>.

The reader will enjoy looking at the geometry of the way the central and simple operation of Exclusive Or is taken apart into the more complex versions of "and" and "or" and how the Geometry holds all these patterns together.

As for the periphery of the Garnet, it is useful to diagram this as a plane graph with the corresponding labels shown upon it. The illustration below exhibits this graph of the rhombic dodecahedron. The rhombic dodecahedron itself does not have a central vertex and the graph below shows precisely the actual vertices of the rhombic dodecahedron and their labels. By comparing with Figure 2, one can see how to bring this graph back into the third dimension. Note how we have all the symmetries apparent in this planar version of the rhombic dodecahedron, but not yet given by space reflection. It requires bringing this graph up into space to realize all its symmetries in geometry.

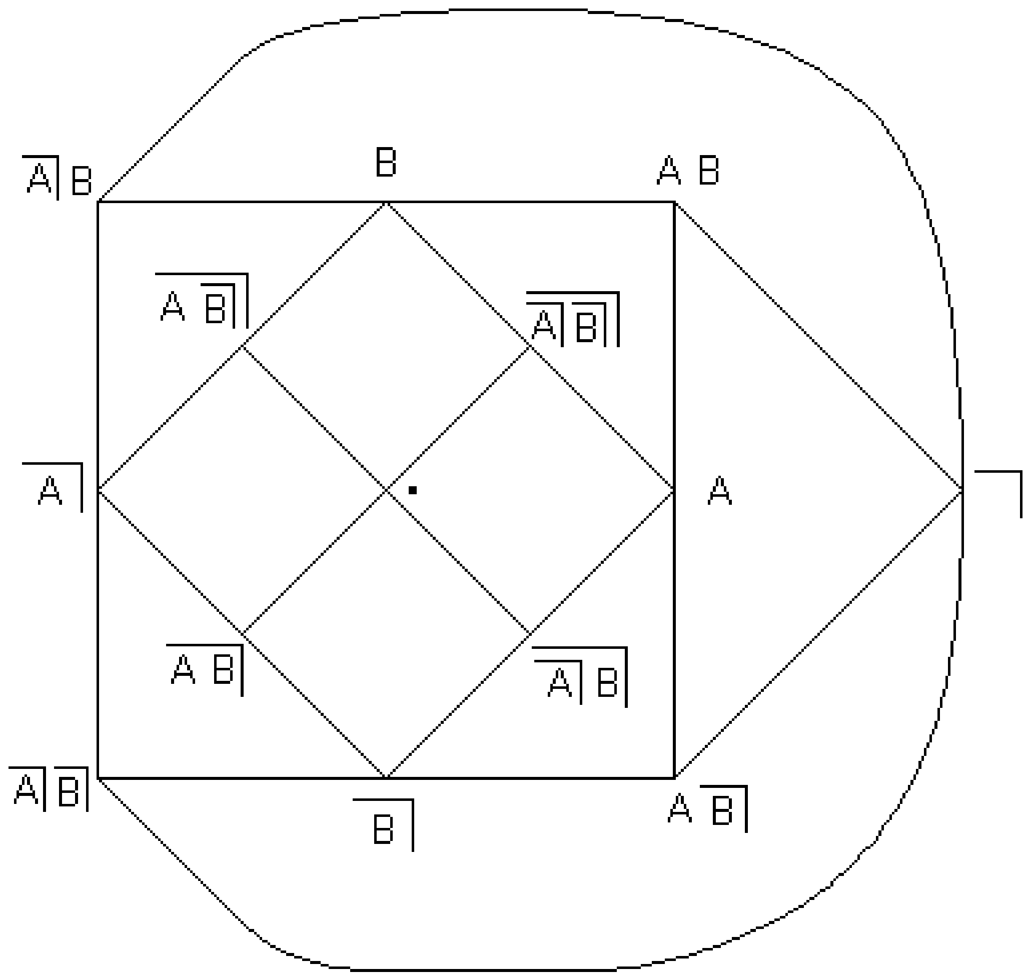



Diagram G2 - Planar Graph of the Rhombic Dodecahedron

Looking at this peripheral structure, we see the genesis of the pattern of the rhombic dodecahedron in relation to the connectives. This graphical pattern can be viewed as the lattice of inclusions of these functions regarded as subsets of a universal set. To see this clearly, view the next diagram where we have labeled the vertices of the graph in standard notation with an upward pointing wedge denoting intersection ("and"), a downward pointing wedge denoting union ("or"), 0 denoting empty set and 1 denoting the universe. Then, going outward from 0, pairs of vertices are connected to vertices denoting the union of their labels until we reach the whole universe which is denoted by 1. This lattice is exactly the graph of the rhombic dodecahedron.

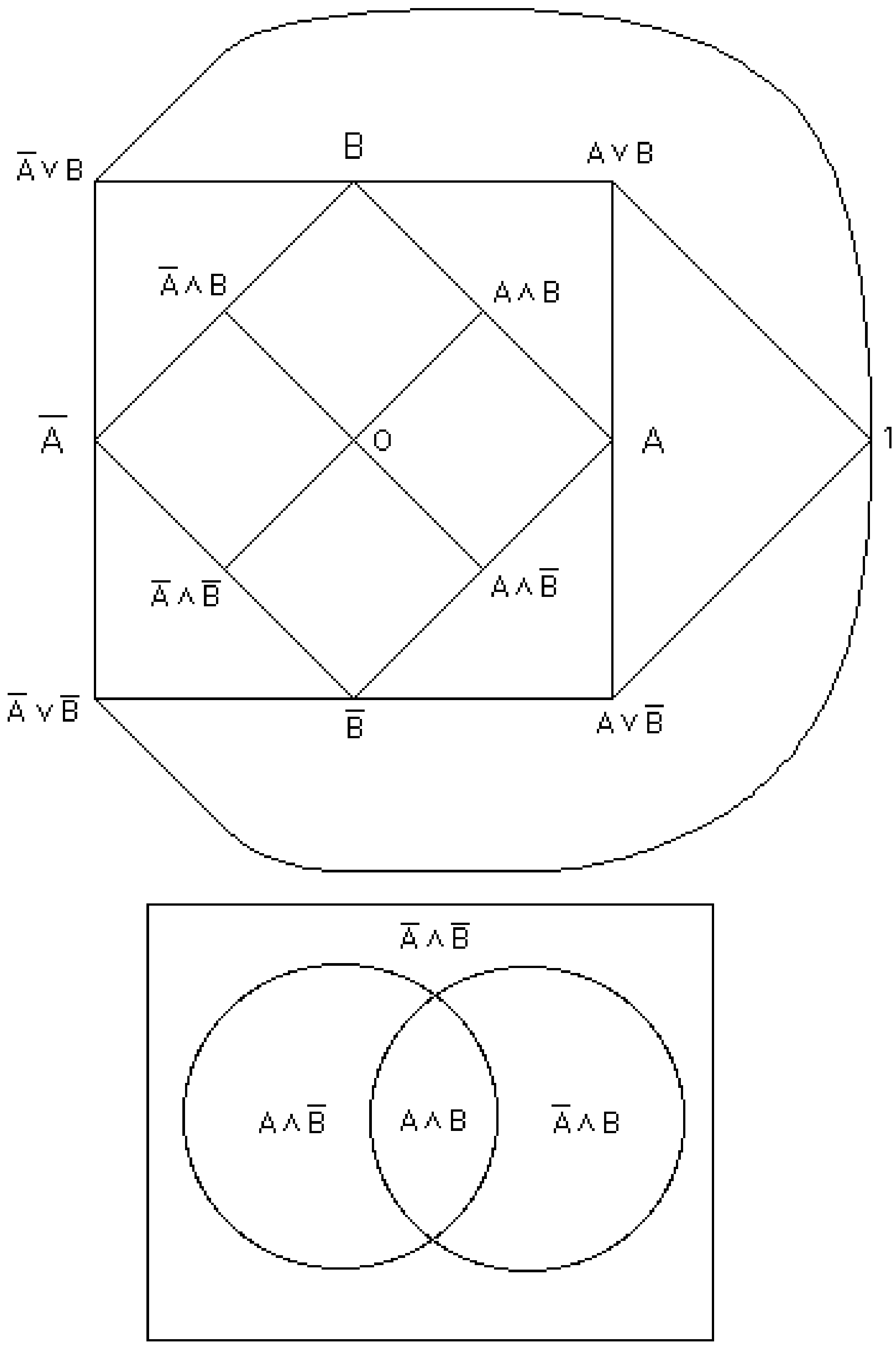



Diagram G3 - The Rhombic Dodecahedron as the Lattice Associated with a Two Circle Venn Diagram

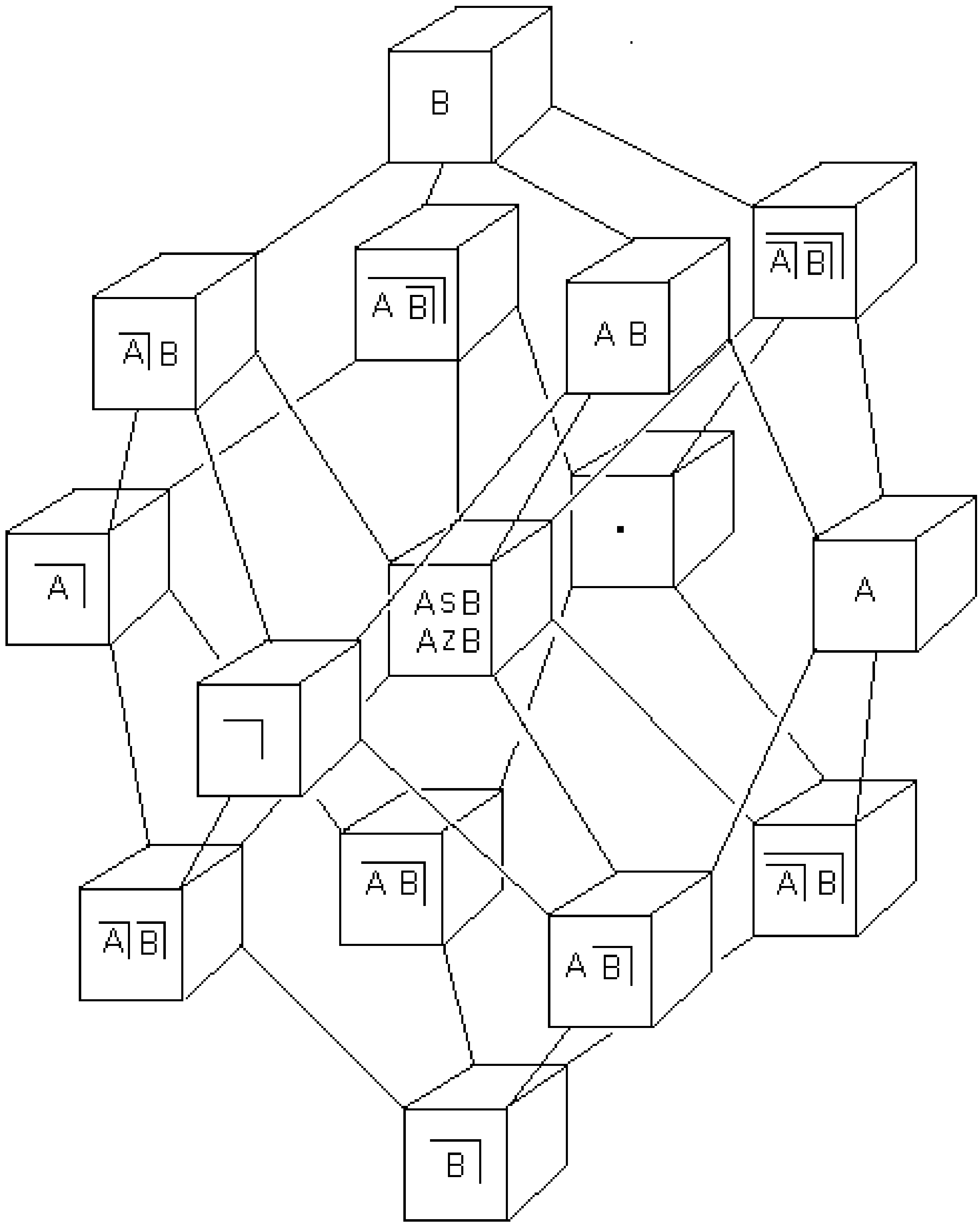



Figure G2 - The Logical Garnet

In this section we have exhibited a version of the Logical Garnet suited to Laws of Form. This appearance of significant Geometry at the very beginning of Logic and Form deserves deeper investigation. The diagrammatic investigations of Peirce, Venn, Carroll, Nicod [Nicod] and Spencer-Brown are all ways of finding geometry in logic, but in Zellweger's Logical Garnet classical three-dimensional geometry appears, and this is an indication that one should think again on the relationship of logic and mathematics.

## XX. Remembering

This work relates at an abstract level with the notions of autonomy and autopoesis inherent in the earlier work of Maturana, Uribe and Varela [V]. There they gave a generalized definition of life (autopoesis) and showed how a self-distinguishing system could arise from a substrate of "chemical" interaction rules. I am sure that the relationship between the concept of the reentering mark and the details of this earlier model was instrumental in getting Francisco to think deeply about Laws of Form and to focus on the Calculus for Self-Reference. Later developments in fractal explorations and artificial life and autopoesis enrich the context of Form Dynamics.

At the time (around 1980) that Francisco and I discussed Form Dynamics we were concerned with providing a flexible framework within which one could have the "eigenforms" of Heinz von Foerster [VF] and also the dynamical evolution of these forms as demanded by biology and by mathematics. Francisco had a deep intuition about the role of these eigenforms in the organizational structure of the organism. This is an intuition that comes forth in his books [V1,V2] and in his other work as well.

There is a more general theme that has been around since that time. It is the theme of "unfolding from a singularity" as in catastrophe theory. In the metaphor of this theme the role of the fixed point is like the role of the singularity. The fixed point is an organizing center, but it is imaginary in relation to the actual behaviour of the organism, just as the "I" of an individual is imaginary in relation to the social/biological context. The Buddhists say that the "I" is a "fill-in". The linguists point out that " I am the one who says "I"." The process that is living never goes to the fixed point, is never fully stable. The process of approximation that is the experiential and experienced I is a process lived in, and existing in the social/biological context. Mind becomes conversational domain and "mind" becomes the imaginary value generated in that domain.

Heinz von Foerster [VF] said "I am the observed link between myself and observing myself."

The fixed point is fundamental to what the organism is not. In the imaginary sense, the organism becomes what it is not.

In those same years, from 1978 until the middle 1990's I had a long and complex correspondence with G. Spencer-Brown that culminated in my paper [Map] about his approach to the four-color map theorem. These conversations also revolved around the nature of mathematics and the nature of the circuit structures in Chapter 11 of Laws of Form.

Since 1980, I have been in remarkable conversation with James Flagg about all topics related to Laws of Form and many other subjects. Conversation with other members of the Chicago Laws of Form Group from the 1970's continue unabated in non-local realms (Jerry Swatez, Paul Uscinski, David Solzman). In the intervening years the notion of locality has changed radically, and yet it is still personal conversation that has the highest value.

## XXI. Epilogue

There is a kind of blinding clarity about these simple ideas near the beginning of Laws of Form. They point to a clear conception of world and organism arising from the idea of a distinction. Nevertheless, if you follow these ideas out into any given domain you will be confronted by, perhaps engulfed by, the detailed complexities of that domain. The non-numerical mathematics acts differently than in traditional numerical models. It acts as an arena for the testing of general principles and as a metaphor that can be used in the face of complexity. One keeps returning to the mystery of how "it" emerged from "nothing".

"Crawling up along the waves of an oscillation Parabel asked Cookie: " I fail to understand what we are doing here." "What do you mean?" she replied. Well, in the beginning there was only void. Right? And then somehow we are crawling our way upward toward stable forms. Where do these forms come from? How can there be anything at all if we began with absolute nothingness. I don't get it. Listen, says Cookie. It's a secret. Actually all this is ... nothing. You see it all begins to look flowing and strong now. Not solid. Not yet. And you can imagine a time when it will even feel solid and real. But look here Parabel it is actually nothing, nothing

at all. Nothing is an opportunity to imagine something. Absolute nothing is the most powerful opportunity of all to imagine anything at all. Because there is really absolutely nothing, the contrast with even a flickering thought of something is enough to make that something seem real! You imagined it all up. Yup! And you have nothing to thank for that."
[F]

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