# Chapter 18

# †Mathematical Techniques and Notation Used in this Book

## 18.1 Vectors and Operators in 3D

For some vector $\vec{a}$, written explicitly in terms of its three components in a geometry with 3 orthonormal basis vectors $(\hat{\alpha}, \hat{\beta}, \hat{\gamma})$:

$$\vec{a} = a_\alpha \hat{\alpha} + a_\beta \hat{\beta} + a_\gamma \hat{\gamma} \ ,$$

where $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\gamma}$, are unit vectors along the $\alpha$-, $\beta$-, and $\gamma$- directions.

Its magnitude squared is written several ways:

$$a^2 = |\vec{a}|^2 = \vec{a} \cdot \vec{a} \ ,$$

and is given by:

$$a^2 = (a_\alpha \hat{\alpha} + a_\beta \hat{\beta} + a_\gamma \hat{\gamma}) \cdot (a_\alpha \hat{\alpha} + a_\beta \hat{\beta} + a_\gamma \hat{\gamma}) = (a_\alpha^2 + a_\beta^2 + a_\gamma^2)$$

This is often a form of confusion since, without context, $a^2$ could stand for $a$ times $a$ or $\vec{a} \cdot \vec{a}$. It should be clear from the context in which it is used.

The magnitude of a vector is:

$$a = |\vec{a}| = +\sqrt{a_\alpha^2 + a_\beta^2 + a_\gamma^2} \ .$$

Note that the positive square root is taken.

Some other useful identities:

$$\begin{aligned}\vec{a}\cdot\vec{b} &= a_\alpha b_\alpha + a_\beta b_\beta + a_\gamma b_\gamma = |\vec{a}||\vec{b}|\cos\theta_{\vec{a},\vec{b}} \\ |\vec{a}+\vec{b}|^2 &= a^2 + 2\vec{a}\cdot\vec{b} + b^2 \end{aligned} \tag{18.1}$$

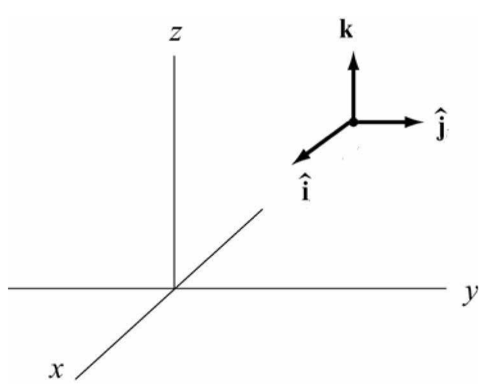
where $\theta_{\vec{a},\vec{b}}$ is the angle between the vectors $\vec{a}$ and $\vec{b}$.

### 18.1.1 Some common coordinate system representations

The common realizations of 3D coordinate ststems are:

| coordinate system | variable names | domain | unit vectors |
|---|---|---|---|
| rectilinear | $(x, y, z)$ | $-\infty < x < \infty$<br>$-\infty < y < \infty$<br>$-\infty < z < \infty$ | $(\hat{x}, \hat{y}, \hat{z})$, or<br>$(\hat{\imath}, \hat{\jmath}, \hat{k})$, or<br>$(\hat{n}_x, \hat{n}_y, \hat{n}_z)$ |
| cylindrical | $(\rho, \phi, z)$ | $0 \le \rho < \infty$<br>$-\infty < z < \infty$<br>$0 \le \phi < 2\pi$ | $(\hat{\rho}, \hat{\phi}, \hat{z})$, or<br>$(\hat{n}_\rho, \hat{n}_\phi, \hat{n}_z)$ |
| spherical-polar | $(r, \theta, \phi)$ | $0 \le r < \infty$<br>$0 \le \theta \le \pi$<br>$0 \le \phi < 2\pi$ | $(\hat{r}, \hat{\theta}, \hat{\phi})$, or<br>$(\hat{n}_r, \hat{n}_\theta, \hat{n}_\phi)$ |

#### The 3D rectilinear coordinate system



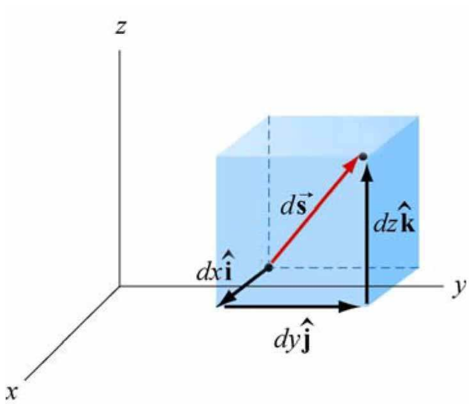

The vector for position in this coordinate system is:

$$\vec{x} = x\hat{x} + y\hat{y} + z\hat{z} \tag{18.2}$$

with the properties:

$$x = \hat{x} \cdot \vec{x} \quad ; \quad y = \hat{y} \cdot \vec{x} \quad ; \quad z = \hat{z} \cdot \vec{x} \tag{18.3}$$

An elemental volume, $\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ in this coordinate system is shown below.

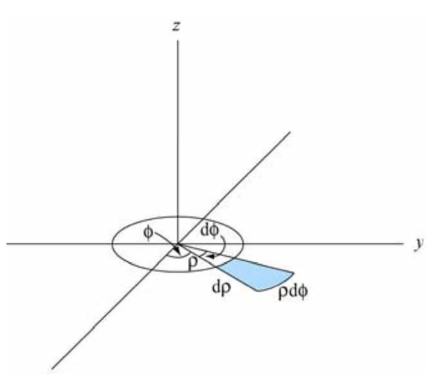



A 3D integral in this coordinate system is represented by:

$$\int \mathrm{d}\vec{x}\, f(x,y,z) \quad \text{or} \quad \int \mathrm{d}x \int \mathrm{d}y \int \mathrm{d}z\, f(x,y,z) \quad \text{or} \quad \iiint \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\, f(x,y,z) \tag{18.4}$$

All forms of (18.4) are employed in this book.

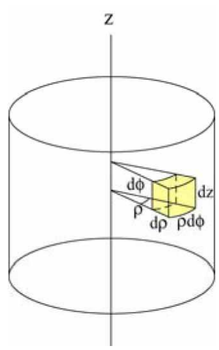
### The 3D cylindrical-planar coordinate system



The vector for position in this coordinate system is:

$$\vec{x} = \rho\hat{\rho} + \rho\hat{\phi} + z\hat{z} \tag{18.5}$$

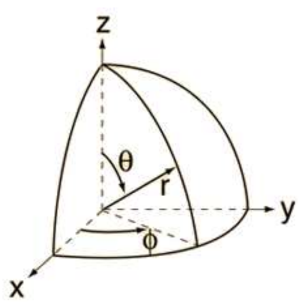
An elemental volume, $\mathrm{d}V = r\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}z$ in this coordinate system is shown below.




A 3D integral in this coordinate system is represented by:

$$\int \mathrm{d}\vec{x}\, f(\rho,\phi,z) \quad \text{or} \quad \int \mathrm{d}\rho \int \rho\,\mathrm{d}\phi \int \mathrm{d}z\, f(\rho,\phi,z) \quad \text{or} \quad \iiint \rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}z\, f(\rho,\phi,z) \tag{18.6}$$

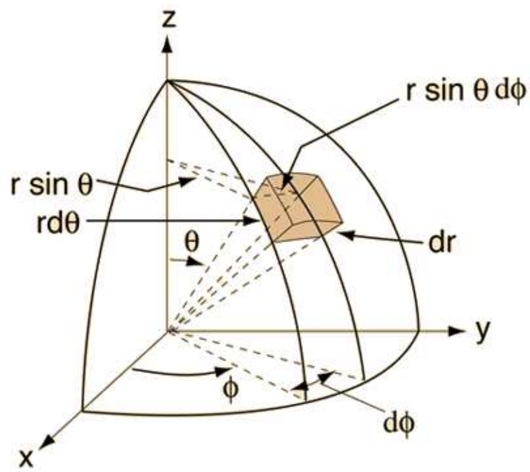
**The 3D spherical-polar coordinate system**




The vector for position in this coordinate system is:

$$\vec{x} = r\hat{r} + r\hat{\theta} + r\sin\theta\hat{\phi} \tag{18.7}$$

An elemental volume, $\mathrm{d}V = r^2\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$ in this coordinate system is shown below.




A 3D integral in this coordinate system is represented by:

$$\int \mathrm{d}\vec{x}\, f(r,\theta,\phi) \quad \text{or} \quad \int \mathrm{d}r \int r\mathrm{d}\theta \int r\sin\theta \mathrm{d}\phi\, f(r,\theta,\phi) \quad \text{or} \quad \iiint r^2 \sin\theta\, \mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi\, f(r,\theta,\phi) \tag{18.8}$$

Yet another form used in this book is

$$\int r^2 \mathrm{d}r\mathrm{d}\Omega\, f(r,\theta,\phi) \tag{18.9}$$

where $\mathrm{d}\Omega$ is the differential element of solid angle.

### Transformations between coordinate systems

Here are the two most common transformations:

$$\begin{aligned} \vec{x} &= \rho\cos\phi\hat{x} + \rho\sin\phi\hat{x} + z\hat{z} \\ &= r\sin\theta\cos\phi\hat{x} + r\sin\theta\sin\phi\hat{y} + r\cos\theta \end{aligned} \tag{18.10}$$

## 18.2 Common Trigonometric Relations

$$\begin{aligned}
\cos(a \pm b) &= \cos a \cos b \mp \sin a \sin b \\
\sin(a \pm b) &= \sin a \cos b \pm \cos a \sin b \\
\cos^2 a &= \frac{1 + \cos(2a)}{2} \\
\sin^2 a &= \frac{1 - \cos(2a)}{2} \\
\sin^2 a + \cos^2 a &= 1 \\
1 + \cos a &= 2\cos^2(a/2) \\
1 - \cos a &= 2\sin^2(a/2) \qquad (18.11)
\end{aligned}$$

## 18.3 Common Hyperbolic Functions

$$\begin{aligned}
\cosh a &= \frac{e^a + e^{-a}}{2} \\
\sinh a &= \frac{e^a - e^{-a}}{2} \\
\cosh(a \pm b) &= \cosh a \cosh b \pm \sinh a \sinh b \\
\sinh(a \pm b) &= \sinh a \cosh b \pm \cosh a \sinh b \\
\cosh^2 a &= \frac{\cosh(2a) + 1}{2} \\
\sinh^2 a &= \frac{\cosh(2a) - 1}{2} \\
\cosh^2 a - \sinh^2 a &= 1 \\
\cosh a + 1 &= 2\cosh^2(a/2) \\
\cosh a - 1 &= 2\sinh^2(a/2) \qquad (18.12)
\end{aligned}$$

## 18.4 Complex Numbers or Functions

$$
\begin{aligned}
i &= \sqrt{-1} \\
z &= x + iy \\
z^* &= x - iy \\
x &= \mathrm{Re}(z) \\
y &= \mathrm{Im}(z) \\
e^{iy} &= \cos y + i \sin y \\
\cos x &= \frac{e^{ix} + e^{-ix}}{2} \\
\sin x &= \frac{e^{ix} - e^{-ix}}{2i} \\
\cos ix &= \cosh x \\
\sin ix &= i \sinh x \\
\cosh ix &= \cos x \\
\sinh ix &= i \sin x \\
\cos z &= \cos x \cosh y - i \sin x \sinh y \\
\sin z &= \sin x \cosh y + i \cos x \sinh y \\
\cos^2 z + \sin^2 z &= 1 \\
\cosh z &= \cosh x \cos y + i \sinh x \sin y \\
\sinh z &= \sinh x \cos y + i \cosh x \sin y \\
\cosh^2 z - \sinh^2 z &= 1
\end{aligned} \tag{18.13}
$$

where $x$ and $y$ are real.

## 18.5 3D Differential Operators in a Cartesian Coordinate System

$$
\begin{aligned}
(\hat{\alpha}, \hat{\beta}, \hat{\gamma}) &= (\hat{x}, \hat{y}, \hat{z}) \\
\vec{\nabla}\psi &= \hat{x}\frac{\partial \psi}{\partial x} + \hat{y}\frac{\partial \psi}{\partial y} + \hat{z}\frac{\partial \psi}{\partial z} \\
\vec{\nabla} \cdot \vec{F} &= \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \\
\vec{\nabla} \times \vec{F} &= \hat{x}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \hat{y}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \hat{z}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) \\
\vec{\nabla} \cdot \vec{\nabla}\psi = \nabla^2\psi &= \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} \qquad (18.14)
\end{aligned}
$$

where $\psi$ is an arbitrary scalar function, and $\vec{F}$ is an arbitrary vector function of $(x, y, z)$.

## 18.6 3D Differential Operators in a Cylindrical Coordinate System

$$
\begin{aligned}
(\hat{\alpha}, \hat{\beta}, \hat{\gamma}) &= (\hat{\rho}, \hat{\phi}, \hat{z}) \\
\vec{\nabla}\psi &= \hat{\rho}\frac{\partial \psi}{\partial \rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial \psi}{\partial \phi} + \hat{z}\frac{\partial \psi}{\partial z} \\
\vec{\nabla} \cdot \vec{F} &= \frac{1}{\rho}\frac{\partial(\rho F_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z} \\
\vec{\nabla} \times \vec{F} &= \hat{\rho}\left(\frac{1}{\rho}\frac{\partial F_z}{\partial \phi} - \frac{\partial F_\phi}{\partial z}\right) + \hat{\phi}\left(\frac{\partial F_\rho}{\partial z} - \frac{\partial F_z}{\partial \rho}\right) + \hat{z}\frac{1}{\rho}\left(\frac{\partial(\rho F_\phi)}{\partial \rho} - \frac{\partial F_\rho}{\partial \phi}\right) \\
\nabla^2\psi &= \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial \psi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2} \qquad (18.15)
\end{aligned}
$$

## 18.7 3D Differential Operators in a Spherical Coordinate System

$$\begin{aligned}
(\hat{\alpha}, \hat{\beta}, \hat{\gamma}) &= (\hat{r}, \hat{\theta}, \hat{\phi}) \\
\vec{\nabla}\psi &= \hat{r}\frac{\partial \psi}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial \psi}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial \psi}{\partial \phi} \\
\vec{\nabla}\cdot\vec{F} &= \frac{1}{r^2}\frac{\partial(r^2 F_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta F_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi} \\
\vec{\nabla}\times\vec{F} &= \hat{r}\frac{1}{r\sin\theta}\left(\frac{\partial(\sin\theta F_\phi)}{\partial\theta} - \frac{\partial F_\theta}{\partial\phi}\right) + \hat{\theta}\left(\frac{1}{r\sin\theta}\frac{\partial F_r}{\partial\phi} - \frac{1}{r}\frac{\partial(rF_\phi)}{\partial r}\right) + \hat{\phi}\frac{1}{r}\left(\frac{\partial(rF_\theta)}{\partial r} - \frac{\partial F_r}{\partial\theta}\right) \\
\nabla^2\psi &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{\mathcal{L}^2(\theta,\phi)\psi}{r^2} \\
\nabla^2\psi &= \frac{1}{r}\frac{\partial^2(r\psi)}{\partial r^2} + \frac{\mathcal{L}^2(\theta,\phi)\psi}{r^2} \quad \text{(alternative form)} \\
\mathcal{L}^2(\theta,\phi)\psi &\equiv \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}
\end{aligned} \tag{18.16}$$

## 18.8 Dirac, Kronecker Deltas, Heaviside Step-Function

$$\begin{aligned}
\int_{x_1}^{x_2} \mathrm{d}x\,\delta(x)f(x) &= f(x_0),\ \text{if}\ x_1 \le x_0 \le x_2\ \text{(Dirac's delta function)} \\
&= 0,\ \text{if}\ x_0 < x_1,\ \text{or},\ x_0 > x_2 \\
\delta_{mn} &= 1,\ \text{if}\ m = n\ \text{(Kronecker's delta)} \\
&= 0,\ \text{if}\ m \ne n \\
\theta(x) &= 1,\ \text{if}\ x \ge 0\ \text{(Heaviside's step function)} \\
&= 0,\ \text{if}\ x < 0
\end{aligned} \tag{18.17}$$

## 18.9 Taylor/MacLaurin Series

Let $f(x)$ be a function of a single variable that is infinitely differentiable. We define $f^{(k)}(a)$ to be the $k^{\text{th}}$ derivative of $f(x)$, evaluated at $x = a$. That is,

$$
\begin{aligned}
f^{(k)}(a) &= \left(\frac{\mathrm{d}^k f(x)}{\mathrm{d}x^k}\right)_{x=a} \\
f^{(0)}(a) &= f(a) \\
f^{(1)}(a) &= f'(a) \\
f^{(2)}(a) &= f''(a) \\
. & \\
. & \\
etc. &
\end{aligned} \tag{18.18}
$$

The *Taylor Series* of $f(x)$ in the vicinity of $x = a$ is

$$
f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k \ . \tag{18.19}
$$

When $a = 0$, the series is known at the *MacLaurin Series*:

$$
f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k \ . \tag{18.20}
$$

**Examples:**

$$
\begin{aligned}
\log(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \cdots \\
e^x &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} \cdots \\
A + Bx + Cx^2 &= A + Ba + Ca^2 + (B + 2Ca)(x-a) + C(x-a)^2 \\
(1-\beta^2)^{-1/2} &= 1 + \frac{\beta^2}{2} + \frac{3}{8}\beta^4 \cdots
\end{aligned} \tag{18.21}
$$

# 18.10 Orthogonal functions

## 18.10.1 Legendre Polynomials

The Legendre Polynomials were first discovered (in 1782!) from the Taylor expansion of a $1/r$ potential:

$$\frac{1}{|\vec{x} - \vec{x}\,'|} = \frac{1}{\sqrt{r^2 + r'^{\,2} - 2rr'\cos\theta}} = \sum_{l=0}^{\infty} \frac{r'^{\,l}}{r^{l+1}} P_l(\cos\theta) \ , \tag{18.22}$$

where $\theta$ is the opening angle between $vecx$ and $\vec{x}\,'$.

With the convention

$$P_l(1) = 1 \tag{18.23}$$

$$\int_0^{\pi} \mathrm{d}\theta \sin\theta P_l(\cos\theta) P_{l'}(\cos\theta) = \int_{-1}^{1} \mathrm{d}z P_l(z) P_{l'}(z) = \frac{2}{2l+1}\delta_{ll'} \qquad \text{(with substition } z = \cos\theta\text{)} \tag{18.24}$$