	WELCOME TO MAT 342 **Applied Complex Analysis** **Spring 2016**

Class Tuesdays and Thursdays, 1:00pm - 2:20pm in Mathematics, Room P-131

Introduction: This is a mathematically rigorous course and most statements will come with complete proofs. Topics covered will include properties of complex numbers, analytic functions with examples, contour integrals, the Cauchy integral formula, the fundamental theorem of algebra, power series and Laurent series, residues and poles with applications, conformal mappings with applications and other topics if time permits.

Text Book: Complex Variables and Applications by James Ward Brown and Ruel V. Churchill, ninth edition, McGraw-Hill, 2013

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Office hours:	Tuesday, Wednesday, Thursday, 10:00am - 11:00am or by appointment

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Grader:	Zili Zhang Math Tower 2-116 E-mail: *zhangzili@math.stonybrook.edu*
Office hours:	Monday 2:00pm - 3:00pm
Math learning center hours:	Tu 4:00pm - 6:00pm (Math Tower Room S-240A)

Grading Policy: The overall numerical grade will be computed by the formula **20% Homework + 30 % Midterm Exam+ 50% Final Exam**

Homework: Homework will be assigned every week. Doing the homework is a *fundamental* part of the course work. Although you might discuss and work together, the solutions need to be handed in alone. You will have one week (Thursdays to Thursdays) to solve the problems. The tasks for the next week will be posted here Thursdays morning at the latest.

Please note: The page numbering in the Stony Brook version of the textbook is not identical to the original! The original page numbers can still be found; They are a bit to the bottom left of the new page numbering (Of course not on the first page of each chapter).

1st assignment: Page 5, prob. 2,5,6,11; Page 8, prob. 8; Page 13, prob 3,4,5; Page 16, prob 2,7; Due Feb. 4

2nd assignment: Page 23, prob. 1; Page 24, prob. 9; Page 30, prob 1,2; Page 31, prob 4,7; Page 34, prob 1,2,3; Page 35, prob 6; Due Feb. 11

3rd assignment: Read section 14 ("The mapping $w=z^2$") thoroughly (pages 40-43); Page 43, prob. 2,3; Page 54, prob. 1,5; Page 55, prob 10,11,13; Stereographic projection; Due Feb. 18
4th assignment: Page 61, prob. 2,3,4; Page 70, prob. 1ad,2ab; Page 71, prob 3,5,6,7; Due Feb. 25
5th assignment: Page 76, prob. 1,2,4; Learn for the Midterm Exam!; Due Mar. 3
6th assignment: Page 84, prob. 1,2; Page 85, prob. 5; Page 89, prob. 4; Page 90, prob. 6,7,10,11,12,13; Due Mar. 10
7th assignment: Page 95, prob. 3,4; Page 96, prob. 5; Page 97, prob. 10; Page 99, prob. 1,3; Page 103, prob. 3,4,8,9; Due Mar. 24
8th assignment: Page 108, prob. 14; Page 111, prob. 6; Page 112, prob. 16; Page 114, prob. 1; Page 119, prob. 1,2,3; Page 123, prob. 1; Page 124, prob. 5,6; Due Mar. 31
9th assignment: Page 133, prob. 4,8,9; Page 135, prob. 13; Page 139, prob. 3; Page 140, prob. 8; Page 147, prob. 2; Page 159, prob. 1,4,5; Due Apr. 7
10th assignment: Page 170, prob. 1abc,4; Page 172, prob. 10; Page 177, prob. 1,4; Page 178, prob. 5,7,8; Due Apr. 14
11th assignment: Page 185, prob. 1,2,7; Page 196, prob. 3,6,8,9; Page 197, prob. 11; Page 205, prob. 1,2; Due Apr. 21
12th assignment: Page 206, prob. 6; Page 219, prob. 4,6,7; Page 220, prob. 8; Page 221, prob. 11; uniform convergence; Due Apr. 28
13th assignment: Page 254, prob. 5,6; Page 264, prob. 1; Page 265, prob. 4; Page 273, prob. 2; Due May 5;

Supplement: Since we didn't finish the last example in the lecture, I wrote it down in details. You can download it here. If you find any typos, errors etc., please let me know.

Review Session: Here are the slides of the review session.

Midterm Exam: Thursday, March 3, 1:00pm-2:20pm in Mathematics, Room P-131

Here is a test Midterm Exam. Here are solutions for the test Midterm Exam. If you find any typos, errors etc., please let me know.

Reminder The Midterm Exam will take place in the usual classroom during lecture time. You will have 80 minutes to work on the exam. I will provide paper for solutions. You only need to bring a pen. Make sure that it is a permanent one (i.e. no pencils or other easily erased pens)! Please be on time. If possible, be in the room several minutes before 1pm. This will make sure that we can start on time. Of course, you are not allowed to use cell phones, your textbook or other notes, or calculators etc. during the exam.

Here is the Midterm Exam. Here are solutions for the Midterm Exam. If you find any typos, errors etc., please let me know.

Final Exam: Monday, May 16, 5:30pm-8:00pm, Mathematics, Room P-131

Here is a test Final Exam. If you have any questions, please do not hesitate to ask! The Final Exam will have the same format and the same number of tasks.

Remark: There was a typo in task 5. It should have been "Prove that there exists some complex number z such that f(z)=z_0". I changed it in the file.

Here are solutions for the test Final Exam. As always: If you find any typos, errors etc., please let me know.

N. B. Use of calculators is not permitted in any of the examiniations.

Disability Support Services (DSS)

If you have a physical, psychological, medical or learning disability that may impact your course work, please contact Disability Support Services, ECC (Educational Communications Center) Building, room 128, (631) 632-6748. They will determine with you what accommodations, if any, are necessary and appropriate. All information and documentation is confidential. Students who require assistance during emergency evacuation are encouraged to discuss their needs with their professors and Disability Support Services. For procedures and information go to the following website: http://www.stonybrook.edu/ehs/fire/disabilities

Academic Integrity

Representing another person's work as your own is always wrong. Faculty are required to report any suspected instances of academic dishonesty to the Academic Judiciary. Faculty in the Health Sciences Center (School of Health Technology & Management, Nursing, Social Welfare, Dental Medicine) and School of Medicine are required to follow their school-specific procedures. For more comprehensive information on academic integrity, including categories of academic dishonesty, please refer to the academic judiciary website at http://www.stonybrook.edu/commcms/academic_integrity/index.html

Critical Incident Management

Stony Brook University expects students to respect the rights, privileges, and property of other people. Faculty are required to report to the Office of Judicial Affairs any disruptive behavior that interrupts their ability to teach, compromises the safety of the learning environment, or inhibits students' ability to learn. Faculty in the HSC Schools and the School of Medicine are required to follow their school-specific procedures.

Task (**Stereographic projection**). *Find an explicit formula for the stereographic projection discussed in the lecture. In other words: Let S^2 be the unit sphere in R^3, i.e.*

$$S^2 = \left\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\right\}$$

and let C be the x-y-plane in $\mathbb{R}^3$, i.e.

$$C = \left\{(x, y, z) \in \mathbb{R}^3 \mid z = 0\right\}.$$

(Note that C and the complex plane $\mathbb{C}$ can easily be identified by mapping a point $(x, y, 0) \in C$ onto $x + iy \in \mathbb{C}$.)

Let $\varphi : S^2 \to C \cup \{\infty\}$ be the stereographic projection defined in the lecture. Find explicit formulae for both φ and $\varphi^{-1} : C \cup \{\infty\} \to S^2$.

Remark: By definition, $\varphi((0, 0, 1)) = \infty$ and $\varphi^{-1}(\infty) = (0, 0, 1)$. Hence, the question is, how does φ map a point $(x, y, z) \in S^2 \setminus \{(0, 0, 1)\}$? Note that $z \neq 1$ for all these points!

Task 1 (**Uniform convergence of a sequence of real functions**). *For $n \in \mathbb{N}$ with $n \geq 3$ define $f_n : [0,1] \to \mathbb{R}$ by*

$$f_n(x) = \begin{cases} nx & \text{if } 0 \leq x < \frac{1}{n} \\ 2 - nx & \text{if } \frac{1}{n} \leq x < \frac{2}{n} \\ 0 & \text{if } \frac{2}{n} \leq x \leq 1 \end{cases}.$$

Prove that $(f_n)_{n \in \mathbb{N}_{\geq 3}}$ converges pointwise but not uniformly to $f : [0,1] \to \mathbb{R}$, $f(x) = 0$.

Task 2 (**Uniform convergence of power series**). *Let $S(z) = \sum_{n=0}^{\infty} z^n$. As seen in the lecture, the radius of convergence of this power series is $R = 1$. Furthermore, we know that $S(z)$ converges absolutely for any $z \in \mathbb{D} = \{z \in \mathbb{C} \ : \ |z| < 1\}$ and that S is uniformly convergent on $\overline{B_r(0)} = \{z \in \mathbb{C} \ : \ |z| \leq r\}$ for any $0 < r < 1$.*

Prove that S is not uniformly convergent on the whole unit disk $\mathbb{D}$.

We did not have the time to finish the second example of how to compute certain improper integrals using residues. Since I think that this is a very important application of complex analysis, I decided to write down the example in detail. Although the example can be found in the textbook, I think it is advantageous to see the same example written out differently.

Before we start to discuss the example, let me again explain the idea. We want to compute improper integrals of the form

$$\int_{-\infty}^{\infty} f(x)\sin(ax)dx \quad \text{or} \quad \int_{-\infty}^{\infty} f(x)\cos(ax)dx$$

or only integrals of such functions starting at 0. Thereby, $a > 0$ is a positive real number. Again, we assume that $f = \frac{p}{q}$ is a real rational function, i.e. we have two polynomials p and q whose coefficients are all real numbers and who do not share a common factor (which just means that p and q do not share a common zero). Furthermore, we assume that q has no real zero but at least one zero in the upper half plane H, which is the set

$$H = \{z \in \mathbb{C} \mid \text{Im } z > 0\}.$$

As in the example with only a rational function, we denote all the zeros of q which lie in H by $z_1, \dots, z_n$ and chose $R > 0$ so large that $|z_j| < R$ for $j = 1, \dots, n$. Let C_R denote the contour which is given by walking along the part of the circle of radius R about the origin in $\overline{H}$ which starts in R and ends in $-R$. Although the parametrisation of this contour is irrelevant for our application, we can always define $C_R : [0, \pi] \to \mathbb{C}$ by $C_R(t) = Re^{it}$. Let $\gamma : [-R, R] \to \mathbb{C}$ be given by $\gamma(t) = t$. This formula parametrises the real interval $[-R, R]$ in the most simple way, from left to the right. Let $C = \gamma + C_R$. Then C is the contour given by the interval $[-R, R]$ and the semicircle of radius R in the upper half plane H. Note that all the zeros of q which lie in H also lie in the interior of C.

If we were to try to apply exactly the same steps we used to compute improper integrals of rational functions, we will soon run into difficulties. The problem is, that we need to find an upper bound for the modulus of the integral

$$\int_{C_R} f(z)\sin(az)dz \quad \text{or} \quad \int_{C_R} f(z)\cos(az)dz.$$

To be more precise: We need an upper bound which tends to 0 as $R \to \infty$. As seen in the case for rational functions, the function f is usually not a problem, but sin and cos are. Recall that for any complex number $z = x + iy \in \mathbb{C}$

$$|\sin(z)|^2 = \sin(x)^2 + \sinh(y)^2 \quad \text{and} \quad |\cos(z)|^2 = \cos(x)^2 + \sinh(y)^2.$$

Hence, both $|\sin(z)|$ and $|\cos(z)|$ tend to infinity as $y \to \infty$ (or $y \to -\infty$). But we can avoid these complications by using a simple trick:

$$e^{iax} = \cos(ax) + i\sin(ax) \quad \text{for all } x \in \mathbb{R}$$

and thus for any $r > 0$

$$\int_{-r}^{r} f(x)e^{iax}dx = \int_{-r}^{r} f(x)\cos(ax)dx + i\int_{-r}^{r} f(x)\sin(ax)dx.$$

Instead of computing the improper integrals from the beginning directly, we compute

$$\int_{-\infty}^{\infty} f(x)e^{iax}dx$$

and obtain the values of the integrals we started with by just taking the real and imaginary parts of the last integral. The advantage of the last integral is that for a point $z = x+iy \in H$ (i.e. $y > 0$), we have

$$\left|e^{iaz}\right| = \left|e^{ia(x+iy)}\right| = \left|e^{iax-ay}\right| = e^{-ay} \le 1$$

since $a > 0$ and $y > 0$. This means that it suffices again to find an upper bound of $|f(z)|$ on C_R; the function e^{iaz} is not interesting anymore.

Let us finally turn to the example mentioned in the lecture. We want to compute the improper integral

$$\int_{0}^{\infty} \frac{\cos(2x)}{(x^2+4)^2}dx.$$

For this, we define

$$f(z) = \frac{1}{(z^2+4)^2},$$

which is just the function in front of $\cos(ax)$ thought of as being a complex function. The function f is analytic in the whole complex plane with the only exceptions being the zeros of the denominator. It is not hard to see that

$$(z^2+4)^2 = 0 \Leftrightarrow z^2+4 = 0 \Leftrightarrow (z = 2i \text{ or } z = -2i).$$

If we write $f(z) = \frac{p(z)}{q(z)}$, with $p(z) = 1$ and $q(z) = (z^2+4)^2$ for all $z \in \mathbb{C}$, we see that p and q do not share a common factor and that the zeros of q are all not real and that the only zero of q in the upper half plane H is $z_0 = 2i$. Since the function e^{i2z} is entire, the only singular values of $f(z)e^{i2z}$ are the singular values of f, which are just the zeros of q. Hence, the only singular value of $f(z)e^{i2z}$ in H is the point z_0. Let $R > 2$ and define the curves

$$\gamma_R : [-R, R] \to \mathbb{C},\ \gamma(t) = t \quad \text{and} \quad C_R : [0, \pi] \to \mathbb{C},\ C_R(t) = Re^{it}.$$

Let now $C = \gamma_R + C_R$. This is the positively oriented contour which consists of the real interval $[-R, R]$ and the semicircle of radius R in $\overline{H}$. The singular point z_0 lies in the interior of C. Thus, we can apply the residue theorem and see that

$$\int_C f(z)e^{i2z}dz = 2\pi i \operatorname*{Res}_{z=z_0} f(z)e^{i2z}$$

but also

$$\int_C f(z)e^{i2z}dz = \int_{\gamma_R} f(z)e^{i2z}dz + \int_{C_R} f(z)e^{i2z}dz = \int_{-R}^{R} f(x)e^{i2x}dx + \int_{C_R} f(z)e^{i2z}dz.$$

Combined, we get

$$\int_{-R}^{R} f(x)e^{i2x}dx = 2\pi i \operatorname*{Res}_{z=z_0} f(z)e^{i2z} - \int_{C_R} f(z)e^{i2z}dz.$$

Note that this equality holds for any $R > 2$, in particular it still holds if we increase the value of R. It remains to compute the residue above and to show that the last integral vanishes if $R \to \infty$.

Let us start with the residue. We have $q(z) = (z^2+4)^2 = (z-2i)^2(z+2i)^2$. If we write

$$\phi(z) = \frac{e^{i2z}}{(z+2i)^2}$$

we see that this function is analytic in H and thus in particular analytic at $z_0 = 2i$. Furthermore,

$$\phi(z_0) = \phi(2i) = \frac{e^{-4}}{-16} \neq 0.$$

Since the function $f(z)e^{i2z}$ can be written as

$$f(z)e^{i2z} = \frac{\phi(z)}{(z-2i)^2},$$

we see that $z_0 = 2i$ is a pole of order 2 of $f(z)e^{i2z}$. Thus, by a theorem of the lecture, we obtain

$$\operatorname*{Res}_{z=z_0} f(z)e^{i2z} = \frac{\phi^{(2-1)}(z_0)}{(2-1)!} = \phi'(z_0).$$

But

$$\phi'(z) = \frac{2ie^{i2z}(z+2i)^2 - e^{i2z}2(z+2i)}{(z+2i)^4} = \frac{2ie^{i2z}(z+2i) - 2e^{i2z}}{(z+2i)^3}$$

which yields

$$\operatorname*{Res}_{z=z_0} f(z)e^{i2z} = \phi'(z_0) = \phi'(2i) = \frac{-8e^{-4} - 2e^{-4}}{-64i} = \frac{5}{i32e^4}.$$

Thus,

$$\int_{-R}^{R} f(x)e^{i2x}dx = \frac{5\pi}{16e^4} - \int_{C_R} f(z)e^{i2z}dz$$

which implies (by taking real parts on both sides)

$$\int_{-R}^{R} f(x)\cos(2x)dx = \frac{5\pi}{16e^4} - \operatorname{Re}\int_{C_R} f(z)e^{i2z}dz.$$

It remains to show that

$$\operatorname{Re} \int_{C_R} f(z) e^{i2z} dz \to 0 \quad \text{as} \quad R \to \infty.$$

As argued in the beginning, this is not very difficult. For a point $z = x + iy$ on the curve C_R, i.e. $x^2 + y^2 = R^2$ and $y \geq 0$, we have

$$\left| f(z) e^{i2z} \right| = \frac{e^{-ay}}{|(z^2+4)^2|} \leq \frac{1}{(R^2-4)^2}$$

and since the length of C_R is πR this yields

$$\left| \operatorname{Re} \int_{C_R} f(z) e^{i2z} dz \right| \leq \left| \int_{C_R} f(z) e^{i2z} dz \right| \leq \frac{\pi R}{(R^2-4)^2} \to 0 \quad \text{as} \quad R \to \infty.$$

Combined, we obtain

$$P.V. \int_{-\infty}^{\infty} f(x) \cos(2x) dx = \frac{5\pi}{16e^4}$$

and since

$$f(x) \cos(2x) = \frac{\cos(2x)}{(x^2+4)^2}$$

is an even function, we get

$$\int_0^{\infty} \frac{\cos(2x)}{(x^2+4)^2} dx = \frac{1}{2} P.V. \int_{-\infty}^{\infty} f(x) \cos(2x) dx = \frac{5\pi}{32e^4}.$$

As a short remark: The proof above also shows that

$$P.V. \int_{-\infty}^{\infty} \frac{\sin(2x)}{(x^2+4)^2} dx = 0.$$

But to get this result we do not need all the computations above. It suffices to observe that the integrand is an odd function. Note that the proof above does not give the value of

$$\int_0^{\infty} \frac{\sin(2x)}{(x^2+4)^2} dx.$$

Review Session for MAT 342: Applied Complex Analysis

Simon Albrecht

Stony Brook, May 5, 2016

Chapter 1: Complex Numbers

- Definition, Sums, Products
  - $z = x + iy \in \mathbb{C}$, $x, y \in \mathbb{R}$, $x = \operatorname{Re} z$, $y = \operatorname{Im} z$, $i^2 = -1$
  - $z = x + iy, w = a + ib \in \mathbb{C}$
    - $z + w = (x + iy) + (a + ib) = (x + a) + i(y + b)$
    - $zw = (x + iy)(a + ib) = xa - yb + i(xb + ya)$
- Algebraic Properties
  - commutative, associative, distributive laws hold
  - $z + 0 = z$, $z \cdot 1 = z$, each $z \neq 0$ is (uniquely) invertible
  - $zw = 0 \Leftrightarrow (z = 0 \text{ or } w = 0)$
  - in other words: $\mathbb{C}$ is a field
- Vectors, Moduli, Triangle Inequality, Complex Conjugates
  - $z = x + iy \neq 0$ can be thought as being a vector from 0 to (x, y) in the plane
  - $|z| := \sqrt{(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2}$ (euclidean length of vector from 0 to $(\operatorname{Re} z, \operatorname{Im} z)$)
  - $|z + w| \leq |z| + |w|$, $|z + w| \geq ||z| - |w||$
  - $\bar{z} = \operatorname{Re} z - i \operatorname{Im} z$, $|z|^2 = z\bar{z}$

- Exponential Form, Products, Powers
  - For $z \neq 0$: $z = r\,e^{i\theta}$, $r = |z| > 0$, $\theta \in \mathbb{R}$
  - $e^{i\theta} = \cos\theta + i\sin\theta$, thus θ is only unique up to adding a multiple of 2π.
  - $z = r\,e^{i\theta}$, $w = s\,e^{i\phi}$, then $zw = rs\,e^{i(\theta+\phi)}$
  - $z^n = r^n\,e^{in\theta}$
- Arguments, Roots of Complex Numbers
  - $\arg z = \{\theta \in \mathbb{R} \ : \ z = |z|e^{i\theta}\} = \theta_0 + 2\pi n$, $n \in \mathbb{Z}$, θ_0 one possible argument of z
  - $z = re^{i\theta} \neq 0$, all n^{th}-roots of z are $c_k = r^{\frac{1}{n}} \exp\left(i\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right)\right)$, where $k = 0, \ldots, n-1$

- Basic Topology:
  - Neighbourhood, Deleted Neighbourhood
    - $\varepsilon > 0$, ε-neighbourhood of $z \in \mathbb{C}$ is

$$B_\varepsilon(z) = \{w \in \mathbb{C} \ : \ |w - z| < \varepsilon\}$$

    - deleted ε-neighbourhood:

$$\dot{B}_\varepsilon(z) = \{w \in \mathbb{C} \ : \ 0 < |w - z| < \varepsilon\} = B_\varepsilon(z) \setminus \{z\}$$

  - $S \subset \mathbb{C}$, $z \in \mathbb{C}$
    - z interior point of S: $\exists\, \varepsilon > 0$ s.t. $\mathcal{B}_\varepsilon(z) \subset S$
    - z exterior point of S: $\exists\, \varepsilon > 0$ s.t. $\mathcal{B}_\varepsilon(z) \cap S = \emptyset$
    - z boundary point of S: z is neither interior nor exterior point of S, i.e. $\forall\, \varepsilon > 0\, (B_\varepsilon(z) \cap S \neq \emptyset$ and $B_\varepsilon(z) \cap (\mathbb{C} \setminus S) \neq \emptyset)$, boundary of S (denoted ∂S) is the set of all boundary points of S

- Basic Topology (cont.)
  - $S \subset \mathbb{C}$
    - S is open if $\partial S \cap S = \emptyset$, i.e. each $z \in S$ is an interior point of S
    - S is closed if $\partial S \subset S$, i.e. $\mathbb{C} \setminus S$ is open
    - If S is open, S is called connected if any two points $z, w \in S$ can be joined by a polygonal line in S
    - S is a domain, if S is non-empty, open, connected
    - a domain together with some (or all) of its boundary points is a region
    - S is bounded if $S \subset B_R(0)$ for some $R > 0$

Chapter 2: Analytic Functions

- Limits

$$\lim_{z \to z_0} f(z) = w_0 \Leftrightarrow$$

$$\forall \varepsilon > 0 \, \exists \, \delta > 0 \, \forall z \in \mathbb{C} : 0 < |z - z_0| < \delta \Rightarrow |f(z) - w_0| < \varepsilon$$

- Riemann Sphere (Limits involving ∞)
  - Riemann Sphere: Wrap $\mathbb{C}$ on a sphere sitting above the origin. Add ∞ as the north pole
  - $\lim_{z \to z_0} f(z) = \infty$, if $\lim_{z \to z_0} \frac{1}{f(z)} = 0$
  - $\lim_{z \to \infty} f(z) = w_0$, if $\lim_{z \to 0} f\left(\frac{1}{z}\right) = w_0$
  - $\lim_{z \to \infty} f(z) = \infty$, if $\lim_{z \to 0} \frac{1}{f\left(\frac{1}{z}\right)} = 0$
- Continuity, Derivatives
  - $f : D \to \mathbb{C}$ continuous at $z_0 \in D$ if $\lim_{z \to z_0} f(z) = f(z_0)$
  - $f : D \to \mathbb{C}$, $z_0 \in D$ interior point of D, f differentiable at z_0 if

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists

- Cauchy-Riemann Equations (both in rectangular and polar coordinates)
  - $f(z) = u(x,y) + iv(x,y)$: $u_x = v_y$ and $u_y = -v_x$
  - $f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$: $ru_r = v_\theta$ and $u_\theta = -rv_r$
- Analytic Functions
  - f is analytic in an open set S if f is differentiable at every $z \in S$.
  - f is entire if f is analytic in $\mathbb{C}$.
  - f analytic in domain D, $f'(z) = 0$ for all $z \in D$, then f is constant
- Harmonic Functions
  - $u(x,y)$ harmonic in a domain $D \subset \mathbb{R}^2$ if

    $$u_{xx}(x,y) + u_{yy}(x,y) = 0$$

    for all $(x,y) \in D$
  - $f = u + iv$ analytic, then both u and v are harmonic

- Identity Theorem / Coincidence Principle
  - An analytic function in a domain D is uniquely determined by its values in a subdomain or on a line segment contained in D.
  - Most general version: $D \subset \mathbb{C}$ domain, $f, g : D \to \mathbb{C}$ analytic. If $\{z \in D \;:\; f(z) = g(z)\}$ has an accumulation point in D, then $f = g$.
- Reflection Principle: Let D be a domain which contains a segment of the real axis and whose lower half is the reflection of the upper half (i.e. $z \in D$ iff $\bar{z} \in D$). Let f be analytic in D. Then $\overline{f(z)} = f(\bar{z})$ for all $z \in D$ if and only if $f(x)$ is real for each point x on the segment.

Chapter 3: Elementary Functions

- The Exponential Function
  - $e^{x+iy} = e^x e^{iy} = e^x \cos(y) + ie^x \sin(y)$
  - $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$
  - exp is entire and $2\pi i$-periodic
- The Logarithmic Function
  - $\log z = \ln|z| + i \arg z$
  - $e^{\log z} = z$
- Branches and Derivatives of Logarithms
  - $\alpha \in \mathbb{R}$, restrict $\arg z$ so that $\alpha < \arg z < \alpha + 2\pi$, then

    $$\log z = \ln|z| + i\theta \,(|z| > 0, \alpha < \theta < \alpha + 2\pi)$$

    is a branch of the logarithm and analytic in the slit plane $\{re^{i\theta} \; : \; r > 0, \, \alpha < \theta < \alpha + 2\pi\}$ with derivative $\frac{1}{z}$
  - principle branch Log for $\alpha = -\pi$

- Power Functions
  - $z \neq 0$, $c \in \mathbb{C}$: $z^c = e^{c \log z}$
  - Given a branch of log, z^c becomes an analytic function in $\{re^{i\theta} \ : \ r > 0, \alpha < \theta < \alpha + 2\pi\}$ with derivative cz^{c-1}
  - principle branch of z^c: choose Log
- Trigonometric Functions, Hyperbolic Functions
  - $\sin z = \frac{e^{iz} - e^{-iz}}{2i} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$
  - $\cos z = \frac{e^{iz} + e^{-iz}}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$
  - $\sinh z = \frac{e^{z} - e^{-z}}{2} = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$
  - $\cosh z = \frac{e^{z} + e^{-z}}{2} = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$
- Inverse Trigonometric and Hyperbolic Functions

Chapter 4: Integrals

- Derivatives of Functions $w : [a, b] \to \mathbb{C}$, $w(t) = u(t) + iv(t)$

$$w'(t) = u'(t) + iv'(t)$$

- Definite Integrals of such Functions

$$\int_a^b w(t)dt = \int_a^b u(t)dt + i \int_a^b v(t)dt$$

- Contours
  - arc: $\gamma : [a, b] \to \mathbb{C}$ continuous. Also $\{\gamma(t) \ : \ t \in [a, b]\}$ is called arc
  - simple arc (Jordan arc): γ is also injective
  - simple closed curve (or Jordan curve): γ is simple except that $\gamma(a) = \gamma(b)$
  - γ is positively oriented, if it is in counterclockwise direction
  - if γ' exists on $[a, b]$ and is continuous, then *gamma* is called differentiable arc
  - if γ is differentiable and $\gamma'(t) \neq 0$ for all t, then γ is called smooth
  - A contour (or piecewise smooth arc) is an arc consisting of a finite number of smooth arcs joined end to end

- Contour Integral: $C : [a, b] \to \mathbb{C}$ contour, $f(C(t))$ piecewise continuous, then

$$\int_C f(z)dz = \int_a^b f(C(t))C'(t)dt$$

- Upper Bounds for Moduli of Contour Integrals
  - length L of contour $C : [a, b] \to \mathbb{C}$ is $L = L(C) = \int_a^b |C'(t)|dt$.
  - C contour of length L, f piecewise continuous on C with $|f(z)| \leq M$ for all $z \in C$, then

$$\left|\int_C f(z)dz\right| \leq LM$$

- Antiderivatives: f continuous in domain D. Then
  - f has antiderivative F
  - Contour integrals of f along contours lying entirely in D only depend on start and end point
  - contour integrals of f along closed contours lying entirely in D are all 0

  are equivalent

- Cauchy-Goursat Theorem: Let C be a simple closed contour and f analytic on C and inside C. Then
$$\int_C f(z)dz = 0.$$
- Simply and Multiply Connected domains
  - A domain D is simply connected if every simple closed contour lying in D only encloses points of D, i.e. "D has no holes".
  - If f is analytic in a simply connected domain D, then $\int_C f(z)dz = 0$ for every closed contour lying in D.
  - A domain D is multiply connected, if it is not simply connected.
  - C simple closed contour in counterclockwise direction, C_k, $k = 1, \ldots, n$, simple closed contours lying entirely in the interior of C, all in clockwise direction, f analytic on all of these contours and in the multiply connected domain spanned by these curves, then
$$\int_C f(z)dz + \sum_{k=1}^{n} \int_{C_k} f(z)dz = 0$$

- Cauchy Integral Formula: Let f be analytic everywhere on and inside a simple closed contour C, taken in positive sense. If z is any point interior to C, then

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta.$$

- Extended Cauchy Integral Formula: C and z as above, $n \in \mathbb{N}$, then

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta.$$

- Consequences
  - Analytic functions have derivatives of all orders.
  - Morera's theorem: Let f be continuous on a domain D. If $\int_C f(z)dz = 0$ for every closed contour C in D, then f is analytic.
  - Cauchy's inequality: f analytic inside and on a positively oriented circle C_R of radius R centred at z_0, $M_r = \max_{|z-z_0|=R} |f(z)|$, then for all $n \in \mathbb{N}$
$$\left|f^{(n)}(z_0)\right| \leq \frac{n! M_R}{R^n}.$$
- Liouville's Theorem and the Fundamental Theorem of Algebra
  - Liouville's theorem: A bounded entire function is constant.
  - Fundamental Theorem of Algebra: Every non constant complex polynomial has at least one zero.
- Maximum Modulus Principle: If f is analytic and not constant in a given domain D, then $|f(z)|$ has no maximum value in D.

Chapter 5: Series

- Sequences, Series, Convergence
  - sequence $(z_n)_{n\in\mathbb{N}}$ converges to z if

$$\forall \varepsilon > 0 \, \exists n_\varepsilon \in \mathbb{N} \, \forall n \geq n_\varepsilon : \, |z_n - z| < \varepsilon$$

  - series: $\sum_{n=1}^{\infty} z_n$, $S_N = \sum_{n=1}^{N} z_n$. Series converges to S if $(S_N)_{N\in\mathbb{N}}$ converges to S
  - series $\sum_{n=1}^{\infty} z_n$ is absolutely convergent if $\sum_{n=1}^{\infty} |z_n|$ converges
  - power series: $\sum_{n=0}^{\infty} z_n(z - z_0)^n$
- Taylor Series
  - f analytic in disk $B_{R_0}(z_0)$, then

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \text{for all} \quad z \in B_{R_0}(z_0)$$

    where

$$a_n = \frac{f^{(n)}(z_0)}{n!}.$$

  - if $z_0 = 0$, the Taylor Series is called Maclaurin Series

- Laurent Series: f analytic in $\{z \in \mathbb{C} \ : \ R_1 < |z - z_0| < R_2\}$, C simple closed, positively oriented contour in the annulus, then for z in that annulus
$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$
where
$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz \quad \text{and} \quad b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{-n+1}} dz$$

- Absolute and Uniform Convergence of Power Series
  - There exists a largest ball in which a power series converges.
  - If a power series converges at $z_1 \neq z_0$, then it is absolutely convergent for every z with $|z - z_0| < |z_1 - z_0|$.
  - R radius of convergence, $R_1 < R$, then the power series is uniformly convergent on $\{z \in \mathbb{C} \ : \ |z - z_0| \leq R_1\}$.
  - Uniform convergence: The choice of n_ε in the convergence statement does not depend on the point z where convergence is investigated.

- Further Properties of Power Series $S(z) = \sum_{n_0}^{\infty} a_n(z - z_0)^n$
  - Power series represent continuous functions on their disk of convergence.
  - Power series are analytic on their disk of convergence with

$$S'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}.$$

  - The integral of a power series along some contour C inside the disk of convergence is

$$\int_C S(z)dz = \sum_{n=0}^{\infty} a_n \int_C (z - z_0)^n dz.$$

  - Power series representations are unique.

Chapter 6: Residues and Poles

- Isolated Singular Points: A singular point z_0 of an analytic function f is isolated if there exists some $\varepsilon > 0$ such that there is no other singular point in $\dot{B}_\varepsilon(z_0)$.
- Residues: z_0 isolated singular point of an analytic function f,

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

Laurent Series of f in $\dot{B}_\varepsilon(z_0)$. The coefficient b_1 is called residue of f at z_0

$$\operatorname*{Res}_{z=z_0} f(z) = b_1 = \frac{1}{2\pi i} \int_C f(z)dz.$$

- Cauchy's Residue Theorem: Let C be a simple closed contour, described in positive sense. If f is analytic inside and on C except for a finite number of singular points z_k $(k = 1, \ldots, n)$ inside C, then

$$\int_C f(z)dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z).$$

- Residue at infinity

$$\operatorname*{Res}_{z=\infty} f(z) = -\operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right]$$

- Types of Isolated Singular Points: z_0 isolated singular point of f
  - Laurent Series $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$
  - Principle Part of Laurent Series: $\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$
  - Removable: $b_n = 0$ for all $n \in \mathbb{N}$
  - Essential: infinitely many $b_n \neq 0$
  - Pole of Order m: $b_m \neq 0$, $b_n = 0$ for all $n > m$
- Residues at Poles: z_0 pole of order m of f
  - There exists a function ϕ which is analytic at z_0 and $\phi(z_0) \neq 0$ such that

$$f(z) = \frac{\phi(z)}{(z - z_0)^m}.$$

  -

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}.$$

- Residues at Poles (cont.)
  - If $f(z) = \frac{p(z)}{q(z)}$, $p(z_0) \neq 0$, $q(z_0) = 0$, $q'(z_0) \neq 0$, then $m = 1$ and

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{p(z_0)}{q'(z_0)}.$$

- Zeros of Analytic Functions
  - z_0 is a zero of order m of f if

$$f(z_0) = f'(z_0) = \ldots = f^{(m-1)}(z_0) = 0 \quad \text{but} \quad f^{(m)}(z_0) \neq 0.$$

  - There exists a function ϕ which is analytic at z_0 and $\phi(z_0) \neq 0$ such that

$$f(z) = (z - z_0)^m \phi(z).$$

  - Zeros of analytic functions are always isolated by the coincidence principle, unless f is constantly zero.

- Zeros and Poles: Suppose that p and q are analytic at z_0, $p(z_0) \neq 0$, q has a zero of order m at z_0. Then $\frac{p(z)}{q(z)}$ has a pole of order m at z_0.
- Behaviour of Functions new Isolated Singular Points: z_0 isolated singular point of f
  - If z_0 is removable, then f is bounded and analytic in $\dot{B}_\varepsilon(z_0)$ for some $\varepsilon > 0$. Also: If a function f is analytic and bounded in $\dot{B}_\varepsilon(z_0)$, then either f is analytic at z_0 or z_0 is removable.
  - If z_0 is essential, then f assumes values arbitrarily close to any given number in any deleted neighbourhood of z_0 (Casorati-Weierstraß).
  - If z_0 is a pole of order m, then

$$\lim_{z \to z_0} f(z) = \infty.$$

Chapter 7: Applications of Residues

- Evaluation of Improper Integrals
  - If $\int_{-\infty}^{\infty} f(x)dx$ converges, then the Cauchy principle value $P.V. \int_{-\infty}^{\infty} f(x)dx$ exists.
  - The inverse is in general not true! But if f is even $(f(x) = f(-x))$, then the inverse holds.
  - Idea: Assume that $f(x) = \frac{p(x)}{q(x)}$, p and q do not share a common factor, q has no real zero but at least one zero in the upper half plane. Let $z_1, \ldots, z_n$ be the zeros of q in the upper half plane. Choose $R > 0$ so big that $|z_j| < R$ for all j. Let C_R be the semicircle of radius R in the upper half plane taken in positive sense and let C be the contour consisting of the interval $[-R, R]$ and C_R, taken in positive sense. Then

$$\int_{-R}^{R} f(x)dx = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z) - \int_{C_R} f(z)dz.$$

  - If $\lim_{R\to\infty} \int_{C_R} f(z)dz = 0$ and f is even, we are done.

- Improper Integrals from Fourier Analysis
  - Want to compute integrals of the form ($a > 0$)

$$\int_{-\infty}^{\infty} f(x)\cos(ax)dx \quad \text{and} \quad \int_{-\infty}^{\infty} f(x)\sin(ax)dx.$$

  - Caution: Same idea as on previous slide does not work. Both sin and cos are unbounded in the upper half plane!
  - Solution: $e^{iax} = \cos(ax) + i\sin(ax)$. Thus,

$$\int_{-R}^{R} f(x)\cos(ax)dx + i\int_{-R}^{R} f(x)\sin(ax)dx = \int_{-R}^{R} f(x)e^{iax}dx.$$

    Also for $z = x + iy$ in the upper half plane

$$|e^{iaz}| = |e^{iax-ay}| = e^{-ay} \leq 1.$$

    Hence, compute the last integral!

Outlook

- Argument Principle
- Rouché's Theorem
- Conformal Mappings
- Riemann Mapping Theorem

Name: ______________________ 03/03/16

ID: ______________________

MAT 342 Applied Complex Analysis
Spring 2016 Midterm Exam Example

1. a) (4 pts each) Write the following complex numbers in polar form.

   1) $2\sqrt{3}-1+i(2+i^3)$ 2) $1+\cos\left(\frac{2\pi}{3}\right)+\cos\left(\frac{4\pi}{3}\right)+i\left(\sin\left(\frac{2\pi}{3}\right)+\sin\left(\frac{10\pi}{3}\right)\right)$

   b) (4 pts each) Write the following complex numbers in rectangular form.

   1) $\frac{3e^{-i\pi}}{\sqrt{(1-i)(1+i)+2}\ e^{i2\pi}}$ 2) $\frac{-1+i}{\sqrt{3}-i}$

2. a) (10 pts) Prove that for any two complex numbers $z, w \in \mathbb{C}$ the following equality holds:

   $$|z-w|^2+|z+w|^2=2(|z|^2+|w|^2)$$

   b) (5 pts) Why is this equality called *parallelogram identity*?

3. a) (5 pts) Define the notion of an *open set* and the notion of a *closed set*.

   b) (5 pts each) Which of the following sets are open, which are closed? Prove your claim!

   1) $\left\{z=1+re^{i\theta} \mid r>0,\, 0\le\theta\le\frac{\pi}{2}\right\}$ 2) $\{z\in\mathbb{C} \mid (\mathrm{Re}(z))^2=\mathrm{Im}(z)\}$

   3) $\left\{z\in\mathbb{C} \mid \left(\frac{\mathrm{Re}(z)}{4}\right)^2+\left(\frac{\mathrm{Im}(z)}{2}\right)^2<1\right\}$

4. a) (5 pts) Define the notion $\lim\limits_{z\to z_0} f(z)=w_0$.

   b) (5 pts each) Find the following limits if they exist:

   1) $\lim\limits_{z\to\infty}\frac{z^3-2z+1}{\pi z^3+i3z^2}$ 2) $\lim\limits_{z\to 1}\frac{z^2+2z-3}{z^2-3z+2}$

5. a) (5 pts) State the *Cauchy-Riemann equations in polar form* for a function $f(z)=u(r,\theta)+iv(r,\theta)$.

   b) (12 pts) Let $D=\{z=re^{i\theta} \mid r>0, -\pi<\theta<\pi\}$ and let $f: D\to\mathbb{C}$, $f(z)=\sqrt{z}$ where $\sqrt{\cdot}$ denotes the principle branch of the square root. Find all points $z\in\mathbb{C}$ in which f is differentiable.

6. a) (5 pts) Define the notion *analytic function*.

   b) (12 pts) Prove that the following function is analytic:

   $f:\mathbb{D}\to\mathbb{C}$, $f(z)=e^{\pi x}\cos(\pi y)+2xy+i\left(e^{\pi x}\sin(\pi y)-x^2+y^2\right)$

MAT 342 Applied Complex Analysis
Spring 2016 Midterm Exam Example
Solutions

1. a) 1)

$$2\sqrt{3} - 1 + i(2 + i^3) = 2\sqrt{3} - 1 + 2i + i^4 = 2\sqrt{3} + 2i = \sqrt{12 + 4}e^{i\arctan\left(\frac{2}{2\sqrt{3}}\right)}$$
$$= 4e^{i\arctan\left(\frac{1}{\sqrt{3}}\right)} = 4e^{i\frac{\pi}{6}}$$

2)

$$1 + \cos\left(\frac{2\pi}{3}\right) + \cos\left(\frac{4\pi}{3}\right) + i\left(\sin\left(\frac{2\pi}{3}\right) + \sin\left(\frac{10\pi}{3}\right)\right)$$
$$= 1 + \left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right) + \left(\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3} + 2\pi\right)\right)$$
$$= 1 + e^{i\frac{2\pi}{3}} + e^{i\frac{4\pi}{3}} = 1 + e^{i\frac{2\pi}{3}} + \left(e^{i\frac{2\pi}{3}}\right)^2$$
$$= \frac{\left(e^{i\frac{2\pi}{3}}\right)^3 - 1}{e^{i\frac{2\pi}{3}} - 1} = \frac{0}{e^{i\frac{2\pi}{3}} - 1} = 0$$

b) 1)

$$\frac{3e^{-i\pi}}{\sqrt{(1-i)(1+i) + 2\ e^{i2\pi}}} = \frac{3e^{-i\pi}}{\sqrt{1+1+2}} = \frac{3}{2}e^{-i\pi} = -\frac{3}{2}$$

2)

$$\frac{-1+i}{\sqrt{3}-i} = \frac{(-1+i)(\sqrt{3}+i)}{(\sqrt{3}-i)(\sqrt{3}+i)} = \frac{-\sqrt{3} - i + i\sqrt{3} - 1}{4} = \frac{-1-\sqrt{3}}{4} + i\frac{\sqrt{3}-1}{4}$$

2. a) Let $z = x + iy$ and $w = a + ib$ be complex numbers. We then have

$$\begin{aligned} |z+w|^2 + |z-w|^2 &= |(x+a) + i(y+b)|^2 + |(x-a) + i(y-b)|^2 \\ &= (x+a)^2 + (y+b)^2 + (x-a)^2 + (y-b)^2 \\ &= x^2 + 2xa + a^2 + y^2 + 2yb + b^2 \\ &\quad + x^2 - 2xa + a^2 + y^2 - 2yb + b^2 \\ &= 2(x^2 + y^2 + a^2 + b^2) = 2(|z|^2 + |w|^2) \end{aligned}$$

b) Geometrically, the vectors $z + w$ and $z - w$ are the diagonals of a parallelogram spanned by the vectors z and w. The equation above now states the well-known fact that the sum of the squares of the lengths of the two diagonals equals the sum of the squares of the lengths of the four sides of the parallelogram.

3. a) A set S of complex numbers is called open if it does not contain any of its boundary points. (Equivalently, every point of S is an interior point of S.)

A set S of complex numbers is called closed if it contains all of its boundary points. (Equivalently, the complement $\mathbb{C} \setminus S$ of S in $\mathbb{C}$ is open.)

b) 1) The set $M_1 = \left\{z = 1 + re^{i\theta} \mid r > 0,\, 0 \leq \theta \leq \frac{\pi}{2}\right\}$ is neither open nor closed: The point $z = 2 = 1 + 1e^{i0}$ lies in M_1. For any $\varepsilon > 0$, the point $z - i\frac{\varepsilon}{2}$ lies not in M_1 since $\operatorname{Arg}(z - i\frac{\varepsilon}{2}) < 0$ but it lies in the ε-neighbourhood of z. Hence, z is a boundary point of M_1 which lies in M_1. Thus, M_1 is not open.

We have $1 \notin M_1$, but for any $\varepsilon > 0$ the point $1 + \frac{\varepsilon}{2} = 1 + \frac{\varepsilon}{2}e^{i0}$ lies in M_1 Hence, 1 is a boundary point of M_1 which does not lie in M_1. This implies that M_1 is not closed.

2) The set $M_2 = \{z \in \mathbb{C} \mid (\operatorname{Re}(z))^2 = \operatorname{Im}(z)\}$ is not open but closed. Let $z \in M_2$, i.e. $z = x + ix^2$ for some $x \in \mathbb{R}$. Let $\varepsilon > 0$ and let $w = z + \frac{\varepsilon}{2} = x + i(x^2 + \frac{\varepsilon}{2})$. Then, $w \notin M_2$ but $w \in B_\varepsilon(z)$. Hence, z is a boundary point of M_2 which implies that M_2 is not open.

To show that M_2 is closed, we will show that any sequence in M_2 which converges in $\mathbb{C}$ in fact converges in M_2. It is easy to see that this is equivalent to the statement that M_2 is closed. Let $(z_n)_{n\in\mathbb{N}}$ be a sequence in M_2. Assume that $z_n \to z = x + iy \in \mathbb{C}$ as $n \to \infty$. Let $z_n = x_n + iy_n$. Then, both $x_n \to x$ and $y_n \to y$ as $n \to \infty$. We have to show that $y = x^2$. Since $z_n \in M_2$, we have $y_n = (x_n)^2$ for all $n \in \mathbb{N}$. Because the function $f : \mathbb{R} \to \mathbb{R}$, $f(t) = t^2$, is continuous, we have $f(x_n) \to f(x) = x^2$ as $n \to \infty$ but also $f(x_n) = (x_n)^2 = y_n \to y$ as $n \to \infty$. Hence, $y = x^2$ which implies that $z \in M_2$. Thus, M_2 is closed.

3) The set $M_3 = \left\{z \in \mathbb{C} \mid \left(\frac{\operatorname{Re}(z)}{4}\right)^2 + \left(\frac{\operatorname{Im}(z)}{2}\right)^2 < 1\right\}$ is open but not closed.

We first prove, that M_3 is open. Let $z = x + iy \in M_3$. Then, $\frac{x^2}{16} + \frac{y^2}{4} < 1$. Let $t = 1 - \frac{x^2}{16} - \frac{y^2}{4}$. Then $t > 0$. Let $\varepsilon = \min\{1, \frac{8t}{2|x|+8|y|+5}\}$. We then have $1 \geq \varepsilon > 0$. Let $w = a + ib \in B_\varepsilon(z)$. Thus, $|x - a|, |b - y| \leq |z - w| < \varepsilon$.

$$
\begin{aligned}
\frac{a^2}{16} + \frac{b^2}{4} &= \frac{(a-x+x)^2}{16} + \frac{(b-y+y)^2}{4} \\
&= \frac{x^2}{16} + \frac{y^2}{4} + \frac{(a-x)^2 + 2x(a-x)}{16} + \frac{(b-y)^2 + 2y(b-y)}{4} \\
&\leq 1 - t + \frac{|a-x|^2 + 2|x||a-x|}{16} + \frac{|b-y|^2 + 2|y||b-y|}{4} \\
&\leq 1 - t + \frac{\varepsilon^2 + 2|x|\varepsilon + 4\varepsilon^2 + 8|y|\varepsilon}{16} \\
&\overset{\varepsilon\leq 1}{\leq} 1 - t + \varepsilon\frac{5 + 2|x| + 8|y|}{16} \leq 1 - t + \frac{t}{2} = 1 - \frac{t}{2} < 1
\end{aligned}
$$

Hence, $w \in M_3$ which implies that M_3 is open.

To see that M_3 is not closed, consider the point $z = x + iy = 4$. Then, $z \notin M_3$ since $\left(\frac{4}{4}\right)^2 + \left(\frac{0}{2}\right)^2 = 1$. Let $\delta > 0$. If $\delta > 4$, then $0 \in B_\delta(z) \cap M_3$. If $\delta < 4$, then $z - \frac{\delta}{2} \in B_\delta(z) \cap M_3$. Hence, z is a boundary point of M_3 which does not lie in M_3. Thus, M_3 is not closed.

4. a) Let $z_0, w_0 \in \mathbb{C}$ and let f be a function. The notion $\lim\limits_{z\to z_0} f(z) = w_0$ means that for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(z) - w_0| < \varepsilon$ whenever $|z - z_0| < \delta$.

   b) 1) By a theorem from the lecture, we need to check whether $\lim\limits_{z\to 0} \frac{\frac{1}{z^3} - \frac{2}{z} + 1}{\frac{\pi}{z^3} + i\frac{3}{z^2}}$ exists. For $z \neq 0$, we have

$$\frac{\frac{1}{z^3} - \frac{2}{z} + 1}{\frac{\pi}{z^3} + i\frac{3}{z^2}} = \frac{1 - 2z^2 + z^3}{\pi + i3z}.$$

Since $\lim\limits_{z\to 0}(1 - 2z^2 + z^3) = 1$ and $\lim\limits_{z\to 0}(\pi + i3z) = \pi \neq 0$, we know that the limit exists with $\lim\limits_{z\to 0} \frac{\frac{1}{z^3} - \frac{2}{z} + 1}{\frac{\pi}{z^3} + i\frac{3}{z^2}} = \frac{1}{\pi}$. Hence, $\lim\limits_{z\to\infty} \frac{z^3 - 2z^2 + 1}{\pi z^3 + i3z^2} = \frac{1}{\pi}$.

   2) We have

$$\frac{z^2 + 2z - 3}{z^2 - 3z + 2} = \frac{(z-1)(z+3)}{(z-1)(z-2)} = \frac{z+3}{z-2}.$$

Hence,

$$\lim_{z\to 1} \frac{z^2 + 2z - 3}{z^2 - 3z + 2} = \lim_{z\to 1} \frac{z+3}{z-2} = \frac{4}{-1} = -4.$$

5. a) Let $f(z) = u(r, \theta) + iv(r, \theta)$. The Cauchy-Riemann equations are fulfilled in $z_0 = r_0 e^{i\theta_0}$ if

$$ru_r(r_0, \theta_0) = v_\theta(r_0, \theta_0) \quad \text{and} \quad u_\theta(r_0, \theta_0) = -rv_r(r_0, \theta_0).$$

   b) Since $\sqrt{\cdot}$ is the principle branch of the squareroot function, we have for every $z = re^{i\theta} \in D$

$$f(z) = \sqrt{r} e^{i\frac{\theta}{2}} = \sqrt{r} \cos\left(\frac{\theta}{2}\right) + i\sqrt{r} \sin\left(\frac{\theta}{2}\right) = u(r, \theta) + iv(r, \theta).$$

Hence, the first order partial derivatives of u and v with respect to r and θ exist and we have

$$\begin{aligned} u_r(r_\theta) &= \tfrac{1}{2\sqrt{r}} \cos\left(\tfrac{\theta}{2}\right) & u_\theta(r, \theta) &= \tfrac{-\sqrt{r}}{2} \sin\left(\tfrac{\theta}{2}\right) \\ v_r(r_\theta) &= \tfrac{1}{2\sqrt{r}} \sin\left(\tfrac{\theta}{2}\right) & v_\theta(r, \theta) &= \tfrac{\sqrt{r}}{2} \cos\left(\tfrac{\theta}{2}\right). \end{aligned}$$

Hence,

$$
\begin{aligned}
ru_r(r,\theta) &= \frac{\sqrt{r}}{2}\cos\left(\frac{\theta}{2}\right) = v_\theta(r,\theta)\\
rv_r(r,\theta) &= \frac{\sqrt{r}}{2}\sin\left(\frac{\theta}{2}\right) = -u_\theta(r,\theta).
\end{aligned}
$$

Thus, the Cauch-Riemann equations are fulfilled throughout D. Since the first order partial derivatives of u and v are continuous throughout D, f' exists everywhere in D.

6. a) A function f defined on an open set S is called analytic, if it is differentiable at any point $z \in S$.

   b) For $z = x + iy \in \mathbb{D}$, we have

$$
u(x,y) = e^{\pi x}\cos(\pi y) + 2xy \quad \text{and} \quad v(x,y) = e^{\pi x}\sin(\pi y) - x^2 + y^2.
$$

Thus, the first order partial derivates of u and v exist and

$$
\begin{aligned}
u_x(x,y) &= \pi e^{\pi x}\cos(\pi y) + 2y = v_y(x,y)\\
u_y(x,y) &= -\pi e^{\pi x}\sin(\pi y) + 2x = -v_x(x,y).
\end{aligned}
$$

Hence, the Cauchy-Riemann equations are fulfilled throughout $\mathbb{D}$. Since the first order partial derivatives of u and v are continuous throughout $\mathbb{D}$, the function f is differentiable throughout $\mathbb{D}$, hence analytic.

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MAT 342 Applied Complex Analysis
Spring 2016
Midterm Exam

1. a) (4 pts each) Write the following complex numbers in polar form.

   1) $(1+i)^4$ 2) $\frac{1+i\sqrt{3}}{1-i\sqrt{3}}$

   b) (4 pts each) Write the following complex numbers in rectangular form.

   1) $2^{\frac{3}{2}}e^{i\frac{\pi}{4}}$ 2) $\frac{27e^{i\frac{3\pi}{4}}}{\frac{3}{\sqrt{2}}+i3\sin\left(\frac{\pi}{4}\right)}$

2. a) (5 pts) Define the *absolute value* $|z|$ of a complex number $z = x + iy$.

   b) (5 pts) Find and sketch the set M of all $z \in \mathbb{C}$ such that

   $$5|z|^2 + 4\mathrm{Re}(z^2) \leq 2\mathrm{Im}(\bar{z}) + 8.$$

   c) (5 pts) Find and sketch the set N of all $z \in \mathbb{C}$ such that $\frac{|z-1|}{|z+1|} = 3$.

3. a) (5 pts) Define the notion *domain*.

   b) (5 pts each) Which of the following sets are domains? Prove your claim!

   1) $\{z \in \mathbb{C} \mid |z| < 1\}$ 2) $\left\{z \in \mathbb{C} \mid \exists n \in \mathbb{N}\ z = \frac{1}{n} + i\right\}$

   3) $\left\{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0,\ \mathrm{Im}(z) > \frac{1}{\mathrm{Re}(z)}\right\}$

4. a) (5 pts) Define the *ε-neighbourhood* of a point $z_0 \in \mathbb{C}$ and the *ε-neighbourhood* of ∞ for some $\varepsilon > 0$.

   b) (5 pts each) Find the following limits if they exist:

   1) $\lim\limits_{z\to\infty} \frac{4z^2}{(z-1)^2}$ 2) $\lim\limits_{z\to 1} \frac{e^z - e}{z-1}$

5. a) (5 pts) Define the derivative $f'(z_0)$ of a function f at a point $z_0 \in \mathbb{C}$.

   b) (12 pts) Let $f(z) = |z|^4$. Find all points $z \in \mathbb{C}$ in which f is differentiable.

6. a) (5 pts) State the *Cauchy-Riemann equations in rectangular form* for a function $f(z) = u(x, y) + iv(x, y)$.

   b) (12 pts) Prove that $f(z) = x^3 - 3xy^2 + e^x\cos(y) + i\left(3x^2y - y^3 + e^x\sin(y)\right)$ is entire.

MAT 342 Applied Complex Analysis
Spring 2016 Midterm Exam
Solutions

1. a) 1) $(1+i)^4 = \left(\sqrt{2}e^{i\frac{\pi}{4}}\right)^4 = 4e^{i\pi}$.

   2)

$$\frac{1+i\sqrt{3}}{1-i\sqrt{3}} = \frac{(1+i\sqrt{3})^2}{(1-i\sqrt{3})(1+i\sqrt{3})} = \frac{1+i2\sqrt{3}-3}{4} = -\frac{1}{2} + i\frac{\sqrt{3}}{2} = e^{i\left(\pi+\arctan\left(\frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}}\right)\right)}$$
$$= e^{i\left(\pi+\arctan\left(-\sqrt{3}\right)\right)} = e^{i\left(\pi-\frac{\pi}{3}\right)} = e^{i\frac{2\pi}{3}}$$

   b) 1) $2^{\frac{3}{2}}e^{i\frac{\pi}{4}} = 2\sqrt{2}\cos\left(\frac{\pi}{4}\right) + i2\sqrt{2}\sin\left(\frac{\pi}{4}\right) = 2+2i$

   2) $\frac{27e^{i\frac{3\pi}{4}}}{\frac{3}{\sqrt{2}}+i3\sin\left(\frac{\pi}{4}\right)} = \frac{27e^{i\frac{3\pi}{4}}}{3\cos\left(\frac{\pi}{4}\right)+i3\sin\left(\frac{\pi}{4}\right)} = 9\frac{e^{i\frac{3\pi}{4}}}{e^{i\frac{\pi}{4}}} = 9e^{i\frac{\pi}{2}} = 9i$

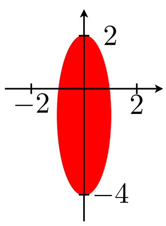
2. a) The absolute value $|z|$ of a complex number $z = x+iy$ is defined by

$$|z| = \sqrt{x^2+y^2}.$$

   b) Let $M = \left\{z \in \mathbb{C} \mid 5|z|^2 + 4\mathrm{Re}\left(z^2\right) \leq 2\mathrm{Im}(\overline{z}) + 8\right\}$. We then have

$$\begin{aligned} z = x+iy \in M &\Leftrightarrow 5|z|^2 + 4\mathrm{Re}\left(z^2\right) \leq 2\mathrm{Im}(\overline{z}) + 8 \\ &\Leftrightarrow 5(x^2+y^2) + 4(x^2-y^2) \leq -2y+8 \\ &\Leftrightarrow 9x^2 + y^2 + 2y \leq 8 \\ &\Leftrightarrow 9x^2 + (y+1)^2 \leq 9 \\ &\Leftrightarrow x^2 + \left(\frac{y+1}{3}\right)^2 \leq 1. \end{aligned}$$

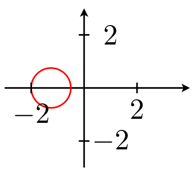
   Hence, M is a filled ellipse centered at $-i$ with major axis in y direction of length 3 and minor axis in x direction of length 1.



c) Let $N = \left\{ z \in \mathbb{C} \mid \left| \frac{z-1}{z+1} \right| = 3 \right\}$. For $z = x + iy \in N$ we then have

$$\begin{aligned}
\left| \frac{z-1}{z+1} \right| = 3 &\Leftrightarrow |z-1| = 3|z+1| \Leftrightarrow |z-1|^2 = 9|z+1|^2 \\
&\Leftrightarrow (x-1)^2 + y^2 = 9(x+1)^2 + 9y^2 \\
&\Leftrightarrow x^2 - 2x + 1 + y^2 = 9x^2 + 18x + 9 + 9y^2 \\
&\Leftrightarrow 8x^2 + 8y^2 + 20x + 8 = 0 \\
&\Leftrightarrow x^2 + y^2 + \frac{5}{2}x + 1 = 0 \\
&\Leftrightarrow x^2 + 2 \cdot \frac{5}{4}x + \frac{25}{16} - \frac{25}{16} + 1 = 0 \\
&\Leftrightarrow \left(x + \frac{5}{4} \right)^2 + y^2 = \left(\frac{3}{4} \right)^2.
\end{aligned}$$

Hence, the set N is a circle centered at $-\frac{5}{4}$ of radius $\frac{3}{4}$.



3. a) A subset D of the complex plane is called domain if D is nonempty, open, and connected.

b) 1) The set $M_1 = \{z \in \mathbb{C} \mid |z| < 1\}$ is a domain: Obviously, $0 \in M_1$. Let $z \in M_1$. Then $\varepsilon = \frac{1-|z|}{2} > 0$. Let $w \in B_\varepsilon(z)$. Then

$$|w| \leq |w - z| + |z| \leq \varepsilon + |z| = \frac{1 + |z|}{2} < 1.$$

Thus, $w \in M_1$ which implies that M_1 is open.

It remains to show that M_1 is connected. Let $z, w \in M_1$. The straight line segment connecting z and w lies completely in M_1. This line segment is $\{tz + (1-t)w \mid t \in [0,1]\}$. Let $t \in [0,1]$. Then

$$|tz + (1-t)w| \leq t|z| + (1-t)|w| < t + (1-t) = 1.$$

Hence, M_1 is connected and thus a domain.

2) The set $M_2 = \left\{ z \in \mathbb{C} \mid \exists n \in \mathbb{N}\, z = \frac{1}{n} + i \right\}$ is not a domain. The point $z = 1 + i$ lies in M_2. But for any $\varepsilon > 0$, the point $1 + \frac{\varepsilon}{2} + i$ lies in $B_\varepsilon(1+i)$. Since $\mathrm{Re}(1 + \frac{\varepsilon}{2} + i) > 1$ but $\mathrm{Re}(z) \leq 1$ for all $z \in M_2$, we have $1 + \frac{\varepsilon}{2} + i \notin M_2$. Hence, M_2 is not open.

3) The set $M_3 = \left\{ z \in \mathbb{C} \mid \mathrm{Re}(z) > 0, \mathrm{Im}(z) > \frac{1}{\mathrm{Re}(z)} \right\}$ is a domain. Since $2+2i \in M_3$, the set is not empty. Let $z = x+iy \in M_3$. Then, $x > 0$ and $y > \frac{1}{x} > 0$. Thus, $t = xy - 1 > 0$. Let $\varepsilon = \min\left\{\frac{x}{2}, 1, \frac{t}{2+2|y|+2|x|}\right\}$. Then $\varepsilon > 0$. Furthermore, let $w = a+ib \in B_\varepsilon(z)$. We then have $|a-x|, |b-y| \leq |z-w| < \varepsilon$. Since $\varepsilon \leq \frac{x}{2}$, we have $a > 0$. We need to show that $b > \frac{1}{a}$. But this holds iff $ba > 1$. We have

$$\begin{aligned} ba &= (b-y+y)(a-x+x) = (b-y)(a-x) + (b-y)x + (a-x)y + yx \\ &= (b-y)(a-x) + (b-y)x + (a-x)y + 1 + t. \end{aligned}$$

It remains to show that $|(b-y)(a-x) + (b-y)x + (a-x)y| < t$ since then $(b-y)(a-x) + (b-y)x + (a-x)y + t > 0$.

$$\begin{aligned} |(b-y)(a-x) + (b-y)x + (a-x)y| &\leq |b-y||a-x| + |b-y||x| + |a-x||y| \\ &\leq \varepsilon^2 + \varepsilon|x| + \varepsilon|y| \overset{\varepsilon \leq 1}{\leq} \varepsilon(1+|x|+|y|) \\ &\leq \frac{t}{2} < t. \end{aligned}$$

Hence, M_3 is open. It remains to show that M_3 is connected. For this, let $z = x+iy$ and $w = a+ib$ be points in M_3. Without loss of generality, we may assume that $a \geq x$. Let $\xi = a+iy$. Since $y > \frac{1}{x} \geq \frac{1}{a}$, we have $\xi \in M_3$. For all $t \in [0,1]$, we have

$$tz + (1-t)\xi = t(x+iy) + (1-t)(a+iy) = tx + (1-t)a + iy$$

and

$$y(tx + (1-t)a) = txy + (1-t)ay > t + (1-t) = 1$$

which implies $y > \frac{1}{tx+(1-t)a}$ and thus $tz + (1-t)\xi \in M_3$. Hence, the whole line segment L between z and ξ lies in M_3. We also have for $t \in [0,1]$

$$t\xi + (1-t)w = t(a+iy) + (1-t)(a+ib) = a + i(ty + (1-t)b).$$

Since

$$a(ty + (1-t)b) = tay + (1-t)ab > t + (1-t) = 1,$$

we also get that the straight line segment K between ξ and w lies completely in M_3. Hence, the polygonal line consisting of L and K connects z and w in M_3. Thus, M_3 is a domain.

4. a) Let $\varepsilon > 0$ and let $z_0 \in \mathbb{C}$. The ε-neighbourhood of z_0 is the set

$$\{z \in \mathbb{C} \mid |z - z_0| < \varepsilon\}$$

and the ε-neighbourhood of ∞ is

$$\left\{ z \in \mathbb{C} \mid |z| > \frac{1}{\varepsilon} \right\}.$$

b) 1)

$$\lim_{z\to\infty}\frac{4z^2}{(z-1)^2}=\lim_{z\to 0}\frac{\frac{4}{z^2}}{\left(\frac{1}{z}-1\right)^2}=\lim_{z\to 0}\frac{4}{(1-z)^2}=4$$

2) By definition, the limit is the derivative of the exponential function at 1. Since the exponential function is entire, this limit exists and since the derivative of the exponential function is the exponential function itself, the limit is $e^1=e$.

5. a) The derivative $f'(z_0)$ of a function f at a point z_0 is the limit

$$\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0},$$

provided that this limit exists.

b) With $z=x+iy$, we can write

$$f(z)=|z|^4=(x^2+y^2)^2=x^4+2x^2y^2+y^4+i0=u(x,y)+iv(x,y).$$

Hence, the partial derivatives of u and v exist with

$$\begin{aligned} u_x(x,y)&=4x^3+4xy^2 & u_y(x,y)&=4x^2y+4y^3\\ v_x(x,y)&=0 & v_y(x,y)&=0. \end{aligned}$$

These partial derivatives are continuous everywhere. Thus, it suffices to find all points in which the Cauchy-Riemann equations are fulfilled. Let $z=x+iy$. The Cauchy-Riemann equations are fulfilled iff

$$\begin{aligned} 4x(x^2+y^2)&=4x^3+4xy^2=u_x(x,y)=v_y(x,y)=0\\ 4y(x^2+y^2)&=4x^2y+4y^3=u_y(x,y)=-v_x(x,y)=0. \end{aligned}$$

But these equations are fulfilled if and only if $x=y=0$. Hence, f is only in 0 differentiable and nowhere else.

6. a) Let $f(z)=u(x,y)+iv(x,y)$ be a function. The Cauchy-Riemann equations in rectangular form are

$$\begin{aligned} u_x(x,y)&=v_y(x,y)\\ u_y(x,y)&=-v_x(x,y). \end{aligned}$$

b) Let $f(z)=x^3-3xy^2+e^x\cos(y)+i\left(3x^2y-y^3+e^x\sin(y)\right)=u(x,y)+iv(x,y)$. Then u and v are partially differentiable with

$$\begin{aligned} u_x(x,y)&=3x^2-3y^2+e^x\cos(y)\\ u_y(x,y)&=-6xy-e^x\sin(y)\\ v_x(x,y)&=6xy+e^x\sin(y)\\ v_y(x,y)&=3x^2-3y^2+e^x\cos(y). \end{aligned}$$

These partial derivatives are continuous everywhere and the Cauchy-Riemann equations are fulfilled everywhere. Thus, f' exists on all of $\mathbb{C}$. Hence, f is entire.

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MAT 342 Applied Complex Analysis
Final Exam Example

May 2016

1. (12 pts, 4 pts each)

   a) Define the notion *complex differentiable.*

   b) Define the *principle branch of the logarithm.*

   c) State *Cauchy's residue theorem.*

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2. (12 pts, 4 pts each)

   a) Find the multiplicative inverse of $3 + 4i$ and write the solution in rectangular form.

   b) Find all $z \in \mathbb{C}$ such that $z^2 = 4i$.

   c) Prove the triangle inequality: For all $z, w \in \mathbb{C}$, the inequality

$$|z + w| \leq |z| + |w|$$

   holds.

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3. (10 pts) Find all $z \in \mathbb{C}$ such that

$$z^4 + z^3 + z^2 + z + 1 = 0.$$

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4. (12 pts) Let f be an entire function such that

$$f(z) = f(z+1) = f(z+i)$$

for all $z \in \mathbb{C}$. Prove that f is constant.

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5. (10 pts) Let p be a polynomial of degree d_p and let q be a polynomial of degree d_q with $\max\{d_p, d_q\} \geq 1$. Assume that q is not constantly 0 and that p and q do not share a common zero. Let $f : \mathbb{C} \setminus \{z \in \mathbb{C} \mid q(z) = 0\} \to \mathbb{C}$ be given by

$$f(z) = \frac{p(z)}{q(z)}.$$

Let $z_0 \in \mathbb{C}$. Prove that there exists some $z \in \mathbb{C}$ such that $f(z) = z_0$.

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6. (12 pts) Find the Laurent series of

$$f(z) = \frac{1}{(z-1)(z-3)}$$

in $\{z \in \mathbb{C} \mid 0 < |z-1| < 2\}$.

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7. (12 pts, 4 pts each) Let

$$f(z) = \frac{1}{(z-2)(z-4)}.$$

Find the contour integrals of f along the circles about the origin of radius 1, 3 and 5, taken in counterclockwise direction.

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8. (20 pts, 10 pts each) Compute both

   a)
$$\int_0^\infty \frac{1}{1+x^4}dx \quad \text{and}$$

   b)
$$\int_{-\infty}^\infty \frac{x\sin(ax)}{x^4+4}dx \quad \text{where } a>0$$

using residues.

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MAT 342 Applied Complex Analysis
Final Exam Example

May 2016

1. (12 pts, 4 pts each)

   a) Define the notion *complex differentiable.*

   Let $S \subset \mathbb{C}$ be an open set and let $f : S \to \mathbb{C}$ be a function. Let $z_0 \in S$. The function f is called (complex) differentiable at z_0 if the limit

   $$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

   exists.

   b) Define the *principle branch of the logarithm.*

   The principle branch of the logarithm is defined by

   $$\text{Log}(z) = \ln(|z|) + i\text{Arg}(z),$$

   where $z \in \mathbb{C} \setminus \{r \in \mathbb{R} \mid r \leq 0\}$ and $-\pi < \text{Arg}(z) < \pi$.

   c) State *Cauchy's residue theorem.*

   Let C be a simple closed, positively oriented contour, and let f be a function which is analytic on C and inside C with the possible exception of finitely many points z_k $(k = 1, \ldots, n)$ inside C. Then

   $$\int_C f(z)dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_0} f(z).$$

Continue on page 2

Name: ______________________ **ID:** ______________

2. (12 pts, 4 pts each)

   a) Find the multiplicative inverse of $3+4i$ and write the solution in rectangular form.

   b) Find all $z \in \mathbb{C}$ such that $z^2 = 4i$.

   c) Prove the triangle inequality: For all $z, w \in \mathbb{C}$, the inequality

   $$|z+w| \leq |z| + |w|$$

   holds.

   a)
   $$(3+4i)^{-1} = \frac{1}{3+4i} = \frac{3-4i}{9+16} = \frac{3}{25} - i\frac{4}{25}.$$

   b) We have $4i = 4e^{i\frac{\pi}{2}}$. Thus, the two complex roots are

   $$\sqrt{4}e^{i\frac{\pi}{4}} = 2e^{i\frac{\pi}{4}} = 2\frac{1}{\sqrt{2}} + i2\frac{1}{\sqrt{2}} = \sqrt{2} + i\sqrt{2}$$

   and
   $$\sqrt{4}e^{i\left(\frac{\pi}{4}+\pi\right)} = -2e^{i\frac{\pi}{4}} = -\sqrt{2} - i\sqrt{2}.$$

   c) Let $z, w \in \mathbb{C}$. Since $|z|^2 = z\overline{z}$, we get

   $$\begin{aligned} |z+w|^2 &= (z+w)(\overline{z+w}) = (z+w)(\overline{z}+\overline{w}) = z\overline{z} + z\overline{w} + w\overline{z} + w\overline{w} \\ &= |z|^2 + z\overline{w} + \overline{z\overline{w}} + |w|^2 = |z|^2 + 2\mathrm{Re}(z\overline{w}) + |w|^2 \\ &\leq |z|^2 + 2|z||w| + |w|^2 = (|z|+|w|)^2. \end{aligned}$$

   Thus,
   $$|z+w| \leq |z| + |w|.$$

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3. (10 pts) Find all $z \in \mathbb{C}$ such that

$$z^4 + z^3 + z^2 + z + 1 = 0.$$

Proof. We have for $z \neq 1$

$$z^4 + z^3 + z^2 + z + 1 = \frac{z^5 - 1}{z - 1}$$

(partial sum of the geometric series). Thus,

$$z^4 + z^3 + z^2 + z + 1 = 0 \Leftrightarrow (z^5 = 1 \text{ and } z \neq 1).$$

Hence, all solutions of the equation are the non-trivial 5^{th} roots of unity, i.e.

$$e^{i\frac{2\pi}{5}} \quad , \quad e^{i\frac{4\pi}{5}} \quad , \quad e^{i\frac{6\pi}{5}} \quad , \quad e^{i\frac{8\pi}{5}}.$$

$\square$

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Name: ______________________________ **ID:** ________________

4. (12 pts) Let f be an entire function such that

$$f(z) = f(z+1) = f(z+i)$$

for all $z \in \mathbb{C}$. Prove that f is constant.

Proof. Let $Q = \{z \in \mathbb{C} \mid 0 \leq \mathrm{Re} z, \mathrm{Im} z \leq 1\}$. Then for any $w \in \mathbb{C}$ there exists some $z \in Q$ such that $f(z) = f(w)$ (write $w = a + ib = (n+s) + i(m+r)$ for some $n, m \in \mathbb{Z}$ and $0 \leq s, r < 1$). Since Q is bounded and closed (i.e. compact) and f is continuous on Q, f is bounded on Q. Due to the argument above, f is bounded on all of $\mathbb{C}$. Thus, f is a bounded entire function which must be constant by Liouville's theorem. $\square$

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Name: ______________________ **ID:** ______________

5. (10 pts) Let p be a polynomial of degree d_p and let q be a polynomial of degree d_q with $\max\{d_p, d_q\} \geq 1$. Assume that q is not constantly 0 and that p and q do not share a common zero. Let $f : \mathbb{C} \setminus \{z \in \mathbb{C} \mid q(z) = 0\} \to \mathbb{C}$ be given by

$$f(z) = \frac{p(z)}{q(z)}.$$

Let $z_0 \in \mathbb{C}$. Prove that there exists some $z \in \mathbb{C}$ such that $f(z) = z_0$.

Proof. We have

$$f(z) = z_0 \Leftrightarrow \frac{p(z)}{q(z)} = z_0 \Leftrightarrow p(z) = z_0 q(z) \Leftrightarrow p(z) - z_0 q(z) = 0.$$

Since p and q do not share a common zero, the zeros of q can't be solutions. But $p - z_0 q$ is a polynomial of degree $\max\{d_p, d_q\} \geq 1$. Hence, it has at least one zero $z \in \mathbb{C}$ by the Fundamental Theorem of Algebra. For this zero, $f(z) = z_0$ holds. $\square$

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6. (12 pts) Find the Laurent series of

$$f(z) = \frac{1}{(z-1)(z-3)}$$

in $\{z \in \mathbb{C} \mid 0 < |z-1| < 2\}$.

Proof. We have for $z \neq 1$ and $z \neq 3$

$$\frac{-1}{2(z-1)} + \frac{1}{2(z-3)} = \frac{-(z-3)+(z-1)}{2(z-1)(z-3)} = f(z).$$

For $z \in \mathbb{C}$ with $0 < |z-1| < 2$, we have $\frac{|z-1|}{2} < 1$ and thus

$$\begin{aligned} f(z) &= \frac{-1}{2(z-1)} + \frac{1}{2(z-3)} = \frac{-1}{2(z-1)} + \frac{1}{2((z-1)-2)} \\ &= \frac{-1}{2(z-1)} + \frac{1}{4}\frac{1}{\frac{z-1}{2}-1} = \frac{-1}{2(z-1)} - \frac{1}{4}\frac{1}{1-\frac{z-1}{2}} \\ &= \frac{-1}{2(z-1)} - \frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{z-1}{2}\right)^n = \frac{-1}{2(z-1)} - \sum_{n=0}^{\infty}\frac{(z-1)^n}{2^{n+2}}. \end{aligned}$$

□

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7. (12 pts, 4 pts each) Let
$$f(z) = \frac{1}{(z-2)(z-4)}.$$
Find the contour integrals of f along the circles about the origin of radius 1, 3 and 5, taken in counterclockwise direction.

Proof. Define curves $\gamma_1, \gamma_3, \gamma_5 : [0, 2\pi] \to \mathbb{C}$ by $\gamma_1(t) = e^{it}$, $\gamma_3(t) = 3e^{it}$, $\gamma_5(t) = 5e^{it}$. These curve parametrise the circles about the origin of radius 1, 3 and 5, all in counterclockwise direction.

As a rational function, f is analytic in the whole plane with the only exceptions being the zeros of the denominator, i.e. f is analytic in $\mathbb{C} \setminus \{2, 4\}$. In particular, f is analytic inside and on γ_1. By the Cauchy-Goursat theorem, this yields
$$\int_{\gamma_1} f(z)dz = 0.$$
Furthermore, we have that 2 lies inside γ_3, but 4 lies outside γ_3. By Cauchy's residue theorem, this yields
$$\int_{\gamma_3} f(z)dz = 2\pi i \operatorname*{Res}_{z=2} f(z)$$
and since both 2 and 4 lie inside γ_5
$$\int_{\gamma_5} f(z)dz = 2\pi i \left(\operatorname*{Res}_{z=2} f(z) + \operatorname*{Res}_{z=4} f(z) \right).$$
Since both 2 and 4 are simple poles of f ($\frac{1}{z-4}$ and $\frac{1}{z-2}$ are analytic and nonzero at 2 and 4, respectively), we get
$$\operatorname*{Res}_{z=2} f(z) = \frac{1}{2-4} = -\frac{1}{2} \quad \text{and} \quad \operatorname*{Res}_{z=4} f(z) = \frac{1}{4-2} = \frac{1}{2}.$$
Thus,
$$\int_{\gamma_3} f(z)dz = -\pi i \quad \text{and} \quad \int_{\gamma_5} f(z)dz = 0.$$
□

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8. (20 pts, 10 pts each) Compute both

   a)
$$\int_0^\infty \frac{1}{1+x^4}dx \quad \text{and}$$

   b)
$$\int_{-\infty}^\infty \frac{x\sin(ax)}{x^4+4}dx \quad \text{where } a>0$$

   using residues.

Proof. For $R>0$, we define $\gamma_1 : [-R,R] \to \mathbb{C}$, $\gamma_1(t)=t$, and $\gamma_2 : [0,\pi] \to \mathbb{C}$, $\gamma_2(t) = Re^{it}$. Furthermore, let $\gamma_R = \gamma_1 + \gamma_2$. This curve consists of the real integral $[-R,R]$ and the semicircle of radius R in the upper half plane, taken in positive orientation.

a) The integrand $f(z) = \frac{1}{z^4+1}$ is analytic in the entire plane with the only exception being its singular points which are the 4^{th} roots of -1, i.e.

$$e^{i\frac{\pi}{4}} \quad , \quad e^{i\frac{3\pi}{4}} \quad , \quad e^{i\frac{5\pi}{4}} \quad , \quad e^{i\frac{7\pi}{4}}.$$

Only $e^{i\frac{\pi}{4}} = \frac{1+i}{\sqrt{2}}$ and $e^{i\frac{3\pi}{4}} = \frac{-1+i}{\sqrt{2}}$ lie in the upper half plane, $e^{i\frac{5\pi}{4}} = \frac{-1-i}{\sqrt{2}}$ and $e^{i\frac{7\pi}{4}} = \frac{1-i}{\sqrt{2}}$ both lie in the lower half plane. There is no singular point on the real line. If we choose $R>1$, then both $e^{i\frac{\pi}{4}}$ and $e^{i\frac{3\pi}{4}}$ lie inside γ_R.

Both points are simple poles of f ($f = p/q$, $p(z)=1$, $q(z) = z^4+1$, $q'(z) = 4z^3$, $q\left(e^{i\frac{\pi}{4}}\right) = 0 = q\left(e^{i\frac{3\pi}{4}}\right)$, $q'\left(e^{i\frac{\pi}{4}}\right) \neq 0 \neq q'\left(e^{i\frac{3\pi}{4}}\right)$). Thus,

$$\operatorname*{Res}_{z=e^{i\frac{\pi}{4}}} f(z) = \frac{1}{4\left(e^{i\frac{\pi}{4}}\right)^3} = \frac{1}{4e^{i\frac{3\pi}{4}}} = \frac{-1}{4}e^{i\frac{\pi}{4}} \quad \text{and} \quad \operatorname*{Res}_{z=e^{i\frac{3\pi}{4}}} = \frac{1}{4e^{i\frac{9\pi}{4}}} = \frac{-1}{4}e^{i\frac{3\pi}{4}}.$$

Using Cauchy's residue theorem, we get

$$\int_{\gamma_R} f(z)dz = 2\pi i\left(-\frac{1}{4}e^{i\frac{\pi}{4}} - \frac{1}{4}e^{i\frac{3\pi}{4}}\right) = \frac{-\pi i}{2\sqrt{2}}(1+i+(-1+i)) = \frac{\pi}{\sqrt{2}}.$$

Furthermore, $|f(z)| \leq \frac{1}{R^4-1}$ for z on γ_2 and $L(\gamma_2) = \pi R$. Thus,

$$\left|\int_{\gamma_2} f(z)dz\right| \leq \frac{\pi R}{R^4-1} \to 0 \quad \text{as} \quad R\to\infty.$$

Thus,

$$\begin{aligned}\frac{\pi}{\sqrt{2}} &= \lim_{R\to\infty}\int_{\gamma_R} f(z)dz = \lim_{R\to\infty}\left(\int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz\right)\\ &= \lim_{R\to\infty}\int_{-R}^{R} f(x)dx + \lim_{R\to\infty}\int_{\gamma_2} f(z)dz = P.V.\int_{-\infty}^{\infty}\frac{1}{x^4+1}dx + 0.\end{aligned}$$

Since $\frac{1}{x^4+1}$ is an even function, we get

$$P.V.\int_{-\infty}^{\infty}\frac{1}{x^4+1}dx = \int_{-\infty}^{\infty}\frac{1}{x^4+1}dx = 2\int_0^{\infty}\frac{1}{x^4+1}dx$$

and hence

$$\int_0^{\infty}\frac{1}{x^4+1}dx = \frac{\pi}{2\sqrt{2}}.$$

b) The function $f(z) = \frac{z}{z^4+4}$ has four singular points at the 4^{th} roots of -4, i.e. at

$$\sqrt{2}e^{i\frac{\pi}{4}} \quad , \quad \sqrt{2}e^{i\frac{3\pi}{4}} \quad , \quad \sqrt{2}e^{i\frac{5\pi}{4}} \quad , \quad \sqrt{2}e^{i\frac{7\pi}{4}}.$$

As in a), only the first two lie in the upper half plane, the other two in the lower half plane, and none on the real line. Write $z_1 = \sqrt{2}e^{i\frac{\pi}{4}}$ and $z_2 = \sqrt{2}e^{i\frac{3\pi}{4}}$. For $R > \sqrt{2}$, both z_1 and z_2 also lie inside γ_R. Since $f(z)e^{iaz}$ has the same singular points as $f(z)$, we get by Cauchy's residue theorem

$$\int_{\gamma_R} f(z)e^{iz}dz = 2\pi i\left(\operatorname*{Res}_{z=z_1} f(z)e^{iaz} + \operatorname*{Res}_{z=z_2} f(z)e^{iaz}\right).$$

With the same argument as in a), we see that z_1 and z_2 are simple poles of $f(z)e^{iaz}$ and the residues are

$$\operatorname*{Res}_{z=z_1} f(z)e^{iaz} = \frac{z_1e^{iaz_1}}{4z_1^3} = \frac{e^{iaz_1}}{4z_1^2} = \frac{e^{iaz_1}}{8e^{i\frac{\pi}{2}}} = \frac{e^{iaz_1}}{8i}$$

and

$$\operatorname*{Res}_{z=z_2} f(z)e^{iaz} = \frac{z_2e^{iaz_2}}{4z_2^3} = \frac{e^{iaz_2}}{4z_2^2} = \frac{e^{iaz_2}}{8e^{i\frac{3\pi}{2}}} = \frac{e^{iaz_2}}{-8i}.$$

Since $z_1 = \sqrt{2}e^{i\frac{\pi}{4}} = 1+i$ and $z_2 = -1+i$, we get

$$\begin{aligned}\int_{\gamma_R} f(z)e^{iaz}dz &= \frac{2\pi i}{8i}\left(e^{iaz_1} - e^{iaz_2}\right) = \frac{\pi}{4}\left(e^{ia-a} - e^{-ia-a}\right)\\ &= \frac{\pi e^{-a}}{4}\left(e^{ia} - e^{-ia}\right) = \frac{\pi e^{-a}2i}{4}\sin(a).\end{aligned}$$

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Since

$$\frac{\pi e^{-a}2i}{4}\sin(a) = \int_{\gamma_R} f(z)e^{iaz}dz = \int_{\gamma_1} f(z)e^{iaz}dz + \int_{\gamma_2} f(z)e^{iaz}dz$$
$$= \int_{-R}^{R} \frac{x}{x^4+4}e^{iax}dx + \int_{\gamma_2} f(z)e^{iaz}dz,$$

we get

$$\frac{\pi e^{-a}}{2}\sin(a) = \mathrm{Im}\left(\int_{-R}^{R} \frac{x}{x^4+4}e^{iax}dx + \int_{\gamma_2} f(z)e^{iaz}dz\right)$$
$$= \int_{-R}^{R} \frac{x\sin(ax)}{x^4+4}dx + \mathrm{Im}\left(\int_{\gamma_2} f(z)e^{iaz}\right)$$

For $t \in [0,\pi]$, we have

$$f(\gamma_2(t))e^{ia\gamma_2(t)} = \frac{Re^{it}}{R^4e^{i4t}+4}e^{iaR(\cos(t)+i\sin(t))} = \frac{re^{it}}{R^4e^{i4t}+4}e^{iaR\cos(t)}e^{-aR\sin(t)}.$$

Since $\sin(t) \geq 0$ for these t and both $a, R > 0$, we get

$$\left|f(\gamma_2(t))e^{ia\gamma_2(t)}\right| \leq \frac{R}{R^4-4}e^{-aR\sin(t)} \leq \frac{R}{R^4-4}$$

which implies with $L(\gamma_2) = \pi R$

$$\left|\mathrm{Im}\left(\int_{\gamma_2} f(z)e^{iaz}dz\right)\right| \leq \left|\int_{\gamma_2} f(z)e^{iaz}dz\right| \leq \frac{\pi R^2}{R^4-4} \to 0 \quad \text{as} \quad R \to \infty.$$

Combined,

$$\frac{\pi e^{-a}}{2}\sin(a) = \lim_{R\to\infty}\int_{-R}^{R} \frac{x\sin(ax)}{x^4+4}dx = P.V.\int_{-\infty}^{\infty} \frac{x\sin(ax)}{x^4+4}dx.$$

Since $\frac{x\sin(ax)}{x^4+4}$ is even, we get

$$\int_{-\infty}^{\infty} \frac{x\sin(ax)}{x^4+4}dx = \frac{\pi e^{-a}}{2}\sin(a).$$

$\square$