Homework 9 Solutions

1) ...try to find an infinite polynomial that represents LN(X) (the "natural" logarithm function)... ... let's find an infinite polynomial for LN(1+X) instead. The derivative of LN(1+X) equals 1/(1+X). Okay, now (a) write down a generating function equal to 1/(1+X)

Using the result from a previous homework that $\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \ldots$

Next, (b) find an infinite polynomial (generating function) whose derivative is the infinite polynomial you just found in part (a) - and so this should be equal to LN(1+X).

Thus, following the suggestion we find that $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots$ since we can check that the (term by term) derivative of $x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots$ is equal to the infinite series we just found in part (a), i.e. $1 - x + x^2 - x^3 + x^4 - \ldots$. A good way to get to this result is just to take the antiderivative of the series we found in part one. You can do it term by term, just like a polynomial. (Note, it's actually not completely trivial that you can "bump up" the rules for antiderivates to an infinite series, but we can easily see that we get the right result when we differentiate.)

Note that unlike the infinite polynomials that we found in class for e^x, Sin(X), and Cos(X) the infinite polynomial for LN(1+X) will only produce results for -1 < X < 1 as you can check. For instance LN(0) is "negative infinity." Try plugging X = -1 into the infinite polynomial you just found for LN(1+X) and check to see that you get a familiar non-converging series as a result! (write down which series you get)

Okay, so plugging -1 in for x in the series $x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots$ gives $-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \ldots$ and given that we know that $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ is the harmonic series whose sum doesn't actually exist ("equals infinity"), then negative this is equivalent to "negative infinity" just as predicted given that LN(0) is "negative infinity" (take a look at a graph of the Natural Log function to see what happens as x approaches 0).

2) Now try plugging X = 1 into the infinite polynomial we found in class for e^x and find (write down) a series whose sum equals the number e (= 2.718281828...) Add up the first 8 terms of the series and see for yourself if this is "believable"!

Calculating the sum of the first 8 terms we have

$$e \approx 1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\frac{1}{6!}+\frac{1}{7!} \approx 2.718254$$

making this a believable infinite sum. (A few of you commented that "therefore" this generating function gives us e. *This is not true.* Yes, this is some nice evidence that we have the correct generating function, but checking that the first eight terms gives a nice looking result does not a proof make!)

4) Finally, please look through the following two pages [Complex Multiplication using a similar triangles perspective] (from Mathematical Connections by Al Cuoco) ...

Note - for this problem just write down your conjecture - you don't need to write down a proof of this - we'll discuss how to actually prove this together in class.

We can get the modulus of wz by multiplying the moduli of w and z (i.e. |wz| = |w||z|), and similarly, we add the arguments to get the argument of wz (i.e. arg(wz) = arg(w) + arg(z)).

Bonus *- if you've worked out a conjecture about the product of wz as compared to w and z, what about division? How does w/z compare to w and z? (hint, in this case note that z is the product of the complex numbers w and w/z, and you should already know about what happens with complex products).*

Here suppose we call the division of **w** by **z** just "**s**", that is **w/z** = **s**. Then it simply follows that **w** = **zs**. We know that the product of **z** and **s** works by taking the product of the lengths (moduli) of **z** and **s**, and the argument for **zs** is the sum of the arguments of **z** and **s**.

Given that w is the product of **z** and **s**, then this means that the length (modulus) of **w** is the product of the lengths/moduli of **z** and **s**. Thus the modulus of **s** i.e. the modulus of **w/z** is simply the modulus of **w** divided by the modulus of **z**.

Finally, the argument of **w** is the sum of the argument of **z** plus the argument of **s**, so likewise, the argument of **s** must be the argument of **w** minus the argument of **z**.

5) In class using complex numbers, we found the two classic trig addition formulas: Cos(A+B) = Cos(A)Cos(B) - Sin(A)Sin(B) and Sin(A+B)= Cos(A)Sin(B)+Sin(A)Cos(B) and then used these to show that Cos(2A) = $(Cos(A))^2$ - $(Sin(A))^2$ or 2 $(Cos(A))^2$ - 1 (known as "Double Angle" formulas). Note that the second form gives Cos(2A) just in terms of Cos(A).

Is there a Triple Angle formula? Find a formula for Cos(3A) that's just in terms of Cos(A) (hint - use the Cos(A+B) formula but do it with Cos(A + 2A) instead, and then rewrite things in terms of Cos(A) by replacing $(Sin(A))^2$ with $(1 - Cos(A))^2$ whenever you can. You'll also need to rewrite Sin(2A) at some point using the Sin(A+B) formula).

Let's work through the triple angle formula using the hints in the problem…

$$\cos(3A) = \cos(A+2A) = \cos(A)\cos(2A) - \sin(A)\sin(2A) =$$
$$= (\cos A)(\cos^2 A - \sin^2 A) - (\sin A)(2\cos A \sin A) =$$
$$= \cos^3 A - \cos A \sin^2 A - 2\cos A \sin^2 A =$$
$$= \cos^3 A - (\cos A)(1-\cos^2 A) - 2(\cos A)(1-\cos^2 A) =$$
$$= \cos^3 A - \cos A + \cos^3 A - 2\cos A + 2\cos^3 A =$$
$$= 4\cos^3 A - 3\cos A$$

Bonus*: is there a Quartic Angle formula for Cos(4A) that's just in terms of Cos(A) (now it's really getting ugly - not for the faint of heart - definitely feel free to use Mathematica as much as you'd like!)*

Yes there is! Take a look at http://www.integraltec.com/math/math.php?f=trig.html and scroll down to the "multiple angle formulas" section – and feel free to browse through some of the rest of this amazing list of trig identities!