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Alexander Zivich
CURVE TRACING

IN

CARTESIAN COORDINATES

BY

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PREFACE

THIS book relates, not to the general theory of curves, but to the definite problem of ascertaining the form of a curve given by its equation in Cartesian coordinates, in such cases as are likely to arise in the actual applications of Analytical Geometry. The methods employed are exclusively algebraic, no knowledge of the Differential Calculus on the part of the reader being assumed.

I have endeavored to make the treatment of the subject thus restricted complete in all essential points, without exceeding such limits as its importance would seem to justify. This it has seemed to me possible to do by introducing at an early stage the device of the Analytical Triangle, and using it in connection with all the methods of approximation.

In constructing the triangle, which is essentially Newton's parallelogram, I have adopted Cramer's method of representing the possible terms by points, with a distinguishing mark to indicate the actual presence of the term in the equation. These points were regarded by Cramer as marking the centres of the squares in which, in New-

ton's parallelogram, the values of the terms were to be inscribed; but I have followed the usual practice, first suggested, I believe, by Frost, of regarding them merely as points referred to the sides of the triangle as coordinate axes. It has, however, been thought best to return to Newton's arrangement, in which these analytical axes are in the usual position of coordinate axes, instead of placing the third side of the triangle, like De Gua and Cramer, in a horizontal position.

The third side of the Analytical Triangle bears the same relation to the geometrical conception of the line at infinity that the other sides bear to the coordinate axes. I have aimed to bring out this connection in such a way that the student who desires to take up the general theory of curves may gain a clear view of this conception, and be prepared to pass readily from the Cartesian system of coordinates, in which one of the fundamental lines is the line at infinity, to the generalized system, in which all three fundamental lines are taken at pleasure.

Lists of examples for practice will be found at the end of each section. These examples have been selected from various sources, and classified in accordance with the subjects of the several sections.

W. W. J.

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CURVE TRACING

I

Equations Solved for One Variable

1. THE equation of a curve is in these pages supposed to be given in Cartesian coordinates ; and the curve is said to be *traced*, when the general form of its several parts or branches is determined, and the position of those which are unlimited in extent is indicated. In the diagrams the coordinate axes will, for convenience, be assumed rectangular ; but the methods are equally applicable to oblique axes. When it is possible to solve the equation for one of the coordinates, so as to express its value in terms of the other, the resulting form of the equation affords the most obvious method of tracing the curve. This may be done for either variable when the equation is of the second degree in both variables, in which case the curve is a *conic*. For example, let the given equation be

$$2x^2 - 2xy + y^2 - 4x + 2y + 1 = 0. \quad (1)$$

Solving for y , we have

$$y = x - 1 \pm \sqrt{(2x - x^2)}. \quad (2)$$

Thus, for any given value of x , we have two values of y ; and if we put

$$y' = x - 1, \quad (3)$$

the equation of the curve becomes

$$y = y' \pm \sqrt{(2x - x^2)}. \quad (4)$$

Diameters

2. Equation (3) represents a straight line, and equation (4) shows that the two ordinates for the curve may be found by adding to and subtracting from the ordinate of the straight line the same quantity. In other words, the chord joining the two points of the curve which have the same abscissa, that is, any chord parallel to the axis of y , is bisected by the straight line. This line is therefore called a *diameter* of the curve. The diameter represented by equation (3) is constructed in Fig. 1.

3. The radical $\sqrt{(2x - x^2)}$, which is half the length of the chord, varies with x , and vanishes for the two values

$$x = 0 \quad \text{and} \quad x = 2;$$

the corresponding points on the diameter are therefore also points of the curve. Writing the radical in the form

$$\sqrt{[x(2 - x)]},$$

it is obvious that all values of x between 0 and 2 give real values to the radical, since they render both of the factors

under the radical sign positive, and that all values exterior to these limits give imaginary values to the radical. Hence all the real points of the curve lie between the straight lines $x = 0$ and $x = 2$, as represented in Fig. 1. These limiting lines are evidently tangents to the curve at the points where they cut the diameter. Putting the radical in the form

$$\sqrt{[1 - (1 - x)^2]},$$

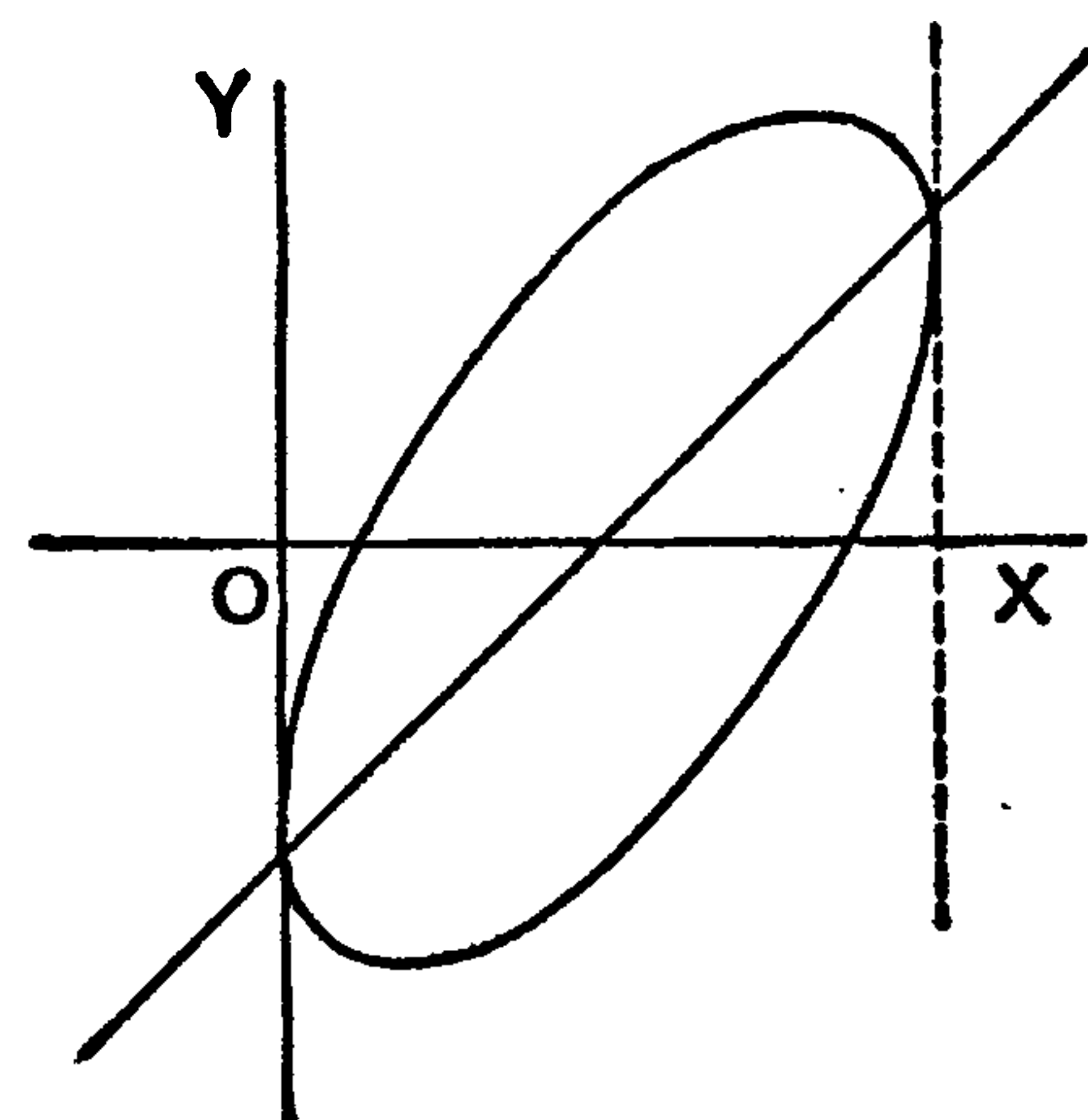


Fig. 1

we see that the maximum value of the radical is unity, and corresponds to the value $x = 1$, or the middle point of the diameter. The lines passing through the corresponding points of the curve parallel to the diameter are obviously tangents. The curve is an ellipse.

4. As a second example let us take the curve

$$y^2 - xy + y - x + 1 = 0. \quad (1)$$

Solving for y ,

$$y = \frac{1}{2}(x - 1) \pm \frac{1}{2}\sqrt{(x^2 + 2x - 3)}.$$

Putting $x^2 + 2x - 3 = 0$, we find that the radical vanishes when $x = -3$, and when $x = 1$; the equation may therefore be written in the form

$$y = \frac{1}{2}(x - 1) \pm \frac{1}{2}\sqrt{[(x + 3)(x - 1)]}. \quad (2)$$

The straight line

$$y = \frac{1}{2}(x - 1) \quad (3)$$

is therefore a diameter, and the lines

$$x = -3 \quad \text{and} \quad x = 1$$

are tangents; but the radical is, in this case, imaginary for values of x between the limiting values, since such values make one of the factors positive and the other negative, and real for values exterior to the limits. Hence the curve consists of two branches touching the lines $x = -3$ and $x = 1$ at their intersections with the diameter represented by equation (3), as in Fig. 2.

Asymptotes

5. Putting the radical in the form

$$\sqrt{[(x + 1)^2 - 4]},$$

we see that its value increases indefinitely as x increases in

numerical value, but that it is always less than $x + 1$. Thus the points of the curve lie between the lines

$$y = \frac{1}{2}(x - 1) \pm \frac{1}{2}(x + 1), \quad (4)$$

that is to say, the lines

$$y = x \quad \text{and} \quad y = -1.$$

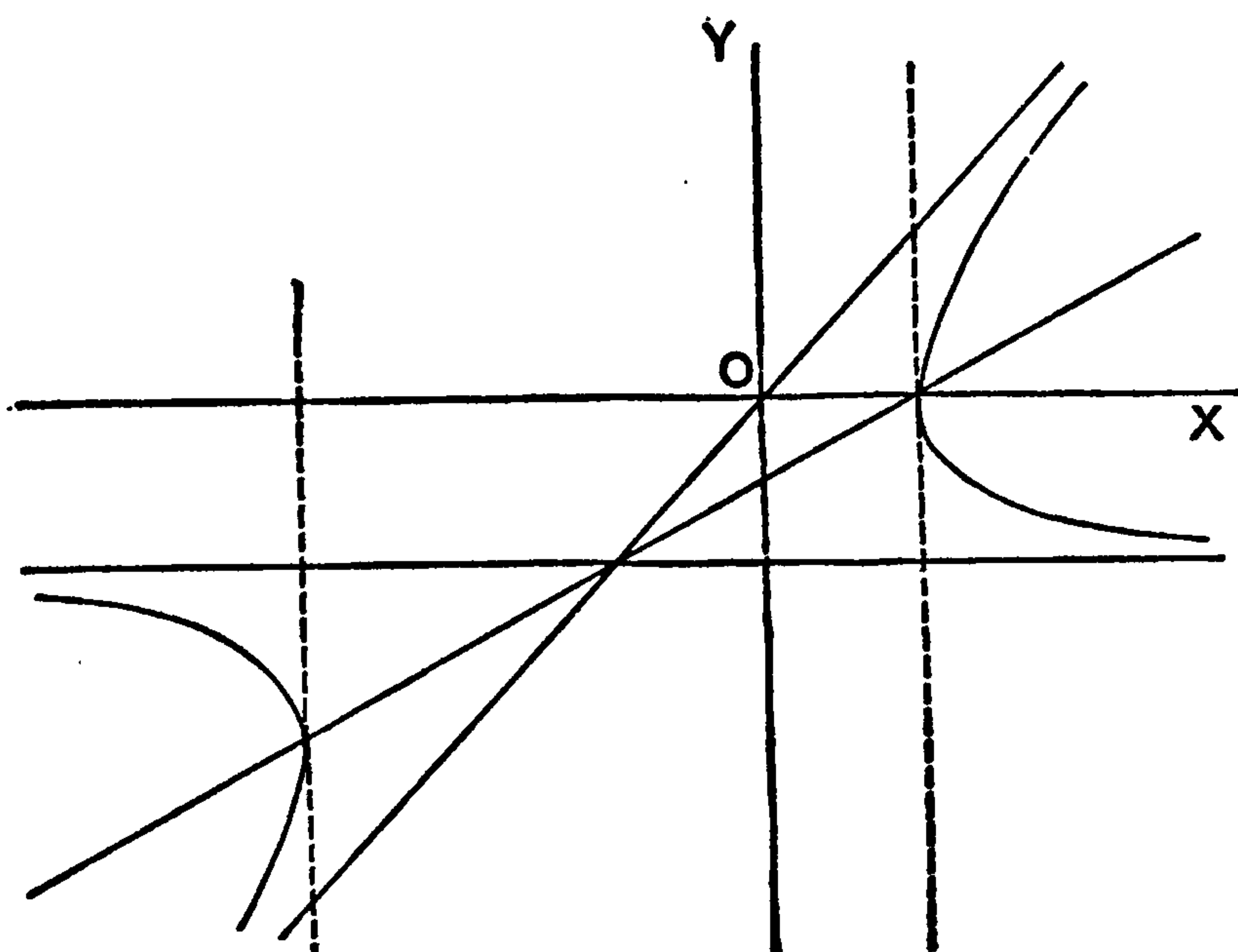


Fig. 2

The curve approaches indefinitely to these lines, since the difference between $x + 1$ and the radical decreases without limit as x increases numerically. The lines are therefore *asymptotes*. The asymptotes and the two branches of the curve are constructed in Fig. 3. The curve is an hyperbola.

6. It is evident from the above method of finding the asymptotes that the position of these lines will not be affected by a change in the value of the absolute term. For example, if the absolute term in equation (1) be changed from 1 to 2, the only change will be that the radical will become $\sqrt{(x+1)^2 - 8}$, and the asymptotes will still be represented by equation (4). Moreover, the value of the absolute term may be such as to make the equation identical with the equation which represents the two asymptotes. In the present instance, the latter equation is $(y-x)(y+1) = 0$, or $y^2 - xy + y - x = 0$, which is equation (1) with the absolute term changed to zero. If, on the other hand, the absolute term is changed to -1 , the radical becomes $\sqrt{(x^2 + 2x + 5)}$ which cannot be made to vanish as in Art. 4, but is always numerically greater than $x+1$. Thus the curve will no longer cut the diameter, as in Fig. 2, but will lie on the other side of the asymptotes. It will, in fact, be the conjugate of the hyperbola drawn in Fig. 2.

7. If we solve equation (1) of Art. 4 for x , we have

$$x = y + \frac{1}{y+1}.$$

This form of the equation shows that for each value of y there is a single value of x , but this value is infinite when $y = -1$. The line $y = -1$ is therefore an asymptote. The value of the fraction $\frac{1}{y+1}$ is positive for all values of y algebraically greater than -1 , and negative for all values less than -1 ; hence the part of the curve above the line $y = -1$ lies on the right of the line $y = x$, and that below the line $y = -1$ on the left of the line $y = x$. As the numerical value of y increases indefinitely, that of the fraction decreases indefinitely, which shows that $y = x$ is also an asymptote.

Parabolas

8. It follows from what is shown in Art. 6 that an equation of the second degree will represent an hyperbola when the terms of the second degree can be resolved into real factors. In this case, they constitute the difference of two squares. On the other hand, the equation represents an ellipse when these terms constitute the sum of two squares, as in equation (1), Art. 1. The intermediate case is that in which the terms form a perfect square. For example, let the equation be

$$x^2 + 2xy + y^2 - 2x - 6y + 1 = 0. \quad (1)$$

Solving for y ,

$$y + x - 3 = \pm 2\sqrt{2 - x};$$

hence

$$y + x - 3 = 0 \quad (2)$$

is the equation of the diameter bisecting chords parallel to the axis of y . The radical is real for all values of x algebraically less than 2, and imaginary for all values greater than 2; hence the curve lies on the left of the line $x = 2$, which it touches at the point (2, 1) where it cuts the diameter.

If we solve for x , we find, in like manner,

$$x + y - 1 = \pm 2\sqrt{y},$$

hence

$$x + y - 1 = 0 \quad (3)$$

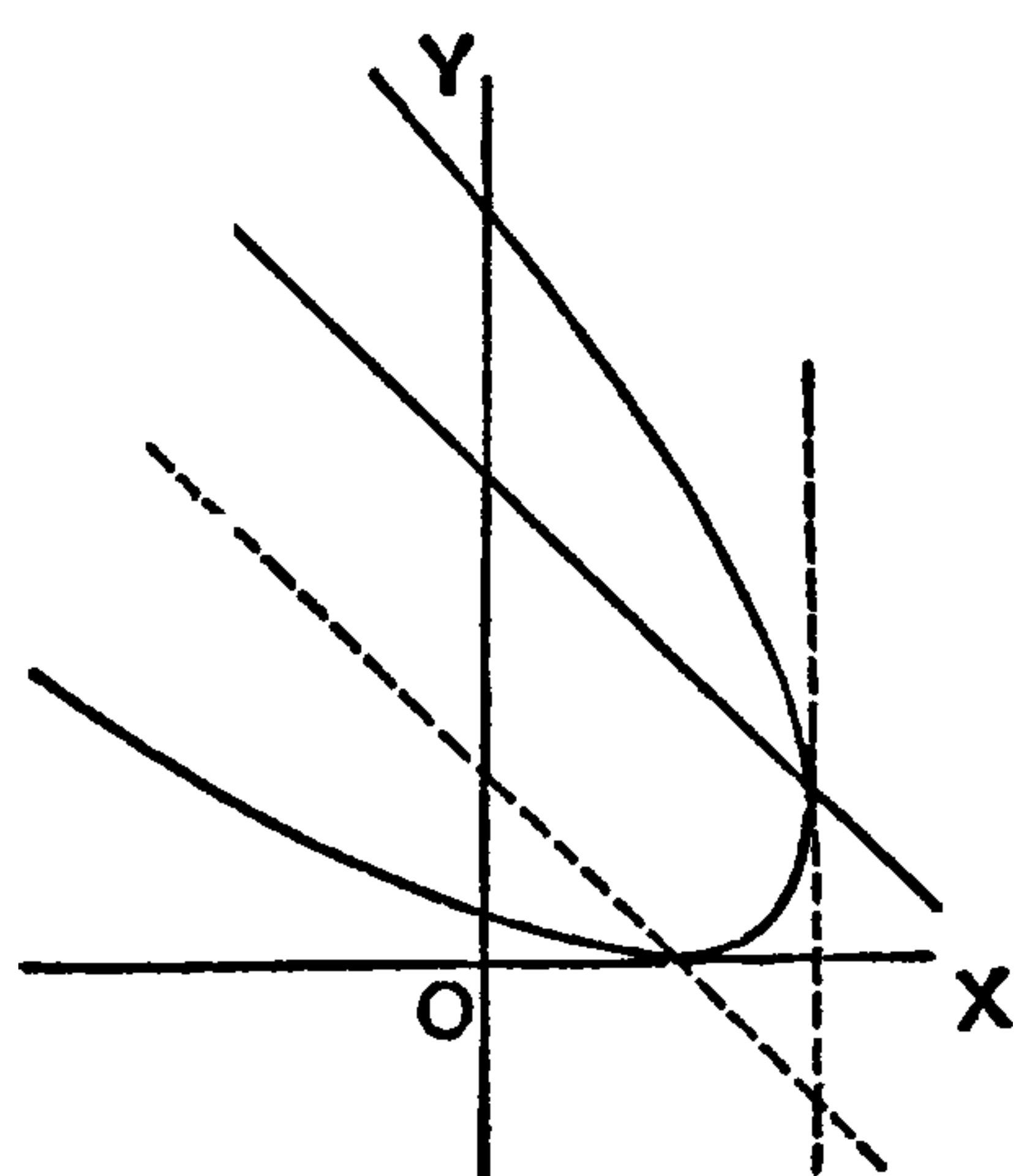


Fig. 3

is the equation of the diameter bisecting chords parallel to the axis of x , and the curve touches the line $y = 0$ at the point (1, 0). The value of the radical in either form of the equation increases without limit, but there is no asymptote. The curve is a parabola.

Curvilinear Diameters

9. We pass now to an example in which the curve is a cubic, although the equation is only of the second degree with respect to either variable. Given the equation

$$2x^2y - x^2 + y^2 + 2x = 0: \quad (1)$$

solving the equation as a quadratic for y , we have

$$y = -x^2 \pm \sqrt{(x^4 + x^2 - 2x)},$$

or, resolving the quantity under the radical sign into factors,

$$y = -x^2 \pm \sqrt{x(x-1)(x^2+x+2)}. \quad (2)$$

Putting $y = 0$ in equation (1), we find that the curve intersects the axis of x at the origin, and at the point $(2, 0)$; and putting $x = 0$, we find that it meets the axis of y at no point except the origin. Equation (2) gives in general two values of y for each value of x , and the corresponding points are equally distant from that point of the parabola

$$y = -x^2 \quad (3)$$

which has the same abscissa. In other words, chords parallel to the axis of y are bisected by this parabola, which is therefore a *diameter*. This diametral parabola is constructed in Fig. 4.

10. The quantity which occurs under the radical sign in equation (2), namely

$$x(x-1)(x^2+x+2),$$

changes sign only twice as x passes through all possible values, since it vanishes only when $x = 0$ and when $x = 1$. It is, in

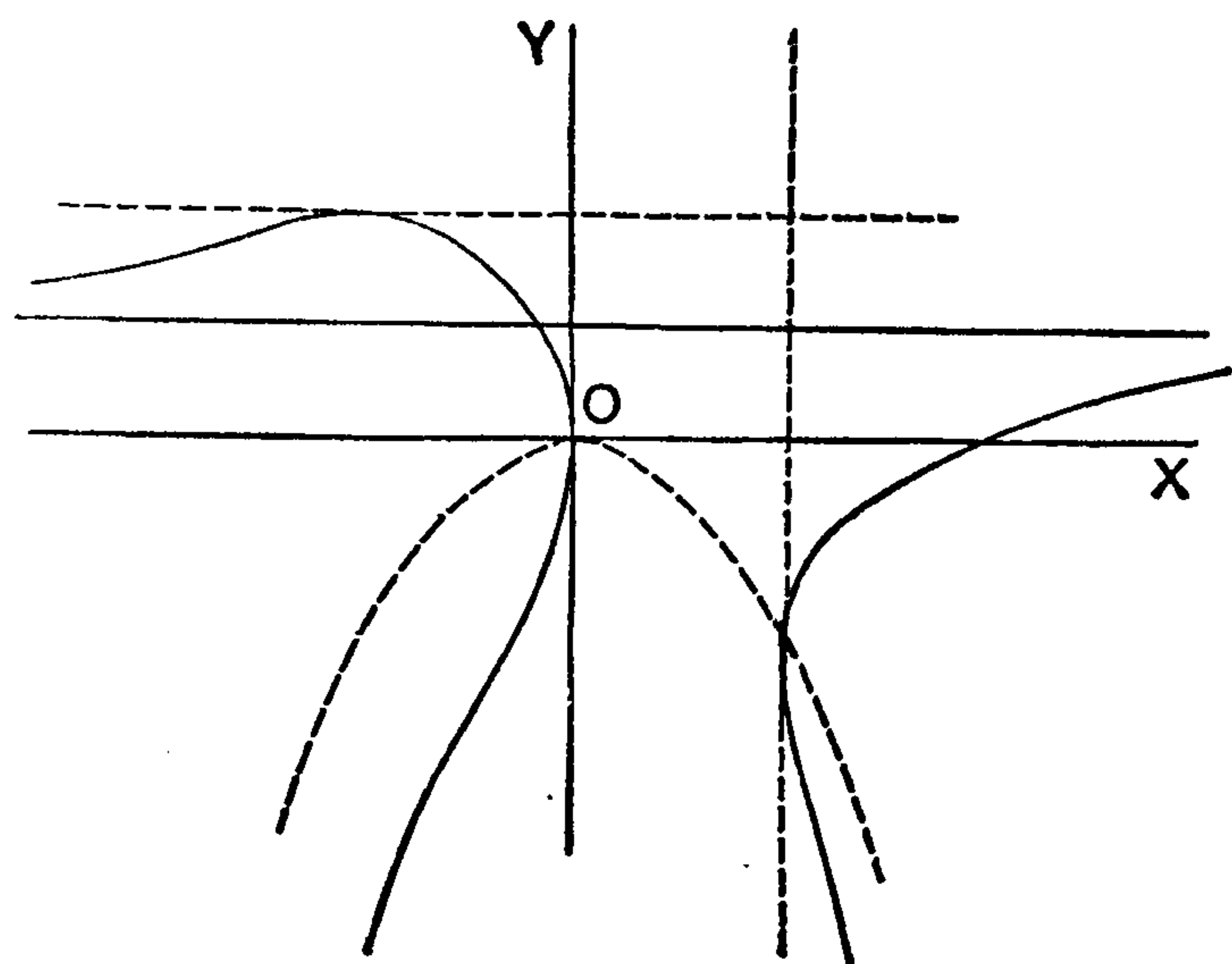


Fig. 4

fact, negative for all values of x between these limits, and positive for all other values of x . There is therefore no part of the curve between the lines $x = 0$ and $x = 1$; and these limiting lines touch the curve at the points where they intersect the diametral curve represented by equation (3); that is, at the origin and at

the point $(1, -1)$. See Fig. 4.

11. If we solve equation (1) for x , we have

$$x = \frac{-1 \pm \sqrt{(1 + y^2 - 2y^3)}}{2y - 1}. \quad (4)$$

One value of x becomes infinite when $2y - 1 = 0$; the other value takes an indeterminate form in equation (4), but, referring to equation (1), it is found to be $x = -\frac{1}{8}$. Hence the line

$$y = \frac{1}{2}$$

is an asymptote, cutting the curve at the point $(-\frac{1}{8}, \frac{1}{2})$, as in the diagram. The expression under the radical sign evidently vanishes for $y = 1$; and, putting the radical in the form

$$\sqrt{(1 - y)(1 + y + 2y^2)},$$

it is seen to be imaginary for all values of y greater than unity, and real for all other values. Hence the curve lies entirely below the line

$$y = 1;$$

and, substituting this value in equation (4), we find that this line touches the curve at the point $(-1, 1)$.

Employment of the Ratio of the Coordinates

12. If we denote the ratio of the coordinates of any point by m , thus

$$m = \frac{y}{x},$$

the value of m may be regarded as a new coordinate of the point, which, in conjunction with one of the other coordinates, will serve to determine the position of the point. The points which have a common value of m are situated upon the straight line

$$y = mx$$

which passes through the origin; and, when a point is determined by the values of m and x , it is virtually determined as the intersection of a line of this character and a line parallel to the axis of y .

If, now, we eliminate y from the equation of a curve by means of the equation $y = mx$, we shall have the equation of the curve in the form of a relation between m and x . When this form of the equation can be solved for x , we can, by giving values to m , determine any desired number of points on the curve, and can trace its form with as much facility as when y is expressed in terms of x .

13. As an illustration, let the equation of the curve be

$$y^3 - x^3 + 2xy + x^2 = 0. \quad (1)$$

Putting mx for y , this becomes

$$(m^3 - 1)x^3 + (2m + 1)x^2 = 0. \quad (2)$$

The factor x^2 in this equation indicates that the curve passes twice through the origin; for a straight line generally cuts the curve in three points, and equation (2) shows that for all lines of the form $y = mx$ two of these points have the abscissa $x = 0$. Rejecting this factor, and solving for x , we have

$$x = \frac{1 + 2m}{1 - m^3}, \quad (3)$$

in which x is expressed in terms of the variable m and denotes the abscissa of the third point in which the line $y = mx$ meets the curve. The ordinate of this point, which we shall denote by P , is

$$y = mx = \frac{m(1 + 2m)}{1 - m^3}; \quad (4)$$

and, assuming the value of m , we may determine the position of P by means of any two of the three coordinates x , y and m .

14. Beginning with $m = 0$, equation (3) gives $x = 1$, and P is at A , the point $(1, 0)$, the line $y = mx$ coinciding with the

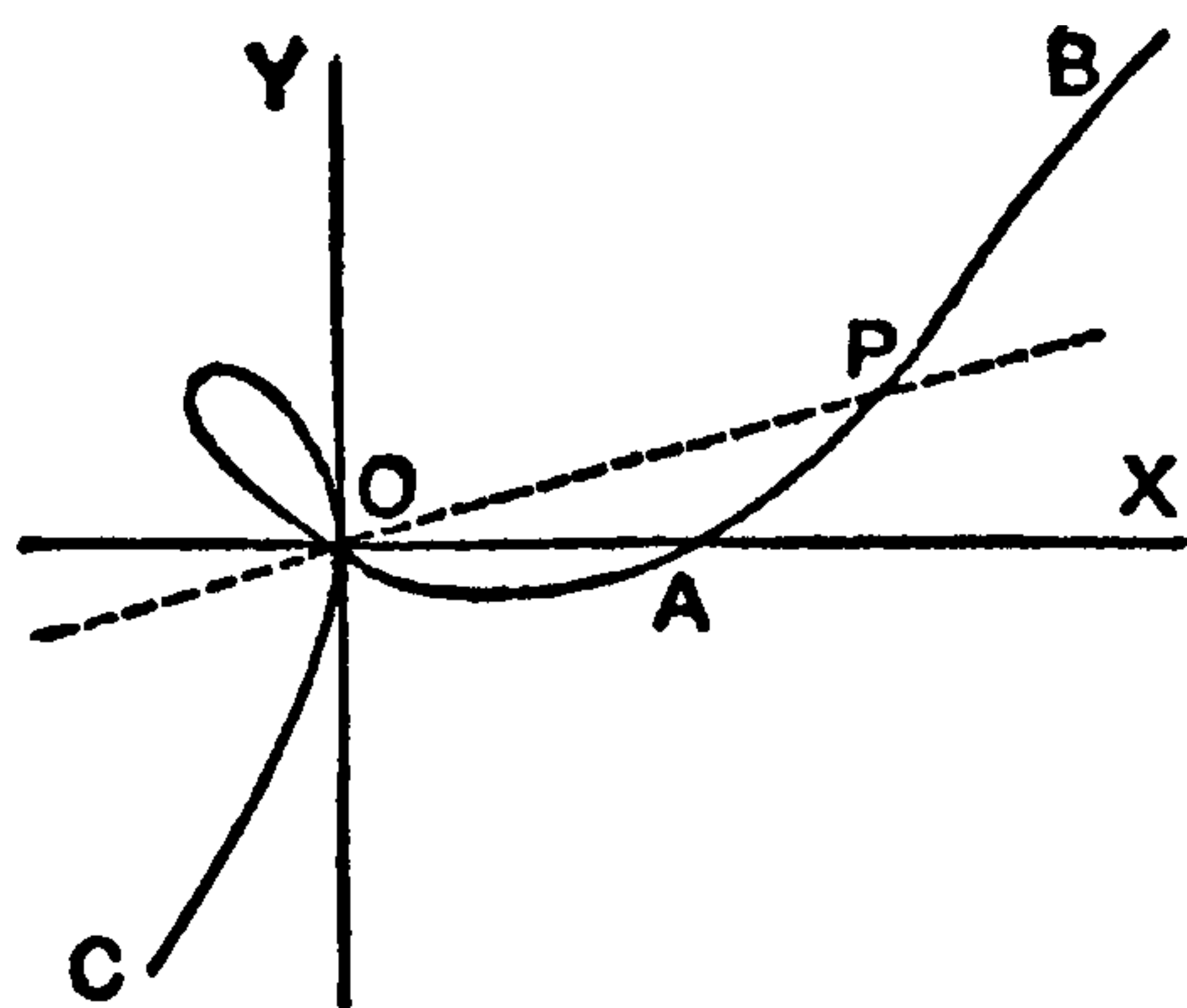


Fig. 5

axis of x ; as m increases, the line rotates about the origin, coinciding with the axis of y when m is infinite. When $m = 1$, the values of x and y are both infinite; thus, as m varies from 0 to 1, x increases from 1 to infinity, P describing the infinite branch AB in Fig. 5. Passing the value

$m = 1$, x changes sign; and as m passes from 1 to ∞ , P describes the infinite branch CO , arriving at the origin when $m = \infty$, since equation (4) then gives $y = 0$.

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Asymptotes

16. A real point at infinity usually corresponds to an asymptote whose direction is determined by the value of m . To illustrate the method of finding its position, let us determine the asymptote to the curve drawn in Fig. 5. The equation of the curve, equation (1), Art. 13, may be written in the form

$$y - x = - \frac{2xy + x^2}{y^2 + xy + x^2}. \quad (1)$$

Now, when the point P recedes indefinitely on the branch AB , x and y become infinite, but $y - x$ may nevertheless have a finite value. To find this value, put mx for y in equation (1), and then make $m = 1$ and $x = \infty$; or, what comes to the same thing, put $y = x$, and $x = \infty$. The process may be expressed thus,

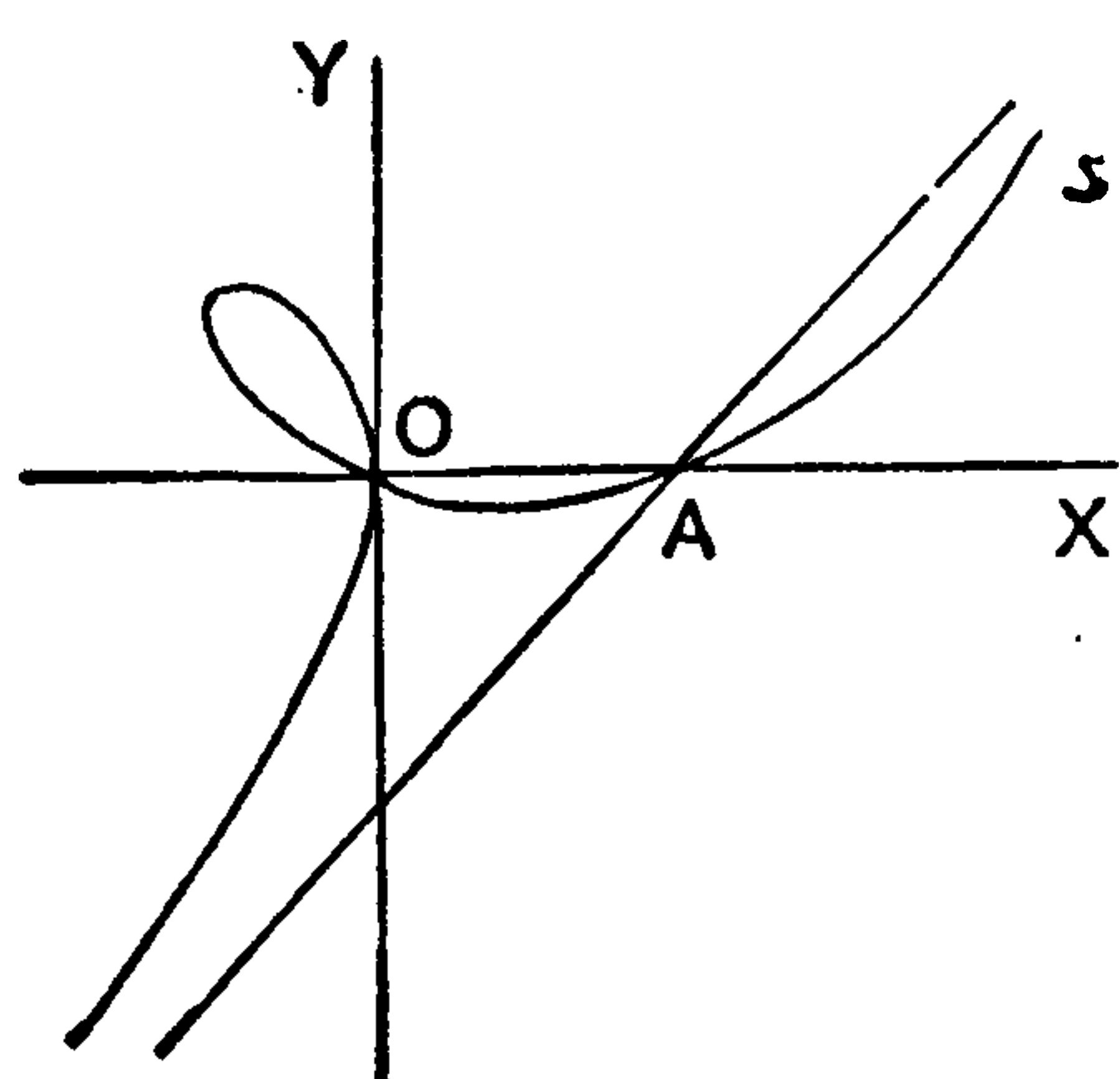


Fig. 6

$$y - x = - \left[\frac{2xy + x^2}{y^2 + xy + x^2} \right]_{y=x=\infty} = -1,$$

in which the suffixed equation is to be understood to mean that the ratio of y to x is one of equality when x is infinite. The result shows that, when this is the case, the quantity $y - x$ approaches the finite limit -1 . If, now, we draw the straight line

$$y - x = -1,$$

in which this quantity has constantly the value -1 , it is evident that the point P moving along the curve approaches indefinitely to this line, which is therefore an asymptote, as represented in Fig. 6.

17. It is obvious that, had the equation contained terms of a degree lower than the second, they would have vanished in the process of finding the asymptote. For example, the curve

$$y^3 - x^3 + 2xy + x^2 + x = 0,$$

whose equation differs from that of the curve drawn in Fig. 6 only by containing the term x , will have the same asymptote; for we have

$$y - x = - \frac{2xy + x^2 + x}{y^2 + xy + x^2} \Bigg]_{y=x=\infty} = - \frac{3x^2 + x}{3x^2} \Bigg]_{\infty} = - 1.$$

Thus, if the equation of the curve is of the n th degree, the asymptotes generally depend only upon the terms of the n th and $(n - 1)$ th degrees.

Symmetry of Curves

18. In some cases, the form of the equation indicates the symmetry of the curve in certain respects. For example, if the equation contains powers of y with even exponents only, the curve is symmetrical to the axis of x ; for, in this case, if the point (a, b) satisfies the equation, the point $(a, -b)$ symmetrically situated with respect to this axis also satisfies the equation. In like manner, if the equation contains powers of x with even exponents only, the curve is symmetrical to the axis of y . Again, if the sum of the exponents of x and y in each term is an even number, or if it is in each term an odd number, the curve is symmetrical with respect to the origin as a centre; for, in this case, if the point (a, b) satisfies the equation, the point $(-a, -b)$ will also satisfy it. If x and y are interchangeable, the curve is symmetrical to the line bisecting the angle between the axes; since, if the point (a, b) satisfies the

equation, the point (b, a) will also satisfy it. These considerations will be useful in the case of some of the following examples.

Examples I

Trace the curves whose equations are given below :

1. $x^2 + xy - y^2 + 3x = 0.$
2. $5x^2 - 4xy + 4y^2 - 8y - 4 = 0.$
3. $x^2 - 2xy + y + 2 = 0.$
4. $x^2 - 2xy + y^2 - 2x - 2y + 1 = 0.$
5. $6x^2 + 4xy + y^2 - 3x - 2y - 2 = 0.$
6. $x^2 - 8xy + 16y^2 - 6x - 12y + 9 = 0.$
7. $xy^2 - x^2 - y^2 = 0.$
8. $x^3 - x - y = 0.$
9. $x^2y - a^2y + a^3 = 0.$
10. $y^3 = x^2(x - a).$
11. $x^3 + x^2 - y^2 = 0.$
12. $x^2y - 2axy + a^2x - a^2y = 0.$
13. $x^4 = y^2(4a^2 - x^2).$
14. $4y^2(x + a) = a(5x + a)^2.$
15. $x^2y^2 = a^2(x^2 + y^2).$
16. $(x - 2a)xy = a(x - a)(x - 3a).$
17. $x^3 - xy^2 + ax^2 + ay^2 = 0.$
18. $x^4 - 3axy^2 + 2ay^3 = 0.$
19. $y^3 - x^3 - y + 4x = 0.$
20. $x^3 + y^3 - x^2 - y^2 = 0.$
21. $x^3 + y^3 - x^2 + 4y^2 = 0.$
22. $x^3 - y^3 + (2y - x)^2 = 0.$
23. $(x + y)(x^2 + y^2) = 2axy.$
24. $y^4 - y^3x + x^3 - 2x^2y = 0.$
25. $y^3 - x^3 + 2xy - y^2 = 0.$
26. $(1 + x^2)y = 1 + x \pm \sqrt{(x - x^3)}.$
27. $y^3 = x^2(x - a).$
28. $6x(1 - x)y = 1 + 3x.$

II

The Analytical Triangle

19. LET two intersecting straight lines be drawn; and let each term in the complete equation of the n th degree be represented by a point whose coordinates with respect to these lines are the exponents of x and y respectively in that term: we shall thus have a triangular arrangement of points, there being $n + 1$ points on each side of the triangle. Thus, if $n = 3$, we have the arrangement given in Fig. 7, in which there are four points on each side of the triangle. The diagram thus formed is called the *analytical triangle*; the lines of reference are the *analytical axes* of x and y respectively. The points upon OA , the analytical axis of x , represent the terms of the equation which do not contain y ; the points on OB represent the terms which do not contain x ; and the points on the third side, AB , of the analytical triangle represent the terms of the n th degree.

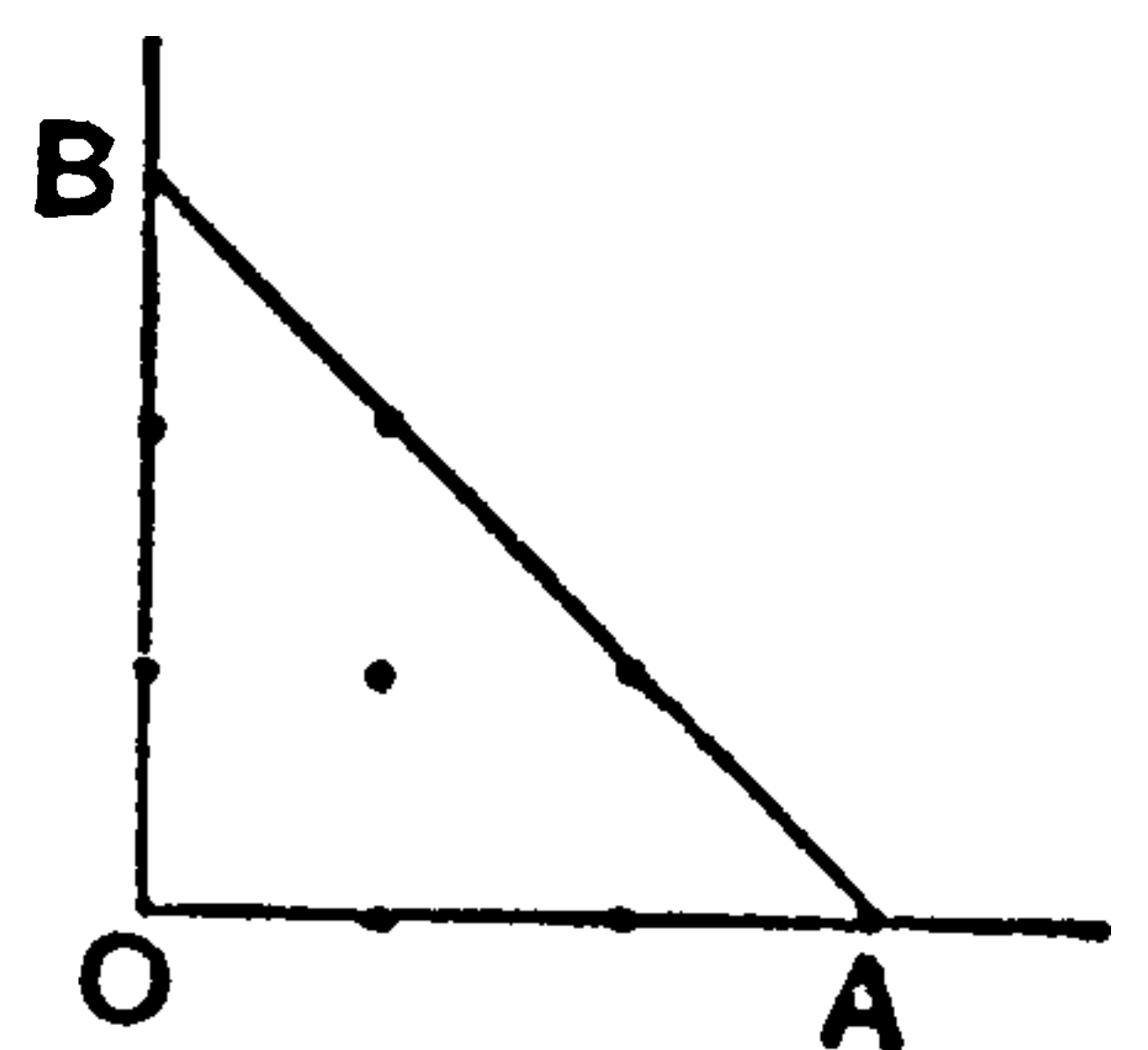


Fig. 7

20. The equation of a given curve is said to be *placed upon the analytical triangle* when the points which represent the terms actually occurring in the equation are marked in some convenient manner. Thus the equation

$$2x^2y - x^2 + y^2 + 2x = 0$$

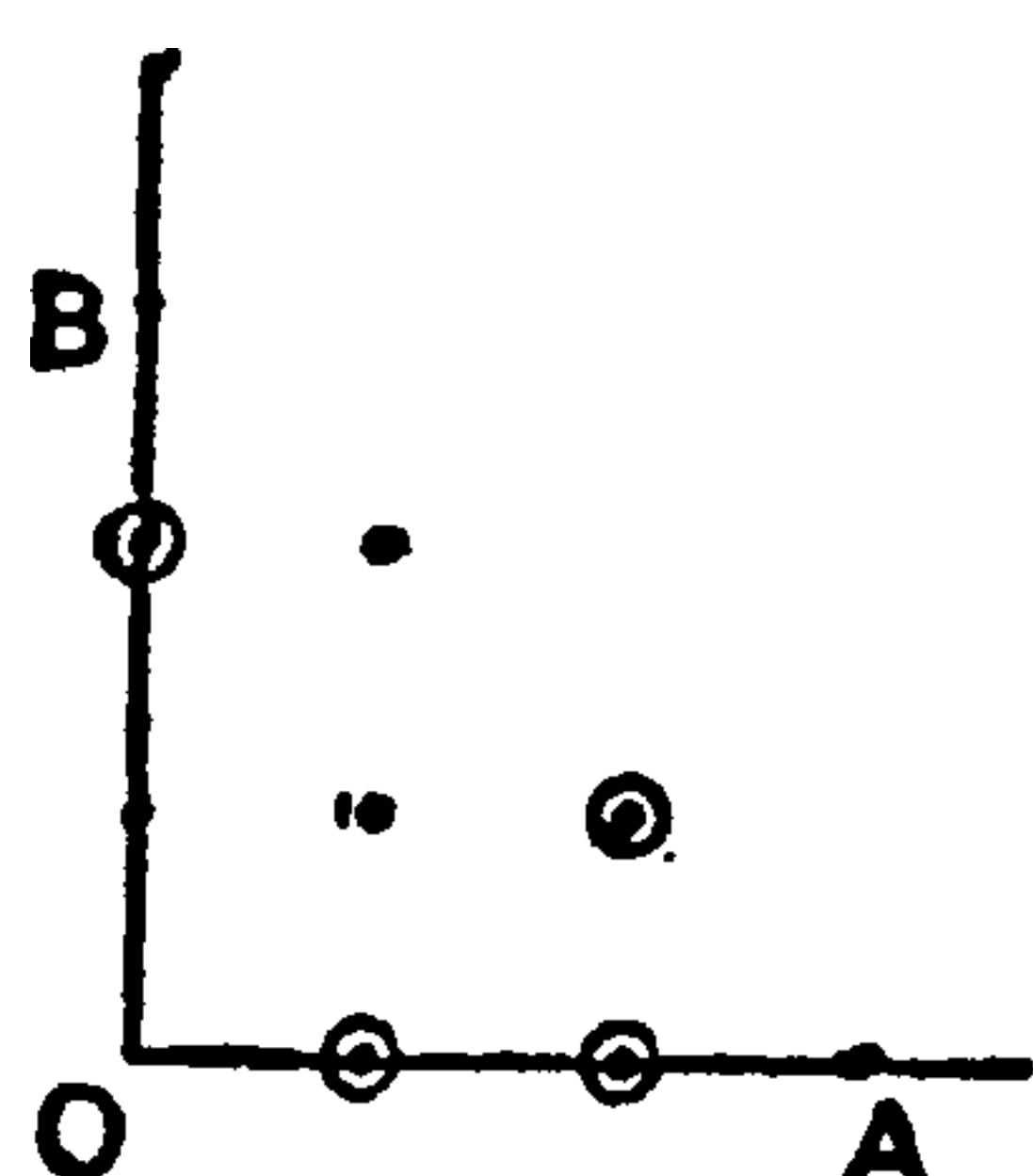


Fig. 8

of the curve traced in Fig. 4, page 8, is placed on the analytical triangle in Fig. 8. In thus placing the equation, no attention is paid to the coefficients, the object

being simply to indicate the presence of certain terms in the equation, and the absence of others. When the values of the terms are required, it is of course necessary to refer to the equation of the curve.

Intersections with the Axes

21. If we put $y = 0$ in the equation of a curve, and suppose x to have a finite value, all the terms except those represented by points on the analytical axis of x will vanish. Hence the result of equating these terms to zero is an equation determining all the finite points in which the curve meets the axis of x . This equation will generally have n roots real or imaginary; thus, if $n = 3$, its form will be

$$A + Bx + Cx^2 + Dx^3 = 0, \quad (a)$$

a cubic equation. If there be no absolute term in the given equation, that is, if in equation (a) $A = 0$, one of the roots will be zero, and the curve will pass through the origin. On the other hand, if the term Dx^3 be absent, the equation reduces to one of the second degree, and determines but two finite intersections with the axis of x ; but, putting equation (a) in the form

$$\frac{A}{x^3} + \frac{B}{x^2} + \frac{C}{x} + D = 0,$$

which is a cubic equation determining three values of $\frac{1}{x}$, we see that, when $D = 0$, one of the values of $\frac{1}{x}$ is zero, and therefore one of the values of x is infinite. Hence, in this case, one of the intersections of the curve with the axis of x is said to be *at infinity*. The geometrical meaning of this is that a straight

line usually cuts the curve in three points ; but, when the line is brought into coincidence with the axis of x , one of these points of intersection recedes indefinitely and disappears. An inspection of Fig. 8 shows that the curve whose equation is there placed upon the analytical triangle cuts the axis of x at infinity, at the origin, and at one other point.

22. In like manner, an inspection of Fig. 8 shows that the curve cuts the axis of y at infinity, and *twice* at the origin ; it is therefore said to meet the axis of y in *two coincident points* at the origin. The geometrical meaning of this is that two of the three points in which a straight line cuts the curve come into coincidence at the origin when the cutting line is brought into coincidence with the axis of y . It is evident that this will happen whenever the curve has two branches passing through the origin, and also whenever the axis of y is a tangent to the curve ; but, since in the present instance the curve cuts the axis of x but once at the origin, we infer that the curve has a single branch passing through the origin and touching the axis of y .

23. It is evident that there must always be at least one of the marked points upon each analytical axis ; for otherwise the equation could be divided throughout by y or by x , and its locus would not be a proper curve of the n th degree, but the combination consisting of the straight line $y = 0$, or $x = 0$, and a curve of the $(n - 1)$ th degree. If there is but one marked point on an analytical axis, it divides the side of the analytical triangle into parts which indicate respectively the number of times the curve cuts the corresponding geometrical axis at the origin and at infinity, and the curve cuts the axis in no other point. But, if there are two or more marked points upon the analytical axis, the number of spaces between the most distant of these points indicates the number of finite points of intersection with

the axis distinct from the origin. When there are two or more of these roots, a pair of them may be imaginary, or they may be real and equal. In the latter case, the curve cuts the geometrical axis in two coincident points, which generally indicates tangency to the axis, as in the case of the two zero roots for y considered in Art. 22.

The Line at Infinity

24. It was shown in Art. 15 that, if we put equal to zero the group of terms of the highest degree in the equation of a curve, the resulting equation gives the values of m , the ratio of y to x , for the points at infinity. These terms are those whose representative points are situated upon the third side, AB , of the analytical triangle; and it is customary to speak of this equation as determining *the intersections of the curve with the line at infinity*. Each of the real intersections generally determines, as shown in Art. 16, the direction of an asymptote.

25. The equation in question is of the general form

$$Lx^n + Mx^{n-1}y + \dots + Ry^n = 0,^* \quad (b)$$

which, being of the n th degree, determines n values of $\frac{y}{x}$; thus, the line at infinity, like an ordinary straight line, is said to cut the curve in n real or imaginary points. If the term Lx^n , represented by the vertex A of the analytical triangle, is wanting

* If the equation of the curve is as usual rendered homogeneous by the introduction of the letter a , which may be regarded as denoting the unit of length, this equation can be derived from the equation of the curve by putting $a = 0$; just as the equation for the intersections with the axis of x is derived by putting $y = 0$. Hence, as $y = 0$ is the equation of the axis of x , so the impossible equation $a = 0$ is regarded as the equation of the line at infinity.

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fore the equation of the asymptote. Compare Art. 11. It is evident that, in this process, all the terms vanish except those containing the highest power of x which occurs in the equation, that is to say, the terms represented by the points adjacent to the vertex A of the analytical triangle; hence the result is arrived at by simply putting the sum of these terms, namely, $2x^2y - x^2$, equal to zero, and rejecting the common factor x^2 .

Parabolic Branches

27. The attempt to find an asymptote parallel to the axis of y in the case of the curve considered in the preceding articles results in an impossible equation. Thus, dividing equation (1) by y^2 , we have

$$2 \frac{x^2}{y} - \frac{x^2}{y^2} + 1 + 2 \frac{x}{y^2} = 0,$$

in which it is impossible to make y infinite while x remains finite; we infer, therefore, that the curve has *parabolic branches* in the direction of the axis of y ; that is, infinite branches which, like the parabola, have no asymptote. This is owing to the absence from the equation of the curve of the term represented by the point on the third side AB of the analytical triangle adjacent to the vertex B . See Fig. 8, page 15. Now, in consequence of the absence of this term, the equation for the intersection with the line at infinity has a double root $x = 0$, so that *the line at infinity is said to meet the curve in two coincident points* where it crosses the axis of y . It is evident that this would also occur if the curve had two asymptotes parallel to the axis of y ; but, since Fig. 8 shows that the curve meets the axis of y but once at infinity, the diagram indicates the parabolic branch.

28. In general, the line at infinity is said to meet the curve

in *two coincident points* whenever the equation for its intersections has a pair of equal roots, and this usually indicates a parabolic branch. For, let the equation of the curve be written in the form

$$P_n + P_{n-1} + P_{n-2} + \dots = 0, \quad (1)$$

where P_n denotes the sum of the terms of the n th degree, P_{n-1} the sum of the terms of the $(n-1)$ th degree, and so on: then, if m_1 is a double root of the equation $P_n = 0$, equation (1) may be written in the form

$$(y - m_1x)^2 Q_{n-2} + P_{n-1} + P_{n-2} + \dots = 0, \quad (2)$$

in which Q_{n-2} is an expression of the $(n-2)$ th degree. Putting $y = m_1x$ in the expressions Q_{n-2} , P_{n-1} , etc., the equation takes the form

$$(y - m_1x)^2 x^{n-2} q_{n-2} + x^{n-1} p_{n-1} + x^{n-2} p_{n-2} + \dots = 0, \quad (3)$$

in which q_{n-2} , p_{n-1} , etc., are numerical quantities. Dividing by x^{n-2} , and making x infinite, it is plain that this gives an infinite value to $y - m_1x$, *except when* $p_{n-1} = 0$. Hence in general we have a parabolic branch, and *the line at infinity is regarded as a tangent to the curve*. This may be further explained as meaning that, whereas the asymptote to an infinite branch is the tangent whose point of contact is at infinity, a parabolic branch is one for which the tangent whose point of contact is at infinity is altogether at an infinite distance.

Parallel Asymptotes

29. The exceptional case mentioned in the preceding article occurs when the substitution $y = m_1x$ reduces the expression P_{n-1} to zero; that is, when P_{n-1} contains $y - m_1x$ as a factor.

When this is the case, the value of $y - m_1x$, when x and y are infinite, may be determined by a quadratic equation. For example, let the equation of the curve be

$$x^3 - 2x^2y + xy^2 + ax^2 - axy - 2a^2x + a^2y = 0, \quad (1)$$

which may be written in the form

$$x(x - y)^2 + ax(x - y) - 2a^2x + a^2y = 0.$$

Dividing by x , and making $y = x$ when x is infinite, we have the equation

$$(x - y)^2 + a(x - y) - a^2 = 0,$$

which gives

$$x - y + \frac{a}{2}(1 + \sqrt{5}) = 0 \quad \text{and} \quad x - y + \frac{a}{2}(1 - \sqrt{5}) = 0,$$

the equations of two parallel asymptotes.

The case here considered of course includes that in which there are no terms of the $(n - 1)$ th degree; that is, when $P_{n-1} = 0$. It is to be noticed that the asymptotes, being determined by a quadratic equation, may be imaginary, in which case there will be no corresponding infinite branches.

30. An inspection of Fig. 9, in which equation (1) is placed

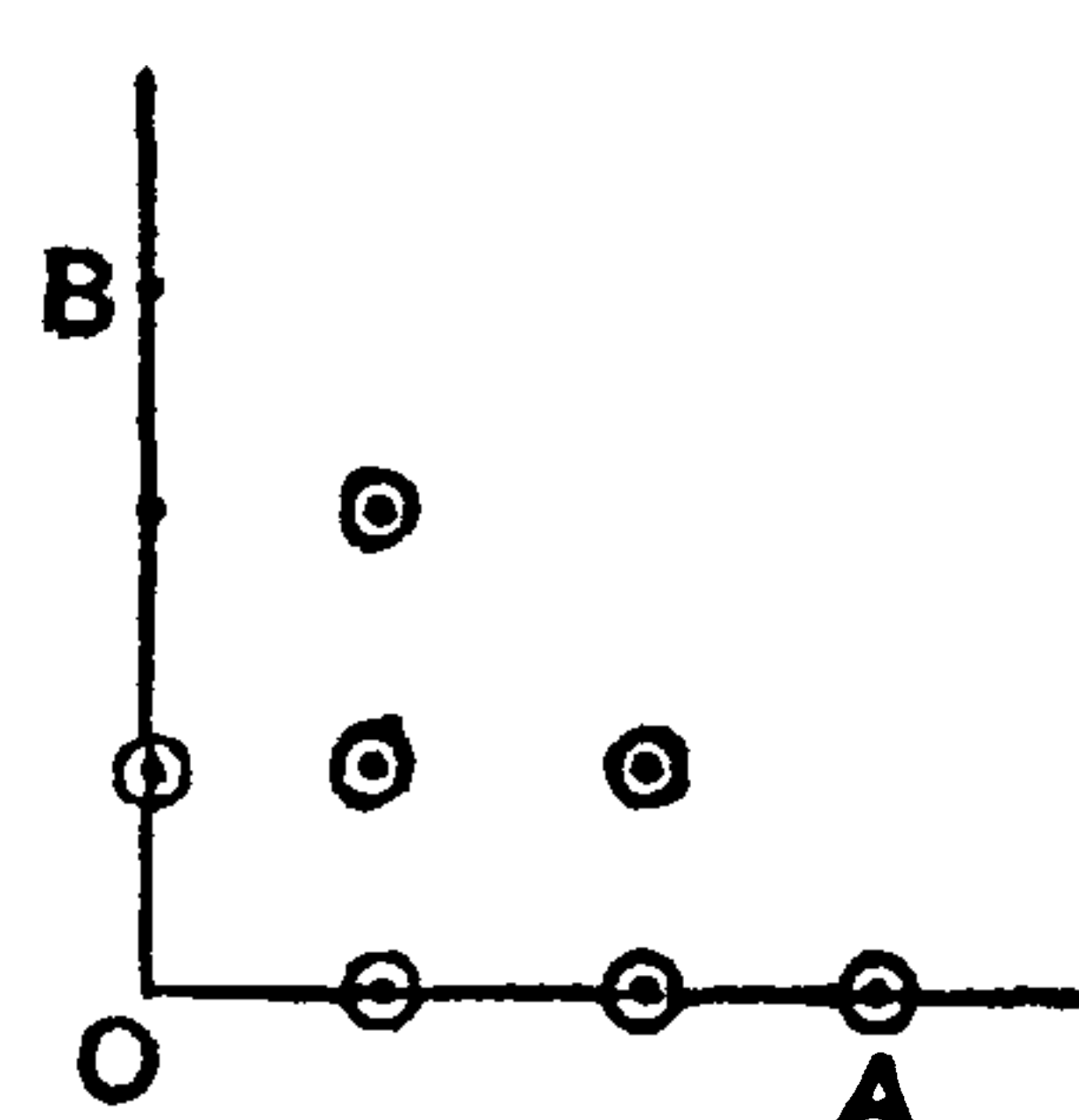


Fig. 9

upon the analytical triangle, shows that the curve cuts the axis of y at the origin and in two coincident points at infinity; and, since it cuts the line at infinity but once in the direction of the axis of y , we infer that the axis of y is a tangent at infinity, that is to say, an asymptote. The

method given in Art. 26 also shows that the asymptote is in this case the axis itself.

31. The intersections with the axis of x are determined by the equation

$$x^3 + ax^2 - 2a^2x = 0$$

or

$$x(x - a)(x + 2a) = 0.$$

The points indicated are constructed in Fig. 10, together with the asymptotes determined in the preceding articles. It is evident that a line parallel to the parallel asymptotes cuts the curve in two points at infinity, and hence each of these asymptotes meets the curve in *three* points at infinity, and in no other point. Thus the curve consists of three disconnected branches, as represented in the diagram.

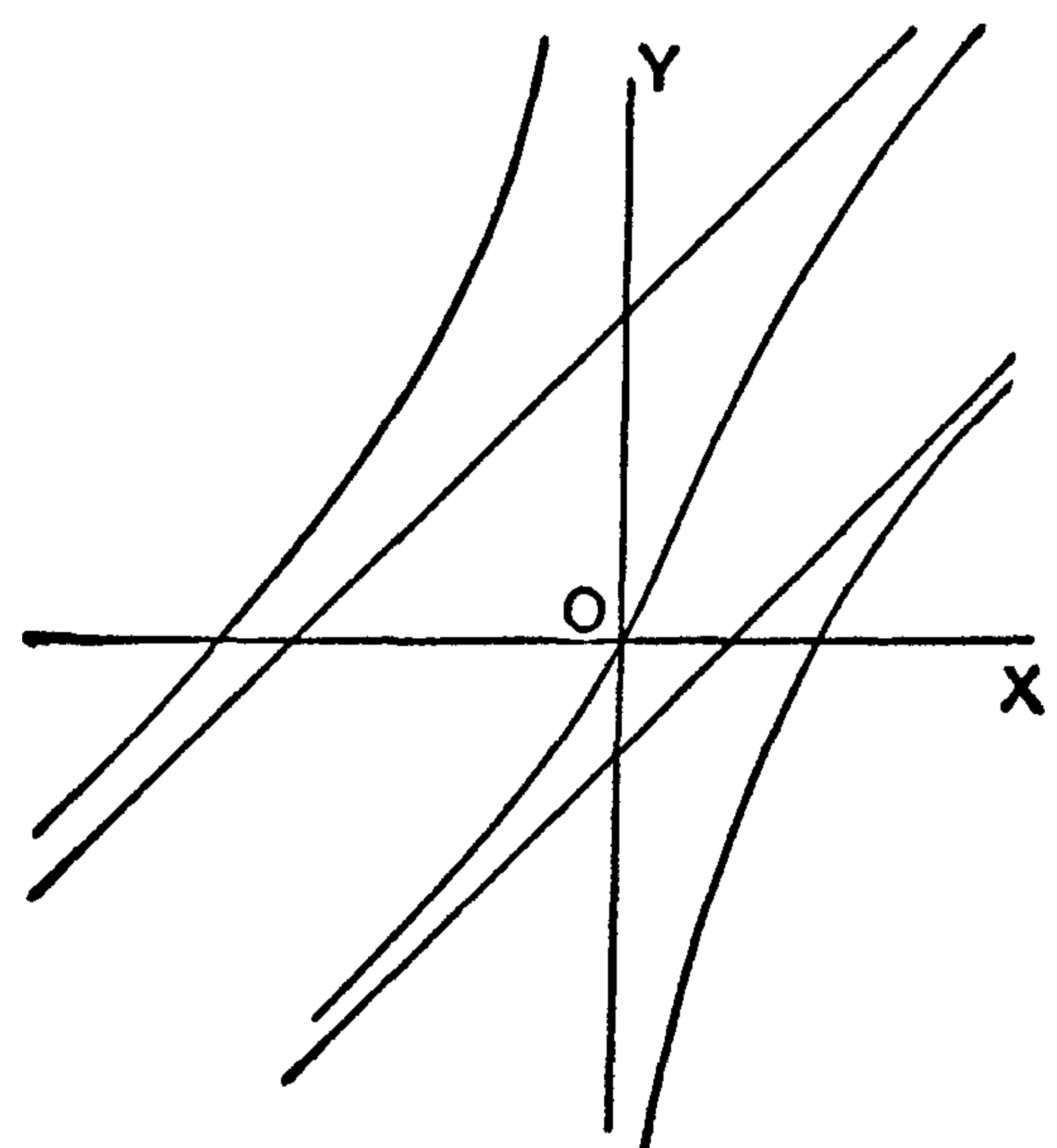


Fig. 10

Tangents

32. The form of the curve is more precisely ascertained by determining its inclination at known points. Dividing the equation of the curve, equation (1), Art. 29, by x , we have

$$x^2 - 2xy + y^2 + ax - ay - 2a^2 + a^2 \frac{y}{x} = 0. \quad (2)$$

Now suppose the point (x, y) moving upon the curve to pass through the origin; the direction of its motion at the instant when it reaches the origin depends upon the value of the ratio $\frac{y}{x}$ at this instant, which may be called the *direction ratio* of the curve at this point. This value is determined by putting $x = 0$ and $y = 0$ in equation (2). The result is

$$-2a^2 + a^2 \left[\frac{y}{x} \right]_{0,0} = 0,$$

in which all the terms of equation (2) have vanished except those derived from the terms of the first degree in equation (1). Hence the direction of the curve at the origin is the same as that of the straight line whose equation is formed by equating to zero the terms represented by the points in Fig. 9 adjacent to the analytical origin. Thus,

$$-2a^2x + a^2y = 0,$$

or

$$y = 2x,$$

is the equation of the tangent at the origin.

33. The tangents at the other points where the curve crosses the axis of x may be found by a method similar to that employed in the preceding article. Grouping together the terms containing the same power of y , equation (1), Art. 29, may be written in the form

$$x(x - a)(x + 2a) + y(a^2 - ax - 2x^2) + xy^2 = 0. \quad (3)$$

Now, when the point (x, y) , moving upon a branch of the curve, passes through the point $(a, 0)$, its direction ratio will be the value of $\frac{y}{x - a}$, the quantity which would be denoted by $\frac{y}{x}$ if the curve were referred to the point $(a, 0)$ as origin. From equation (3), we derive

$$\frac{y}{x - a} = \frac{x(x + 2a)}{2x^2 + ax - a^2 - xy}; \quad (4)$$

whence, putting $x = a$ and $y = 0$ in the second member, we have

$$\left. \frac{y}{x - a} \right]_{a, 0} = \frac{3}{2},$$

which determines the direction of the curve at the point $(a, 0)$. Hence the straight line

$$2y = 3(x - a),$$

which passes through this point in the same direction, is the required tangent. In like manner, the equation of the tangent at $(-2a, 0)$ is found to be

$$5y = 6(x + 2a).$$

34. It was shown in Art. 17 that the position of an asymptote generally depends only upon the terms of the n th and of the $(n - 1)$ th degrees; that is to say, when the equation of the curve is placed on the analytical triangle, the asymptote is determined by the terms represented by the marked points upon the side AB of the analytical triangle and upon the adjacent parallel line. The process exemplified in the preceding article shows that, in like manner, the position of the tangent at a point of intersection with the axis of x generally depends only upon the terms not containing y and those containing the first power of y , which terms are represented by the marked points on the analytical axis of x and on the adjacent parallel line. If the latter terms be wanting, the value of $\left. \frac{y}{x - a} \right]_{a, 0}$ will be infi-

nite, and the tangent will be the line $x = a$, parallel to the axis of y . For example, the curve

$$x^3 - y^3 + ax^2 + ay^2 = 0$$

cuts the axis of x at the point $(-a, 0)$, and the equation does not contain the first power of y ; accordingly, we find

$$\left. \frac{y}{x + a} \right]_{-a, 0} = \left. \frac{x^2}{y(y - a)} \right]_{-a, 0} = \infty,$$

thus the line $x = -a$ is the tangent to the curve.

35. If the equation determining the intersections with the axis of x has a double root, the curve is said to meet the axis of x in *two coincident points*. The numerator of the value of the direction ratio, found as in Art. 33, will in this case contain the factor $x - a$, supposing $(a, 0)$ to be the intersection in question; hence its value will generally be zero, and the axis of x will be the tangent. Thus, in the case of the curve

$$x^3 + xy^2 - 2ax^2 + a^2x - a^2y = 0,$$

the equation for the intersections with the axis of x is

$$x(x - a)^2 = 0,$$

of which $x = a$ is a double root, and we have

$$\left. \frac{y}{x - a} \right]_{a,0} = \left. \frac{x(x - a)}{a^2 - xy} \right]_{a,0} = 0.$$

Hence $y = 0$ is the equation of the tangent, and the two coincident intersections indicate tangency to the axis of x .

Nodes

36. If, however, when $x = a$ is a double root, the terms containing the first power of y be wanting, or if they contain $x - a$ as a factor, the value of $\left. \frac{y}{x - a} \right]_{a,0}$ can be determined by a quadratic equation, as in the following example. Let the equation of the curve be

$$x^3 - y^3 - a(4x^2 - 3xy + 2y^2) + a^2(4x - 6y) = 0. \quad (1)$$

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When the roots are equal, we have a *cusp*, or point at which two branches meet with a common tangent. When the roots are imaginary, there are no real tangents; hence we have an isolated point or *acnode*. It will be noticed that the case of parallel asymptotes, explained in Art. 29, is analogous to the node; accordingly, a curve with parallel asymptotes is said to have a *node at infinity*. If the asymptotes, or tangents at the node at infinity, are found to be coincident, we have a cusp at infinity; and if they are found to be imaginary, the curve is said to have an *acnode at infinity*.

Intersection of a Curve with a Tangent

38. Since a tangent to a curve meets the curve in two coincident points at the point of contact, it can meet the curve in only $n - 2$ other points, the curve being of the n th degree. Hence, if we eliminate one of the variables between the equations of the curve and a tangent, the result will be an equation of the $(n - 2)$ th degree. This circumstance facilitates the determination of particular points which may be useful in tracing the curve. In particular, if the curve be a cubic, the equation will be of the first degree, and the single point of intersection determined will have rational coordinates. For example, the equation of the tangent at the origin to the curve constructed in Fig. 10, page 23, was found in Art. 32 to be

$$y = 2x.$$

Substituting in the equation of the curve, equation (1), Art. 29,

$$x^3 - 2x^2y + xy^2 + ax^2 - axy - 2a^2x + a^2y = 0,$$

we have

$$x^3 - ax^2 = 0.$$

Rejecting the factor x^2 , which corresponds to the two coincident points at the origin, we have $x = a$; hence the point $(a, 2a)$ is a point of the curve.

Intersection of a Curve with an Asymptote

39. Since an asymptote is a tangent whose point of contact is at infinity, the remarks made in the preceding article apply also to the intersection of a curve with its asymptotes. In particular, a cubic cuts each of its asymptotes in a single point. When the cubic has three real asymptotes, it is convenient to determine the three points of intersection at once by combining with the equation of the curve the equation which represents the three asymptotes. This equation will be a cubic agreeing with the equation of the curve in the terms of the third and of the second degree, for it follows from Art. 17 that curves having the same asymptotes must agree in the terms of the n th and $(n - 1)$ th degrees. If, therefore, we subtract this equation from that of the given curve, we shall have an equation of the first degree which must be satisfied by each of the three intersections; in other words, the equation of a straight line cutting the three asymptotes in the required points.

40. For example, let the equation of the curve be

$$x^2y + xy^2 + 2x^2 - 3xy - 2y^2 = 0.$$

The equations of the asymptotes are

$$x - 2 = 0, \quad y + 2 = 0,$$

and

$$x + y - 3 = 0.$$

Multiplying these equations, we have, for the equation of the three asymptotes in combination,

$$xy(x + y) + 2x^2 - 3xy - 2y^2 - 10x + 2y + 12 = 0.$$

Hence the equation of the straight line passing through the points of intersections is

$$5x - y - 6 = 0.$$

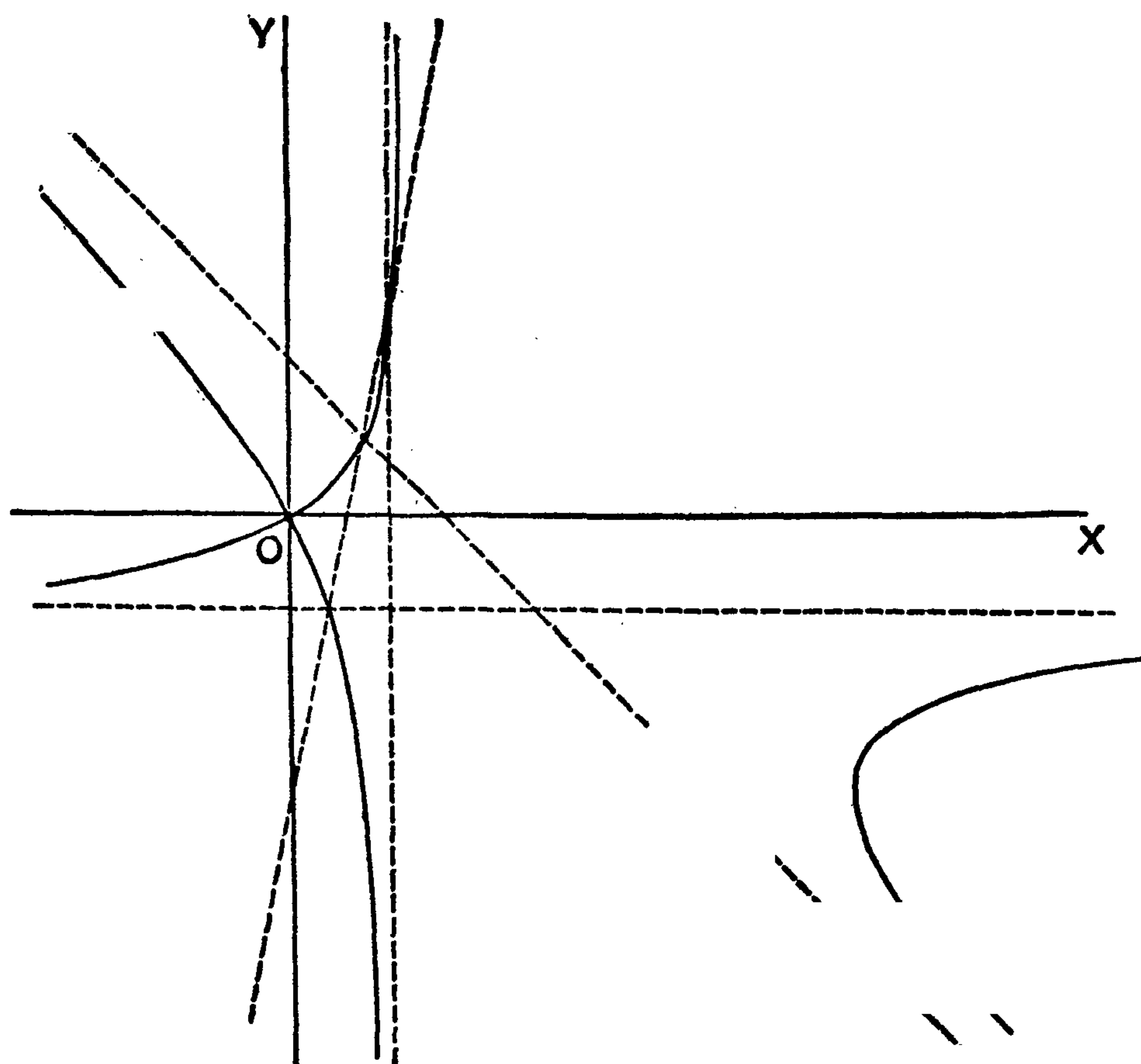


Fig. 11

Combining this equation with the separate equations of the asymptotes, we find the points of intersection to be $(2, 4)$, $(\frac{4}{5}, -2)$ and $(\frac{3}{2}, \frac{3}{2})$. The asymptotes and the line of intersection are constructed in Fig. 11. The curve

has a node at the origin, at which the tangents are

$$2x^2 - 3xy - 2y^2 = 0,$$

or

$$x - 2y = 0, \quad \text{and} \quad 2x + y = 0;$$

hence the form of the curve is that indicated in the figure.

Examples II

1. $x(x^2 + y^2) + a(y^2 - 2x^2) = 0.$
2. $y^3 - x^2y + xy + 2x^2 = 0.$
3. $x^2y - y^2x + x^2 - 4y^2 = 0.$

-
4. $x^2y^2 + 3xy^3 + x^2 - xy - 2y^2 = 0.$
 5. $x^2y - y^2x + x^2 - 4y^2 = 0.$
 6. $x^5 - 2a^3x^2 + 5a^3xy - 2a^3y^2 + y^5 = 0.$
 7. $x(x^2 + y^2) + x^2 - y^2 + x - y = 0.$
 8. $(x^2 - a^2)(y^2 - b^2) = a^2b^2.$
 9. $x^2(y - x) + xy + x^2 + x + 2 = 0.$
 10. $(x^2 - y^2)^2 = a^2(x^2 + y^2).$
 11. $x(y + x)^2 + a^2y = 0.$
 12. $x^3 - xy^2 + 3axy + ay^2 = 0.$
 13. $xy^2 + 4(x + y) = 8.$
 14. $x^3 - 4xy^2 - 3x^2 + 12xy - 12y^2 + 8x + 2y + 4 = 0.$
 15. $(y - 2x)(y^2 - x^2) - a(y - x)^2 + 4a^2(x + y) - a^3 = 0.$
 16. $(y^2 - x^2)(y - 2x) + a(4y^2 - 9x^2) = 0.$
 17. $(y - x)^2(y + 2x) + 4(y - x)^2 - 16(y + 2x) = 0.$
 18. $(x^2 - y^2)(x - a) + a^2y = 0.$
 19. $y(y - x)^2(y - 2x) + 3a(y - x)x^2 - 2a^2x^2 = 0.$
 20. $x^2y^2 - 3xy^2 - 4x^2 + 2y^2 = 0.$

III

Approximate Forms of Curves

41. WHEN the equation of a curve is placed upon the analytical triangle, the absence of the term represented by the analytical origin indicates that the curve passes through the origin of coordinates ; and we have seen in Art. 32 that the result of equating to zero the terms of the first degree is the equation of the tangent to the curve at the origin, that is to say, the straight line which approximates most closely to the curve at that point. Accordingly, if only one of the terms of the first degree occurs in the equation of the curve, the tangent is one of the coordinate axes ; and it is now to be shown that we can, in this case, with equal facility determine a closer approximation to the form of the curve at the origin.

42. For example, let the given equation be

$$x^2y + ay^2 - a^2x = 0, \quad (1)$$

which is placed upon the analytical triangle in Fig. 12. The tangent at the origin is the line $x = 0$, the axis

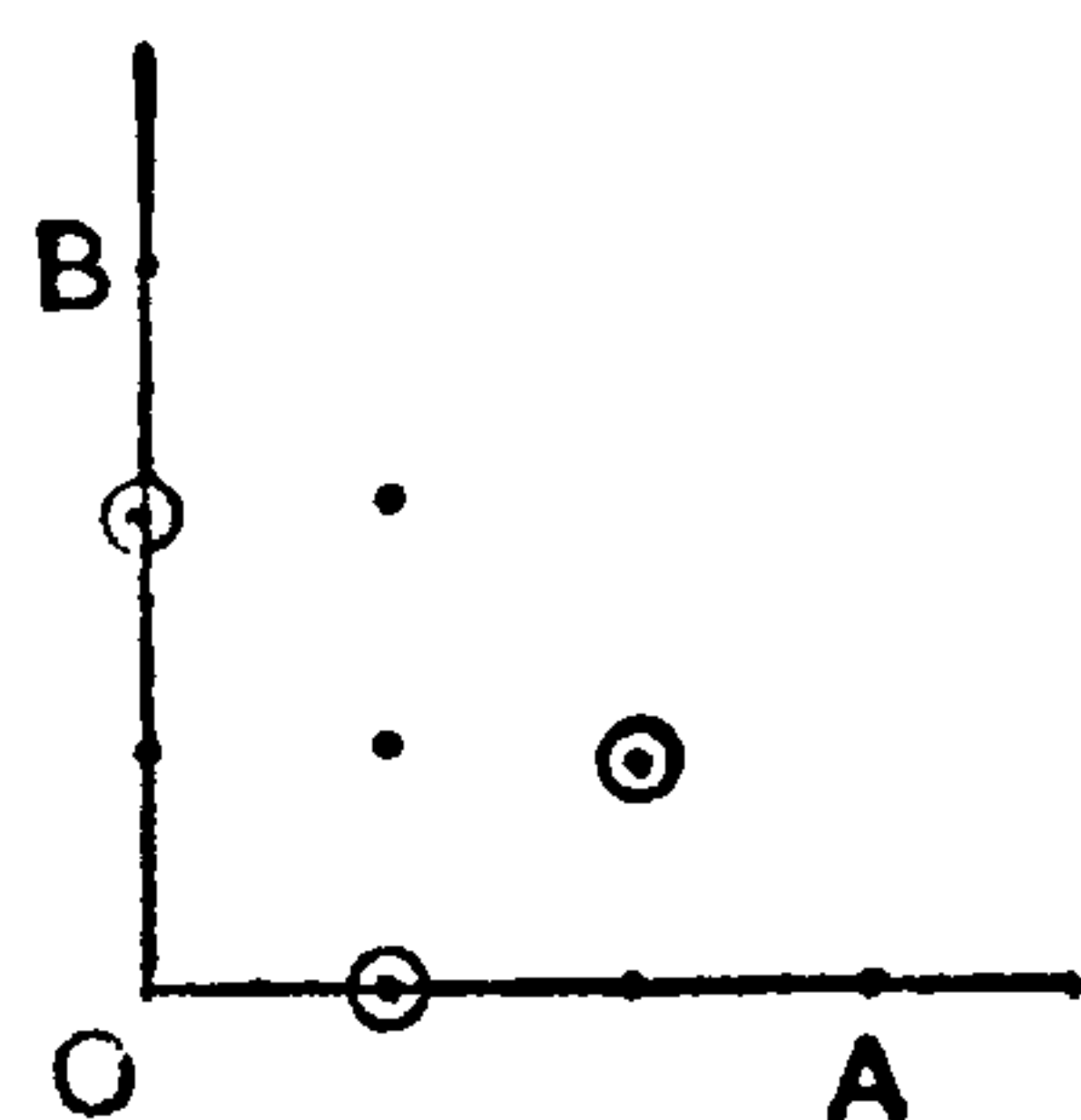


Fig. 12

of y . The ratio $\frac{y}{x}$ being infinite at the origin, the

ratio $\frac{y^2}{x}$ may have a finite value when x and y

vanish simultaneously. Dividing equation (1) by

x , we have

$$xy + a \frac{y^2}{x} - a^2 = 0, \quad (2)$$

and, making x and y each equal zero in this equation, we have

$$\left[\frac{y^2}{x} \right]_{0,0} - a = 0; \quad (3)$$

hence the equation

$$y^2 - ax = 0, \quad (4)$$

in which this ratio has constantly the same value as that assigned by equation (3), represents a curve approximating to the form of the given curve at the origin. This approximating curve is the parabola $y^2 = ax$, which touches the axis of y and lies on its right; hence the form of the curve at the origin is as represented in Fig. 13.

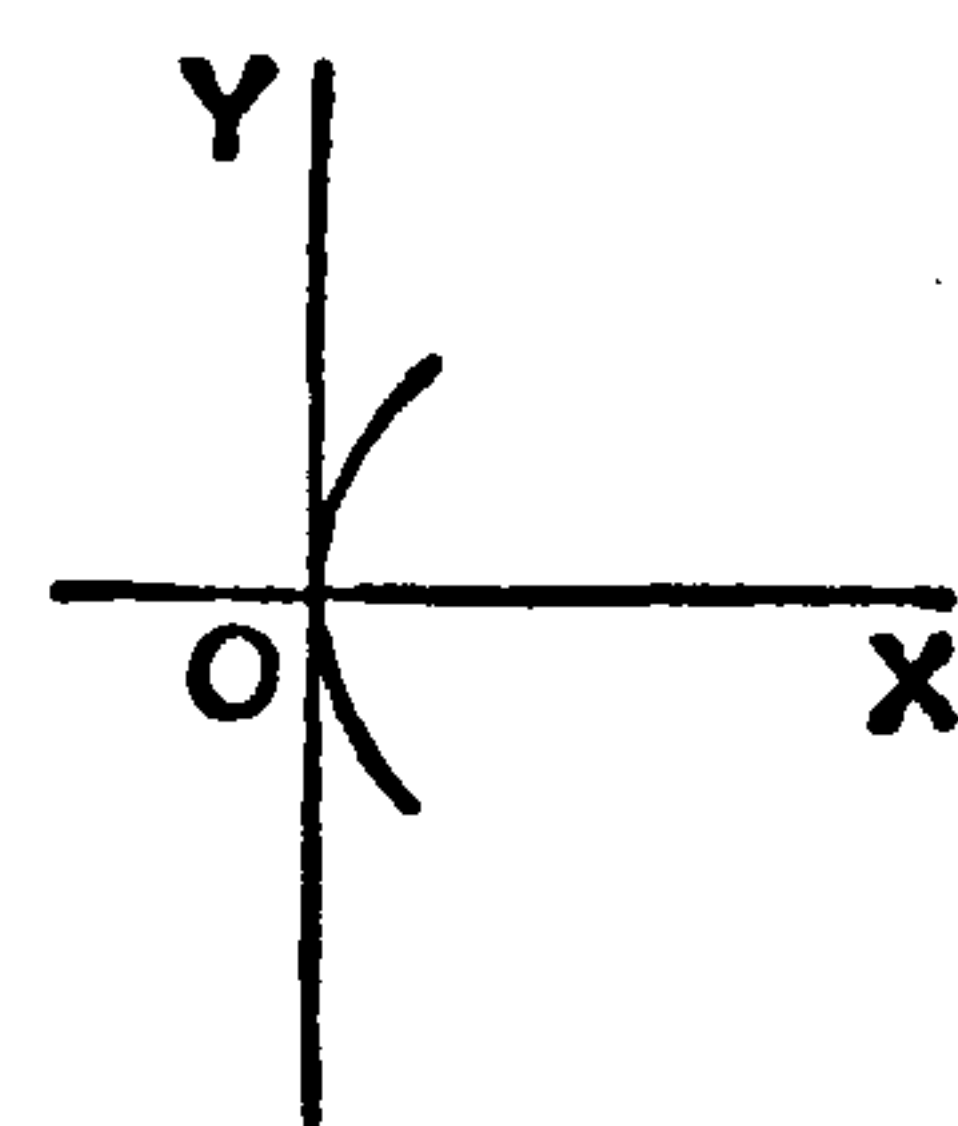


Fig. 13

Approximate Forms at Infinity

43. The absence of the term represented by the vertex A of the analytical triangle indicates, as explained in Art. 25, that the curve passes through the intersection of the axis of x with the line at infinity; and it is shown in Art. 26 that, when this is the case, the result of putting equal to zero the terms containing the highest power of x which occurs in the equation is the equation of the asymptote parallel to the axis of x . It will be convenient to refer to this point at infinity as the point A ; thus, as in the case of the origin, the equation of the tangent at the point A consists of the terms represented by the points of the analytical triangle adjacent to the vertex A . Accordingly, when, as in Fig. 12, the term represented by the point on the analytical axis of x adjacent to the vertex A is absent, the tangent at A is the line $y = 0$, the axis of x , as explained in Art. 30; but, in this case, as in the analogous case at the origin, we can with equal facility determine a closer approxima-

tion to the form of the curve at the infinitely distant point A . Since, for a point P moving on the infinite branch passing through A , y becomes zero as x becomes infinite, the value of xy may remain finite. Dividing the equation of the curve, equation (1), Art. 42, throughout by x , we have

$$xy + a \frac{y^2}{x} - a^2 = 0.$$

Making $x = \infty$ and $y = 0$ in this equation, we find

$$xy]_{\infty, 0} - a^2 = 0.$$

Hence the hyperbola

$$xy - a^2 = 0 \quad (5)$$

approximates closely to the remote portions of the infinite branch passing through A . This hyperbola lies in the first

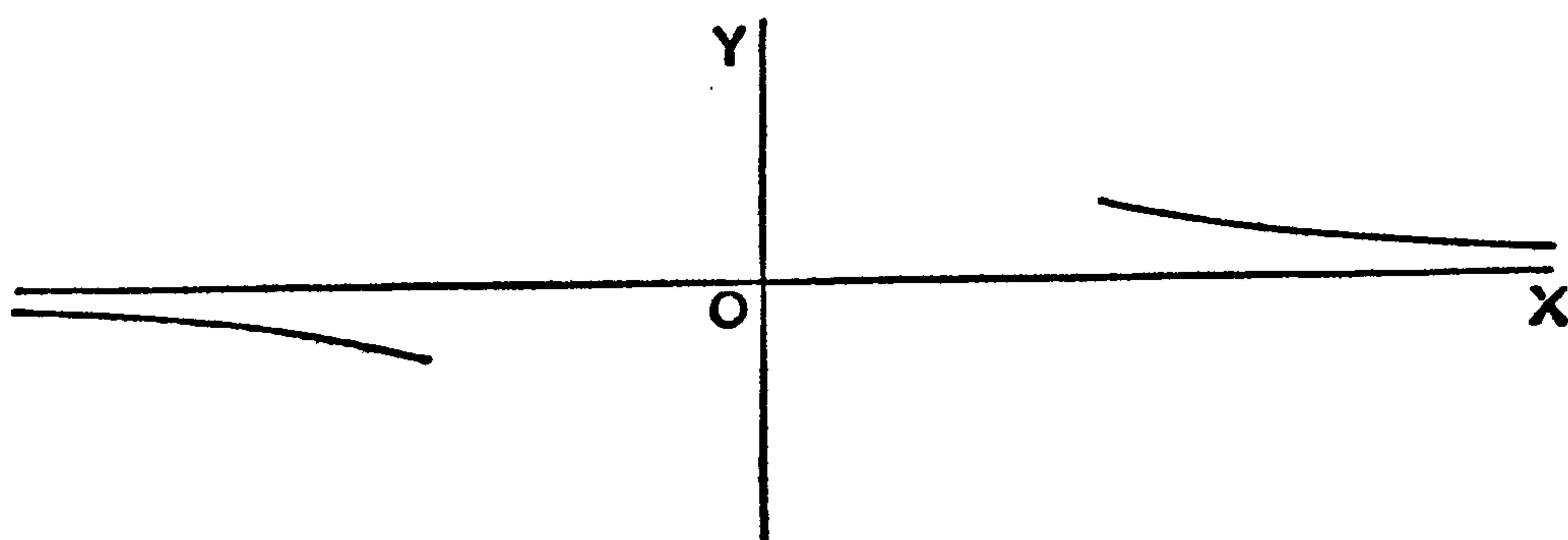


Fig. 14

and third quadrants: we infer, therefore, that the infinite branch having the axis of x as an asymptote lies above the axis on the right and below it on the left, as in Fig. 14.

44. Fig. 12 shows that the curve passes also through the intersection of the line at infinity with the axis of y , which we shall call the point B ; and the absence of the term represented by the point adjacent to B on the side AB of the analytical triangle indicates, as explained in Art. 27, that the line at infinity is the tangent at B ; in other words, that the infinite branch passing through B is parabolic. In this case also we can determine an approximation to the form of the infinite

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46. The general form of the curve being indicated by the positions of the approximating curves, it is of course unneces-

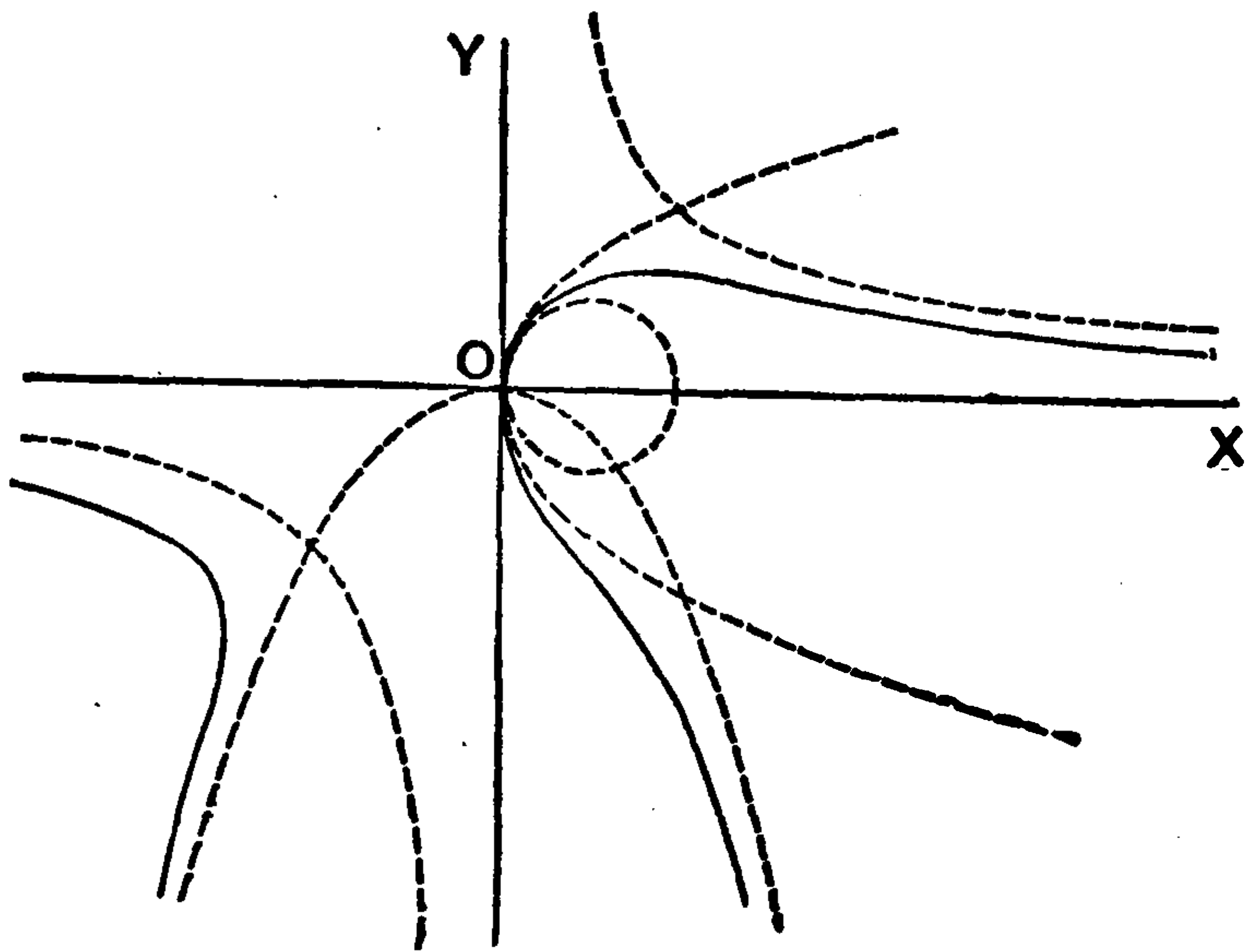


Fig. 15

sary to construct the latter, unless an accurate drawing is desired. It is, however, useful to notice that, when the approximating curve at the origin is a common parabola, the radius of curvature at the origin is one half the coefficient of the coordinate whose first power occurs, when the

coefficient of the square is unity. For example, equation (4), Art. 42, is

$$y^2 = ax.$$

The equation of the circle whose radius is $\frac{1}{2}a$, and which touches the axis of y at the origin, is

$$y^2 + \left(x - \frac{a}{2}\right)^2 = \frac{a^2}{4},$$

or

$$y^2 + x^2 - ax = 0.$$

The method of Art. 42 gives for this circle the same approximating parabola as for the given curve. The circle is therefore as good an approximation as the parabola is to the given curve.

General Method of Determining the Equations of Approximate Curves

47. The process of finding the equations of approximating curves, illustrated in the preceding articles, consists in the

rejection of certain terms of the given equation, which vanish at the origin or at one of the infinitely distant points A or B . These three points, which are called *the fundamental points*, may be regarded as corresponding to the three vertices of the analytical triangle. We shall now show that, two of the marked points, when the equation of a curve is placed upon the analytical triangle, being joined by a straight line, *if there are no marked points on the same side of the line with one of the vertices of the analytical triangle, the terms represented by the points situated upon this line will constitute the equation of an approximate curve at the corresponding fundamental point.*

48. We shall prove this proposition first when the vertex in question is the analytical origin. Since there is at least one marked point upon each analytical axis, the line must in this case cut both analytical axes, like the line CD in Fig. 16. Let p, q be the analytical coordinates of one of the two points, and $p - r, q + s$ those of the other; so that the terms represented by these points are of the form

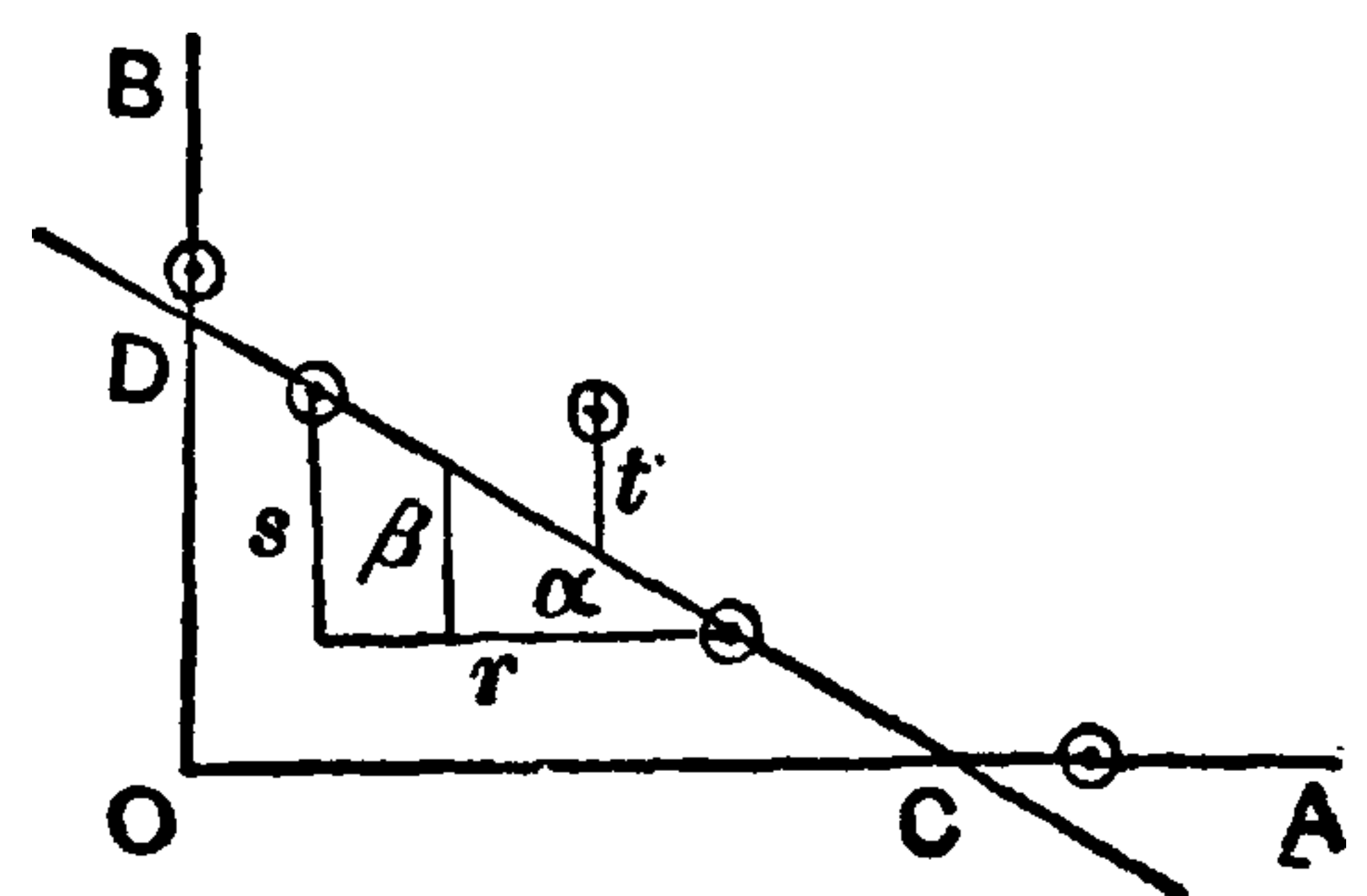


Fig. 16

$$Lx^p y^q + Mx^{p-r} y^{q+s}.$$

Represent the analytical coordinates of any other point upon the line by $p - a, q + \beta$, in which a and β may be fractional. Let us suppose that the equation contains a term represented by this point; then the terms represented by the three points will be of the form

$$Lx^p y^q + Mx^{p-r} y^{q+s} + Nx^{p-a} y^{q+\beta}, \quad (1)$$

in which $N = 0$ if the supposed term does not actually occur

in the equation. Now let the equation of the curve be divided through by $x^p y^q$; these terms then become

$$L + M \frac{y^s}{x^r} + N \frac{y^\beta}{x^\alpha}. \quad (2)$$

From Fig. 16 we have, by similar triangles,

$$\frac{\beta}{s} = \frac{\alpha}{r};$$

so that, putting γ for the value of either member of this equation, we may write $\beta = \gamma s$, $\alpha = \gamma r$, and the terms (2) may be written

$$L + M \frac{y^s}{x^r} + N \left(\frac{y^s}{x^r} \right)^\gamma. \quad (3)$$

If, then, we suppose the ratio $\frac{y^s}{x^r}$ to remain finite when the point P , moving upon the curve, passes through the origin, the ratio $\frac{y^\beta}{x^\alpha}$ will also remain finite.

49. Now consider any marked point not situated upon the line: such a point is by hypothesis on the upper side of the line; it therefore represents a term containing a higher power of y than it would if, while containing the same power of x , it were represented by a point upon the line. Hence, after the equation is divided by $x^p y^q$, the term will be of the form

$$N \left(\frac{y^s}{x^r} \right)^\gamma y^t,$$

where t is positive; and, when x and y are each put equal to zero, this term will vanish. The resulting equation will there-

fore contain only the terms which are represented by points on the line; hence these terms constitute the equation of an approximate form at the origin.

50. In the next place, the line having the same position as in Fig. 16, let us suppose all the marked points not upon the line to be upon the lower side of the line, as in Fig. 17. Then, using the same notation as before, when the equation is divided through by $x^r y^q$, the terms represented by points on the line will take the same form as in expression (3), Art. 48. Every other term contains a lower power of y than it would if, while containing the same power of x , it were represented by a point on the line; it will therefore take the form

$$N\left(\frac{x^s}{y^r}\right)^y y^{-t}.$$

Now, if we suppose the ratio $\frac{y^s}{x^r}$ to remain finite when x and y both become infinite, this term will vanish. The resulting equation will, as before, consist only of the terms represented by points on the line; but it will in this case represent a form at infinity.

51. Since the side AB of the analytical triangle must contain at least one marked point, it must meet the line CD between its intersections, C and D , with the analytical axes. If $r = s$, AB will coincide with CD , and the process is in fact the same as that given in Art. 15 to determine the value of the ratio of y to x when x and y are infinite. But suppose that s is less than r , as represented in Fig. 17; in this case, when x and y are both infinite and $\frac{y^s}{x^r}$ is finite, $\frac{y}{x}$ is readily seen to be

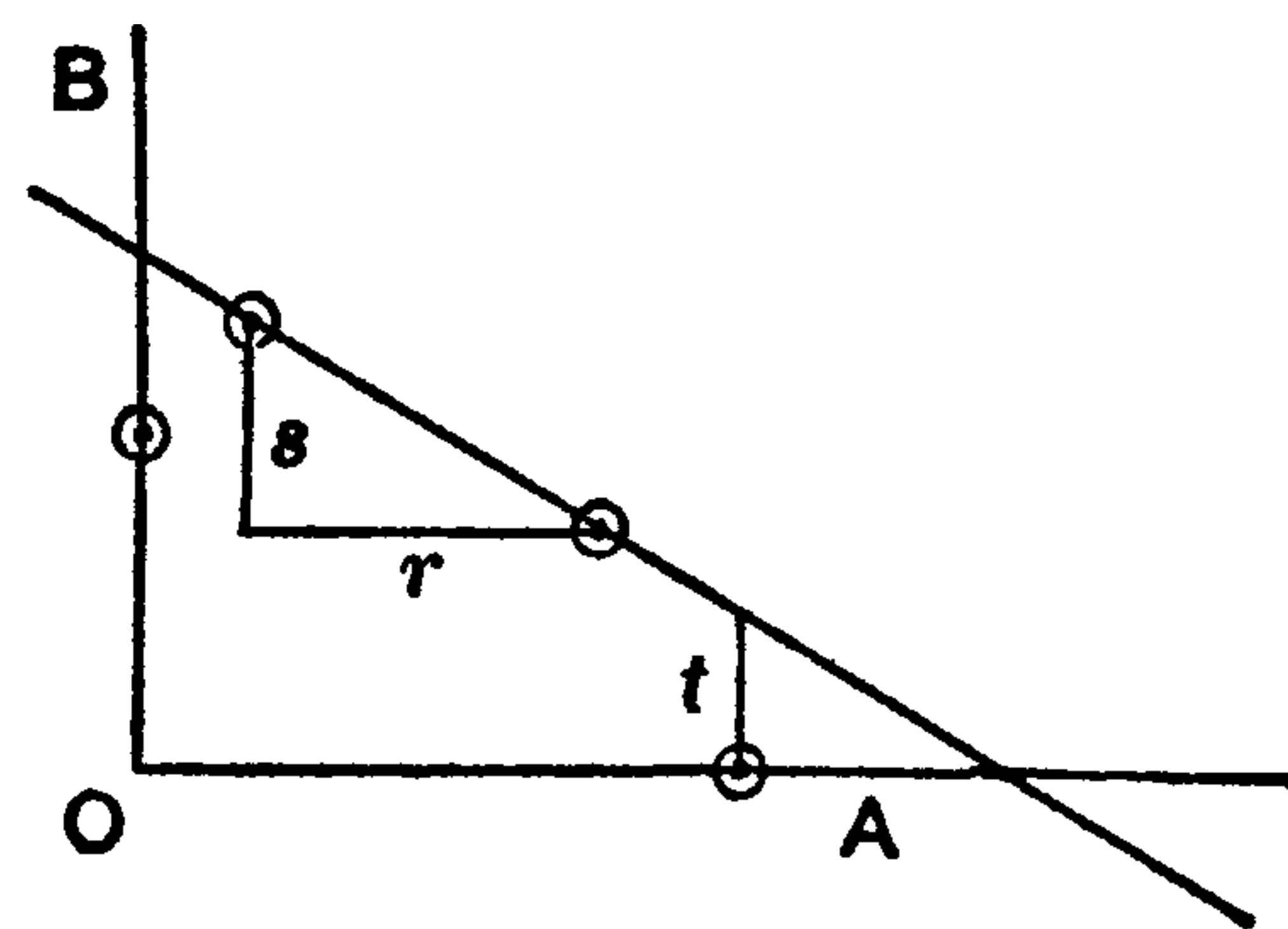


Fig 17

infinite, so that the infinite point on the branch in question is the fundamental point B . Accordingly we find that in this case (see Fig. 17) B is the vertex of the analytical triangle which is on that side of the line CD on which there are no marked points, in conformity with the enunciation of the proposition in Art. 47. Since x and y are both infinite, the result indicates a parabolic branch at the fundamental point B ; and, in like manner, if s be greater than r , we shall have a parabolic branch at the fundamental point A .

52. In the third place, let the line cut one of the analytical axes produced; for example, let it cut the analytical axis of y below the origin, as in Fig. 18. Since the analytical axis of

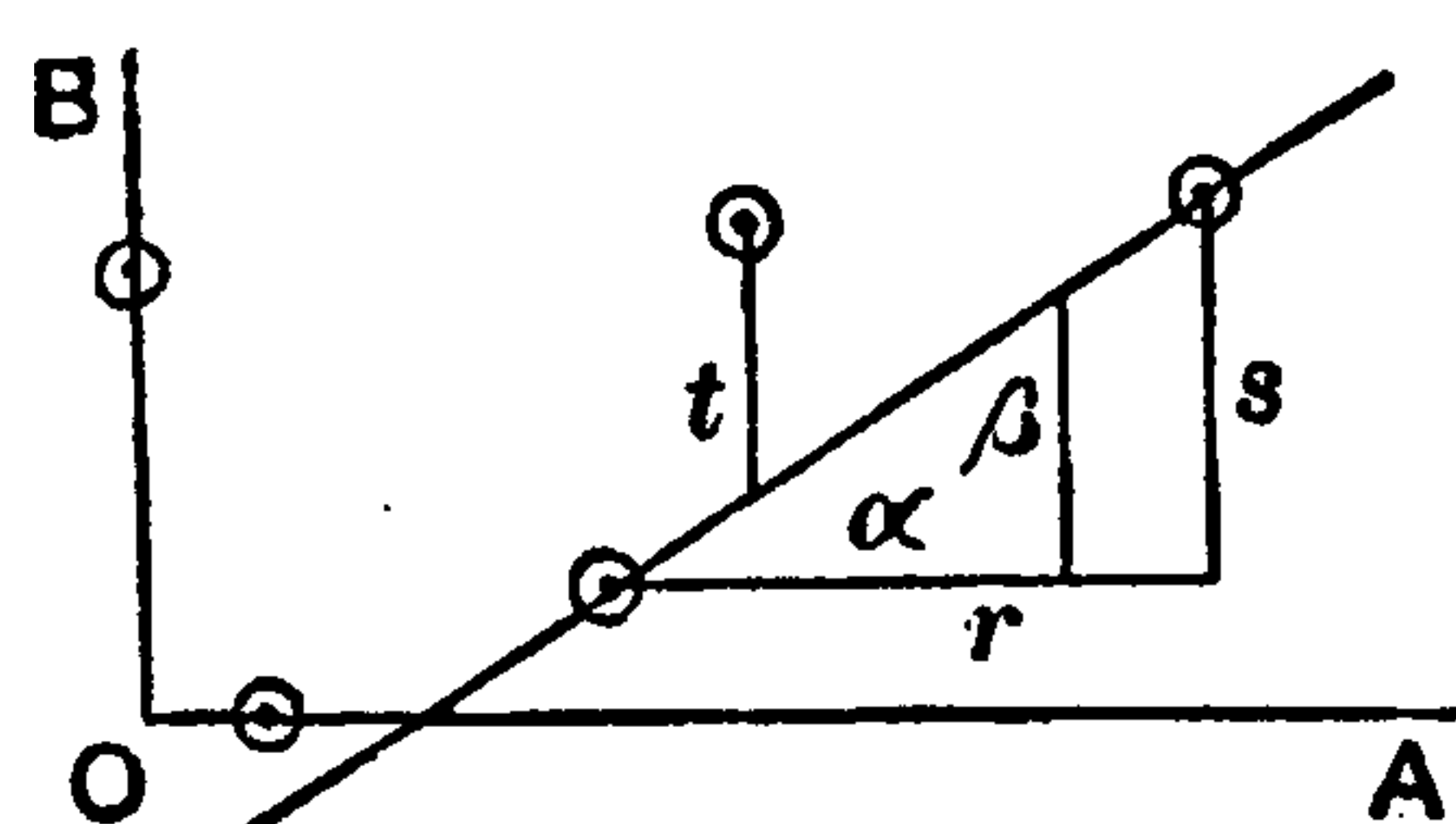


Fig. 18

y must contain at least one marked point, all the marked points not on the line are by hypothesis above it; thus A is the vertex to be considered, and the curve passes through the fundamental point A . Let the analytical

coordinates of the points be p, q and $p + r, q + s$; the terms represented being of the form

$$Lx^p y^q + Mx^{p+r} y^{q+s}.$$

Denoting the analytical coordinates of any other point of the line by $p + a, q + \beta$, the term which would be represented by that point is of the form

$$Nx^{p+a} y^{q+\beta}.$$

Dividing the equation through by $x^p y^q$, the terms in question become

$$L + Mx^r y^s + Nx^a y^\beta, \quad (1)$$

which, since $\frac{\beta}{s} = \frac{a}{r}$, may be written in the form

$$L + Mx^r y^s + N(x^r y^s)^{\frac{a}{r}}. \quad (2)$$

Now, when the point (x, y) recedes from the origin indefinitely on the branch passing through the fundamental point A , if we suppose that, as x becomes infinite, y becomes zero but $x^r y^s$ remains finite, $x^a y^b$ will also remain finite.

53. Now consider the term represented by any marked point not on the line. Such a point is by hypothesis above the line; hence the term contains a higher power of y than it would if, while containing the same power of x , it were represented by a point on the line. Thus, after the equation is divided through by $x^p y^q$, the term will take the form

$$N(x^r y^s)^t y^t,$$

and, when y becomes zero, it will vanish. The resulting equation will therefore, as before, contain only the terms represented by points on the line; and it will in this case be an approximate form at the fundamental point A .

54. Finally, suppose the line to be parallel to one of the analytical axes, for example to the axis of y , so that the curve passes through the fundamental point A . Then, using the same notation as in the preceding articles, we shall have $r = 0$, and expression (1), Art. 52, will become

$$L + M y^s + N y^b.$$

In this case we must regard y as remaining finite when x becomes infinite; and, since all the marked points not on the line are on the left of it, the terms represented by these points contain powers of x lower than x^p ; hence, when x is made infinite after dividing by x^p , these terms will vanish as before. The resulting equation will determine one or more finite values of y when x is infinite; that is, one or more asymptotes parallel to the axis of x , as in Art. 26.

The Analytical Polygon

55. If, when the equation of a curve has been placed upon the analytical triangle, the marked points are joined by straight lines in such a manner as to form a convex polygon exterior to which there is no marked point, the result is called the *analytical polygon*. This polygon will have at least one vertex lying in each side of the analytical triangle, and it may have a side lying in either side of the triangle. In the case of such a side of the polygon, the result of equating the corresponding terms to zero determines simply the intersections of the curve with one of the axes or with the line at infinity. In all other cases, the equation corresponding to a side of the polygon determines an approximate form at one of the fundamental points, in accordance with the theorem proved in the preceding articles. In the example given in Art. 42, the polygon reduces to a triangle having no side coinciding with a side of the analytical triangle; the curve accordingly passes through each fundamental point, and each side of the polygon gives an approximate form at one fundamental point.

Construction of the Approximating Curves

56. If a side of the analytical polygon when produced cuts both axes, as in Figs. 16 and 17, and contains no marked point except its extremities, the equation of the corresponding approximating curve will be of the form

$$Lx^p y^q + Mx^{p-r} y^{q+s} = 0,$$

(see Art. 48), or, rejecting a common factor,

$$Lx^r + My^s = 0. \quad (1)$$

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58. Again, let the equation corresponding to a side of the polygon be

$$3x^3 - a^2y = 0,$$

or

$$a^2y = 3x^3.$$

Since x has the same sign as x^3 , x and y must have the same

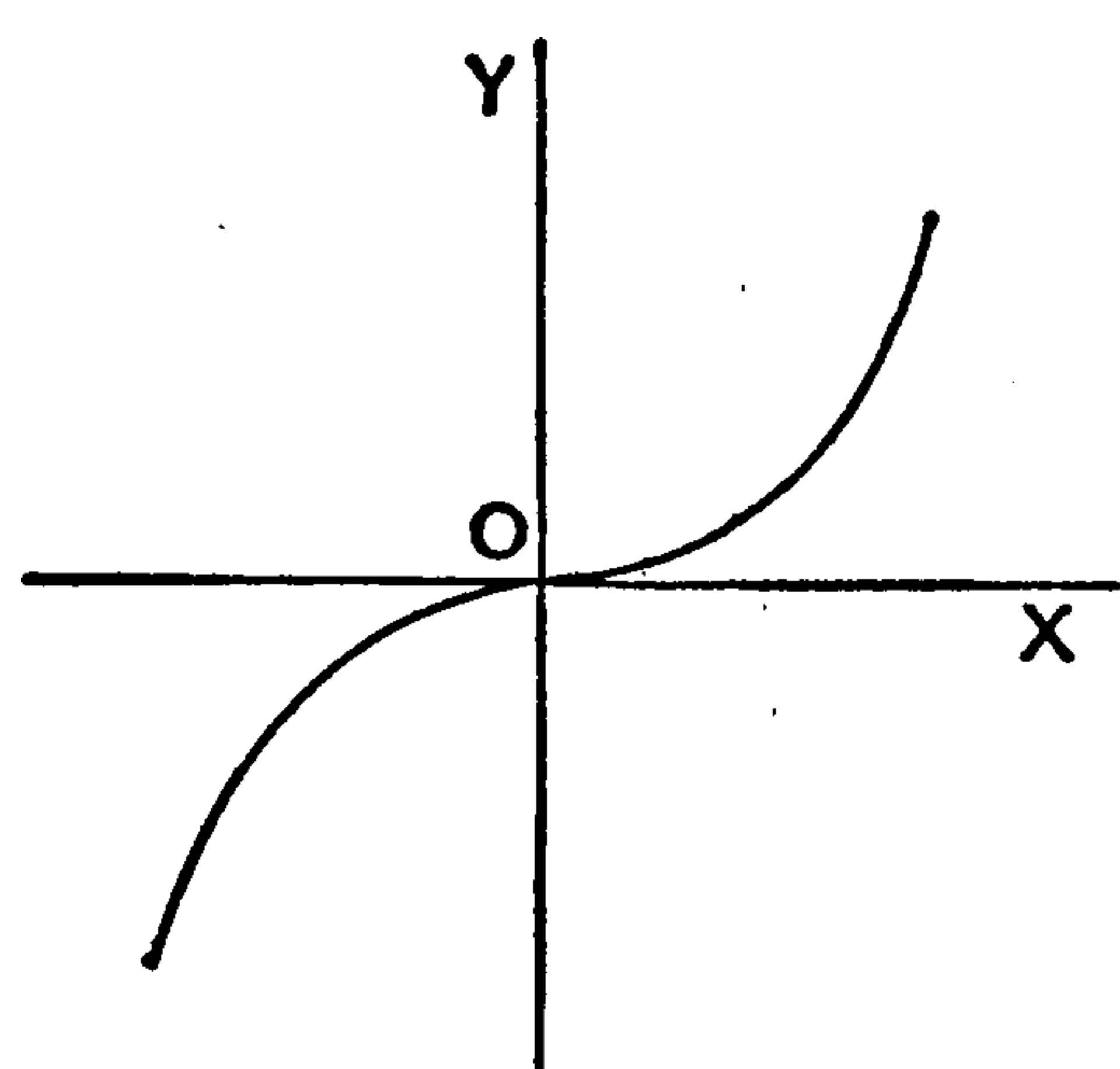


Fig. 20

sign, and the curve lies in the first and third quadrants. The axis of x is the tangent at the origin; hence the form is that indicated in Fig. 20, the curve being in this case a *cubical parabola*. If this be an approximate form at the origin, the given curve has a *point of inflection* at that point; but, if it be a form at infinity, the given

curve has parabolic branches in the first and third quadrants tending to parallelism to the axis of y , and it is said to touch the line at infinity at the fundamental point B .

59. If the side of the polygon when produced cuts one of the axes produced, as in Fig. 18, and contains no marked points except its extremities, the equation of the approximating curve is of the form

$$Lx^py^q + Mx^{p+r}y^{q+s} = 0,$$

or

$$L + Mx^ry^s = 0. \quad (1)$$

Supposing, as before, that r and s are prime to one another, equation (1) represents one of the family of curves known as *the hyperbolas*, having both axes as asymptotes. If $r = 1$ and $s = 1$, we have the common hyperbola, as in Fig. 15, page 36. Since r and s cannot both be even, the curve will always be possible; it will consist of two branches, and it is easy to

determine the quadrants in which these branches lie. For example, let the equation be

$$xy^2 + 2a^3 = 0,$$

or

$$xy^2 = -2a^3.$$

Since y^2 is positive whatever be the sign of y , x must be negative; hence the branches lie in the second and third quadrants, as in Fig. 21. If this be an approximate form for the fundamental point A , it indicates that the given curve has branches approaching as an asymptote the left end of the axis of x on both sides; but, if it be an approximate form for the fundamental point B , it indicates that the curve has branches approaching both ends of the axis of y on the left side.

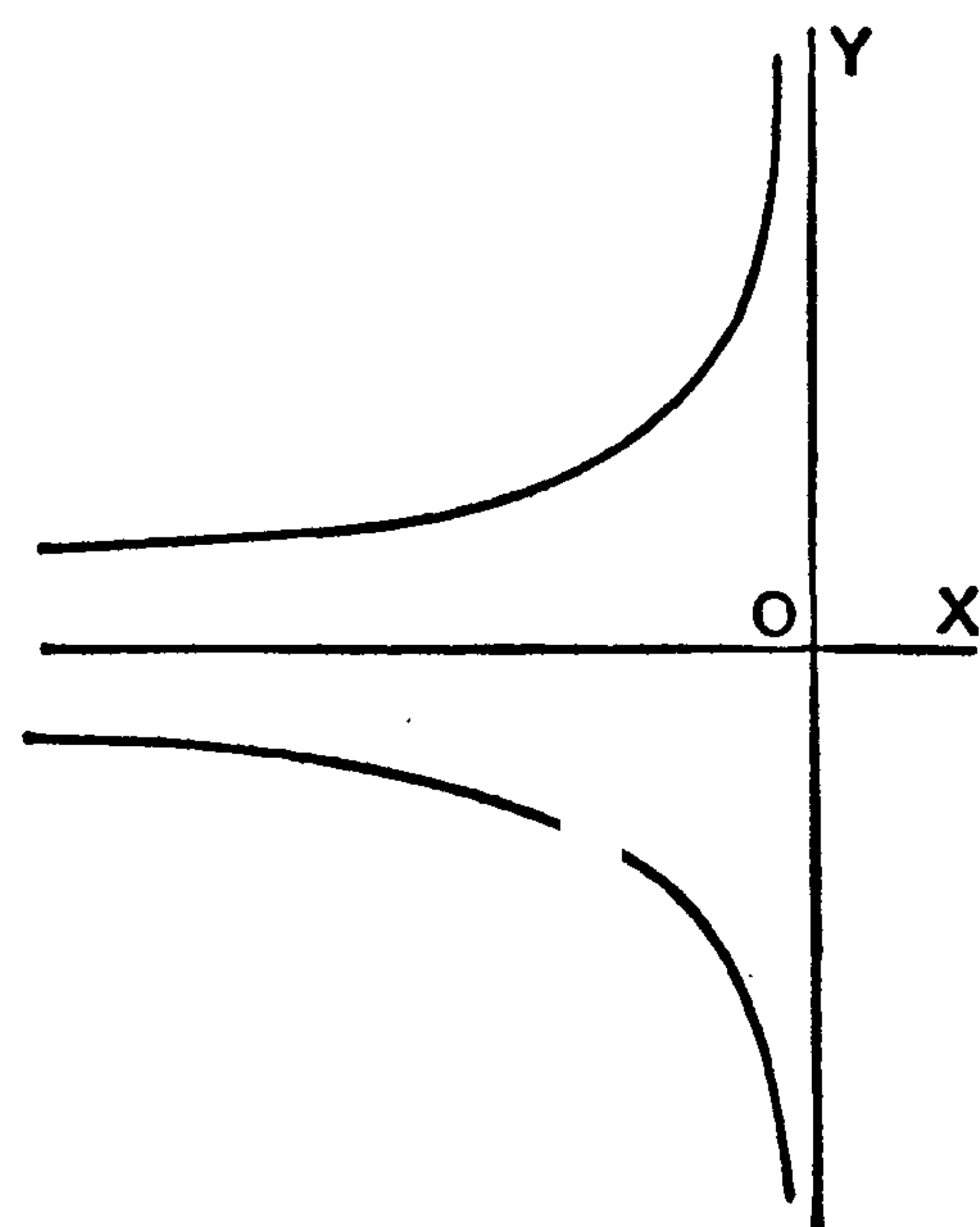


Fig. 21

Sides of the Polygon Representing more than One Approximate Form

60. When a side of the polygon contains any other point of the analytical triangle except its extremities, the corresponding equation represents two or more approximate forms at the same fundamental point. For example, let the equation of the given curve be

$$2y^5 - 5xy^2 + x^5 = 0,$$

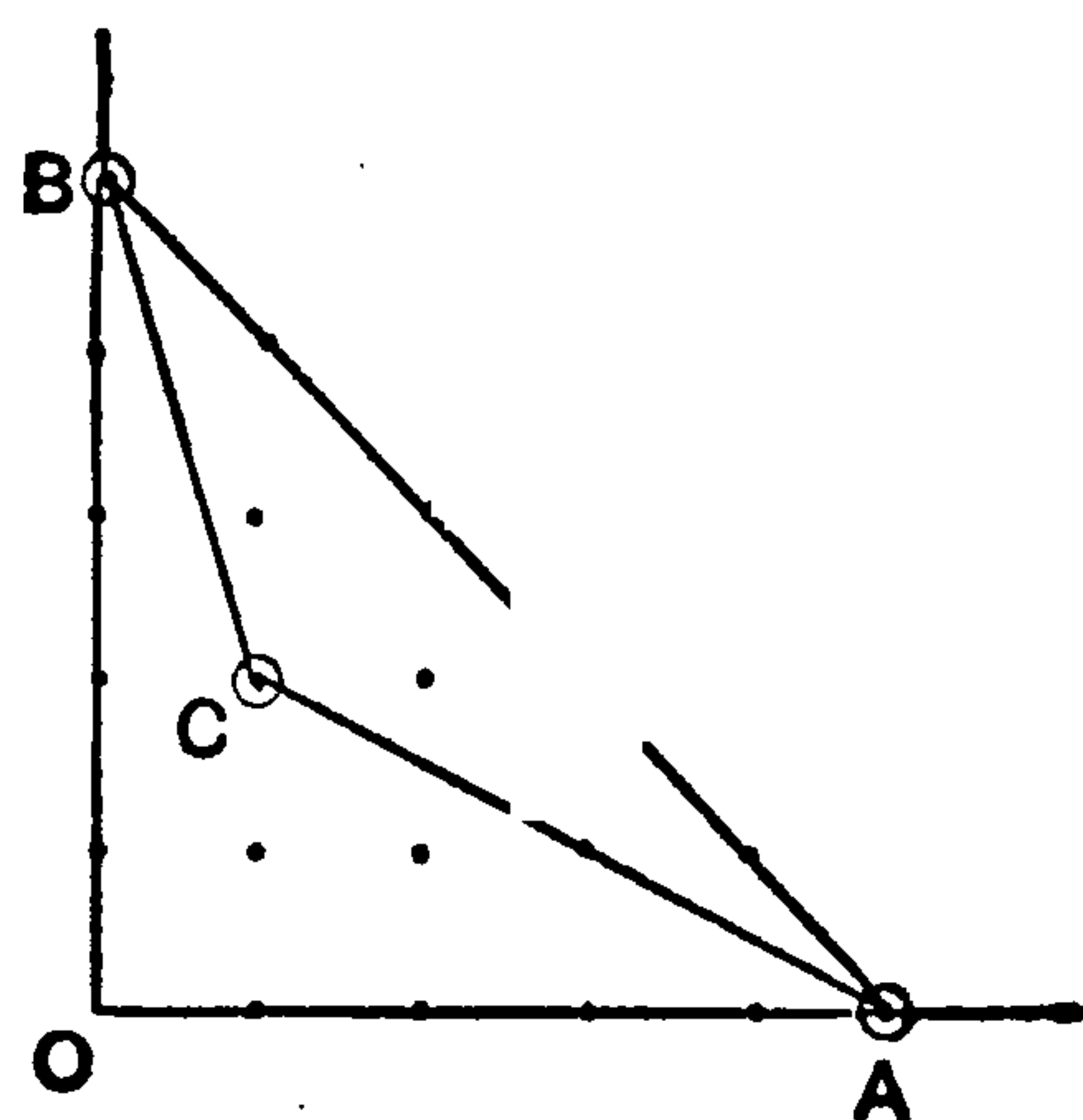


Fig. 22

which is placed upon the analytical triangle in Fig. 22. The side AC of the analytical polygon, which in this case reduces

to a triangle, passes through the point whose analytical co-ordinates are (3, 1). The corresponding equation is

$$-5xy^2 + x^5 = 0,$$

or

$$y^2 = \frac{1}{5}x^4,$$

which may be regarded as a quadratic equation to determine the ratio of y to x^2 . Solving, we have

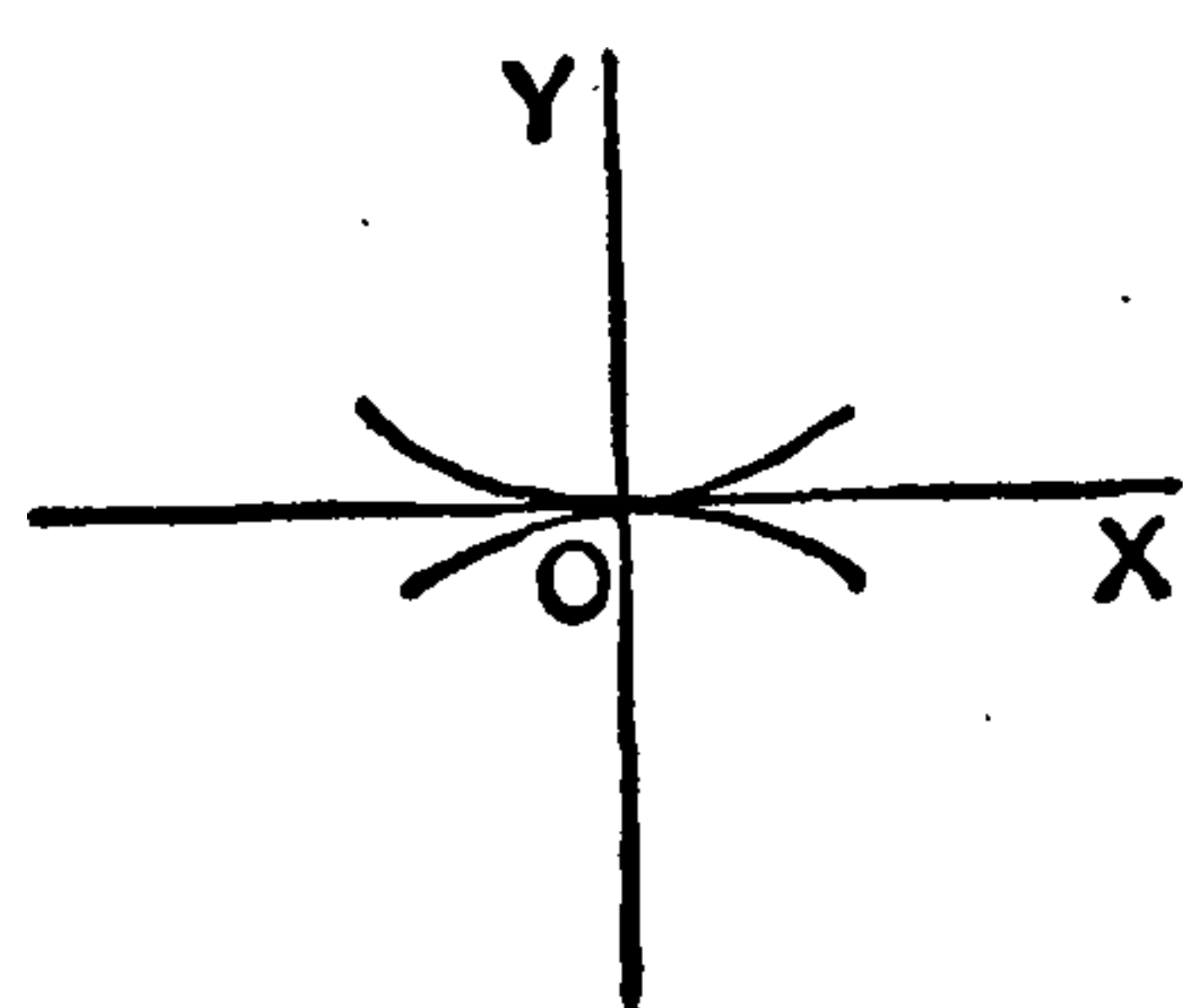


Fig. 23

$$y = \pm \sqrt{\frac{1}{5}} \cdot x^2,$$

the equations of two approximate forms at the origin. The form at the origin indicated by this side is, therefore, that shown in Fig. 23.

61. The side BC in Fig. 22 gives the approximate form

$$y^3 = \frac{5}{2}x,$$

which indicates another branch passing through the origin, touching the axis of y , and lying in the first and third quadrants. The side AB gives

$$\sqrt[5]{2} \cdot y + x = 0$$

for the only real point at infinity, and the absence of terms represented by points on the adjacent parallel line (see Art. 17) shows that the asymptote passes through the origin. The curve does not cut the axes or the asymptote except at the origin, and it is symmetrical in opposite quadrants; hence its form is that represented in Fig. 24.

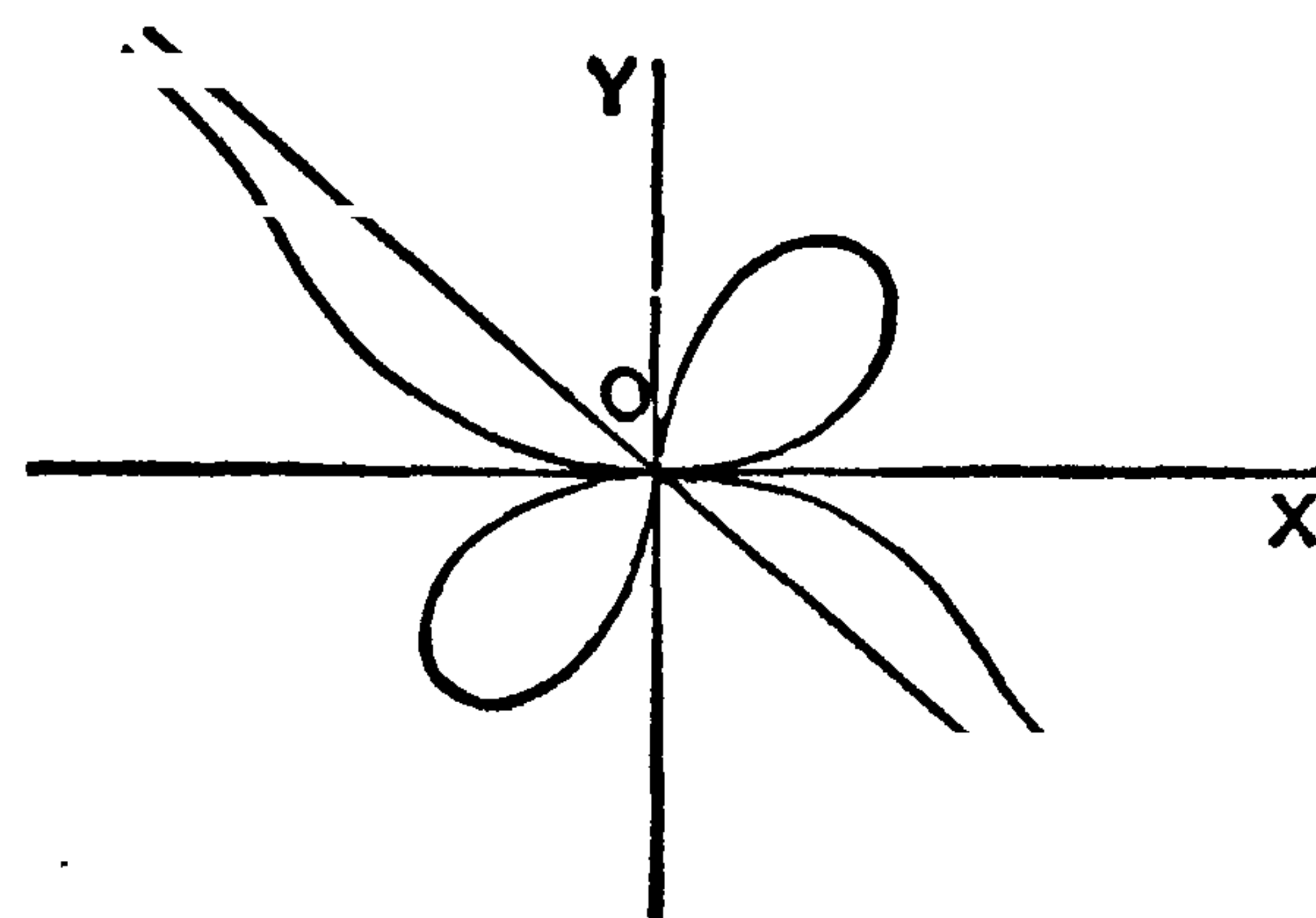


Fig. 24

62. In general, whenever a side of the analytical polygon is divided into segments by points having integral coordinates, whether they be marked points or not, the number of these segments indicates the degree of the corresponding equation, and hence the number of approximate forms represented by the side.

63. The equation corresponding to a side consisting of two or more segments may have a pair of imaginary roots. For example, let the equation of the curve be

$$x^2y^4 + 2a^2xy^3 - a^5y + a^6 = 0, \quad (1)$$

for which the analytical polygon is drawn in Fig. 25. The side OC of the analytical polygon gives, for the approximate forms at the fundamental point A ,

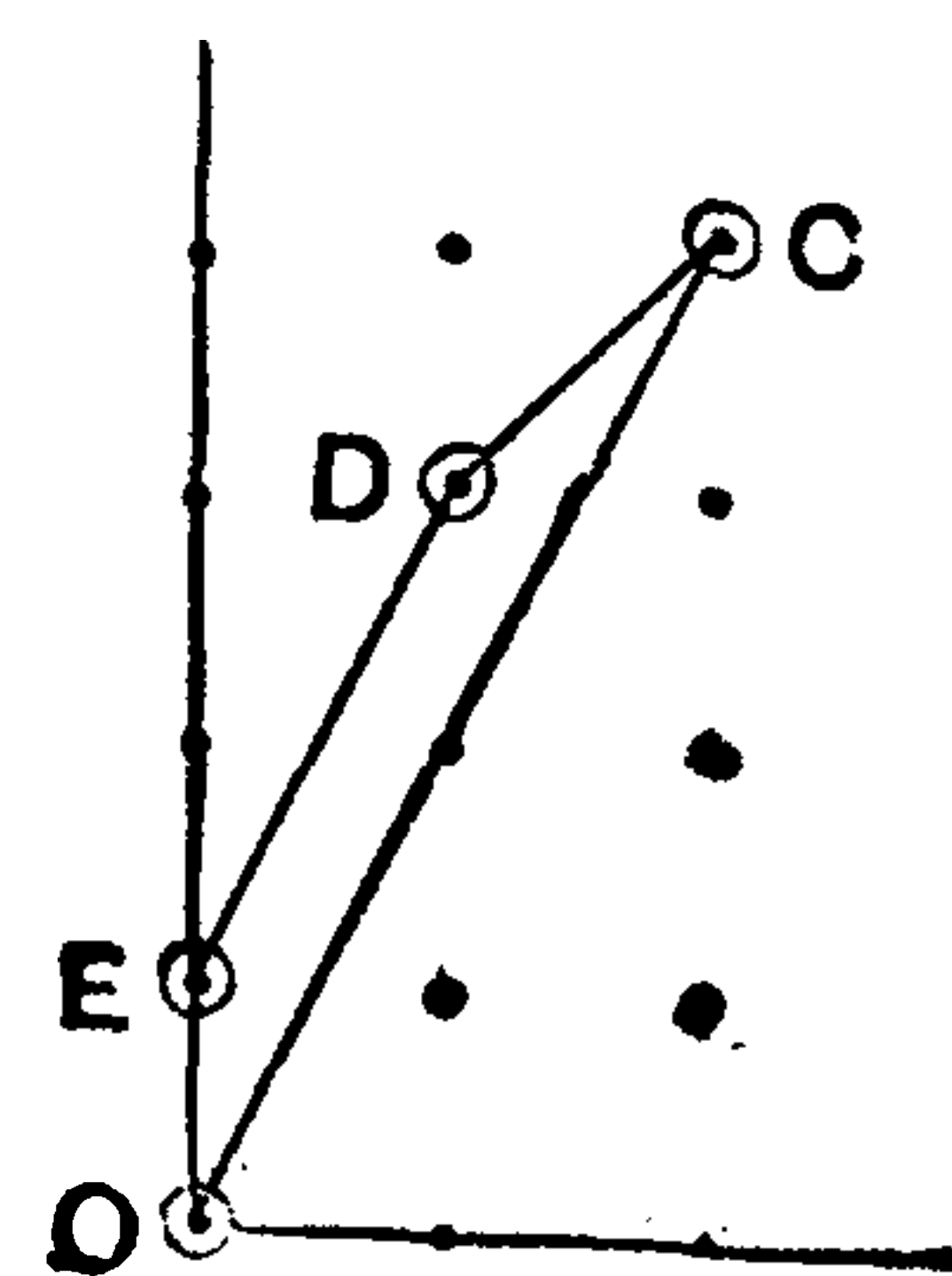


Fig. 25

$$x^2y^4 + a^6 = 0, \quad (2)$$

which is impossible, because it gives imaginary values to the product xy^2 . There are, therefore, no infinite branches in the direction of the axis of x .

The side CD gives, for the fundamental point B , the approximate form

$$xy = -2a^2, \quad (3)$$

indicating branches in the second and fourth quadrants having the axis of y for an asymptote. The side DE gives, for the same fundamental point,

$$2xy^2 = a^3, \quad (4)$$

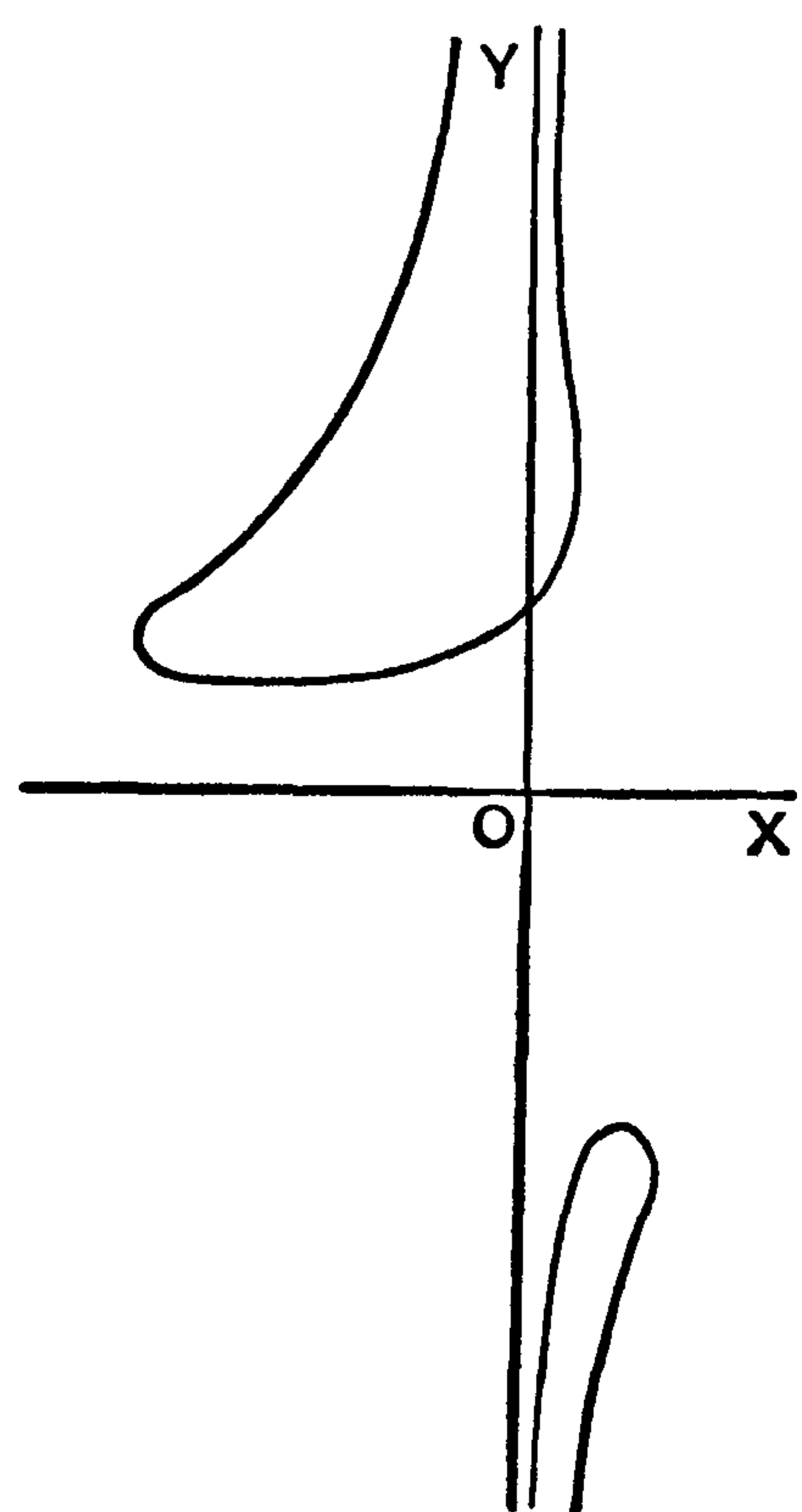


Fig. 26

indicating branches in the first and fourth quadrants, which it is readily seen are much closer to the axis

of y than those corresponding to equation (3). The side EO gives $y = a$ for the intersection with the axis of y . The curve has therefore the form represented in Fig. 26. The position of the branches is more precisely determined by solving the equation for x and determining the limiting values of y ; these are found to be

$$y = \frac{-1 \pm \sqrt{5}}{2}a.$$

Examples III

1. $y^4 + 2axy^3 - ax^3 = 0.$
2. $x^3y - 3xy^3 - x^3 + 2y^2 + y = 0.$
3. $y^4 - y^3x + x^3 - 2x^2y = 0.$
4. $y^4 - 16x^4 + x^2 - 4xy = 0.$
5. $x^4 - y^4 + a^2xy = 0.$
6. $x^6 + a^2y^4 + a^2x^3y + a^3xy^2 = 0.$
7. $x^4 - x^2y + y^3 = 0.$
8. $x^4 + x^2y^2 - 6ax^2y + a^2y^2 = 0.$
9. $x^5 + bx^4 - a^3y^2 = 0.$
10. $y^2x - x^3 + ax^2 - ay^2 = 0.$
11. $y^4 + 2axy^2 - x^4 = 0.$
12. $(x^2 + y^2)^3 = 4a^2x^2y^2.$
13. $2x(x - y)^2 - 3a(x^2 - y^2) + 4a^2y = 0.$
14. $x^5 + y^5 - 5ax^3y = 0.$
15. $x^5 - 5x^2y - 3xy^2 + y^5 = 0.$
16. $x^4 + y^4 - axy^2 = 0.$
17. $x^5 + 2a^2x^2y - a^3y^2 = 0.$
18. $y^5 + ax^4 - a^2xy^2 = 0.$
19. $x^2y^2(x^2 + y^2 + a^2) = c^6.$
20. $y^4 + 2x^2y^2 + 8axy^2 - 8ax^3 = 0.$
21. $(x^2 + y^2)(y - 2x)^2 = 4a^2y^2.$
22. $y^2(y^2 - x^2) = 2a^2x^2.$
23. $x^3 - xy^2 + ay^2 = 0.$
24. $2x^2(x^2 + y^2) - 4ay^3 - 4a^2x^2 - 6a^2y^2 + a^4 = 0.$
25. $y^4 = a(y^2 - x^2)(y - 2x).$

IV

Second Approximations

64. WE have seen that, when a side of the analytical polygon coincides with a side of the analytical triangle, the corresponding equation gives only the points of intersection with one of the fundamental lines, that is with one of the coordinate axes, or with the line at infinity ; what is really determined in the latter case being the direction of an asymptote. The method of obtaining the position of the tangent line or *first approximation* to the curve at one of these points of intersection is explained in Arts. 16 and 33. But, when the intersection in question is one of the fundamental points, we have a side of the polygon which gives at once the first approximation or tangent at the fundamental point.

We have also seen that, if the side of the polygon is parallel to one side of the analytical triangle, so that it corresponds to the same number of intersections with two of the fundamental lines, it gives the first approximation only ; but otherwise the fundamental line with which the side indicates the greatest number of intersections is itself the tangent, and the side gives a *second approximation*, determining, in fact, upon which side of the tangent the curve lies in the neighborhood of the fundamental point. We have now to show how this second approximation can be determined in those cases in which the side of the polygon gives only the first approximation,

65. Let us take for illustration the curve whose equation is

$$x^2y^3 + xy^3 - x^2 + x - y = 0, \quad (1)$$

for which the analytical polygon is drawn in Fig. 27. The side CD gives, for the tangent at the origin, the equation

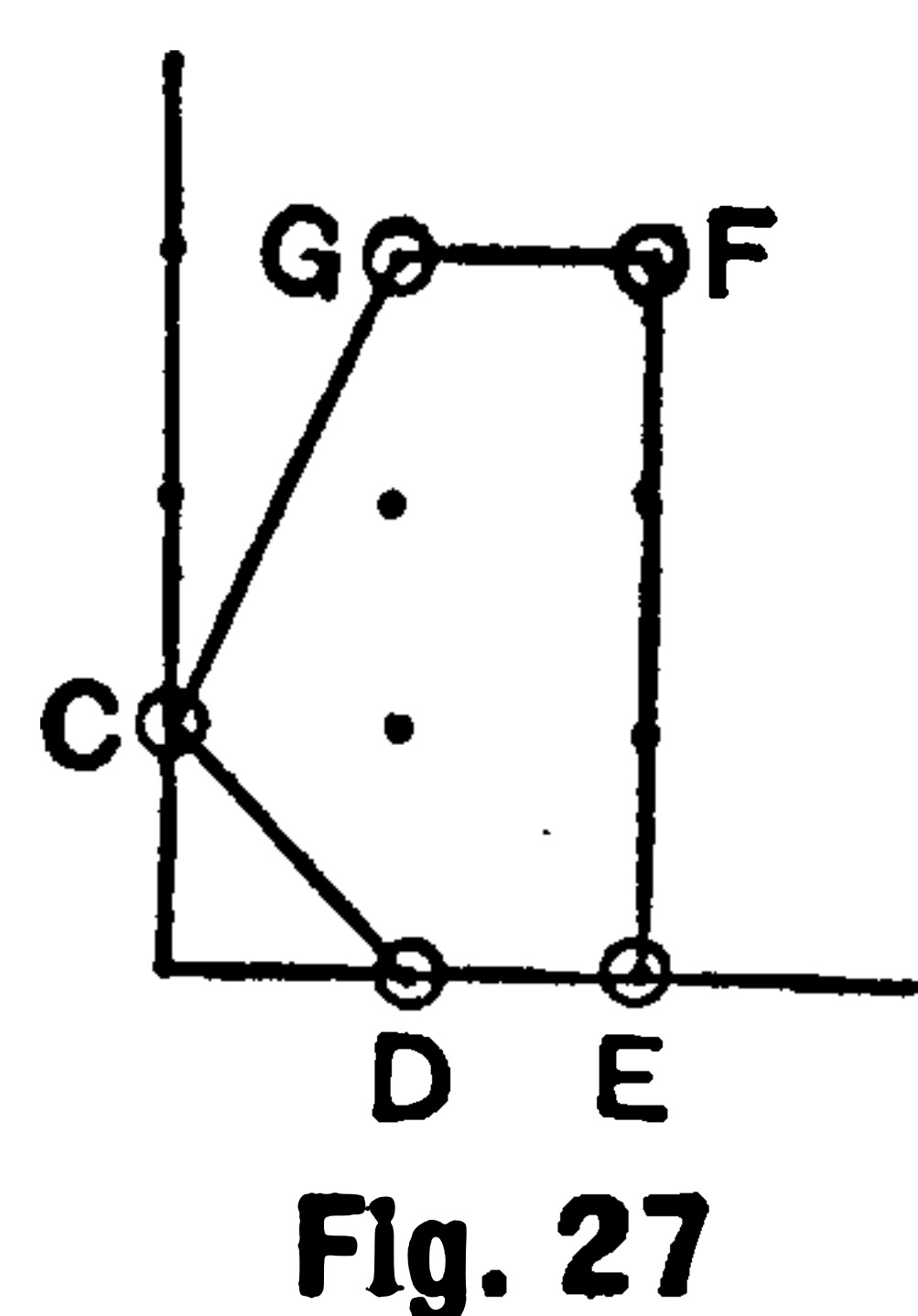


Fig. 27

$$x - y = 0. \quad (2)$$

Now the value of the quantity $x - y$, when (x, y) is a point of the curve, determines the position of the curve with respect to the tangent; since, obviously, if this quantity is positive, the curve is on the right of the tangent, while if it is negative, the curve is on the left of the tangent. In the process of finding the equation of the tangent, as in

Art. 32, we show that at the origin the ratio $\frac{y}{x}$ has the value

unity; therefore $\frac{x - y}{x} = 0$ at that point, and $\frac{x - y}{x^2}$, which is the ratio this last quantity bears to x , may have a finite value when $x = y = 0$. Dividing equation (1) by x^2 , we have

$$\frac{x - y}{x^2} - 1 + y^2 \frac{y}{x} + y^3 = 0,$$

in which, making $y = x = 0$, we have

$$\frac{x - y}{x^2} = 1; \quad (3)$$

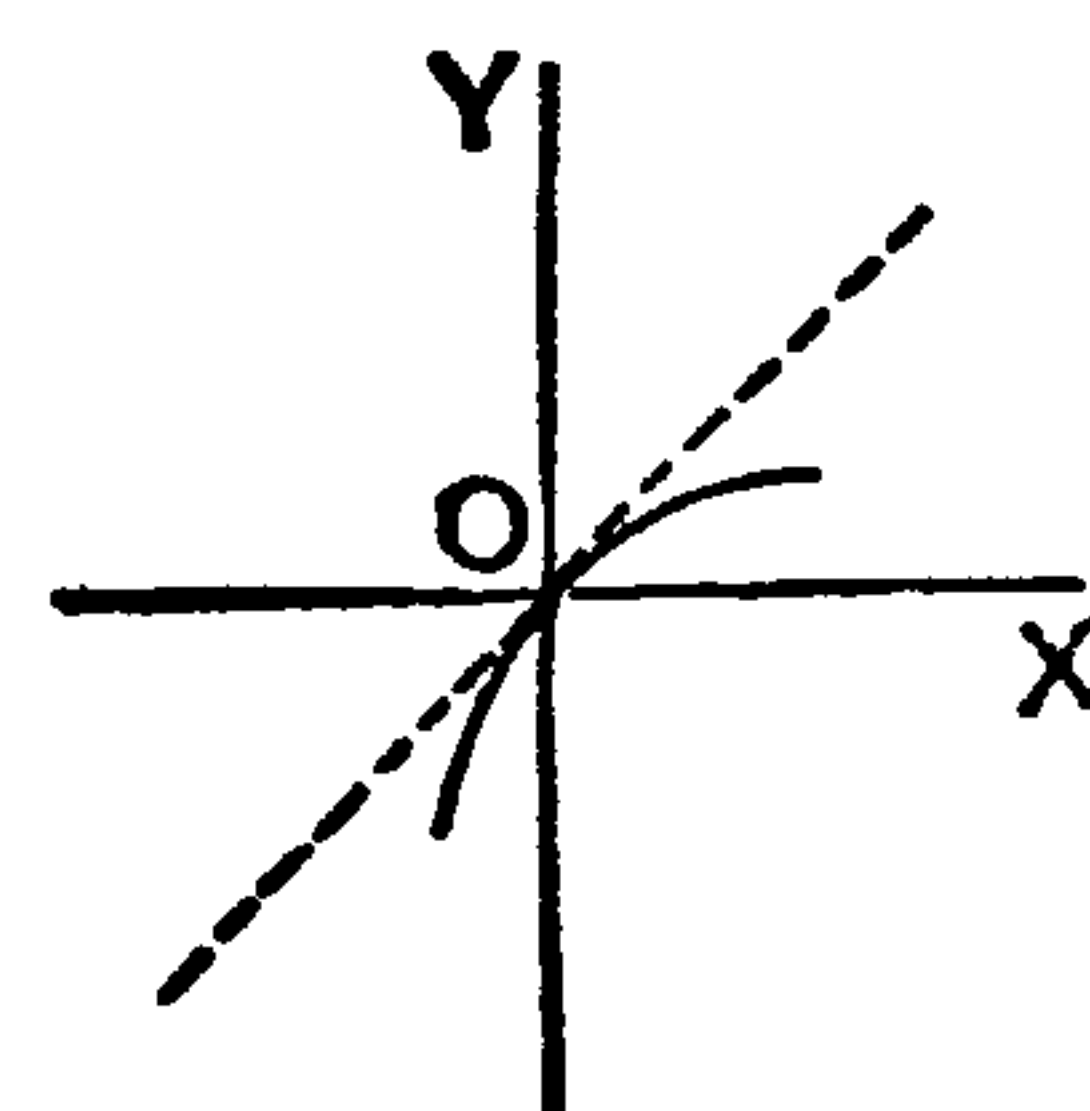


Fig. 28

and, since x^2 is essentially positive, we infer that $x - y$ is positive on both sides of the origin. Thus the form of the curve at the origin is that indicated in Fig. 28.

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67. The side EF in Fig. 27 gives, for the tangents at the fundamental point A , that is to say, the asymptotes parallel to the axis of x ,

$$y^3 - 1 = 0,$$

of which $y = 1$ is the only real root. Arranging the terms according to powers of x , equation (1) may be written

$$x^2(y^3 - 1) + x(y^3 + 1) - y = 0. \quad (4)$$

Now, in finding the asymptote $y - 1 = 0$, as in Art. 26, we show that $y - 1$, which measures the distance of the curve from the asymptote, vanishes when x becomes infinite; hence $x(y - 1)$ may have a finite value when x increases without limit. From equation (4) we have

$$x(y - 1)(y^2 + y + 1) + y^3 + 1 - \frac{y}{x} = 0,$$

in which, putting $y = 1$ and $x = \infty$, we have

$$3x(y - 1) + 2 = 0, \quad (5)$$

which therefore represents a second approximation to the form

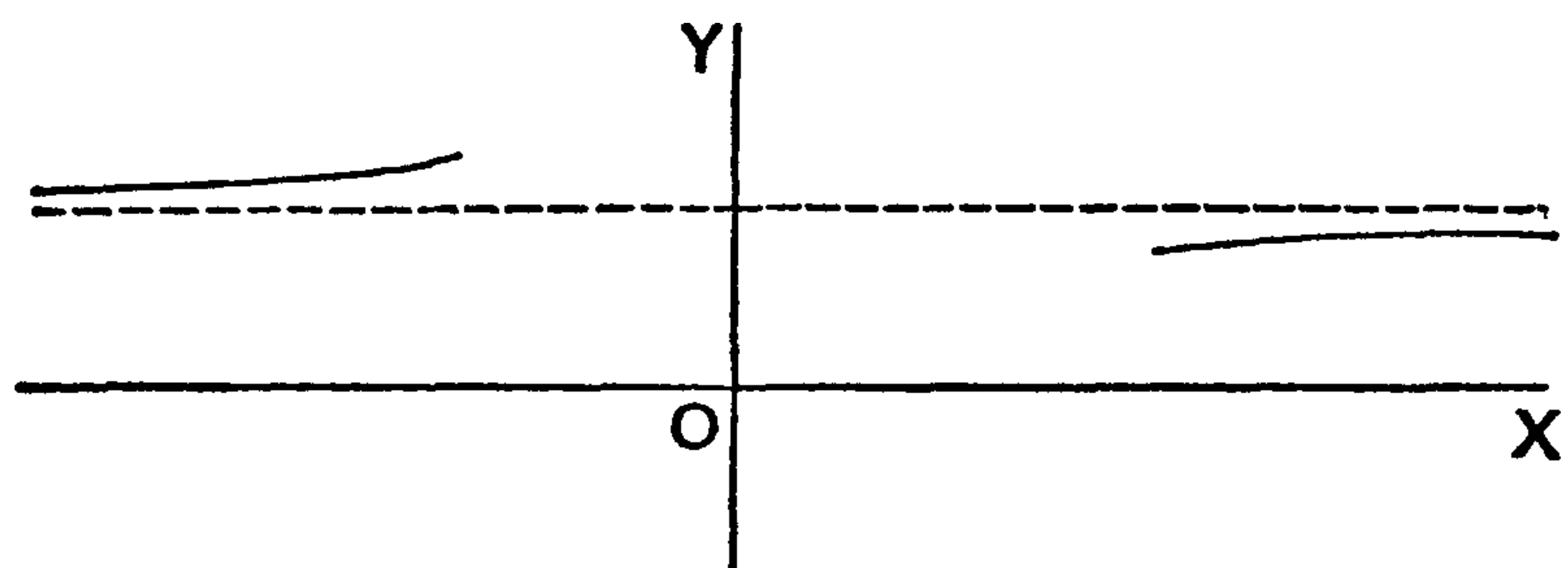


Fig. 29

of the curve when x is very great. In equation (5), $y - 1$ is positive when x is negative, and negative when x is positive; hence we infer that the position of the curve with respect to its asymptote is that represented in Fig. 29.

68. In like manner, the side FG of the polygon gives the

asymptote $x = -1$. Arranging the terms of the equation according to powers of y , we have

$$y^3x(x+1) - y - x^2 + x = 0. \quad (6)$$

Dividing by the highest power of y which occurs except in the first term, the equation takes the form

$$y^2x(x+1) - 1 - \frac{x^2}{y} + \frac{x}{y} = 0,$$

from which we see that, if we make y infinite and $x = -1$, $y^2(x+1)$ will have a finite value: thus

$$-y^2(x+1) - 1 = 0 \quad (7)$$

is the second approximation. Since y^2 is essentially positive, this equation shows that $x+1$ is negative both for positive and for negative values of y ; in other words, the branches to which $x = -1$ is the asymptote lie on its left at both ends of the asymptote.

69. To complete the construction of the curve, we have the side GC of the polygon giving, for another approximate form at B ,

$$xy^2 = 1,$$

which shows that the curve has branches asymptotic to the axis of y in the first and fourth quadrants. The side DE gives the intersection $(1, 0)$ with the axis of x , and the tangent at this point is found to be the line whose equation is

$$x + y = 1.$$

The results thus far determined are constructed in Fig. 30.

70. The curve is of the fifth degree; but, since there are three asymptotes parallel to the axis of x , although two of them are imaginary, a line parallel to the axis of x meets the curve three times at the fundamental point A , which is accordingly said to be a triple point of the curve.

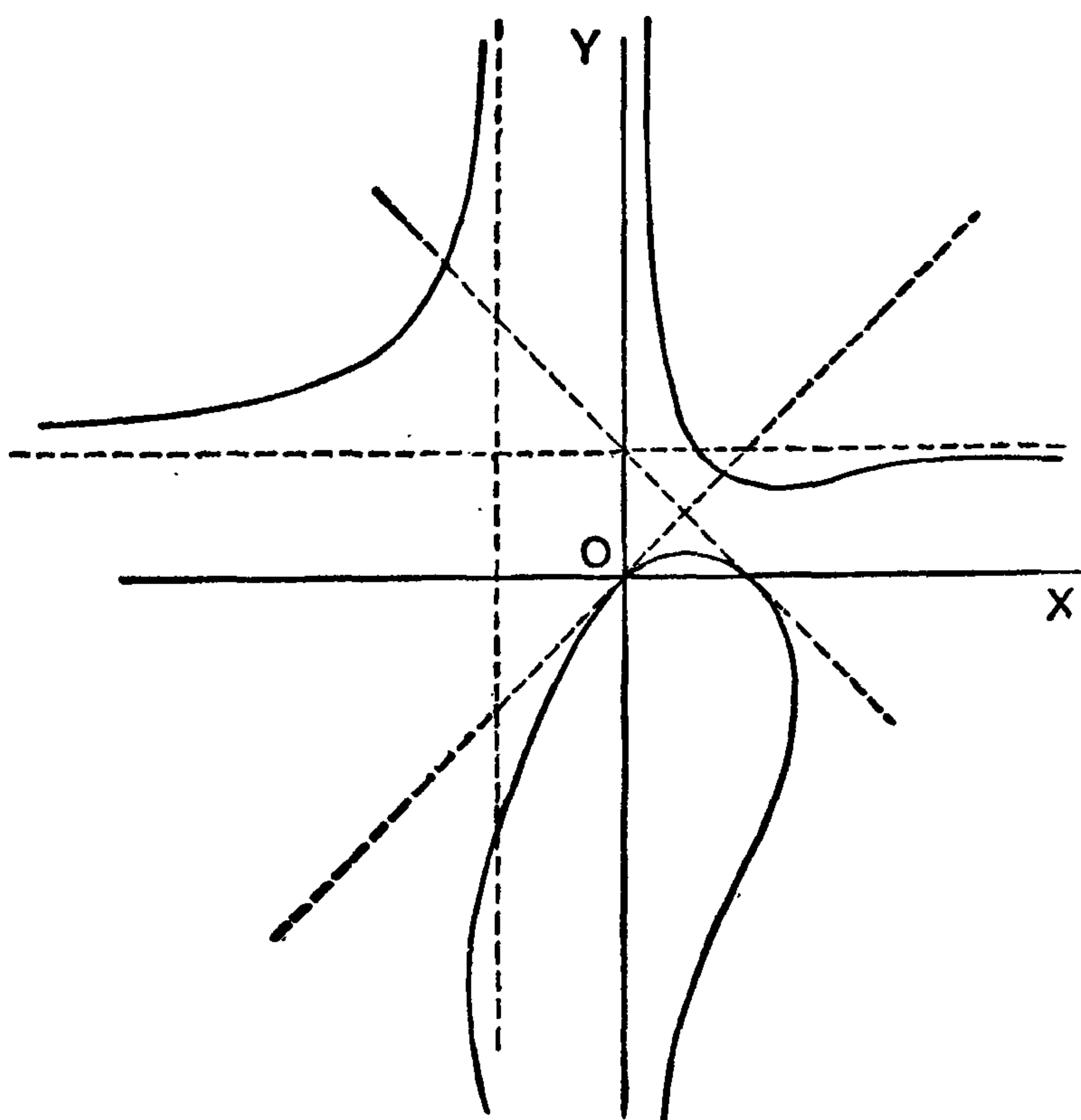


Fig. 30

Hence the asymptote $y = 1$, which is a tangent at A , can meet the curve in only one other point. Accordingly, putting $y = 1$ in the equation of the curve, equation (4), Art. 67, we obtain

$$2x - 1 = 0,$$

an equation of the first degree, giving the point $(\frac{1}{2}, 1)$ as the

intersection of the curve with the asymptote.

71. In like manner, a line parallel to the axis of y meets the curve twice at the fundamental point B , and can cut the curve in only three other points. But the axis of y itself cuts the curve in only one other point, and meets the branch to which it is the tangent at B in three coincident points at the point of contact, like the tangent at an ordinary point of inflexion. The asymptote $x = -1$ has the same character; and accordingly, putting $x = -1$ in the equation of the curve, we find

$$-y - 2 = 0,$$

giving the single point of intersection $(-1, -2)$.

72. There can be no doubt as to the method of joining the branches now determined, except in the case of those in the

first quadrant. To remove this remaining uncertainty, let us consider the intersection of the curve with the tangent line whose equation is

$$x + y = 1.$$

The result of eliminating x from the equation of the curve by means of this equation is

$$y^3 - 3y^2 + 2y - 1 = 0,$$

an equation which is easily shown to have no negative roots. But, writing it in the form

$$(y - 1)^3 = y,$$

it obviously has no positive root less than two. Its only real root is in fact about 2.33. Hence the branches must be joined as in Fig. 30.

Selection of the Terms which Determine the next Approximation

73. It will be noticed that the additional terms employed, when we proceed to the next approximation above that given by the side of the polygon, as in Arts. 65 and 67, are those represented by points situated upon the next adjacent line parallel to the side of the polygon, exactly as in the case of finding the first approximation when the side of the polygon gives only points of the curve. Compare Art. 34. If this parallel line contains no marked points, or if the terms represented are divisible by the factor corresponding to the side of the polygon, we must employ the terms on the nearest parallel line for which they are not so divisible. This is illustrated, in the case of both classes of sides, in the following example.

74. Let the equation of the curve be

$$x^2y^2 + x^3 - y^3 + x^2 - y^2 = 0, \quad (1)$$

for which the analytical polygon is drawn in Fig. 31. The side CD gives the tangents at the origin

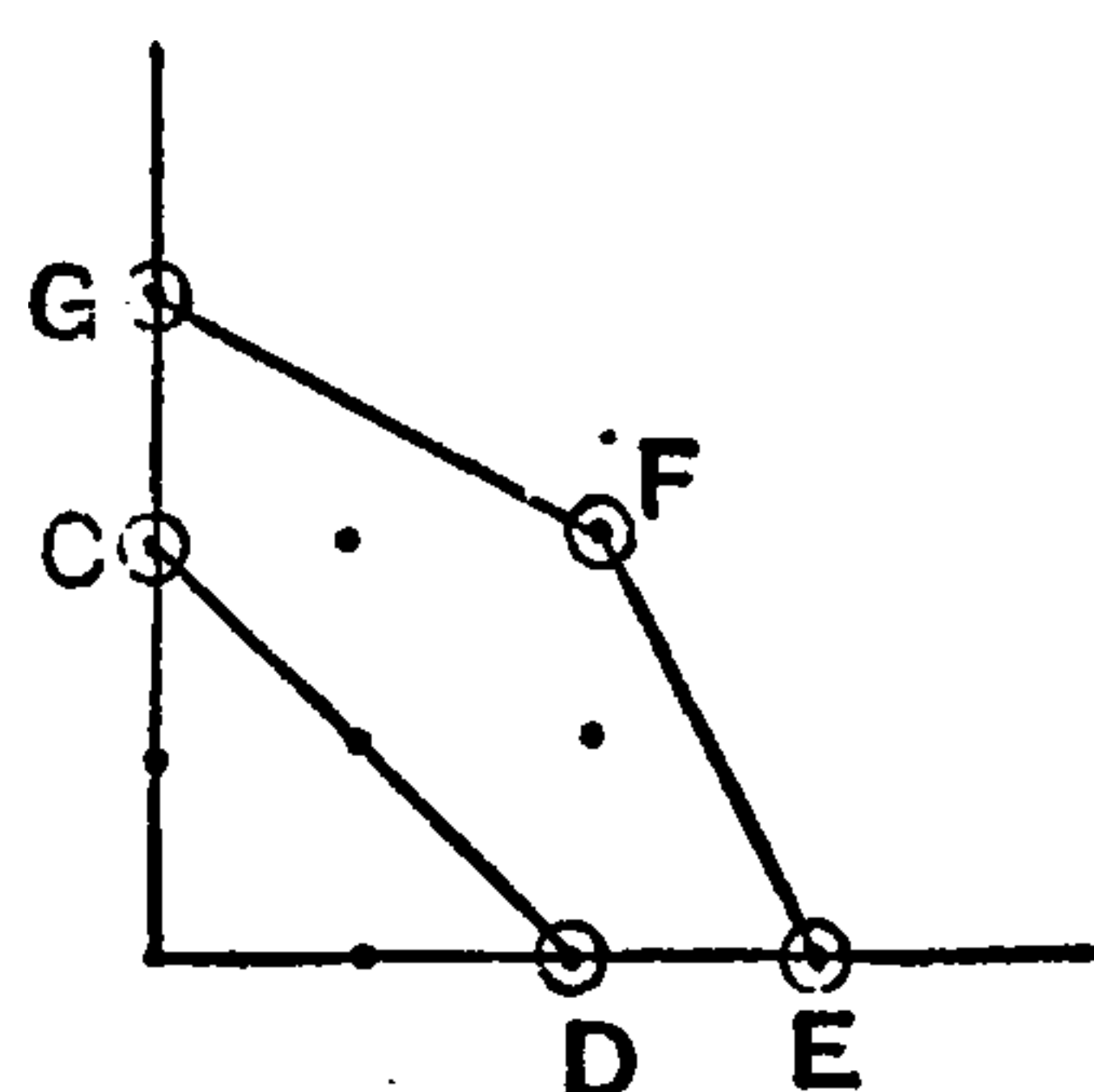


Fig. 31

$$x - y = 0 \quad \text{and} \quad x + y = 0.$$

The terms represented by the next parallel line are

$$x^3 - y^3 = (x - y)(x^2 + xy + y^2). \quad (2)$$

Now in the case of the tangent $x - y = 0$, we have, after putting $x = y$ in the coefficients of $x - y$ in equation (1),

$$2(x - y) + 3x(x - y) + x^3 = 0.$$

This equation shows that, when $x = y = 0$, $x - y$ has a finite ratio to x^3 ; for $x(x - y)$ bears a vanishing ratio to $x - y$. Thus the term arising from expression (2) vanishes, and we have at the origin

$$2 \frac{x - y}{x^3} + 1 = 0;$$

hence

$$2(y - x) = x^3 \quad (3)$$

is the approximate form at the origin of the branch whose tangent is $x = y$. Equation (3) shows that the curve lies above the tangent in the first and below it in the third quadrant.

The second approximation to the branch whose tangent is $x + y = 0$ is found, by putting $y = -x$ in equation (1), to be

$$2(x + y) + 2x^2 = 0,$$

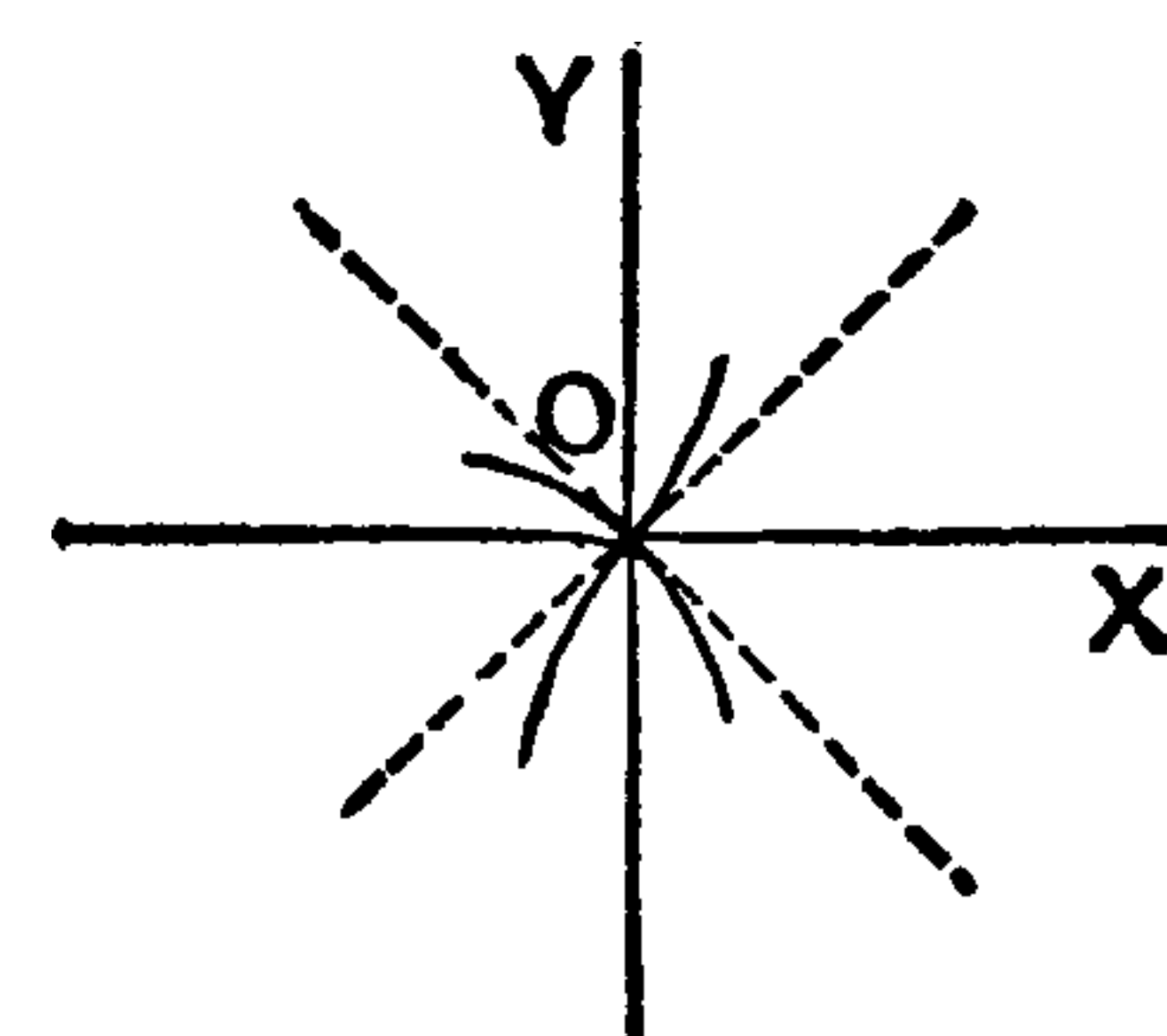


Fig. 32

which shows that the curve lies below this tangent on both sides of the origin. Hence the complete form at the origin is that indicated in Fig. 32.

75. Consider next the side DE of the polygon, which by itself gives only the point of the curve $(-1, 0)$. Writing equation (1) according to powers of y ,

$$x^2(x + 1) + y^2(x^2 - 1) - y^3 = 0. \quad (4)$$

There are no marked points on the parallel line adjacent to the side DE of the polygon, which shows that $x = -1$ is the tangent, as in Art. 34. Moreover, the expression $y^2(x^2 - 1)$, corresponding to the next parallel line, contains the factor $x + 1$, which vanishes at the point in question. Hence it is necessary to include the term represented by the point G ; thus, putting $x = -1$ in the coefficients of y and $x + 1$ in equation (4),

$$x + 1 - 2y^2(x + 1) - y^3 = 0.$$

In this equation, $x + 1$ bears a finite ratio to y^3 at the point $(-1, 0)$; for $y^2(x + 1)$ bears a vanishing ratio to $x + 1$. Hence the approximate form at this point is

$$y^3 = x + 1;$$

and, since y and $x + 1$ have the same sign, the form of the branch is that indicated in Fig. 33.

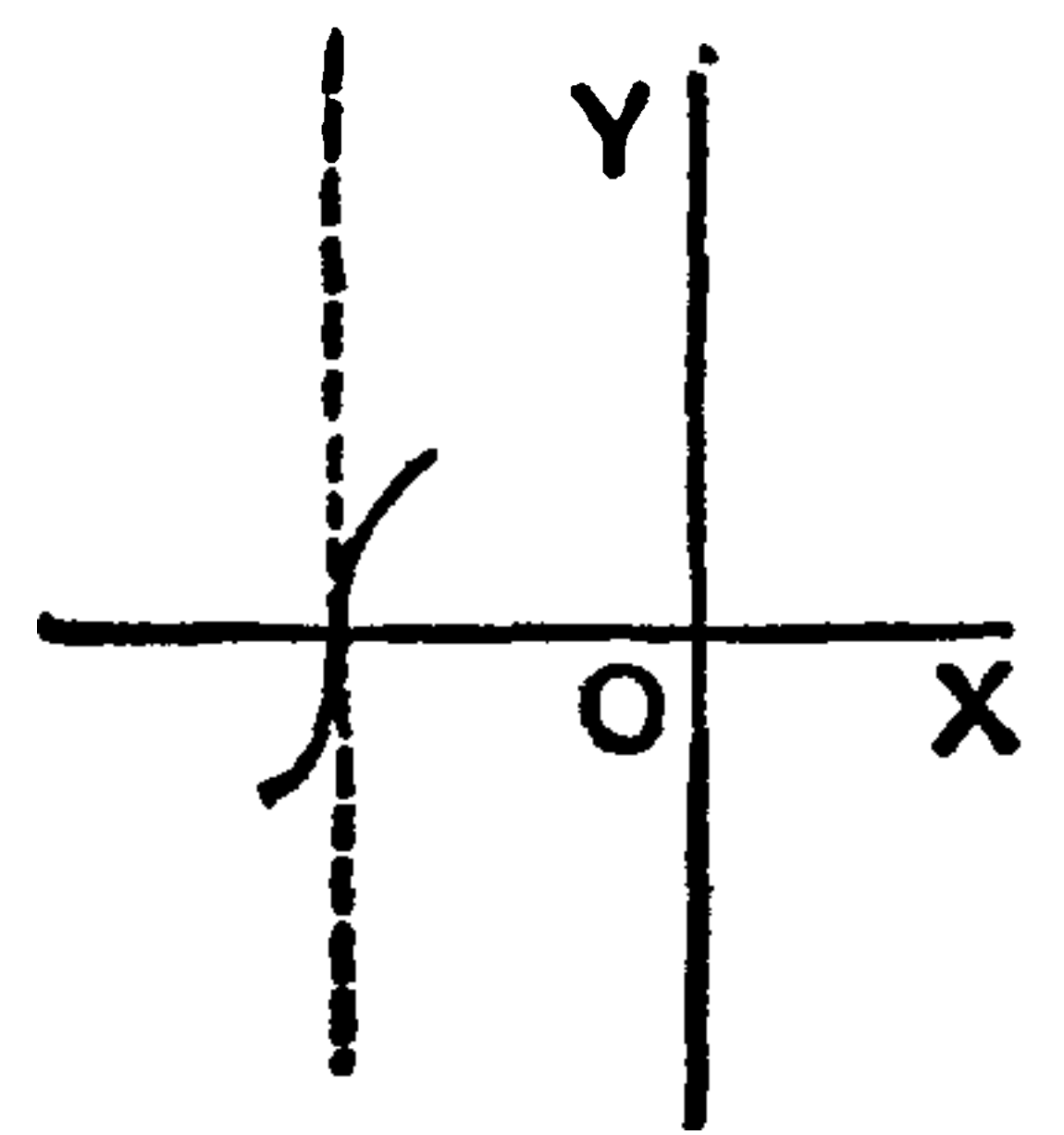


Fig. 33

76. To complete the construction of the curve, we have the side FG , giving

$$x^2 = y,$$

which indicates parabolic branches in the first and second quadrants in the direction of the axis of y . The side EF gives

$$y^2 = -x,$$

hence the curve has parabolic branches in the direction of the axis of x in the second and third quadrants.

The side GC gives the intersection $(0, -1)$. Proceeding as in Art. 75, the tangent at this point is

$$y = -1,$$

and the second approximation is

$$y + 1 = 2x^2,$$

which shows that the curve lies above the tangent, and that the radius of curvature at this point is $\frac{1}{4}$. See Art. 46.

77. The inflexional tangent $y = x$ in Fig. 32 meets the branch to which it is tangent in three coincident points, and crosses the other branch at the node: it therefore meets the curve in four coincident points at the origin; and, since the curve is a quartic, it cannot meet the curve again. The inflexional tangent $x = -1$ (Fig. 33) meets the curve in three coincident points at the point of contact, and meets it also at the fundamental point B ; hence the curve cannot again cross this tangent. It is therefore necessary to join the branches determined as indicated in Fig. 34.

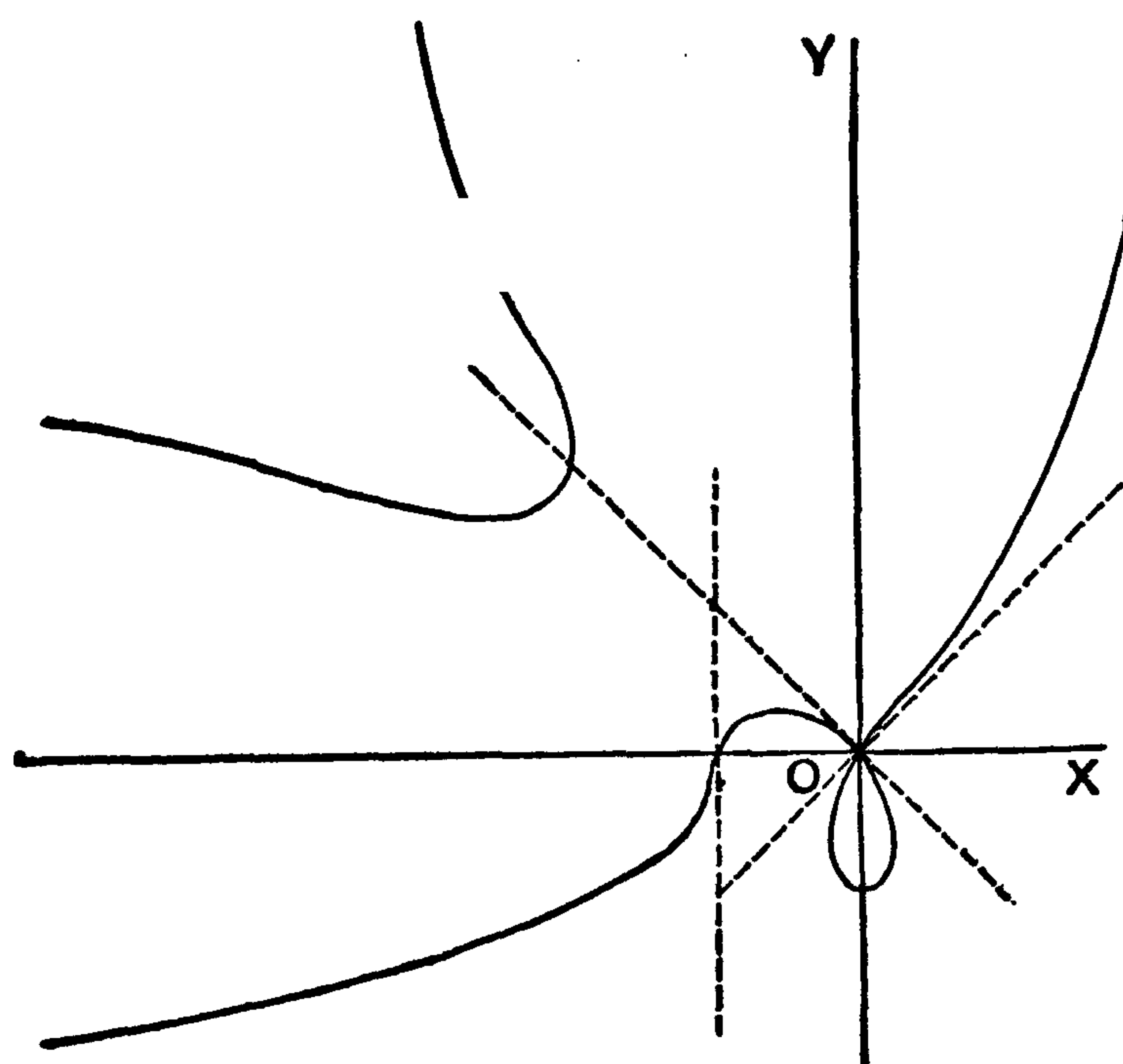


Fig. 34

The tangent $y = -x$ meets the curve in three coincident points at the origin; and, putting $y = -x$ in the equation of the curve, the fourth point of intersection is found to be $(-2, 2)$, thus determining a point on the infinite branch in the second quadrant. The line $x = -2$ will be found to touch the curve at this point, and to meet it again at $(-2, -1)$.

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may have a finite value at the point $(1, 0)$. To ascertain this value, divide equation (2) by y^2 , thus

$$\frac{2x(x-1)}{y^2} - \frac{x^3 + x^2}{y} + x^2 - y = 0.$$

Before putting $y = 0$ in this equation, we must substitute for x its approximate value $1 + y$, derived from the first approximation, in the numerators of the fractions, and retain all the terms which have a finite value when $y = 0$. Thus we have

$$\frac{2(1+y)(x-1)}{y^2} - \frac{1 + 3y + \dots + 1 + 2y + \dots}{y} + 1 = 0,$$

or

$$\frac{2(x-1) - 2y}{y^2} + \frac{2(x-1)}{y} - 5 + 1 = 0;$$

and, since $\frac{x-1}{y} = 1$ at the point $(1, 0)$, this becomes

$$\frac{x-1-y}{y^2} = 1. \quad (4)$$

Hence we infer that $x-1-y$ is positive in the neighborhood of the point $(1, 0)$, and the form of the curve is therefore that represented in Fig. 36.

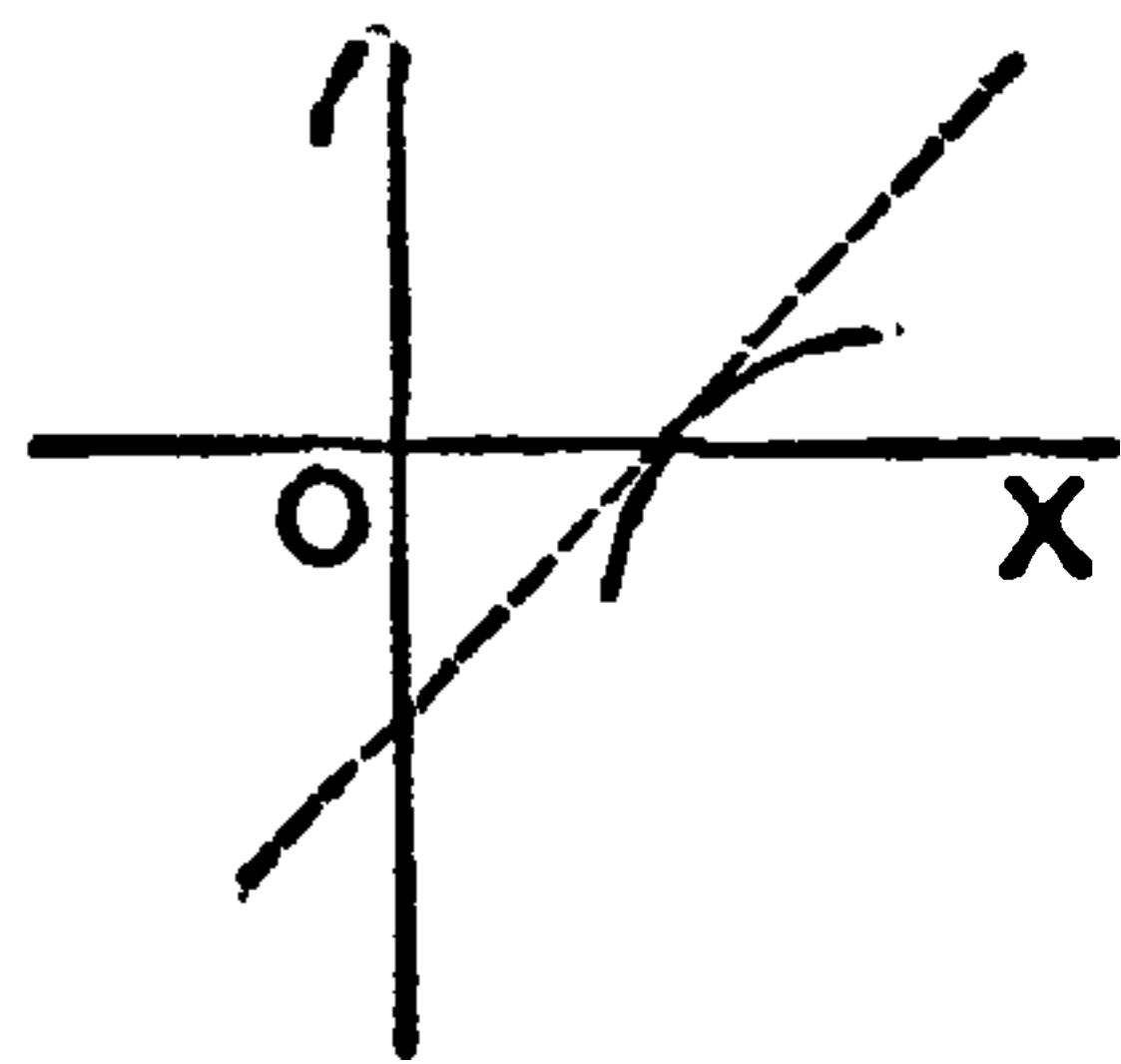


Fig. 36

Equation (4) may be used to determine the radius of curvature, as in Art. 66.

79. We may, in a similar manner, determine a second approximation when the side of the polygon gives only an intersection with the line at infinity and the first approximation gives the asymptote. For example, the side EF in Fig. 35 gives

$$\left[\frac{y}{x} \right]_{\infty} = 1,$$

determining a point at infinity. The equation of the curve is

$$x^2y(y - x) - (y^3 + yx^2) + 2x^2 - 2x = 0, \quad (5)$$

and the asymptote is found by dividing by x^3 and then making $y = x = \infty$, while $y - x$ is assumed to have a finite value. The result is

$$y - x - 2 = 0, \quad (6)$$

the equation of the asymptote. Now, since equation (6) shows that the quantity $y - x - 2$ vanishes when x is infinite, the quantity

$$x(y - x - 2)$$

may have a finite value when the point (x, y) recedes indefinitely upon the branch to which equation (6) represents the asymptote. To ascertain this value, divide equation (5) by x^2 , and put $y = x + 2$, as given by the first approximation, before putting x infinite. Thus we have

$$(x + 2)(y - x) - \frac{x^3 + 6x^2 + \dots + x^3 + 2x^2}{x^2} + 2 - \frac{2}{x} = 0;$$

and, rejecting terms which vanish when x is infinite, this becomes

$$x(y - x - 2) + 2(y - x) - 8 + 2 = 0,$$

or, since $y - x = 2$ when x is infinite,

$$x(y - x - 2) = 2. \quad (7)$$

Hence we infer that $y - x - 2$ is positive when x is positive and great, and negative when x is negative and great; the curve is therefore above the asymptote for distant points in the first quadrant, and below it for distant points in the third quadrant. See Fig. 37, page 64, in which the curve is traced.

80. It should be noticed that, in the processes illustrated in Arts. 78 and 79, while the terms employed in the first approx-

imation are those represented by points on the side of the polygon and on the adjacent parallel line, the additional terms introduced in the succeeding approximation are represented by points on the next parallel line.

Asymptotic Parabolas

81. A process of successive approximation, similar to that employed in the preceding articles, enables us to determine more accurately the position of a parabolic branch at one of the fundamental points A or B . Such a branch is indicated by the side FG of the analytical polygon in Fig. 35. Let the equation of the curve be written in the form

$$y^2(x^2 - y) - x^3y - x^2y + 2x^2 - 2x = 0, \quad (8)$$

the terms included in the first expression being those corresponding to the side of the polygon. These terms alone produce terms with finite values when the equation is divided by y^3 ; the result being

$$\frac{x^2}{y} \Big]_{\infty} = 1 \quad \text{or} \quad \frac{x^2 - y}{y} \Big]_{\infty} = 0, \quad (9)$$

from which we infer that the parabola

$$x^2 = y \quad (10)$$

is an approximate form at the fundamental point B . Now, since it follows from equation (9) that $\frac{y}{x} \Big]_{\infty} = \infty$, the product

$$\frac{y}{x} \cdot \frac{x^2 - y}{y} \quad \text{or} \quad \frac{x^2 - y}{x}$$

may have a finite value at B . This will, in fact, be found to be the case when the equation contains terms represented by points of the analytical triangle situated upon the parallel line adjacent to the side of the polygon in question. In Fig. 35

there is one marked point so situated, namely the point E ; accordingly, dividing equation (8) by y^2x , we have

$$\frac{x^2 - y}{x} - \frac{x^2}{y} - \frac{x}{y} + 2\frac{x}{y^2} - \frac{2}{y^2} = 0,$$

in which the only additional term that does not vanish at the fundamental point B is that represented by the point E . The result is that at B

$$\left[\frac{x^2 - y}{x} \right]_{\infty} - 1 = 0; \quad (11)$$

hence we infer that the parabola

$$x^2 - y - x = 0 \quad (12)$$

is a closer approximation to the curve than that represented by equation (10).

82. Equation (11) shows that $\frac{x^2 - y - x}{x} = 0$ at the fundamental point B ; hence the product of this quantity by x , namely

$$x^2 - y - x,$$

may have a finite value at B . To find this value, divide equation (8) by y^2 , thus

$$x^2 - y - \frac{x^3}{y} - \frac{x^2}{y} + 2\frac{x^2}{y^2} - 2\frac{x}{y^2} = 0. \quad (13)$$

In order to retain all the terms of this equation which have finite values at B , it is necessary in the infinite term $\frac{x^3}{y}$ to substitute for x^2 its value $y + x$ as given by the second approximation, equation (12). The result is that at B

$$x^2 - y - \frac{x(y + x)}{y} - 1 = 0,$$

or

$$x^2 - y - x - \frac{x^2}{y} - 1 = 0;$$

hence the parabola

$$x^2 - y - x - 2 = 0 \quad (14)$$

is a still closer approximation than that represented by equation (12). This curve is called the *asymptotic parabola*.

83. Putting the equation of this parabola in the form

$$(x - \frac{1}{2})^2 = y + 2\frac{1}{4},$$

we see that its vertex is at $(\frac{1}{2}, -2\frac{1}{4})$ and its parameter is unity.

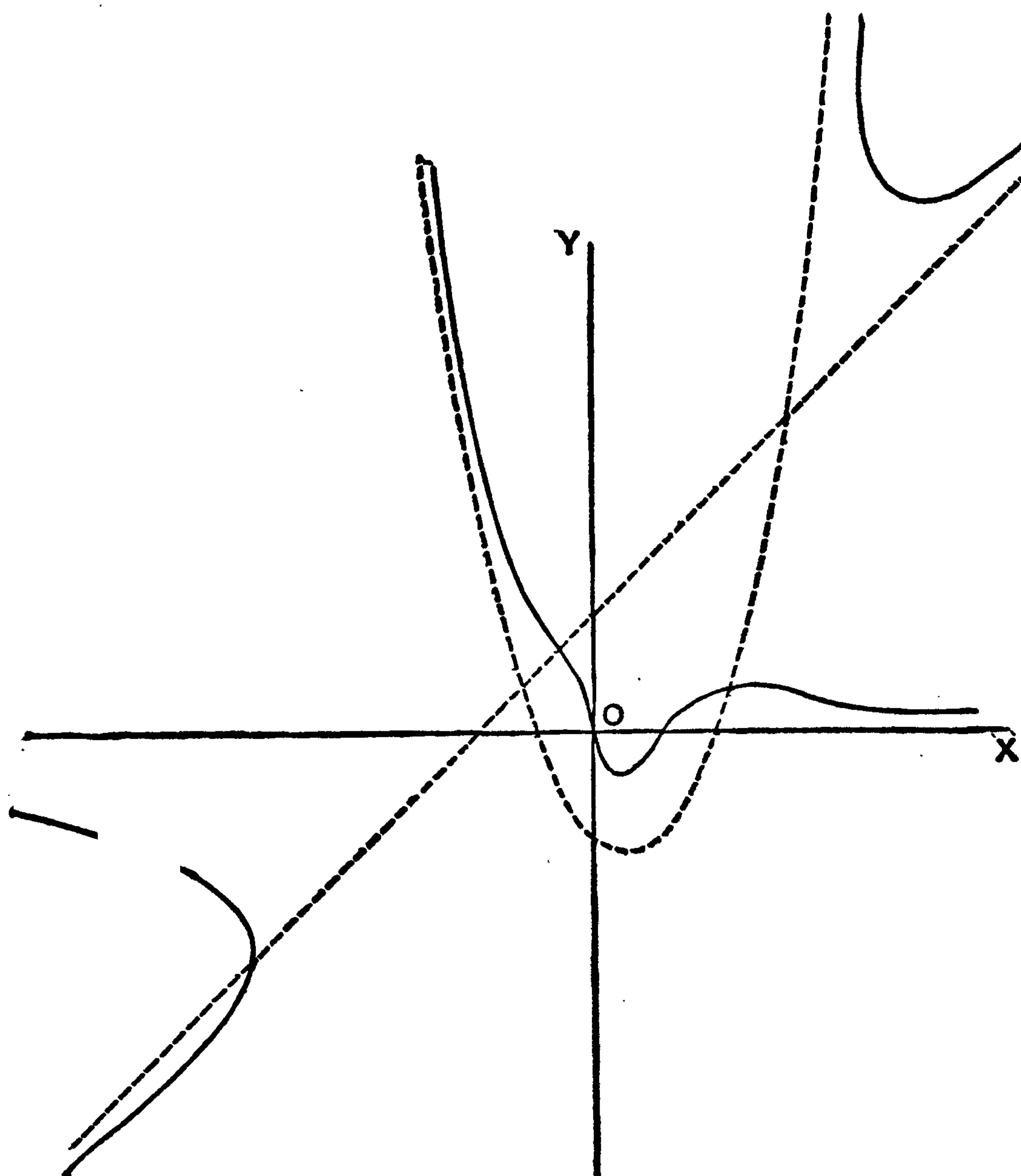


Fig. 37

The parabola is constructed in Fig. 37, together with the branches determined in Arts. 78 and 79, and those correspond-

ing to the remaining sides of the analytical polygon in Fig. 35. In joining the branches, it is obvious that the oblique asymptote must be crossed in the fourth quadrant, and also in the second quadrant; and, since this asymptote cannot be crossed again, the remaining branches must be joined as in the figure.

84. The process given in Arts. 81 and 82 may be continued in order to determine the value of the quantity

$$x(x^2 - y - x - 2),$$

which may be finite at the fundamental point B , since by equation (14) the quantity in the parenthesis vanishes at B . Multiplying equation (13) by x , we have the equation of the curve in the form

$$x(x^2 - y) - \frac{x^2(x^2 + x)}{y} + 2\frac{x^3}{y^2} - 2\frac{x^2}{y^2} = 0.$$

The last two terms vanish at B ; and, employing the value of x^2 given by equation (14), we have at B

$$x(x^2 - y) - \frac{(y + x + 2)(x^2 + x)}{y} = 0,$$

$$x(x^2 - y) - (x^2 + x) - \frac{(x + 2)(y + 2x + 2)}{y} = 0,$$

$$x(x^2 - y) - (x^2 + x) - (x + 2) - \frac{2(x^2 + 3x + 2)}{y} = 0,$$

and, rejecting vanishing terms,

$$x(x^2 - y - x - 2) = 4. \quad (15)$$

By equation (14), the quantity $y - x^2 + x + 2$ is the difference between the ordinate of the curve and the corresponding

ordinate of the asymptotic parabola. Hence we infer from equation (15) that the curve is below the parabola in the first quadrant, and above it in the second quadrant, as represented in Fig. 37; the distance measured in the direction of the axis of y being, when x is great, about $\frac{4}{x}$. *

Examples IV

1. $x^2y^3 + xy^4 - y^3 - x^2 + x = 0.$
2. $x^4 - y^4 + 2ax^3y + a^2y^2 + a^4 = 0.$
3. $x^4 - 2ay^3 - 3a^2y^2 - 2a^2x^2 + a^4 = 0.$
4. $x^3 + x^2 - y^2 + 4y = 0.$
5. $x^4 - 2ay^3 - 2a^2x^2 + 3a^2y^2 = 0.$
6. $(y^2 - 4a^2)(x^2 + y^2) + 4a^4 = 0.$
7. $x(x^2 + y^2)(y + 2x) + 4ax^3 + 2ay^3 = 0.$
8. $xy^2 + yx^2 + ay^2 - a^2x = 0.$
9. $(y - x)^2(y + 2x) = 2y(y + 11x).$
10. $y^4 - 2ay^3 + 2ay^2x - 4a^2xy + a^2x^2 = 0.$
11. $x^2y^2 = a(x^3 + y^3).$
12. $y(x + y)^2 - 4ax(x + y) + 4a^2y = 0.$
13. $x^6 - x^4y^2 - 2a^2xy^3 - a^4x^2 + 3a^5y = 0.$
14. $(x - y)(x^2 - 4y^2) = 2ax^2 - 10axy + 20ay^2 + 24a^2x.$
15. $y^4 - 2axy^2 - 3a^2y^2 - 2a^3x = 0.$
16. $x^2y^2 - 2xy^3 - 2ax^2y + 2axy^2 + a^2(x - y)^2 = 0.$
17. $x^4 + x^3y + x^2y^2 - 2axy^2 - ay^3 - a^2x^2 + a^2y^2 = 0.$
18. $x^3y - x^2y^2 - x^3 - x^2y + xy^2 + y^3 + x^2 = 0.$

* This result, as well as that found in Art. 79, equation (7), gives a good approximation only when the value of x is considerable. For instance, when $x = 5$, the ordinate of the parabola is 18, and that of the rectilinear asymptote is 7; the values of y for the curve are about 15.5 and 9.4, differing from the former by 2.5 and 2.4 respectively, whereas the differences given by equations (15) and (7) would for this value of x be .8 and .4 respectively.

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this line, that is to say, a parabolic branch. It is to be noticed that in these cases the process of approximation determines at once the side of the tangent on which the curve lies, and in the case of the line at infinity it determines in which of two opposite quadrants the parabolic branches lie. For example, take the curve whose equation is

$$x^4 - 2x^2y^2 + y^4 - 2ax^2y + ay^3 = 0, \quad (1)$$

for which Fig. 38 gives the analytical polygon. The side AB gives for the intersection with the line at infinity

$$x^4 - 2x^2y^2 + y^4 = 0$$

or

$$(x + y)^2(x - y)^2 = 0,$$

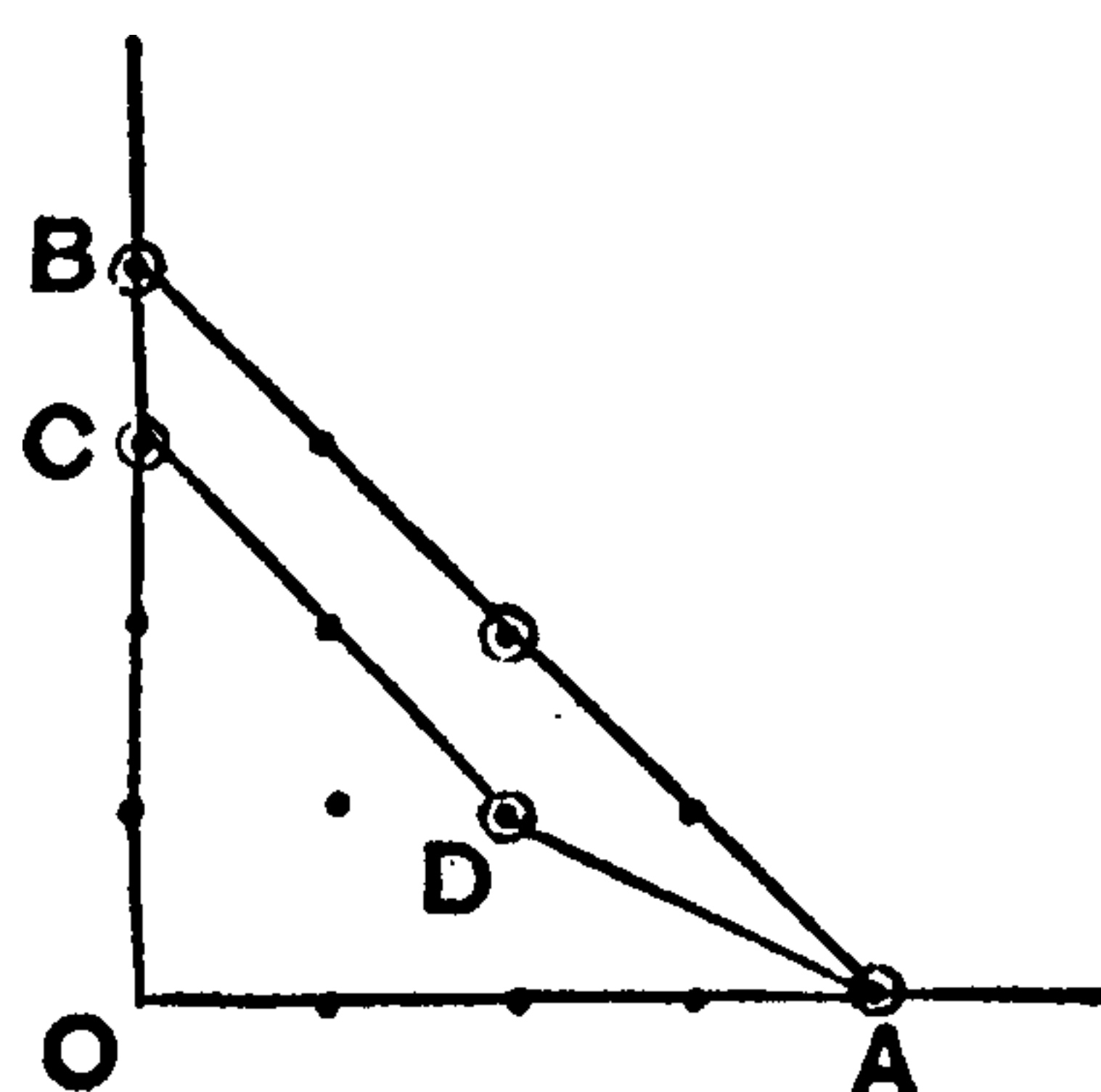


Fig. 38

an equation having two pairs of equal roots.

Now an infinite branch for which $\frac{y}{x} \Big|_{\infty} = 1$ must lie in the first or in the third quadrant, and from equation (1) we have

$$(x - y)^2 = \frac{2ax^2y - ay^3}{(x + y)^2},$$

in which, putting $y = x$, we have, when x is infinite,

$$(x - y)^2 = \frac{ax}{4}.$$

The corresponding branches are parabolic because $x - y$ is infinite when x is infinite; and, since this quantity is real only when x is positive, the branches are in the first quadrant.

87. The construction of the curve is readily completed ; for the side AD of the polygon gives the form at the origin

$$x^2 = 2ay,$$

the side CD gives the tangents

$$y = \pm \sqrt{2} \cdot x,$$

and the side BC gives the point $(0, -a)$. Moreover, the equation containing powers of x with even exponents only, the curve is symmetrical to the axis of y ; hence its form is that indicated in Fig. 39.

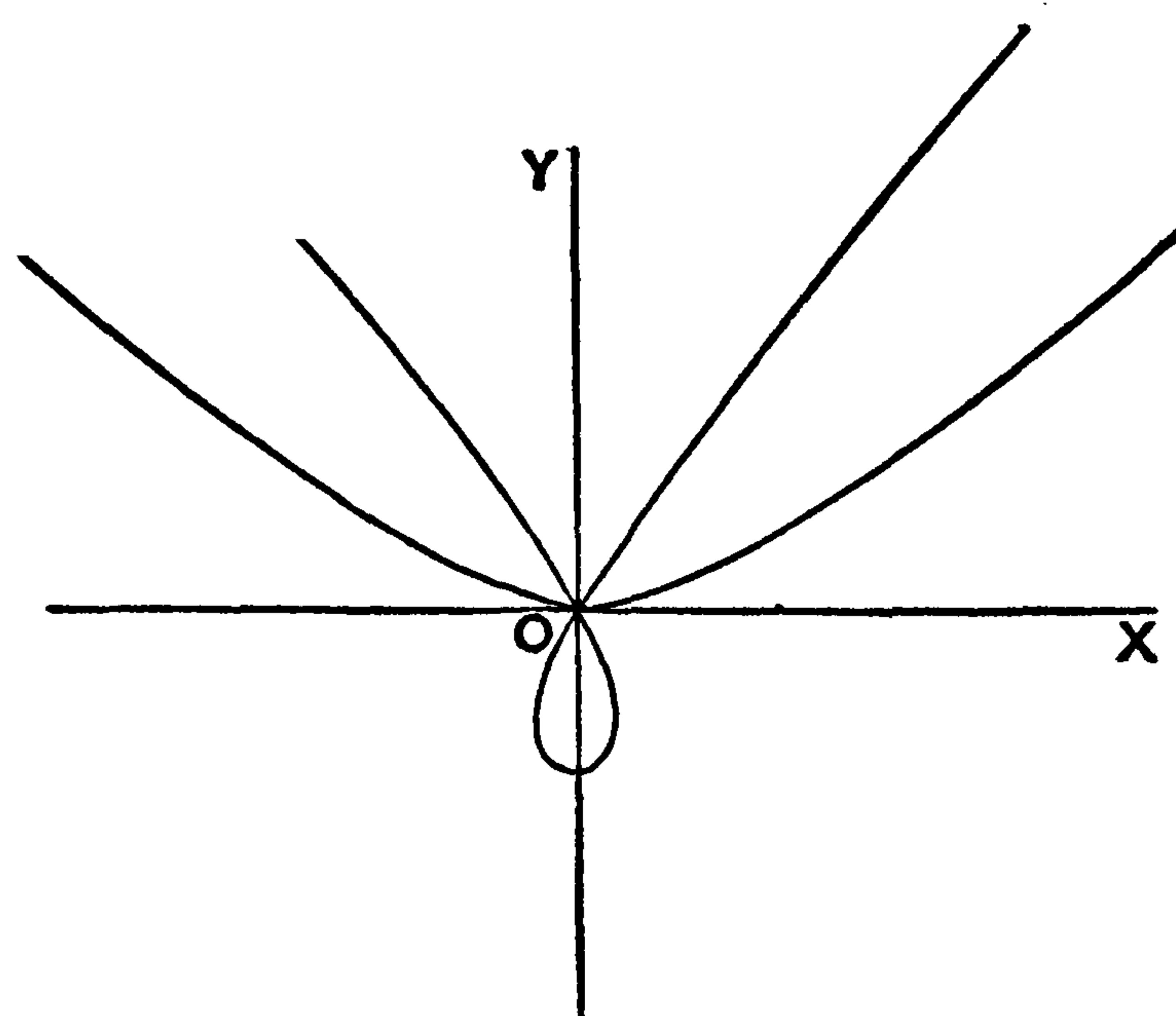


Fig. 39

88. Supposing, as in Art. 86, that the equation giving the intersection with one of the fundamental lines has equal roots, the singular case, corresponding to a node, is indicated by the occurrence of a quadratic equation for the first approximation, as in Art. 36, the roots of this quadratic being real or imaginary according as the node is a crunode or an acnode. In the case of the line at infinity, the singular case is that of parallel asymptotes ; and these may be real forming a crunode at infinity, as in Art. 29, or imaginary forming an acnode at infinity, there being in the latter case, of course, no corresponding infinite branches.

Cusps

89. Let us now suppose that equal roots occur in the quadratic equation determining the first approximation or tangents at a node. This will usually constitute a case in which the curve is real for values of x on one side, and imaginary for

values of x on the other side, of the particular value of x in question. For example, let the equation of the curve be

$$x^2(x^2 + y^2) - 2ax^3 + 2ax^2y - axy^2 + a^2(x - y)^2 = 0, \quad (1)$$

for which Fig. 40 gives the analytical polygon. The side CD gives, for the tangents at the node at the origin, the equation

$$(x - y)^2 = 0,$$

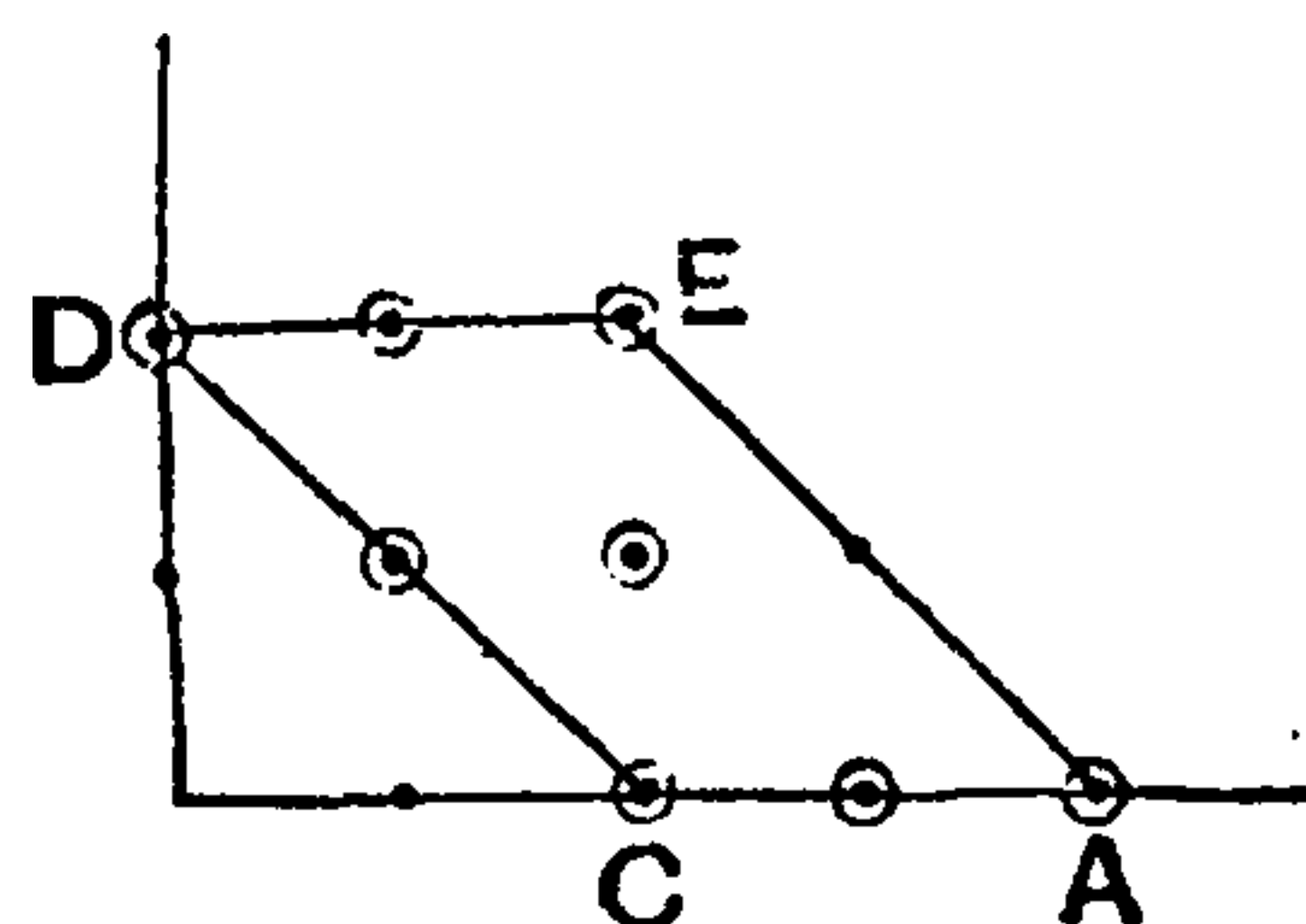


Fig. 40

which has equal roots, showing that both tangents coincide with the line

$$y = x. \quad (2)$$

Proceeding to the next approximation, by dividing equation (1) by x^3 and making $x = y = 0$, we have at the origin

$$\frac{a^2(x - y)^2}{x^3} - a = 0;$$

hence we infer that near the origin the quantity $y - x$ is real when x is positive, and imaginary when x is negative. Moreover, in the former case, this quantity, which is the difference between corresponding ordinates of the curve and the tangent, has two values, one positive and the other negative. Hence the curve lies on the right of the origin and on both sides of the tangent, the two branches forming a *cusp* at the origin, as in Fig. 41.

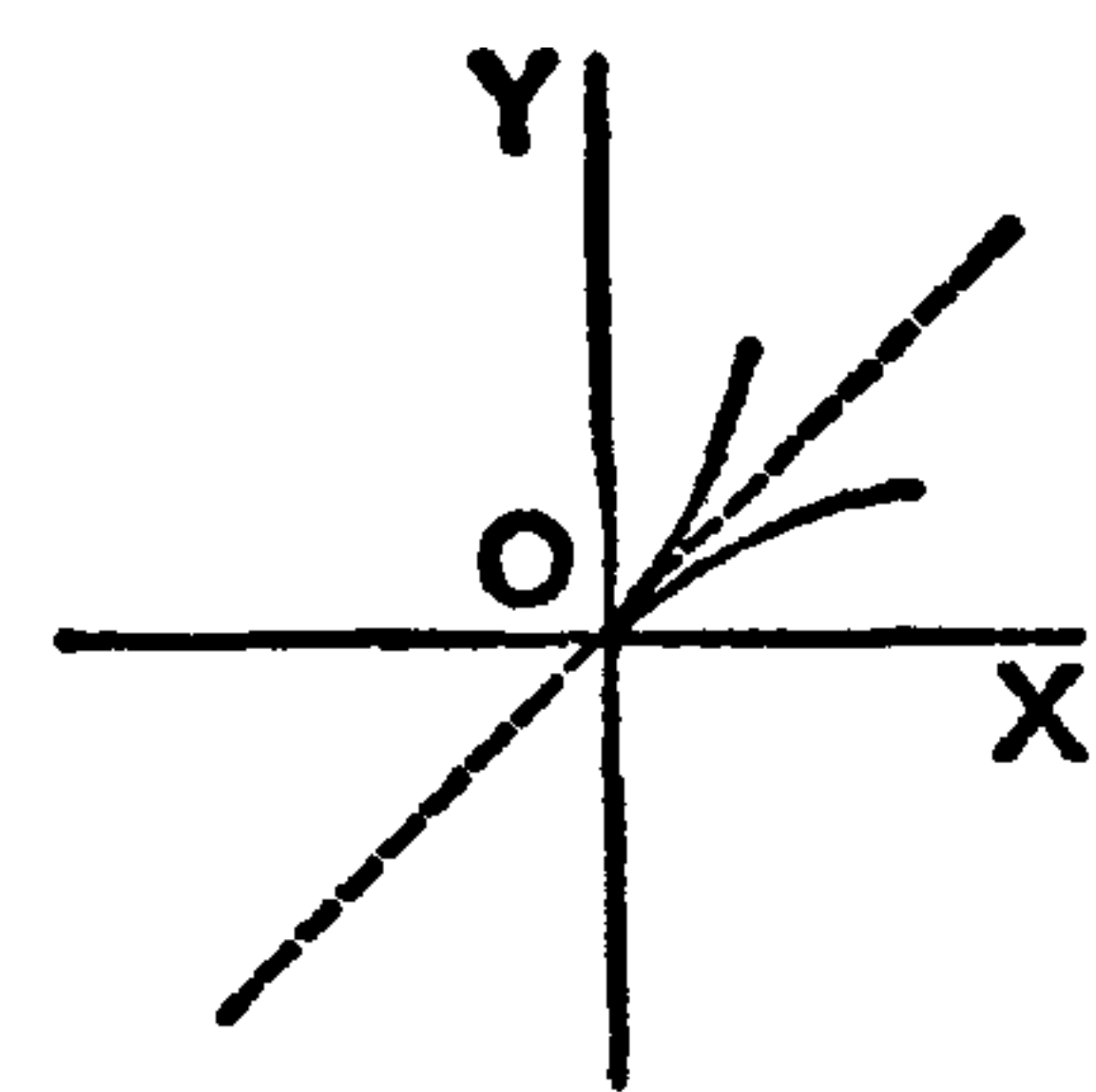


Fig. 41

90. The side CA of the polygon in Fig. 40 gives, for the intersection with the axis of x , the equation

$$x^2 - 2ax + a^2 = 0,$$

which has a pair of equal roots. Arranging equation (1) according to powers of y , it is

$$x^2(x - a)^2 + 2ax(x - a)y + (x^2 - ax + a^2)y^2 = 0,$$

and the occurrence of the factor $x - a$ in the coefficient of y renders this a quadratic equation for the ratio of y to $x - a$ at the point $(a, 0)$, as in Art. 36; hence there is a node at this point.

Putting $x = a$, the quadratic is

$$(x - a)^2 + 2(x - a)y + y^2 = 0,$$

which has equal roots; hence the point is a cusp at which the tangent is the line

$$x - a + y = 0. \quad (3)$$

91. The tangent at a cusp, like a tangent at an ordinary node, meets the curve in three coincident points at the point of contact; thus, the curve being a quartic, the tangent $y = x$ can meet the curve in only one other point, which is found to be $(\frac{1}{2}a, \frac{1}{2}a)$, and the tangent (3) passes through the same point.

The curve has no infinite branches; for the side AE of the polygon gives

$$x^2 + y^2 = 0,$$

indicating imaginary intersections with the line at infinity; and the side DE gives, for the parallel asymptotes or tangents at the fundamental point B ,

$$x^2 - ax + a^2 = 0,$$

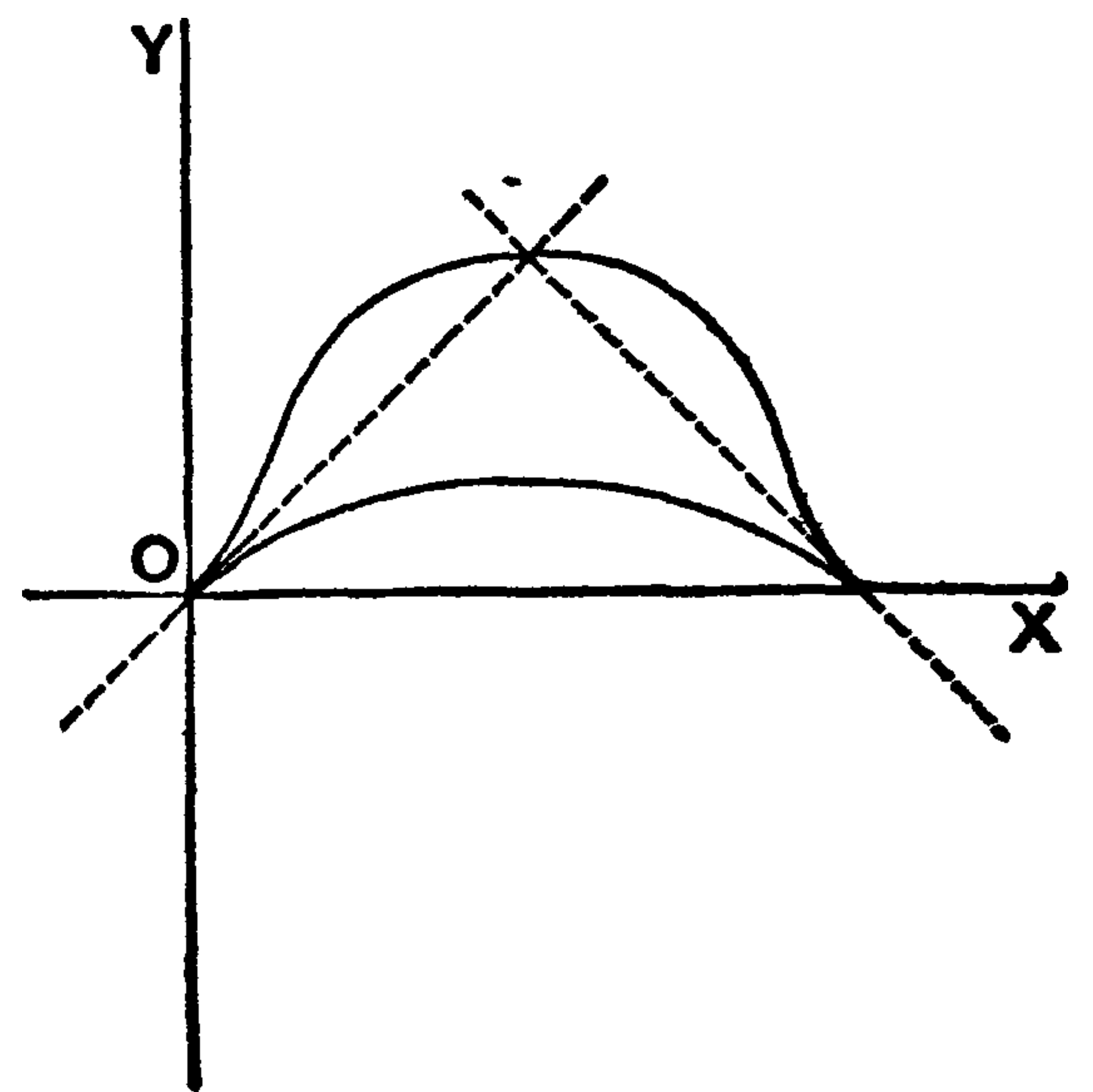


Fig. 42

which has imaginary roots, so that B is an acnode.

The shape of the curve is therefore that shown in Fig. 42.

Tacnodes

92. When equal roots occur in the equation determining the tangents at a node, we have seen in Art. 89 that we ordinarily have a cusp; but here also a singular case, analogous to that explained in Art. 85, may arise; that is to say, the quantity which measures the distance of the curve from the tangent may be real on both sides of the node, or imaginary on both sides of the node. For example, let the equation of the curve be

$$x^2y^2 + 2ay(x^2 - y^2) + a^2(x - y)^2 = 0, \quad (1)$$

for which Fig. 43 gives the analytical polygon. The side CD , indicating a node at the origin, gives an equation with equal roots, and the line

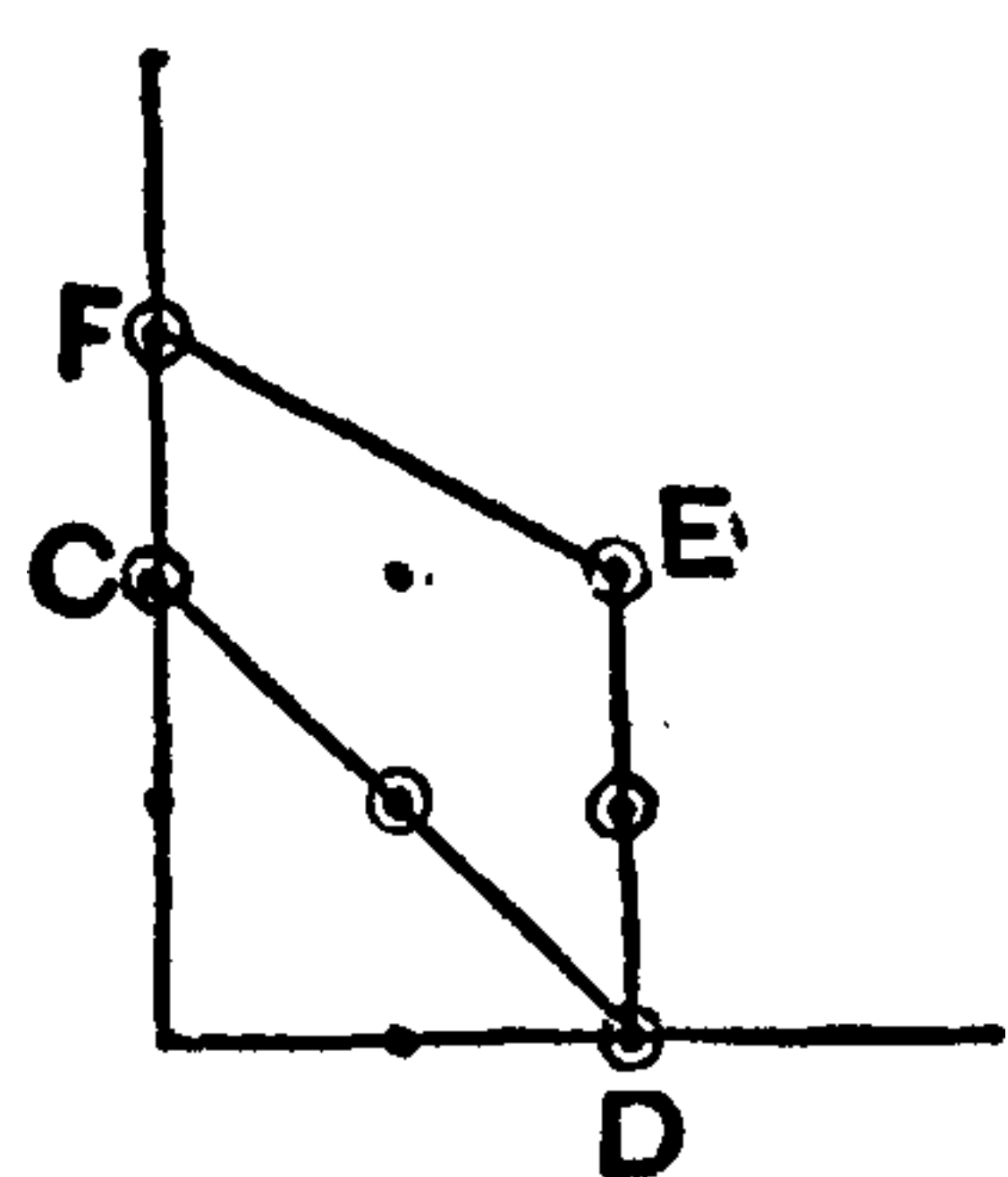


Fig. 43

$$y = x \quad (2)$$

is the only tangent at the origin, as in Art. 89. But, since in this case $x - y$ occurs as a factor in the terms of the third degree also, we no longer obtain a finite ratio between $(x - y)^2$ and x^3 ; in fact, putting in equation (1) $y = x$, as determined by the first approximation, we have

$$x^4 + 4ax^2(x - y) + a^2(x - y)^2 = 0,$$

a quadratic equation for the ratio of $x - y$ to x^2 . Solving, we obtain real roots, namely

$$a(x - y) = (-2 \pm \sqrt{3})x^2; \quad (3)$$

hence the quantity $x - y$ is real on both sides of the origin, having two values, both of which are negative. Thus the form of the curve at the origin is that represented in Fig. 44. The two branches having a common tangent but different curvatures are said to form a

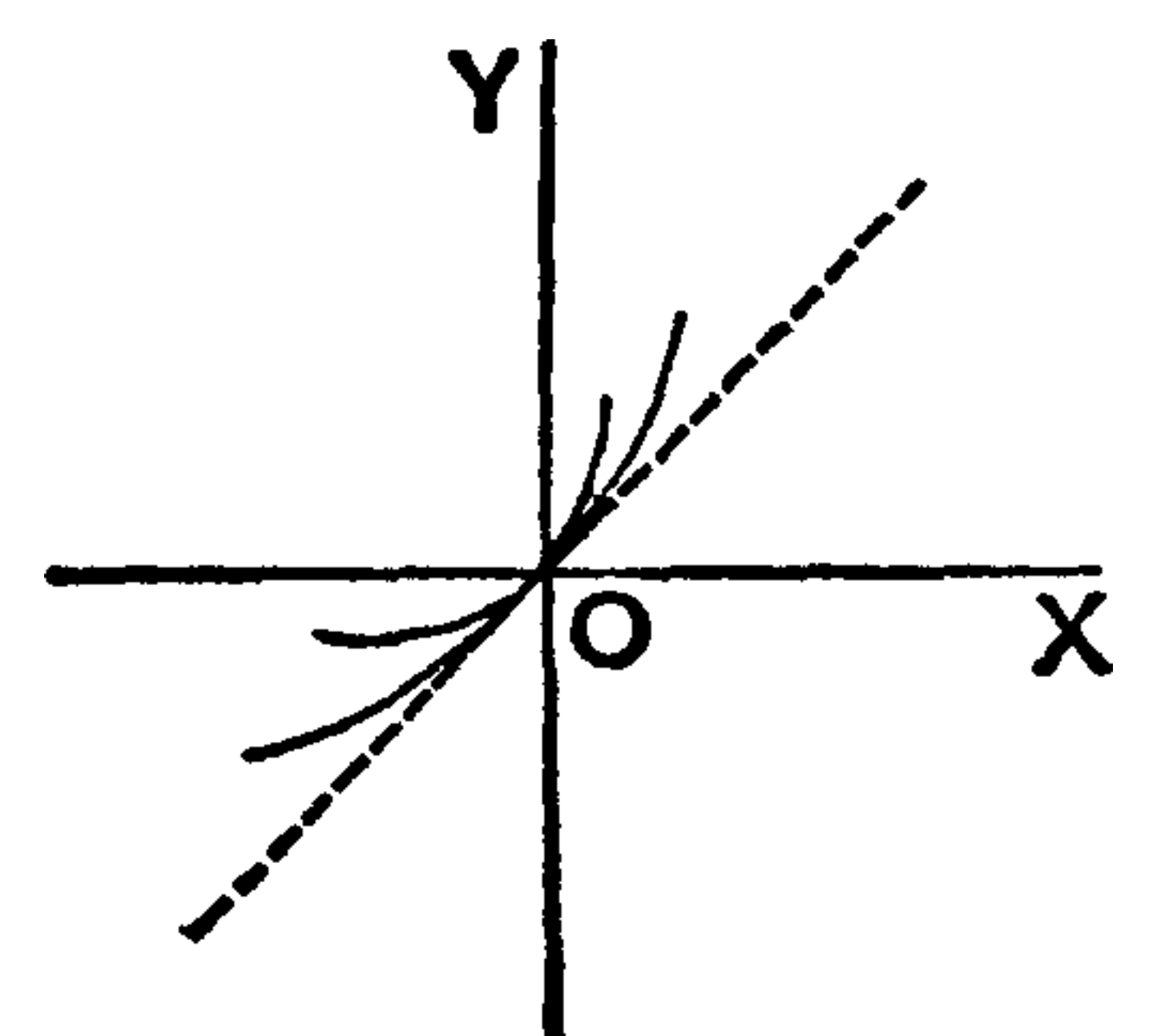


Fig. 44

tacnode. We have already had in Art. 60 an example of a tacnode in which the common tangent is one of the coordinate axes.

Cusps at Infinity

93. The line DE in Fig. 43 corresponds to a node at the fundamental point A ; the equation is

$$y^2 + 2ay + a^2 = 0,$$

which has equal roots; hence the curve has two tangents at A , or asymptotes, coincident with the line

$$y + a = 0.$$

Arranging the terms according to powers of x , the equation of the curve is

$$x^2(y + a)^2 - 2a^2xy - 2ay^3 + a^2y^2 = 0.$$

The next approximation is found, as in Art. 67, by dividing by x and then making $x = \infty$ and $y = -a$ as determined by the first approximation. The result is

$$x(y + a)^2 + 2a^3 = 0;$$

whence we infer that for the infinite branches x must be negative, and that $y + a$ will have two values, one positive and the other negative.

Hence the curve has two infinite branches approaching the left end of the asymptote, one on each side of it, as in Fig. 45, forming a cusp at the fundamental point A ,

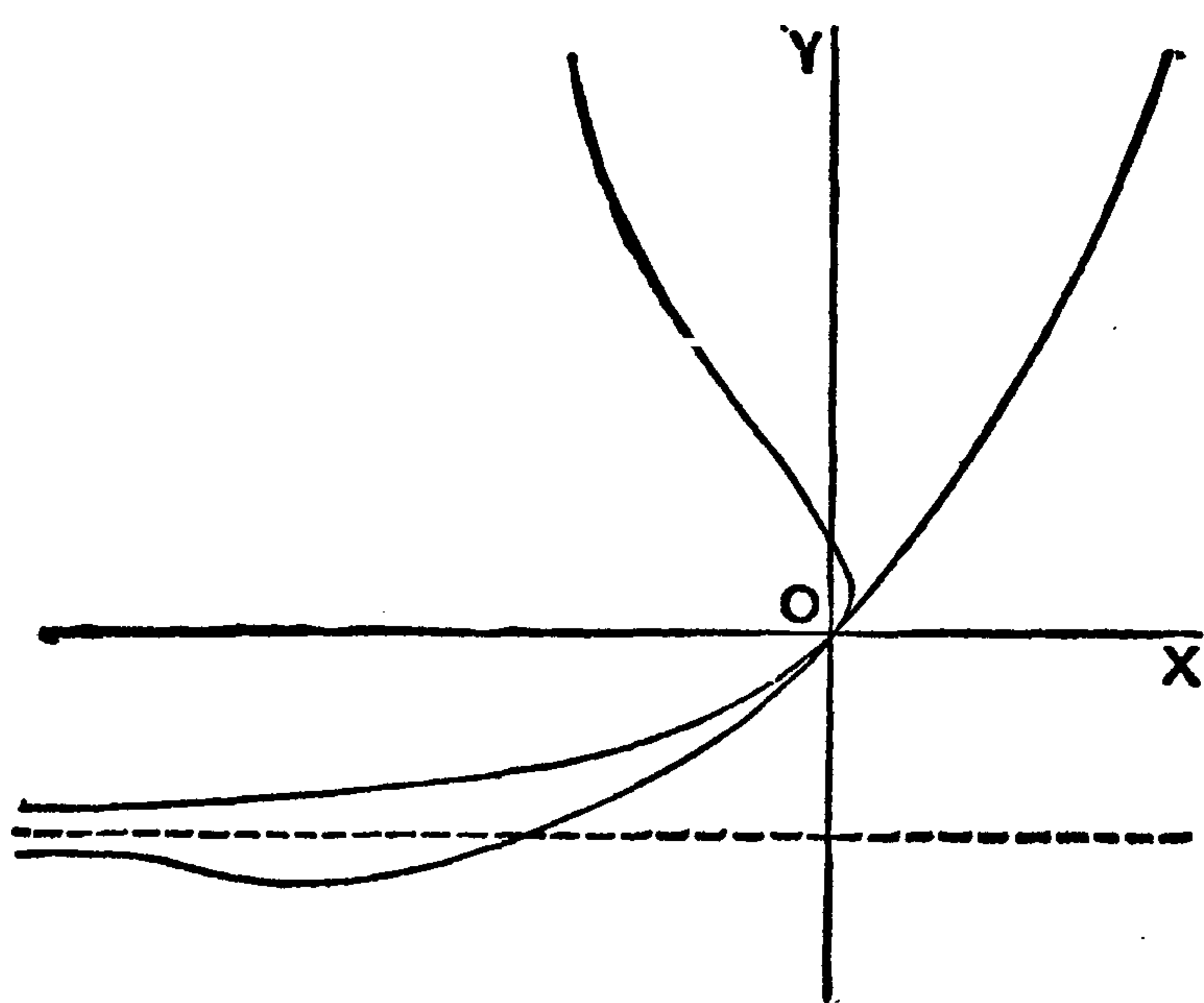


Fig. 45

94. To complete the construction of the curve, we have the side EF giving parabolic branches in the first and second quadrants, and the side FC giving the point $(0, \frac{1}{2}a)$. The tangent $y = x$ meets the curve in four coincident points at the origin, and therefore cannot meet it again. The asymptote, being a tangent at a cusp, meets the curve in three coincident points at infinity, and the fourth point of intersection is found to be at $(-\frac{3}{2}a, -a)$.

Ramphoid Cusps

95. When a quadratic equation presents itself for the second approximation at a node where the tangents are coincident, if the roots are real and distinct, as in Art. 92, the two branches having the common tangent have different curvatures; and these branches are real upon both sides of the node, forming a tacnode. If the roots were imaginary, there would, of course, be no branches, and the node would be an isolated point of the curve.

96. Suppose, now, that this equation has equal roots; the two approximate forms determining the curvature come into coincidence, and it will usually be found that the curve is real on one side and imaginary on the other side of the node, as in the case of the ordinary cusp. For example, let the equation of the curve be

$$x^4 - 2x^2y - 2xy^2 + y^2 = 0, \quad (1)$$

which is placed upon the analytical triangle in Fig. 46. The side AC , which represents a second approximation at the origin, the axis of x being the tangent, gives the quadratic

$$x^4 - 2x^2y + y^2 = 0,$$

which has equal roots, and

$$x^2 - y = 0 \quad (2)$$

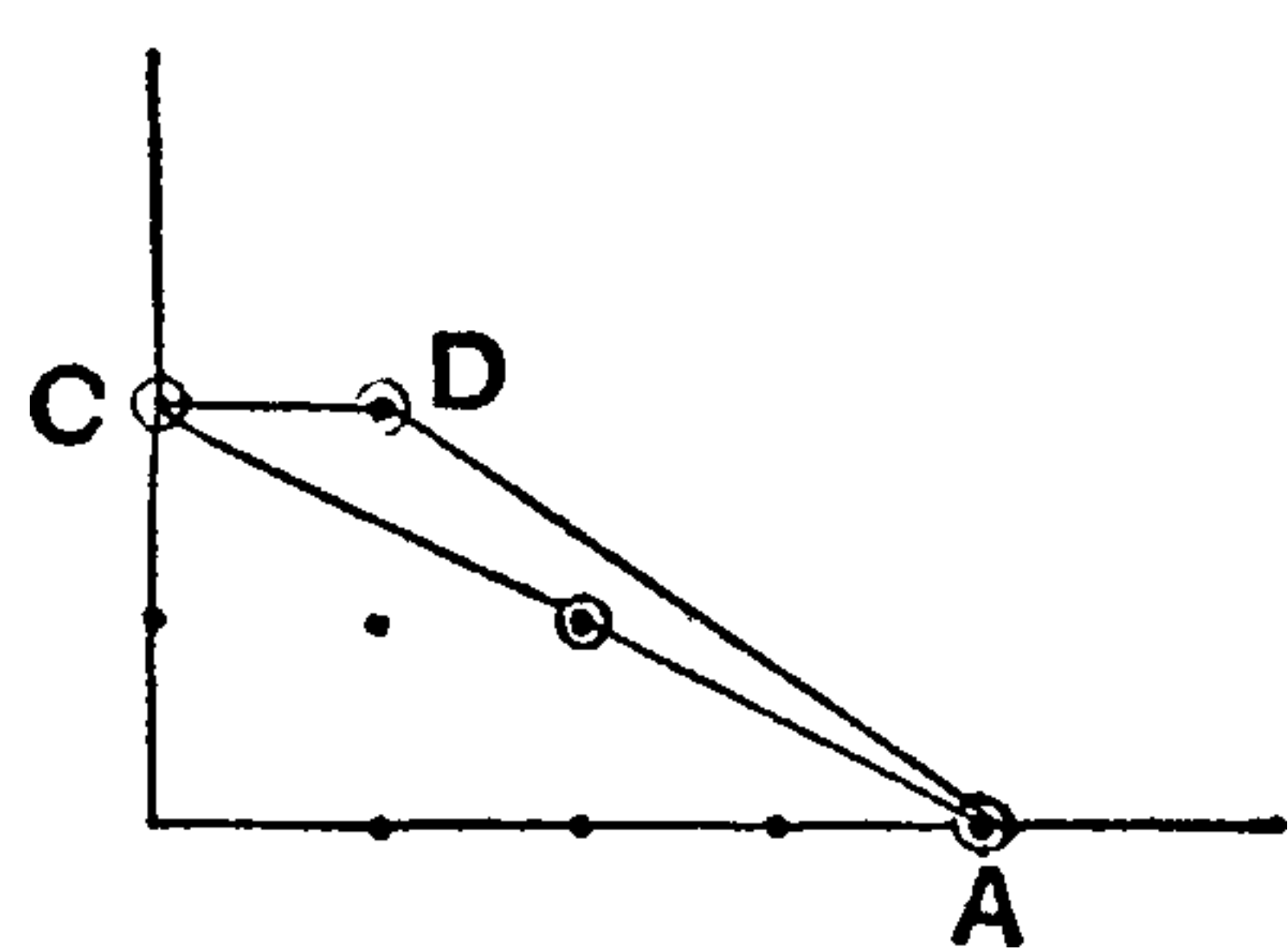


Fig. 46

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end. A branch of the latter character may be regarded as an arc of the curve terminated in each direction by one of the points in which the curve meets the line at infinity. But we may regard the infinite branch as continuous at one of these points with the other infinite branch which terminates at that point, thus making the curve to consist entirely of reentrant branches or *circuits*, which may or may not be cut by the line at infinity. For example, the conic consists always of a single circuit, which in the case of the hyperbola is cut into two arcs by the line at infinity.

99. A circuit is odd or even according as it is cut in an odd or an even number of points by a straight line. Obviously an odd circuit must cut every straight line in at least one point, and in particular it must cut the line at infinity in at least one point. A curve of odd degree must contain at least one odd circuit. Thus a cubic contains always an odd circuit which cuts the line at infinity at least once, and it may in addition contain an even circuit, which last may be a closed curve or oval not cutting the line at infinity.

100. A circuit containing a crunode consists of two parts or loops, the two extremities of a loop being at a common point but not having a common tangent. For example, in Fig. 48 both extremities of the infinite branch in the first quadrant are at the fundamental point B ; but it does not form a complete circuit, because the tangents at the extremities are the asymptote and the line at infinity respectively. The branch in the fourth quadrant is the other loop completing the circuit. An acnode is a point at which an oval vanishes. Thus the nodal variety of a curve is intermediate between varieties which differ in the number of circuits. As an illustration, take the curve whose equation is

$$(x^2 + y^2 - a^2)^2 = c^4 - 4a^2y^2.$$

The curve has no infinite branches, and is symmetrical to both axes. For its intersections with the axis of x , we have

$$x^2 - a^2 = \pm c^2,$$

or

$$x = \pm \sqrt{(a^2 \pm c^2)};$$

and for its intersections with the axis of y ,

$$(y^2 + a^2)^2 = c^4,$$

or

$$y = \pm \sqrt{(\pm c^2 - a^2)}.$$

The curve always cuts the axis of x at the distances $\pm\sqrt{(a^2 + c^2)}$ from the origin; and if $c < a$, it cuts this axis also at the distances $\pm\sqrt{(a^2 - c^2)}$, and does not cut the axis of y . In this case the curve consists of two ovals. If $c > a$, the curve cuts the axis of x only at the distances $\sqrt{(a^2 + c^2)}$ and it cuts the axis of y at the distances $\sqrt{(c^2 - a^2)}$ from the origin. In this case it consists of a single oval. In the intermediate case, when $c = a$, there is a crunode at the origin. The three varieties of the curve, which is known as the Cassinian, are drawn in Fig. 49. The nodal case is the lemniscata.

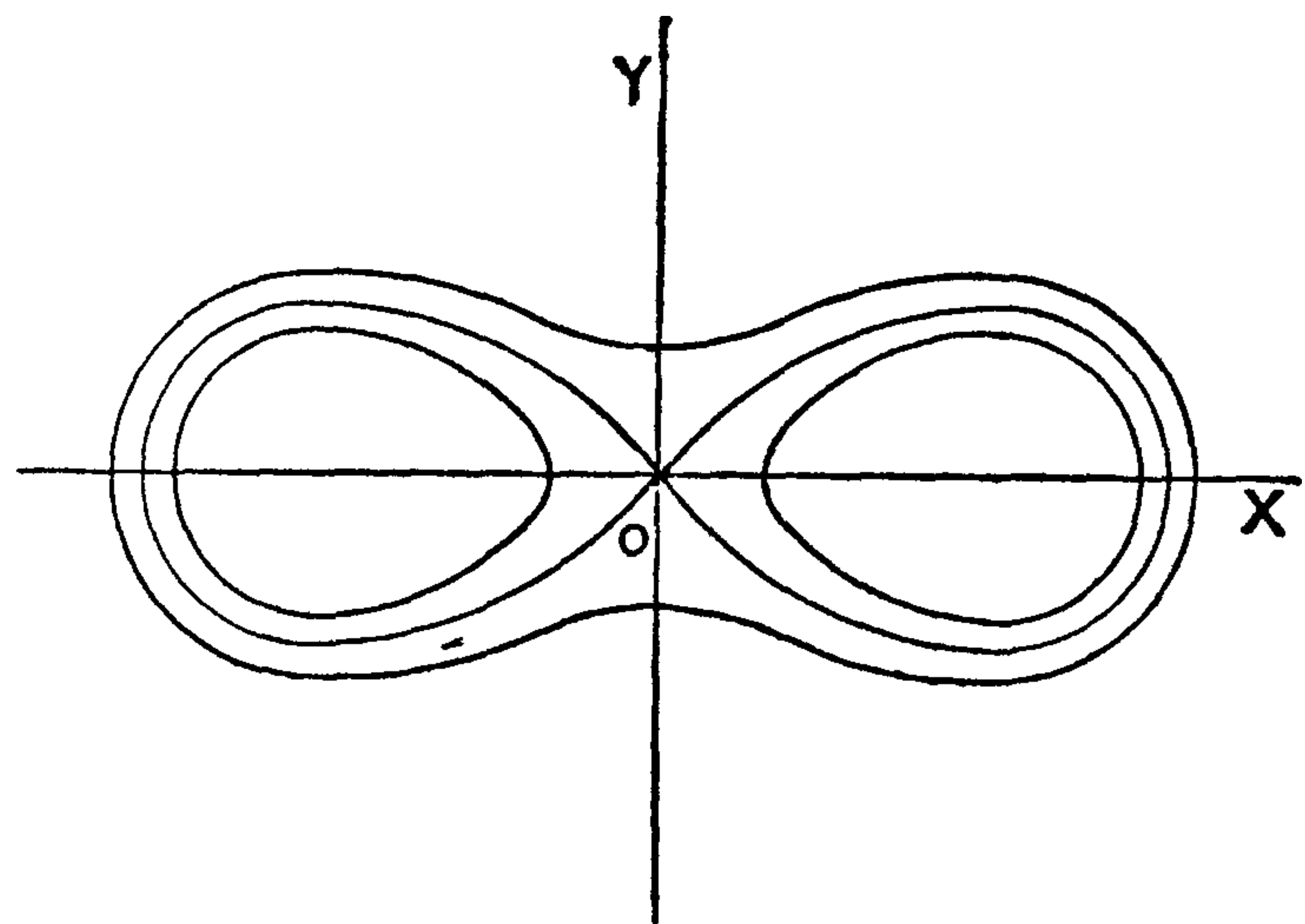


Fig. 49

101. It is evident that a circuit which does not cut either of the fundamental lines, namely the coordinate axes and the line at infinity, will not be detected if we employ only the method of the analytical polygon. The portion of the plane in which such a circuit may possibly lie may generally be greatly limited by the consideration of the intersections of the curve with a straight line moving in such a manner as to sweep over

the entire plane. For example, in Fig. 48 the curve is a quartic and the origin is a node; hence a line passing through the origin meets the curve twice at the origin and in two other points. If the line revolves in the positive direction from coincidence with the axis of x through 180° , it sweeps over the entire plane, and we notice that the two points are real until the line arrives at a position in which it is tangent to the branch in the fourth quadrant. Hence, although there can obviously be no branch in the fourth quadrant other than that drawn, we might still suspect the existence of an oval in the second quadrant.

But, if we put $y = mx$ in the equation of the curve, and solve for x , as in Art. 13, we find

$$x = m(1 + m) \pm m\sqrt{\{m(m + 2)\}};$$

whence we see that the two values of x are real until $m = -2$, and imaginary for all values of m between -2 and 0 . Hence there is no other circuit. The limiting value, $m = -2$, gives $x = 2$; therefore $(2, -4)$ is the point of contact of a tangent passing through the origin.

Auxiliary Loci

102. When the equation of a curve is in such a form that each of its members is readily resolved into simple factors, the loci of the results of equating these factors separately to zero can frequently be used in constructing the curve in the manner illustrated below.

For example, let the equation of the curve be

$$y(y^2 - ax) = (y^2 + a^2)(x - a). \quad (1)$$

The locus of $y = 0$ is the axis of x , and that of $y^2 - ax = 0$ is the parabola constructed in Fig. 50. It is to be noticed that the value of the expression in the first member for the point

(x, y) vanishes and changes sign whenever the point crosses either of these lines, being positive in two of the four regions into which the lines divide the plane, and negative in the other two.

The locus of $x - a = 0$, corresponding to a factor in the second member, is for distinction constructed by a dotted line in the figure. The factor $y^2 + a^2$ is always positive, and there is no corresponding locus; thus the expression in the second number is positive for all points on the right of the dotted line, and negative for all points on its left.

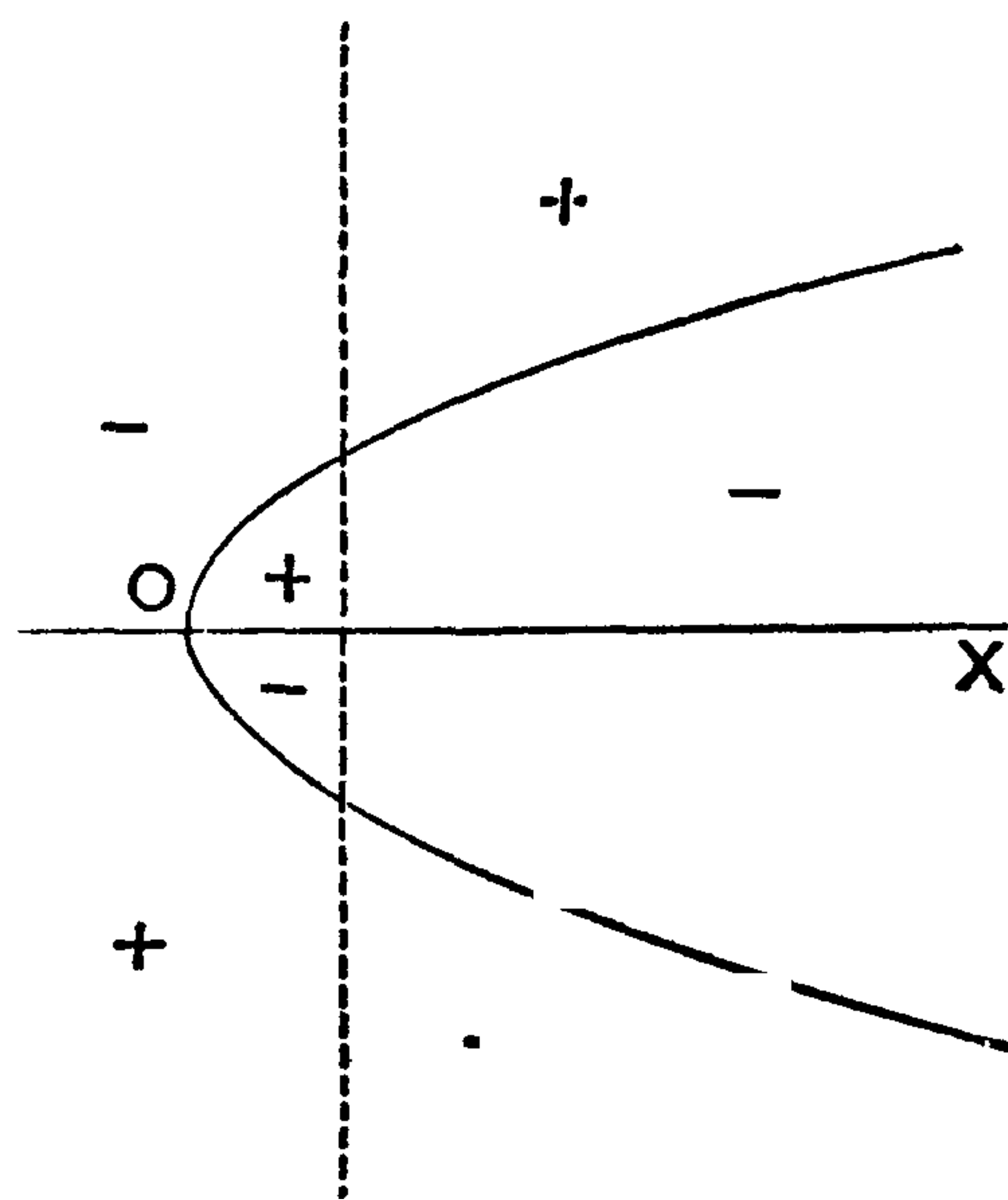


Fig. 50

103. Since each member of equation (1) reduces to zero at a point where the dotted line intersects a full line, it is evident that the curve passes through every such point; it is also evident that the curve cannot meet either of the lines at any other point. Moreover, if we mark each of the eight regions into which the lines divide the plane by the sign $+$ or $-$, according as the expressions in the first and second members of the equation have like or unlike signs in that region, it is evident that no part of the curve can lie in a region marked $-$. Thus at each of the points of intersection a branch of the curve passes from one of the vertically opposite regions marked $+$ to the other.

104. The points of intersection are $(a, 0)$, (a, a) and $(a, -a)$. The inclination of the curve at either of these points is readily determined in a manner equivalent to that which we employ when a curve passes through the origin. Thus, for the point $(a, 0)$, making $x = a$ and $y = 0$ in the coefficients of the factors which vanish at this point in equation (1), we have

$$-a^2y = a^2(x - a); \quad (2)$$

hence

$$y = a - x$$

is the equation of the first approximation or tangent at this point.

105. Again, at the point (a, a) we have, in like manner,

$$y^2 - ax = 2a(x - a), \quad (3)$$

which is in fact the equation of a parabola approximating to the given curve. If we desire only to determine the inclination at (a, a) , we write equation (3) in such a form that $y - a$ or $x - a$ is a factor of each term, thus

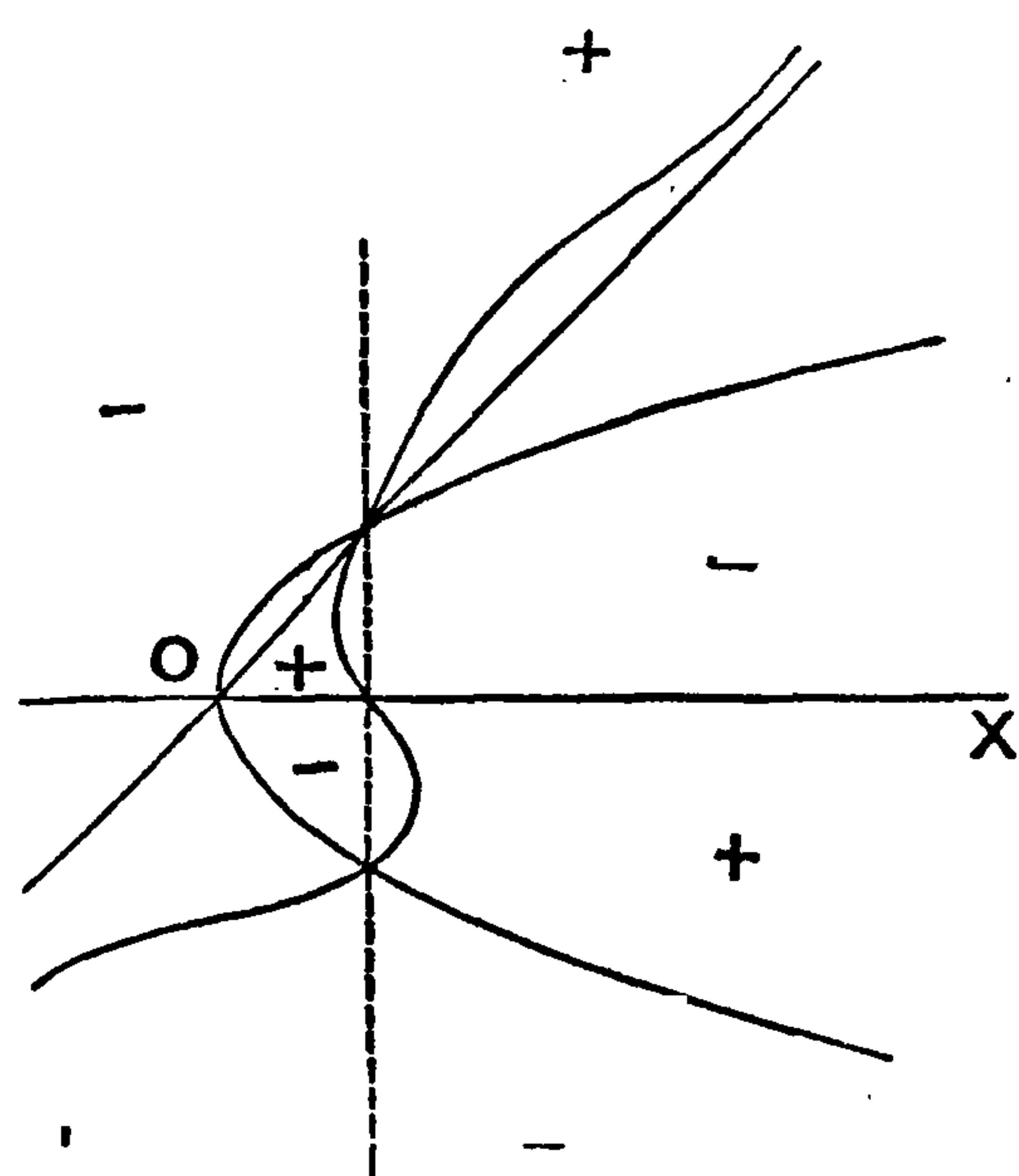


Fig. 51

$$y^2 - a^2 + a(a - x) = 2a(x - a);$$

and, again putting $x = a$ and $y = a$, we have

$$2(y - a) = 3(x - a)$$

for the equation of the tangent. The process is equivalent to transferring the origin to the point (a, a) and retaining in the result only the terms of the first degree.

In like manner we find the tangent at $(a, -a)$ to be

$$2(y + a) = x - a.$$

106. When, as in the present case, only two branches enter a region, it is evident that these branches must be continuous, as represented in Fig. 51. The construction of this branch, which forms a complete circuit, is readily completed after determining the asymptote, which is found to be the line $x = y$. The fundamental point A is an acnode; and, considering straight

lines passing through this point, that is to say, lines parallel to the axis of x , it is evident that the curve contains no other circuit.

Loci representing Squared Factors

107. In the preceding example each of the auxiliary loci represents a single factor, and separates regions oppositely marked. If, now, one of the loci represents a squared factor, the expression containing this factor vanishes but does not change sign, when the point (x, y) crosses this locus; hence the adjacent regions separated by it will be similarly marked. Such a locus may be regarded as formed by two auxiliary loci of the same system coming into coincidence. Hence, at the point where it crosses a single locus of the other system the curve meets the single locus in two points which have come into coincidence; that is, it touches the single locus; in fact, it lies on both sides of the double locus, and on that side of the single locus where the adjacent regions are marked $+$.

108. To illustrate, let us take the curve whose equation is

$$4y^2(x - a)^2(2a - x) = x(x^2 - y^2 - 4ax + 5a^2)^2. \quad (1)$$

The auxiliary loci for the first member are the axis of x , the line $x = a$ and the line $x = 2a$, which are drawn as full lines in Fig. 52. The loci for the second member are the axis of y and the rectangular hyperbola

$$x^2 - y^2 - 4ax + 5a^2 = 0,$$

or

$$(x - 2a)^2 - y^2 + a^2 = 0, \quad (2)$$

which are drawn as dotted lines. The origin is a point at which a single factor meets a double factor, and accordingly the curve

touches the axis of y , lying on its right side where the regions are marked $+$. In fact, if we put $x = 0$ and $y = 0$ in the coefficients of y^2 and x in equation (1), we have

$$8a^3y^2 = 25a^4x,$$

the process being equivalent to that of finding the approximate form at the origin.

Again, the single locus $x = 2a$ cuts the hyperbola, which is a double locus, in the points $(2a, \pm a)$; hence the curve touches the line $x = 2a$ at these points.

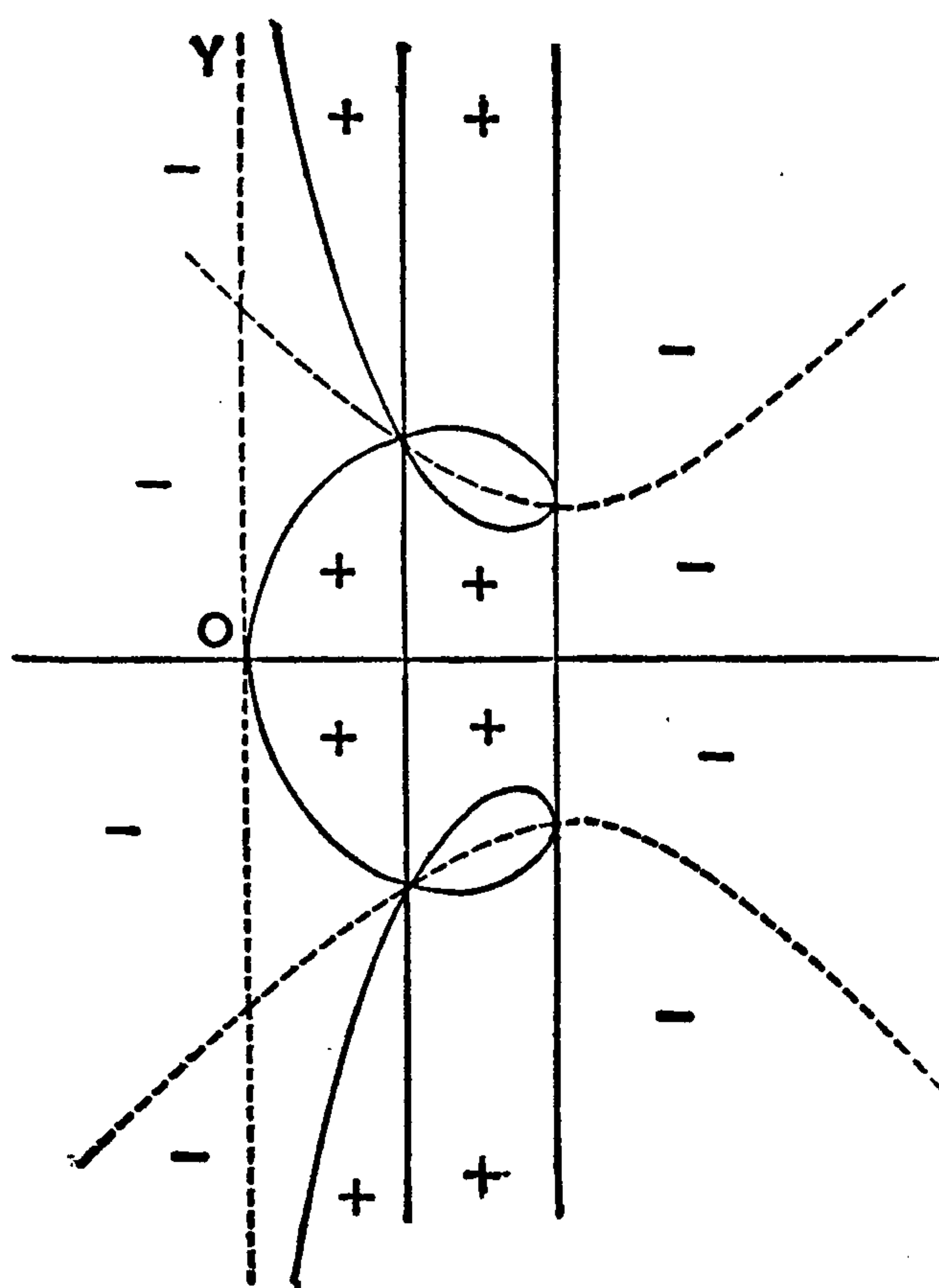


Fig. 52

109. At a point where two double loci, one belonging to each system, meet, the four regions are all similarly marked; and, supposing them marked $+$, we have two branches crossing each of the loci and forming a crunode. Thus, in the present example, the double locus $x = a$ cuts the hyperbola, equation (2), in the points $(a, \pm a\sqrt{2})$; hence these points are nodes. To find the tangents at the upper node, we put $x = a$ and $y = a\sqrt{2}$ in the coefficients of the factors which vanish at this point in

equation (1); the result is

$$8a^2(x - a)^2 = (x^2 - y^2 - 4ax + 5a^2)^2,$$

or

$$2\sqrt{2} \cdot a(x - a) = \pm (x^2 - y^2 - 4ax + 5a^2).$$

Proceeding as in Art. 105, we have

$$\pm 2\sqrt{2} \cdot a(x - a) = (x - a)^2 - 2a(x - a) - (y^2 - 2a^2),$$

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