The Pythagorean Theorem

In right angled triangles, the square on the side subtending the right angle is equal to the squares on the sides containing the right angle.

Prop. I.47, Euclid's *Elements*

The Pythagorean Theorem may well be the most famous theorem in mathematics, and is generally considered to be the first great theorem in mathematics. Pythagoras lived from about 572 B.C. until about 500 B.C., but "his" theorem appears to have been known to the Babylonians at least a thousand years earlier, and to the Hindus and Chinese of Pythagoras' time. However, no proofs are given in these early references, and it is generally accepted that Pythagoras or some member of his school was the first to give a proof of the theorem. The nature of Pythagoras' proof is not known, and there has been much conjecture as to the method he used. Most authorities feel that a dissection proof such as the following was most likely.

Denote the legs and hypotenuse of the given right triangle by a, b,and c, and form two squares, each having side a+b, as in Figure 1. Dissect these squares as shown, noticing that each dissection includes four triangles congruent to the original triangle. The theorem follows by subtracting these four triangles from each square. An important part of this proof is the assertion that the central figurein the second dissection is indeed a square: Can you prove this? What geometry is required?

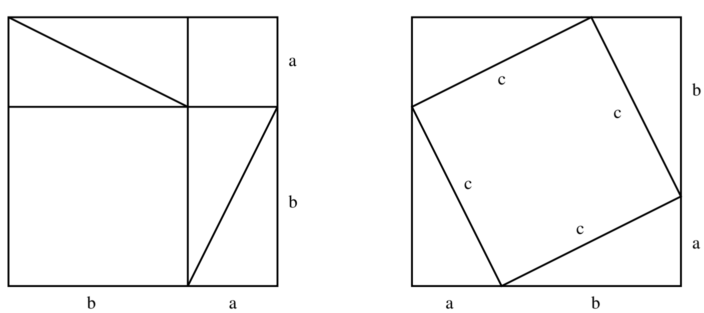

Figure 1: A dissection proof

In addition to its claims as the first and most famous of the great theorems of Mathematics, the Pythagorean Theorem is also probably the theorem with the most proofs. E. S . Loomis has collected 370 proofs of this theorem in his book, *The Pythagorean Proposition*. Two more proofs will be given here, the first by James A. Garfield, done when he was a member of the House of Representatives in 1876 (five years before he became the 20th President of the United States), and the second by Euclid in his *Elements*, written about 300 B.C.

James A. Garfield's Proof: Denote the legs and the hypotenuse of the right triangle by a, b, and c, and form the trapezoid shown in Fig. 2. Compute the area of the trapezoid in two ways, directly using the usual formula, and as the sum of the areas of the three right triangles into which the trapezoid can be dissected. Equating these and simplifying gives:

$$(a+b)(a+b)/2 = ab/2 + ab/2 + c^2/2$$

$$a^2 + 2ab + b^2 = 2ab + c^2$$

$$a^2 + b^2 = c^2$$

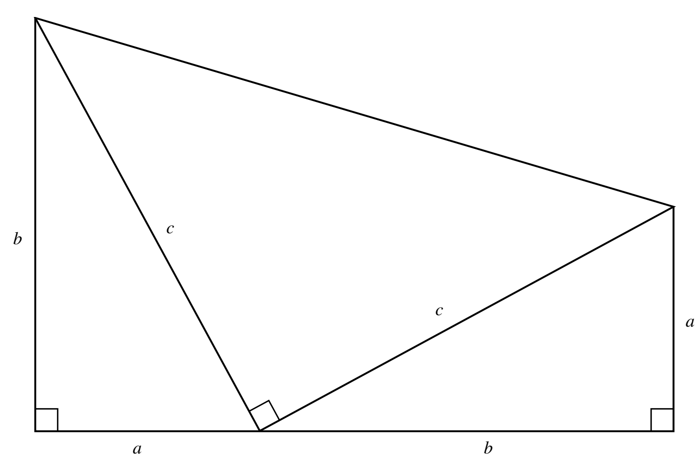

Figure 2: James A. Garfield's proof

Euclid's Proof: The Pythagorean Theorem is proposition 47 in Book I of Euclid's *Elements* and the proof refers to some of the earlier propositions. These should be looked up by the interested reader. Details should also be filled in.

Suppose $\triangle$ABC is a right triangle with $\angle$BAC = 90°. Construct the squares BDEC on BC, AGFB on AB, and AHKC on AC (by I.46). Through point A draw AL parallel to BD, and also draw lines FC and AD (by I.31 and Post. 1). See Figure 3.

Now, $\angle$BAC = 90° and $\angle$BAG = 90°, so GAC is a straight line (by I.14).

$\angle$DBC and $\angle$FBA are right angles, and thus are equal. Adding $\angle$ABC to both yields $\angle$DBA = $\angle$FBC. Furthermore, AB = FB and BD = BC, and so triangles ABD and FBC are congruent (by I.4).

Now, triangle ABD and rectangle BDLM share the same base and lie within the same parallels, and so the area of the rectangle is twice the area of the triangle. The same reasoning applies to triangle FBC and rectangle (square) ABFG since it was shown above that GAC is a straight line. (I.41 is used here.)

However, the congruence of triangles proved above leads us to the fact that the areas of BDLM and ABFG are equal.

The above reasoning should now be repeated to arrive at the fact that the areas of MLEC and ACKH are equal. (The student should draw appropriate auxiliary lines and fill in the details).

For notational convenience, the area of a figure will be indicated by the vertex-notation of the figure, i.e., area(ABC) = ABC. Thus, BDLM = ABFG and MLEC = ACKH. Adding these yields

$$\mathrm{BDLM} + \mathrm{MLEC} = \mathrm{ABFG} + \mathrm{ACKH}$$

which becomes

$$\mathrm{BCED} = \mathrm{ABFG} + \mathrm{ACKH}$$

and the theorem is proved.

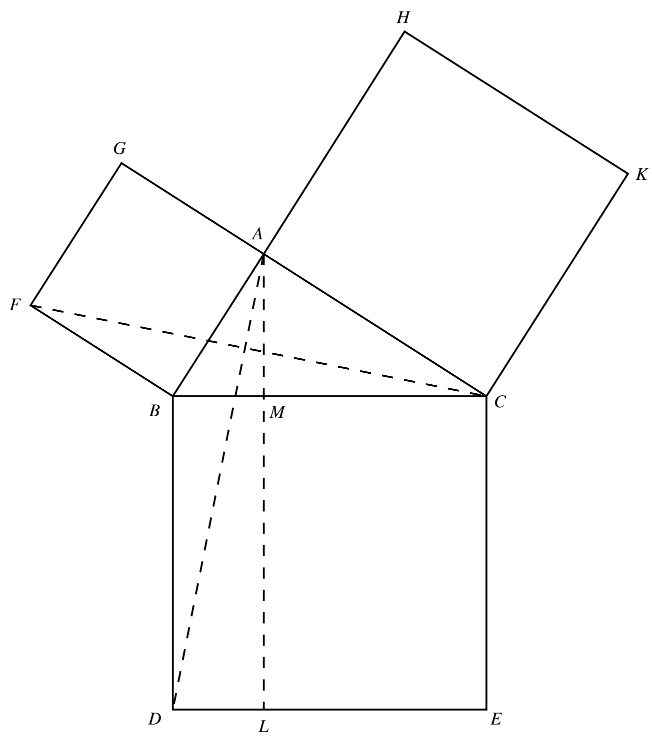

Figure 3: Euclid's proof

In some modern textbooks, many of the exercises following the proof of the Pythagorean Theorem require not the theorem itself, but the still unproved converse. To Euclid's credit, in the Elements the proposition immediately following the Pythagorean Theorem is its converse. Prove the following.

> If in a triangle, the square on one of the sides be equal to the squares on the remaining two sides of the triangle, the angle contained by the remaining two sides of the triangle is right.

Hint: If in triangle ABC, $\angle$BAC is to be proved to be a right angle, construct a line $\perp$ to AC at A, extending to D, such that AD = AB. Then prove triangles ABC and ADC congruent.