OPEN-ENDED QUESTIONS FOR MATHEMATICS

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AND
PROVIDED AS A SERVICE OF THE ARSI RESOURCE COLLABORATIVE UNIVERSITY OF KENTUCKY

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This packet contains open-ended questions for grades 4, 5, and 8 as well as open-response questions for Algebra I / Probability / Statistics and Geometry. The questions were developed with two separate intentions.

Before stating these intentions, let's examine the differences – as used in this packet – between "open-ended" and "open-response." In this set of materials, open-ended refers to a question or problem which has more than one correct answer and more than one strategy to obtain this answer. Open-response refers to a question or problem that may only have one correct answer or one strategy to obtain the answer. In both open-ended and open-response mathematics problems, students are expected to explain or justify their answers and/or strategies.

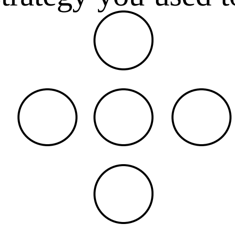
Now for the intentions for the use of these questions. The questions identified for grades 4, 5, and 8 should be used as classroom practice questions. Students can either work with them as members of cooperative groups or the teacher can use the questions for demonstration purposes to illustrate proper use of problem solving strategies to solve problems – as practice either for CATS or for other problem solving situations that students may encounter. The problems are not intended to be ones that can be solved quickly or without thought. However, the challenge provided by these questions should elicit classroom discussion about strategies that may or may not be obvious to the average student. Each of the questions is correlated to the Core Content for Assessment for Grade 5 (the grade 4 and grade 5 questions) or for Grade 8 (the grade 8 questions). If a teacher receiving a copy of these questions does not have the Core Content for Assessment coding page, she/he may contact either the ARSI Teacher Partner in his/her district, the ARSI office (888-257-4836), or the ARSI website of the University of Kentucky resource collaborative at http://www.uky.edu/OtherOrgs/ARSI/curriculum.html then click on *Assessments*.

The high school questions were developed as part of professional development provided to mathematics teachers on how to adapt textbook or other problem sources into open-ended questions. As presently configured, many of these questions can be used in classrooms for assessment purposes. However, the teacher should consider modifying the problems to provide additional practice to their students on how to answer open-ended questions. Assistance in helping teachers in this modification can be found on the Kentucky Department of Education website at http://www.kde.state.ky.us/ or through professional development provided by ARSI or the Regional Service Center support staff in mathematics.

If you have any questions about the use of these materials, please contact the ARSI Resource Collaborative at the University of Kentucky (888-257-4836).

GRADE 4 OPEN-ENDED QUESTIONS

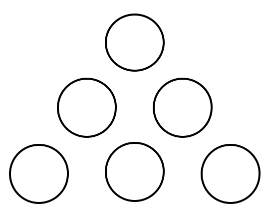
1. Place the digits 1, 2, 3, 4, and 5 in these circles so that the sums across and vertically are the same. Describe the strategy you used to find your solution(s).



2. Levinson's Hardware has a number of bicycles and tricycles for sale. Johnnie counted a total of 60 wheels. How many bikes and how many trikes were for sale? Show how you got your answer in more than one way.

3. Melanie has a total of 48 cents. What coins does Melanie have? Is more than one correct answer possible?

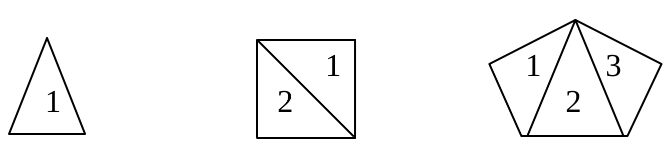
4. Using each of 1, 2, 3, 4, 5, and 6 once and only once, fill in the circles so that the sums of the numbers on each side of the three sides of the triangle are equal. How was the strategy you used for problem #1 above similar to the strategy you used to solve this problem?



5. A rectangle has an area of 120 cm^2. Its length and width are whole numbers.
   a. What are the possibilities for the two numbers?
   b. Which possibility gives the smallest perimeter?

6. The product of two whole numbers is 96 and their sum is less than 30. What are the possibilities for the two numbers?

7. a. Draw the next three figures in this pattern:



3 sides | 4 sides | 5 sides

   b. How many triangles are in a figure with 10 sides?

8. Study the sample diagram. Note that

$2 + 8 = 10 \qquad 5 + 3 = 8 \qquad 2 + 5 = 7 \qquad 8 + 3 = 11.$

$\boxed{2}$	10	$\boxed{8}$
7		11
$\boxed{5}$	8	$\boxed{3}$

Complete each of these diagrams so that the same pattern holds.

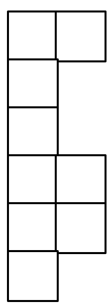
(a)

$\boxed{}$	15	$\boxed{}$
11		20
$\boxed{}$	16	$\boxed{}$

(b)

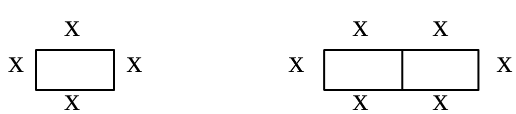
$\boxed{}$	12	$\boxed{}$
10		21
$\boxed{}$	19	$\boxed{}$

9. Nine square tiles are laid out on a table so that they make a solid pattern. Each tile must touch at least one other tile along an entire edge. One example is shown below.



a. What are the possible perimeters of all the figures that can be formed?
b. Which figure has the least perimeter?

10. In the school cafeteria, 4 people can sit together at 1 table. If 2 tables are placed together, 6 people can sit together.



a. How many tables must be placed together in a row to seat: 10 people? 20 people?
b. If the tables are placed together in a row, how many people can be seated using: 10 tables? 15 tables?

11. a. Fill in the blanks to continue this dot sequence in the most likely way.

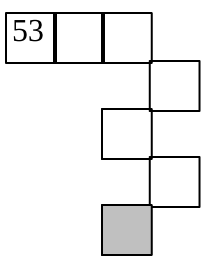
b. What is the number sequence for this pattern?

12. Here is the start of a 100 chart.

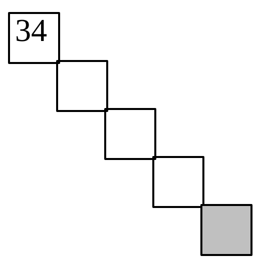
1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30

Shown below are parts of the chart. Without extending the chart, determine which numbers should go in the shaded squares.

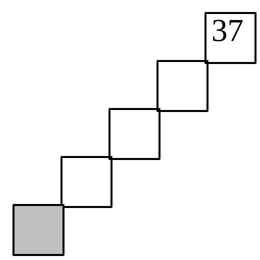
a.



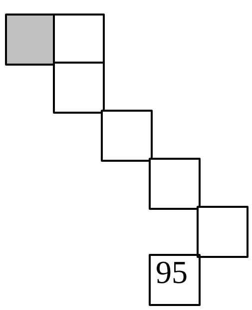
b.



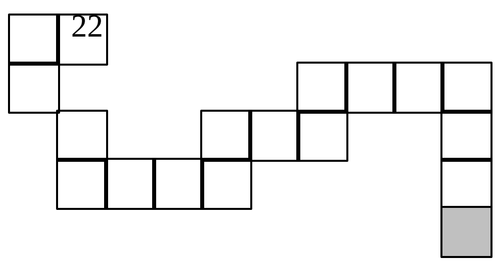
c.



d.

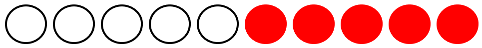


e.

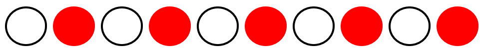


13. The biggest animal in the world is the blue whale. Some blue whales have grown as long as 109 feet and have weighed 150 tons. A baby whale gains about 200 pounds a day.
   a. How many pounds are in a ton?
   b. How long will it take for a baby to weigh its first ton, if it was 400 pounds at birth?

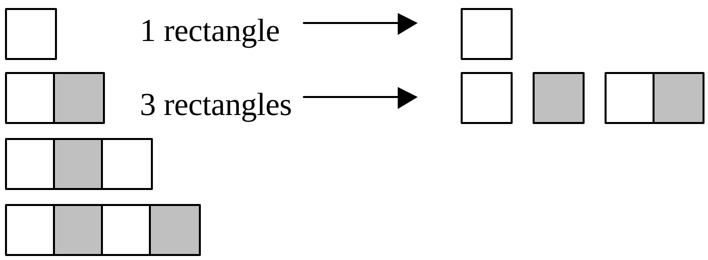
14. Ten white and ten red discs are lined up as shown.



Switching just two adjacent discs at a time, what is the least number of moves you can make to achieve the white, red, white, red, … arrangement shown here? Explain the strategy you used to solve the problem.

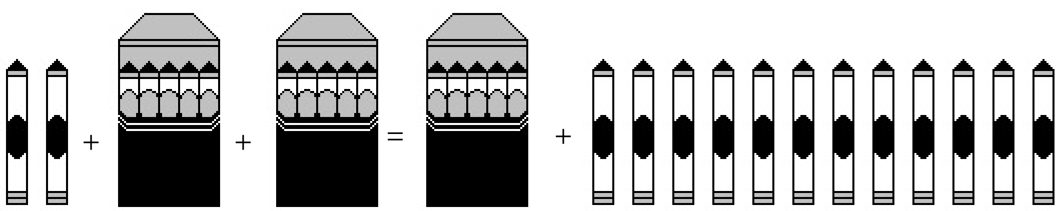


15. How many rectangles are there in each of these figures?



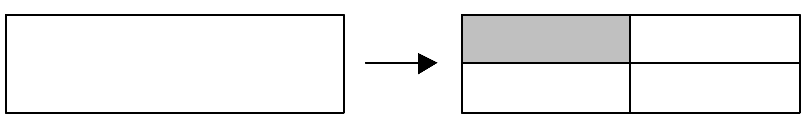
16. A basket starfish has more than 80,000 arms. If each arm needed a glove, how many pairs of gloves would you need?

17. How many crayons are there in each box? Each box has the same number of crayons and:

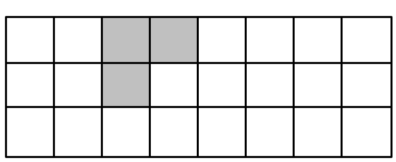


18. In a box of red, yellow, and blue color chips, all but 4 are red, all but 4 are yellow, and all but 4 are blue. How many color chips are in the box altogether?

19. One fourth of the rectangle on the left below is shade by dividing it into fourths.

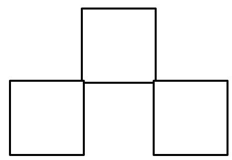


The shaded part below is one-fourth of some rectangle. Draw the rectangle.



20. Three teachers have groups practicing different skits for open house. Mr. Jones has four groups containing 2, 3, 4, and 5 students. Mrs. Smith has groups of 4, 5, 6, and 7. Mrs. Philips has groups of 6, 7, 8, and 9. If each teacher wants to have the same number of students to supervise, then which group should be moved to another classroom? Explain your answer.

21. The castle gardener has a square for each type of rose. There are three roses in her garden. The queen has requested a new rose to be planted. If the gardener moves just three garden planks, one time only, she can get four squares. How can this be accomplished?



22. Jim has six American coins that total $1.15. With his coins, Jim cannot give Ann exact change for a dollar. He cannot give Tim exact change for a fifty-cent piece. He cannot give Sean change for a quarter, or Jill change for a dime, or Cindy change for a nickel. What are Jim's six coins?

23. What is the greatest 3-digit number whose digits total 13? Justify your answer.

24. Sally and Jim each have a bag of hard candy. Sally said, "Jim, if you give me 5 pieces of candy from your bag, I'll have as many pieces as you." Jim laughed and answered, "No, you give me 5 of yours and I'll have twice as many as you." How many pieces did they each have to begin with?

25. A farmer fenced a square plot of ground. When he finished, he noted that there were five fence posts on each of the sides. How many posts are used to fence the plot?

26. Twice the product of 6×5 is three times as great as this number. What is the number?

27. Can you arrange four 5's so that they equal six? (Hint: You must use fractions.)

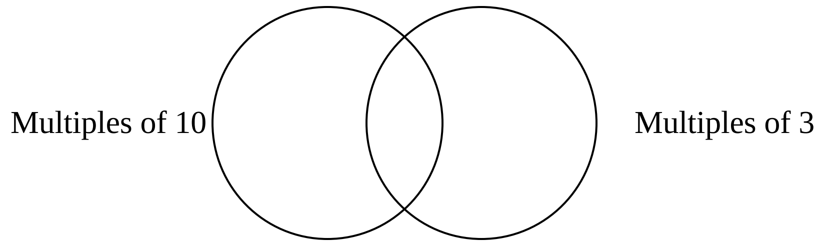
28. Why are 1997 pennies worth almost twenty dollars?

29. a. These numbers belong together in a group: 25, 40, 110, 55
These numbers do not belong in the group: 33, 71, 4, 106
Which of these numbers belong in the group? 75, 205, 87, 43
What is the rule?

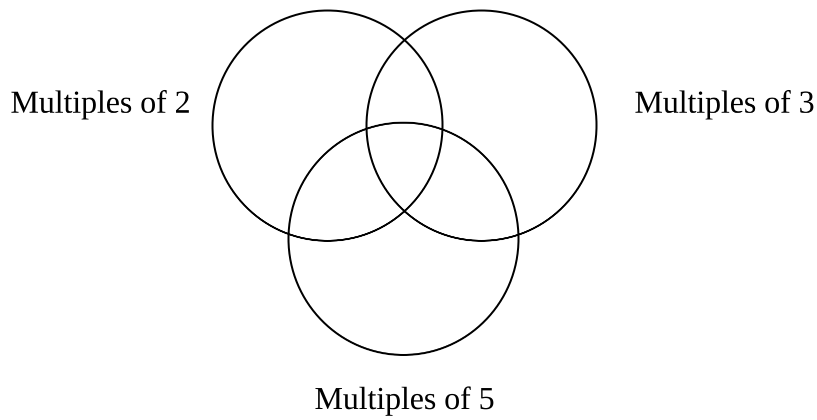
    b. These numbers belong together in a group: 16, 9, 49, 64
These numbers do not belong in the group: 40, 12, 77, 28
Which of these numbers belong in the group? 102, 36, 25, 50
What is the rule?

30. Write the numbers in each circle that are described by the attribute written outside the circle. Numbers in overlapping regions must meet the requirements of **each** circle. Justify why you placed the numbers in the overlapping regions you did.

a.

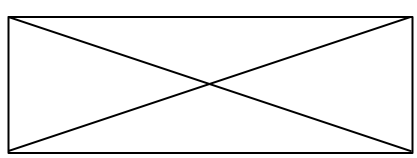


b.



31. Jacob, Jon, and Amanda were building a rectangular snow wall one block thick. They made 24 blocks of snow and couldn't decide what the dimensions should be. Give them a list of the different size walls, using whole blocks.

32. a. How many triangles can you find in the figures below?
    b. How many different edges are used in these triangles?
    c. If the area of each of the 4 smallest triangles is the same and this area is 1 square unit, what is the area of each triangle in the figure?



33. John was looking through his math book and discovered that after page 44, the next page was 51. How many page sheets were torn out of his book? Provide an explanation that will justify your answer.

34. Bob's family of three was driving to Nashville. They were going to stay overnight, sightsee during the next day, and return home in the evening. They had to pay for dinner, breakfast, and lunch. They were to sleep at Grandma's house. Breakfast at McDonald's was $2.32 each. Lunch at Kentucky Fried Chicken was $3.29 each. Dinner at Wendy's was $4.89 each. Was $40 enough money to pay for their food?

35. Jill's mother limited her Nintendo playing to 10 hours per week. She played on only four days, a different amount of time each day. On Saturday, she played twice as much as on Wednesday. She didn't play on Monday, Tuesday, or Thursday. On Friday, she played the least of the days she played. If the times were all different and there were not any partial hours, how many hours did she play on each day?

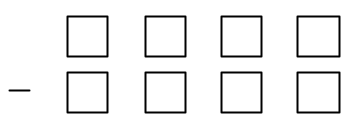
36. To encourage John to work harder in math his mother said she would pay him 10 cents for each right answer and subtract 5 cents for each wrong answer. If he earned 20 cents after doing 32 problems, how many problems did John get right? How many did he get wrong? How many would he have to get right to earn more than a dollar?

37. Joan went fishing. On the first cast she hooked a fish 80 feet from the boat. Each time she reeled in 10 feet of line, the fish would take out 5 feet. How many times did she have to reel in to get the fish to the boat?

38. "If you have a square and you cut off one corner, how many corners do you have left?" asked Mrs. Wheeler. "Easy," answered Tony. "Three." "Wrong, Tony!" cried Donnie. Where did Tony go wrong? Explain (including a sketch).

39. An Egyptian pyramid has a square base and four triangular faces. Use clay or Play-Doh to make a pyramid model. If you make different "plane cuts" through all 4 of the triangles -- without cutting through the square -- which different shapes will the cuts make? A plane cut is kind of like a cheese cutter -- it will be straight, not curved.

40. Use the digits 0, 1, 2, 3, 4, 5, 6, 7 to find the smallest answer possible in this problem. No negative numbers are allowed.



41. Bill received $12 to feed a neighbor's cat for 3 days. At this pay rate, how many days will he have to feed the cat to earn $40? The neighbor's family is going on vacation for 3 weeks next summer. Bill wants to earn enough money to buy a CD player that costs $89. Will he have enough money? Explain.

42. A farmer milks 16 cows every hour. The size of his herd varies as new calves are born and he sales off some of his stock. He never has less than 48 cows nor more than 64. How many hours does it take him to milk his cows? Explain.

43. Jane's mom needs to buy enough gas to fill up her car's tank. She will need 15 gallons. There is a Shell station 3 blocks from her home that sales the gas for \$1.19/gallon. There is a Speedway station one mile away that sales the gas for \$1.07/gallon. How much money will it cost her to buy gas at each station? How much will she save by going to Speedway?

44. Bradley works as a traveling salesman and leases a car for his business. The leasing company has a base rate for the first 15,000 miles per year, but has an additional charge of \$500 + \$0.25/mile for each mile over the 15,000. Bradley has already driven 4300 miles in the first three months. If he drives a similar amount for the rest of the year, how much will he have to pay extra?

45. A clock loses 2 minutes every 8 hours. Alicia's mom plans to set the alarm at 11 p.m. on Sunday night to get Alicia up each morning. Alicia has to get up each day no later than 7 a.m. What time should her mom set the alarm to be sure that Alicia isn't late to school any day, Monday through Friday?

GRADE 4 SOLUTIONS

1. *Extend a pattern* (E-1.2.2) (E-4.2.1)

2		3		2
1 3 5	or	2 1 5	or	1 5 4
4		4		3

2. *Recognize/extend/find rules for number patterns* (E-4.2.1, E-4.2.2, E-4.3.2)

Bikes	Trikes	Bike Wheels	Trike Wheels	Total
30	0	60	0	60
27	2	54	6	60
24	4	48	12	60
etc.				

3. *Recognize/extend/find rules for number patterns; add, subtract, multiply amounts of money* (E-4.2.1, E-4.2.2)

Pennies	Nickels	Dimes	Quarters	Total
48	0	0	0	48
43	1	0	0	48
38	2	0	0	48
38	0	1	0	48
33	1	1	0	48
33	3	0	0	48
etc.				

4. *Recognize/extend/find rules for number patterns* (E-4.21.)

		6						1		
	1		2		or		6		5	
5		3		4		2		4		3

5. *Perimeter and area* (E-2.2.5; M-2.2.5)
   a. If we assume that length must be longer than width, then the possibilities are:

Length	120	60	40	30	24	20	15	12
Width	1	2	3	4	5	6	8	10
Area	120	120	120	120	120	120	120	120
Perimeter	242	124	86	68	58	52	46	44

   b. 12 cm by 10 cm appears to be the rectangle with the smallest perimeter. (Actually, the rectangle with the smallest perimeter is a square each of whose sides is $\sqrt{120}$, or 10.954451…)

6. *Factors* (E-1.2.7; M-1.2.4) 4 and 24, 6 and 16, 8 and 12.

7. *Identify/create patterns; extend/find rules for number patterns* (E-4.2.1, E-4.2.2, E-4.1.1) b. 8, or n-2

8. *Recognize/extend/find rules for number patterns* (E-1.1.5, E-1.2.2, E-4.1.1, E-4.2.1, E-4.3.1)
   There are multiple solutions for each, for example:

(a)	7	15	8		6	15	9		4	15	11
	11		20	or	11		20	or	11		20
	4	16	12		5	16	11		7	16	9
(b)	3	12	9		2	12	10		6	12	6
	10		21	or	10		21	or	10		21
	7	19	12		8	19	11		4	19	15

9. *Perimeter* (E-2.2.5, E-1.1.5) a. 12, 14, 16, 18, 20 b. A square (3 x 3)

10. *Identify/create patterns in real-life situations; extend/find rules for number patterns* (E-4.1.1, E-4.2.1, E-4.3.1)
    a. 10 people = 4 tables, or [(p-2)/2]
       20 people = 9 tables
    b. 10 tables = 22 people, or 2t + 2
       15 tables = 32 people

11. *Recognize/extend/find rules for number patterns* (E-4.2.1, E-4.3.2)
    b. 2, 5, 8, 11, 14, …, 3n-1 (add 3 to find the next term)

12. *Recognize/extend/find rules for number patterns* (E-4.2.1)
    a. 95 b. 78 c. 73 d. 42 e. 70

13. *Use customary units of measure; multistep story problems using combination of operations; division by multiples of 10* (E-2.1.5, E-1.2.2)
    a. 2000 pounds in a ton
    b. 2000 - 400 = 1600; 1600 ÷ 200 = 8 days

14. *Recognize/extend number patterns* (E-4.2.1, E-4.2.2, E-4.3.1)
    10 (Note: The results are the triangular numbers, i.e., two of each color requires 1 switch; three of each color requires 3 switches; 4 of each color requires 6 switches; etc.)

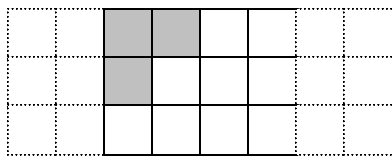
15. *Recognize/extend number patterns* (E-4.2.1, E-4.2.2, E-4.3.1)
    6 and 10
    (Note: The results are the triangular numbers, i.e., 1, 3, 6, 10, 15, …, $\frac{n(n+1)}{2}$)

16. *Division by 1-digit divisor; division with 0's in quotient* (E-1.2.2)
    80,000 ÷ 2 = 40,000 pairs of gloves

17. *Explore the use of variables and open sentences to express relationships; multistep problems using a variety of operations* (E-4.2.3, E-4.1.2)
There are 10 crayons in each box.

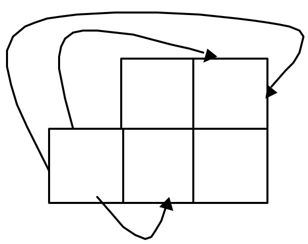
18. *Draw conclusions/make inferences based on data* (E-3.1.3, E-3.3.1)
6 chips: 2 red, 2 yellow, 2 blue.

19. *Use models to solve real world problems involving fractions; relate fractions using models to represent equivalencies* (E-1.1.5; M-1.1.6)
One solution:



20. *Column addition; multistep story problems using combination of operations* (E-1.2.2)
Mr. Jones has 14 students; Mrs. Smith has 22 students; Mrs. Philips has 30 students. Move the group of 8 from Mrs. Philips to Mr. Jones.

21. *Introduce transformations - translation (slide)* (E-2.2.3)

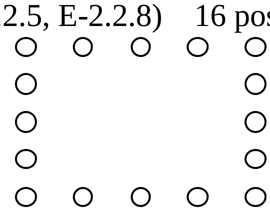


22. *Add, subtract, multiply amounts of money; make change; collect/organize/interpret data* (E-2.2.6, E-3.2.2)
A half-dollar, a quarter, and four dimes

23. *Place value* (E-1.1.4, E-1.2.9, E-1.3.3) 940 (9 + 4 + 0 = 13)

24. *Multistep story problems using combination of operations* (E-4.2.3)
Sally had 25 pieces; Jim had 35.

25. *Perimeter.* (E-1.1.5, E-2.2.5, E-2.2.8) 16 posts.



26. *Multiplication facts; multiples of 10.* (E-1.2.2, E-4.2.3) 20

27. *Numerator/denominator; multistep problems; introduce improper fractions.* (E-1.3.1, E-1.3.3; M-1.3.1)
    $\frac{55}{5} - 5 = 6$

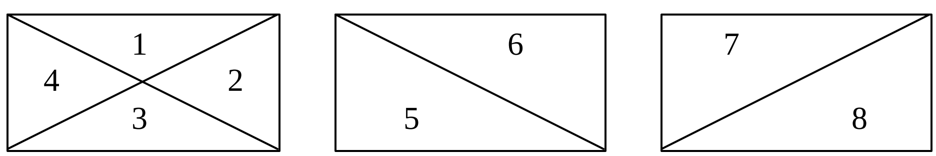
28. *Decimal place value with concrete models.* (E-1.1.4)
    One thousand, nine hundred and ninety-seven pennies are worth \$19.97, which is almost \$20.

29. *Multiples/ square numbers; represent and describe mathematical relationships through looking for a pattern.* (E-1.1.3, E-4.2.1)
    a. 75; 205 -- multiples of 5
    b. 36; 25 -- square numbers

30. *Multiples* (E-1.1.3)
    a. Multiples of 10: 10, 20, 30, …; multiples of 3: 3, 6, 9, 12, …; multiples of both 3 and 10: 30, 60, 90, …
    b. Multiples of 2: 2, 4, 6, 8, …; Multiples of 3: 3, 6, 9, …; multiples of 5: 5, 10, 15, …; multiples of both 2 and 3: 6, 12, 18, …; multiples of both 3 and 5: 15, 30, 45, …; multiples of both 2 and 5: 10, 20, 30, …; multiples of all 3 (2, 3, and 5): 30, 60, 90, …

31. *Factor pairs.* (E-1.2.7)
    Each wall is 1 block thick, then the remaining dimensions are:
    1 block x 24 blocks; 2 blocks x 12 blocks; 3 blocks x 8 blocks; 4 blocks x 6 blocks

32. *Triangles; edges; area of triangle* (E-2.1.1)
    a.



    b. 12 edges in the first figure above; 1 new edge in the second figure (both triangles share a commonly labeled edge); 1 new edge in the third figure -- a total of 14 edges.
    c. Area 1 = 1; area 2 = 1; area 3 = 1; area 4 = 1; area 5 = 2; area 6 = 2; area 7 = 2; area 8 = 2.

33. *Compare and order whole numbers; identify/create patterns in real-life situations.* (E-1.2.9)
    3 pages: (45 & 46; 47 & 48; 49 & 50).

34. *Add, subtract, and multiply amounts of money.* (E-1.2.3) Yes.

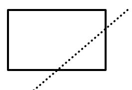
Breakfast was	3 x \$2.32 = \$6.96
Lunch was	3 x \$3.29 = \$9.87
Dinner was	3 x \$4.89 = \$14.67
	Total = \$31.50

35. *Multistep story problems involving combination of operations.* (E-1.2.2)
Jill played 1 hour on Friday, 2 hours on Wednesday, 3 hours on Sunday, and 4 hours on Saturday.

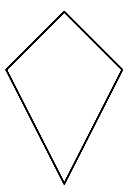
36. *Choosing the appropriate problem solving strategy; add/subtract/multiply amounts of money*. (E-1.2.3)
If he got 12 right and missed 20, then he earned 20 cents. He would need to have at least 18 right to earn more than a dollar.

37. *Identify/describe/create patterns in real-life situations.* (E-3.2.2) 15

38. *Vertices of angles.* (E-2.1.1) The sheet of paper originally had 4 vertices (corners). When a cut was made, it actually created an additional vertex. The sheet now has 5 vertices.



39. *Introduce trapezoid; parallel; (kite).* (M-2.1.2; E-2.1.1)
A square, trapezoid, and kite (quadrilateral in which there are exactly two pairs of congruent sides -- see figure below).



1) if the cut goes through all 4 triangles parallel to the base, a square is produced;
2) if the cut is made through one pair of opposite faces that are parallel to each other and to the base, then a trapezoid is produced;
3) if the cuts are made from one edge towards the opposite edge, which is a different distance from the base in such a way that the other two vertices are both equidistant from the base, then a kite is produced.

Note: Because of its difficulty, this problem would be better used as a performance event with students working together than as an open response question.

40. *Place value; add/subtract 3- and 4-digit numbers.* (E-1.1.4, E-1.2.2, E-1.3.3)

$$\begin{array}{r} 4012 \\ -\underline{3765} \\ 247 \end{array}$$

NOTE: The remaining problems are designed to be done in cooperative groups -- not for individual students.

41. *Multistep story problems; add/subtract/multiply amounts of money.* (E-1.2.3)
$40 - 10 days. For 3 weeks (21 days), Bill will earn $84 -- not enough for an $89 CD player.

42. *Multistep story problems; choosing the appropriate problem solving strategy*. (E-4.2.1, E-1.2.2)
    3 – 4 hours.

43. *Multistep story problems; add/subtract/multiply amounts of money*. (E-1.2.3)
    Shell - $17.85; Speedway - $16.05; savings - $1.80

44. *Multistep story problems; add/subtract/multiply amounts of money*. (E-1.2.3)
    $500 + $550 = $1050

45. *Multistep story problems; choosing the appropriate problem solving strategy; add/subtract amounts of time.* (E-2.2.6)
    13 eight-hour intervals between 11 p.m., Sunday and 7 a.m., Friday.
    13 x 2 = 26 minutes. 7:00 - 0:26 = 6:34 a.m. The alarm needs to be set for 6:34 a.m. to be sure that Alicia gets up by 7 a.m. on Friday.

GRADE 5 OPEN-ENDED QUESTIONS WITH SOLUTIONS

(E-1.1.4, E-1.3.3) Place Value

1. My thousands digit is three times my hundreds digit. The sum of my tens digit and one digit is one less than my thousands digit. My hundreds digit is two less than my tens digit. What 4-digit number am I?

   Ans. Could be either 9353 or 6241

(E-1.1.4, E-1.3.3) Compare/ Order Whole Numbers

2. a) The digit in the tens thousands place is less than the digit in the hundred thousands place. The thousands place digit is larger than the digit in the millions place. The digit in the ones place is at least twice as large as the digit in the tens place. Which park could I be?

Park	Number of Acres
Danali	4,716,726
Gates of the Arctic	7,523,888
Katmai	3,716,000
Kobuk Valley	1,750,421
Lake Clark	2,636,839
Glacier Bay	3,225,284

   b) Give another clue that would produce only one park as the solution.

   Ans. a) Danali or Lake Clark
   b) Varies

(E-1.1.5) Problem Solving Make a Drawing

(E-3.2.6)

3. a) On a hiking trip, Bob and his two friends hiked 15 miles the first day, 10 miles the second, and 7 miles the third. If they walked a total of 60 miles over the 5-day trip and did not walk more than 18 miles on any one day, how many miles could they have walked each of the last two days?

   b) How many miles did they average per day?

   c) If they could hike at a top speed of 6 miles per hour on flat terrain, but had to slow down when climbing hills, could they have made the trip in a single 10-hour day of sunshine? Support your answer.

   Ans. a) 18 one day and 10 the other, or
   17 one day and 11 the other, or
   …….
   14 one day and 14 the other

c) No. Varies, but should include fact that 6 mph for 10 hours would give 60 miles, but would assume that the entire trip was over flat terrain - unlikely.

(E-1.2.6) Estimating Sums

4. Methods of estimating sums include front-end, clustering, finding a range, and rounding. Estimate each of the following, indicate which method you used for each sum, and explain why you used that method.

a) 456 + 453 c) 43 + 47 + 41 + 42 + 44
b) 906 + 214 d) 127 + 149 + 182

(E-1.3.3, E-1.2.2) Add Whole Numbers

5. a) Use each of the digits 2 through 8 once to write a 4-digit number plus a 3-digit number that gives the greatest possible sum and support your case that it will be the largest.

b) Use the same digits to find the sum of a 4-digit number and a 3-digit number that gives the smallest sum.

(E-4.2.1) Deriving Rules

c) Generalize rules for producing the greatest and least sums for a 4-digit plus a 3-digit number. How would the rules differ, if any, when adding a 4-digit number to another 4-digit number?

Ans. a) $\begin{array}{r} 8753 \\ \underline{+\ 642} \\ 9395 \end{array}$ b) $\begin{array}{r} 2357 \\ \underline{+\ 468} \\ 2825 \end{array}$ c) Varies

(E-1.2.6) Estimating Differences

6. Jane used front-end digits to estimate a difference. The minuend was greater than 800 but using front-end its estimate was 800. The estimated difference was 300. (a) What is the least possible number the actual subtrahend could have been? (b) What is the greatest possible number for the subtrahend?

Ans. a) $\begin{array}{l} 801 \\ \underline{-\ \ \ }(402) \\ 399 \end{array}$ b) $\begin{array}{l} 899 \\ \underline{-\ \ \ }(598) \\ 301 \end{array}$

(E-1.2.6) Estimating Differences

7. Suppose you estimate a difference between two 3-digit numbers using front-end digits. Give an example in which:
a) the estimate is less than the actual difference
b) the estimate is greater than the actual difference
Ans. Varies, for example
a) 425 - 215 (actual = 210, estimate = 200)
b) 425 - 275 (actual = 150, estimate = 200)

(E-1.2.2, E-1.3.3) Subtracting Whole Numbers

8. a) Using each of the digits 0 through 7 once to fill in the spaces below so that the difference is 8 in the hundreds place, 8 in the tens place, and 9 in the ones place.

$$\begin{array}{r} \square\square\square\square \\ \underline{\ \ \square\square\square} \\ \square\,8\,8\,9 \end{array}$$

b) Is there more than one solution to this problem? Explain briefly.

(E-1.3.3)

c) Is there more than one solution in which the digit in the thousands place of the difference is 0?

Ans. a) 7531 or 1246

$$\begin{array}{r} 7531 \\ \underline{-\ 642} \end{array} \quad \text{or} \quad \begin{array}{r} 1246 \\ \underline{-\ 357} \end{array}$$

b) Yes, explanation varies.
c) No.

(E-4.2.3) Problem Solving Write an Equation

9. Marlene has a board game in which it is possible to land on spaces worth 1, 3, 5, or 7 points each.
   a) Write an equation to show how it is possible to score a total of 10 points in exactly 4 moves.
   b) Are all scores between 5 and 35 possible with exactly 5 moves? If not, which scores are possible?

Ans. a) $3(3) + 1 = n$
b) No, only the odd numbers are possible.

(E-4.3.1) Problem Solving Write an Equation

10. The math symbols in these equations are missing. Make each equation true by adding two or more symbols (+, -, x, /, or =) to each. For example,
    Problem: 4 2 2 Solutions: 4 = 2 + 2 or 4 - 2 = 2

a) 8 8 4 4
b) 18 9 2 36
c) 2 14 20 4 1

Ans. a) 8 / 8 = 4 / 4
b) 18 + 9 x 2 = 36
c) 2 + 14 = 20 - 4 x 1

(E-4.2.3) Missing Addends

11. There were 42 band members. When the chorus joined them for a performance, there were 77 performers. Which equation below would be best to use to solve this problem? Explain how you made this decision.

a) $77 = 42 - n$ c) $77 + 42 = n$
b) $77 = 42 + n$ d) $77 + n = 42$

Ans. (a). Explanation varies.

(E-1.1.3) Multiples

(E-1.2.8)

12. a) Is the least common multiple of an even and an odd number, even, or odd? Give an example to support your answer.
b) If one number is a divisor of a second number, what is the least common multiple? Give an example.
c) If two numbers do not have any common divisor, what is the least common multiple? Support your answer with an example.

(E-1.1.3) Multiples

13. The Genie and the Jewels.
"Welcome to the World of the Orient," spoke the genie to Aladdin. "You see before you diamonds, rubies, and emeralds. Two diamonds are worth as much as three rubies. Five rubies are worth as much as nine emeralds. Make a pile of diamonds, and then make a pile of emeralds. If you can do this so that the two piles have exactly the same worth, you may keep them all!" How many diamonds should be in the diamond pile, and how many emeralds should be in the emerald pile? Explain your answer.

Ans. Any solution which is a multiple of 10 diamonds and 27 emeralds.

(E-1.1.5) Problem Solving Draw a Diagram

14. A parking lot permits either cars or motorcycles. All together the vehicles parked in a particular day have 60 wheels. Use your reasoning and problem solving skills to find how many cars and how many motorcycles there could be in the parking lot this day.

Ans.

Cars	Wheels	Motorcycles	Wheels	Total
0	0	30	60	60
1	4	28	56	60
2	8	26	52	60
…………				
15	60	0	0	60

(E-1.1.5)
Problem
Solving
Make a
Drawing

15. The Carnival Committee is having their annual meeting to plan for this year's carnival. During this meeting, 8 people will meet to decide upon the location, while another 8 people will meet to discuss which booths will be present this year. You have been selected to set up for this meeting. You have been given 7 square tables and 16 chairs. The criteria for seating arrangements include: the two groups of 8 people each want to be separated and each person will require the use of one side of the table. How will you arrange the tables and chairs for this meeting?

Ans. Possible answers include:

```
  X X           X X X
X☐☐X          X☐☐☐X
X☐☐X           X X X
  X X
```

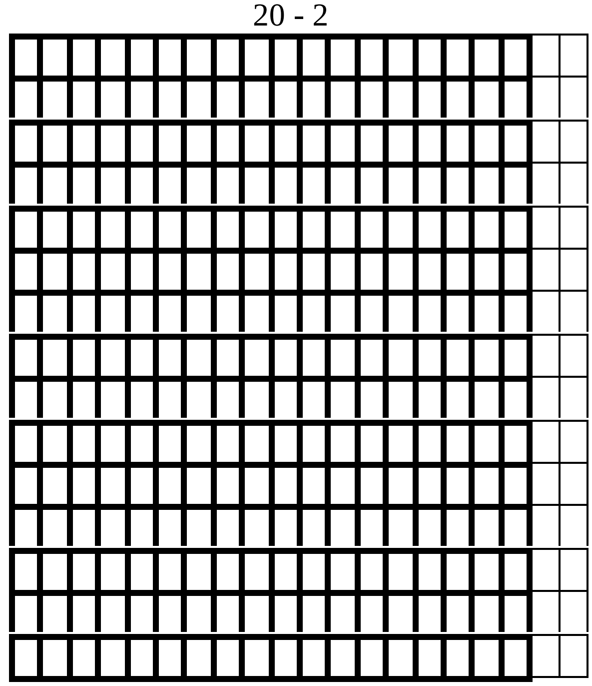
(E-1.1.5)
Mental Math

16. You have to use mental arithmetic to multiply 15 x 18. Use (a) the distributive property; (b) a sketch; and, (c) grid paper to explain how the problem was worked.

Ans. a) 15x18 = 15x (20-2) = (15x20)-(15x2) = 300 - 30 = 270
b) & c)

20 - 2

15



(E-4.1.2, E-4.2.3) Choose the Operation

17. Cinemark Theater has three categories for ticket prices. Adult tickets sell for $6.50 after 3pm or $3.25 for a matinee (before 3pm) and Under 12 tickets sell for $3 at any time. The theater has a maximum capacity of 408 seats. What is the greatest amount that could be collected at:
    a) an 8pm showing?
    b) a 1pm showing?
    c) Explain how much you think would be collected at a matinee showing of a children's movie.

    Ans. a) $2,652
    b) $1,326
    c) Varies

(E-1.2.6) Estimating Products

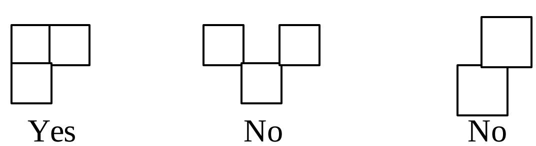
18. When you multiply large numbers, is there any way to estimate how many digits an answer will have? Look at the following examples and see if you can find a pattern that will help you to answer this question. Describe the pattern and then predict how many digits will be in (f).
    a) 10 x 100 = 1,000
    b) 100 x 100 = 10,000
    c) 100 x 1,000 = 100,000
    d) 104 x 4,211 = 437,944
    e) 989 x 8,657 = 8,561,773
    f) 405 x 4,508 = ?

    Ans. Yes, you can predict the number of digits by finding one less than the total number of places in both numbers being multiplied as well as a front-end estimate of the product of both numbers. Thus (f) would have 7 digits since (3 digits + 4 digits) - 1 = 6 and 4 x 4 = 16 or one additional decimal place.

(E-2.2.5) Perimeter and Area

19. You can use color tiles to build rectangles. If you have 16 tiles, what is the perimeter of the rectangle that can be built? What other perimeters are possible if you can build shapes other than rectangles, but the shapes must all be made with tiles that share sides (see example)?

    Example:



Ans. a) Perimeter can be either 16 units (for a square) or 20 units for a 2x8 rectangle.
b) Other possible perimeters are even numbers from 18 to 34.

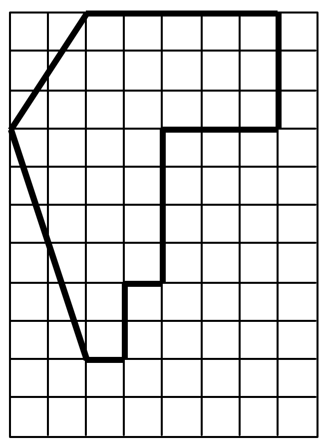
(E-2.3.3, E-2.2.5) Perimeter and Area

20. a) If you double the length of a rectangle and leave the width the same, how does the area change? the perimeter?
b) If you double both the length and width of a rectangle, how does the area change? the perimeter?

Ans. a) Area doubles and perimeter increases by an amount that is twice the original length.
b) Area quadruples and the perimeter increases by an amount that is twice the original length and twice the original width.

(M-2.2.2) Area

21. Explain how to find the area of the following shape in at least two ways.



Area = 34 square units

Ans. a) By counting squares and half-squares
b) By subdividing into rectangles and triangles and finding area of each by counting length/width/height and using formulas.

(E-1.1.3, E-1.2.2) Dividing Whole Numbers

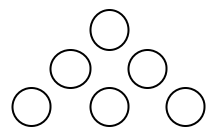
22. Explain in at least two ways whether there is a remainder or not when:
a) you divide an even number by an odd number
b) you divide an even number by an even number
c) you divide an odd number by an even number
d) you divide an odd number by an odd number

Ans. Using grids, manipulatives, numerical examples, etc.

a) remainder
b) no remainder
c) remainder
d) no remainder if the divisor is a factor of the dividend, otherwise there is a remainder

(E-4.2.1)
Problem Solving
Guess & Check

23. Place the numbers 1, 2, 3, 4, 5, and 6 in the spaces so that each side will give the same sum.



Ans.

```
    1                 2                 5
  4   6     or      5   3     or      3   1
5   2   3         4   1   6         4   2   6
Sum = 10          Sum = 11          Sum = 12
```

(E-4.1.2,
E-4.2.2)
Missing Factors

24. Problems such as 3 x n = 84 can be solved by several methods, including guess-and-check, fact families, and a table of multiples. Solve this problem and explain how you solved it using at least two of these methods.

Ans. n = 28 Fact Family: 84 ÷ 3 = 28

n	1	2	3	4	5	25	26	27	28
Multiples of 3	3	6	9	12	15	75	78	81	84

(E-4.2.2)
Problem Solving
Logical Reasoning

25. There are six new babies in the nursery lying in cribs one through five. Ingrid is working today and has no idea what any of the baby's names are. When the new parents come to pick up their babies she needs to make sure to give them the right ones. She knows that the baby's names are Kelsie, Kevin, Klyde, James, and Janie. She has found a few clues to help her name each baby.

Kelsie and Janie are the only girl's names.
Baby 1 has a sign in its crib that has a "J" on it.
Babies 2 and 5 are twins and their names start with the same letter.
Baby 1 is a girl.
The twins' names have the same number of letters.
Baby 3 is a boy.
Kelsie and Klyde are lying next to each other.

Ans. Baby 1 = Janie, Baby 2 = Kevin, Baby 3 = James, Baby 4 = Kelsie, and Baby 5 = Klyde.

(E-4.2.2)
Problem Solving
Logical Reasoning

26. Four fifth grade teachers must be assigned to rooms for this school year. The following requirements must be met for the room assignments:
    - Mrs. A has to be in room #2 or #3
    - Mrs. B can't be in room #2 or #4
    - Mrs. C has to be next to Mrs. B
    - Mrs. D can't be next to Mrs. C

    Ans. Possible solutions include Room 1-B; Room 2-C; Room 3-A; Room 4-D, or Room 1-D; Room 2-A; Room 3-B; Room 4-C, etc.

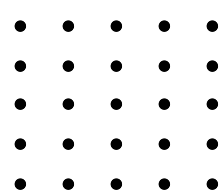
(E-4.2.1)
Problem Solving
Find a Pattern

27. There are less than 15 houses on one side of a street that are numbered 2, 4, 6, etc. Mrs. Murphy lives in one of these houses. The numbers of all the houses numbered below hers have the same sum as all those numbered above hers.
    a) How many houses are there on her side of the street?
    b) What is her house number?

    Ans. a) 8, since 2+4+6+8+10 = 30 and 14+16 = 30
    b) Mrs. Murphy's house number is 12

(E-4.2.1)
Problem Solving
Find a Pattern

28. Consider the following array of dots.

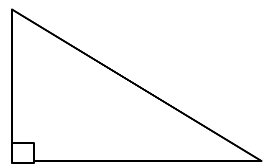


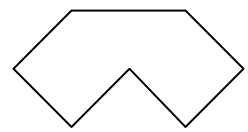
    a) How many squares can you form by connecting these dots?
    b) How many contain the center dot of the array in their interior?
    c) Write an explanation of how you found your answers and describe any patterns you found.

    Ans. a) length of side
    1x1 = 16
    2x2 = 9
    3x3 = 4
    4x4 = 1
    Total = 30
    b) 0 + 1 + 4 + 1 = 6
    c) Varies

(E-4.1.3)
Graphing
Points

29. Imagine you are talking to a student in your class on the telephone who has been absent from school for an illness. Part of the homework that you are trying to explain to him requires him to draw some figures. The other student cannot see the figures because he hasn't received the worksheet. Write a set of directions that you would use to help the other student to draw the figures exactly as shown below.

a) 

b) 

Ans. Varies, but may include coordinates, geometric terms, etc.

(M-1.2.1)
Multiplying
Fractions

30. Kids Bake 'em Cookies (yields 24 cookies)

1 and 1/4 cup flour
1/8 tsp. baking soda
1/8 tsp. salt
1/2 cup salted butter, softened
1/2 cup honey
1 cup chocolate chips
1/4 cup white sugar

Mix flour, soda, and salt in a bowl. Mix butter, sugar, and honey in a separate bowl. Add the flour mixture and chocolate chips to this bowl. Bake at 300° for 18-20 minutes.

It is Saturday afternoon and you have decided to bake some cookies. The recipe that you have makes 24 cookies, but you only want to make 12. Figure out how much of each ingredient is needed for preparing the recipe for only half of the servings. Explain how you arrived at your answers and then rewrite the recipe for that number of servings so that you and your family members can use the adjusted recipe in the future.

Ans. 3/4 cup flour
1/16 tsp. baking soda
1/16 tsp. salt
1/4 cup salted butter, softened
1/4 cup honey
1/2 cup chocolate chips
1/8 cup white sugar

(E-4.2.2, E-3.2.2) Problem Solving Collecting Data

31. Sharon, planning a party, bought some compact discs, pizzas, ad helium-filled balloons. The CDs cost \$15 each, the pizzas cost \$10 each, and the balloons cost \$5 each. If she spent a total of \$100, how many CDs, pizzas, and balloons did she buy?

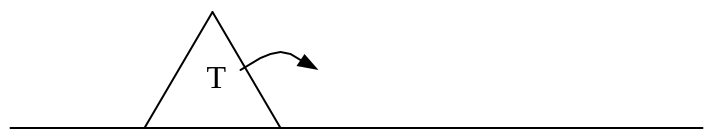
Ans. The assumption is she bought at least one of each item. The answers vary, for example:

CD	Pizza	Balloon
1	8	1
1	7	3
1	6	5

………………………

(E-2.2.3, E-1.1.3) Multiples (Transformations)

32. A demonstration model of a green pattern block (triangle) has side length of 6cm. It is rolled to the right a number of times. If the triangle stops so that the letter "T" is again in the upright position, what possible distances could it have rolled?



Ans. Multiple of 18 distance, i.e., 18, 36, 48, …cm.

(E-1.2.2) Add, Subtract, Multiply, Divide Whole Numbers

33. Choose any number. Multiple that number by 2. Add 5. Multiply by 50. Subtract 365. Add another number. Add 115. What do you get?

Ans. Varies, but will always be the first number selected followed by the second number selected. For example, if the first number chosen is 15 and the other number is 25, then the final result is 1525.

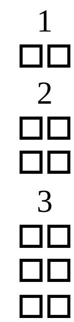
(E-1.2.3) Add Decimals

34. Find a word worth \$0.60 when A=\$0.01, B=\$0.02, C=\$0.03, etc.

Ans. Varies, e.g., SMUG = 0.19 + 0.13 + 0.21 + 0.07 = \$0.60

(E-4.2.1, E-4.2.2)

35. Look at the pattern on the right.
   a) How many □s will be needed for the 16th design?
   b) Describe the number pattern.
   c) Can a design be made using exactly 79 □s? Why or why not?
   d) How many □s are needed for the 67th design?
   e) Which design can be made with 84 □s?
   f) The outstanding dimensions of the rectangles created in designs 1-5 are listed below. Fill in the missing dimensions.

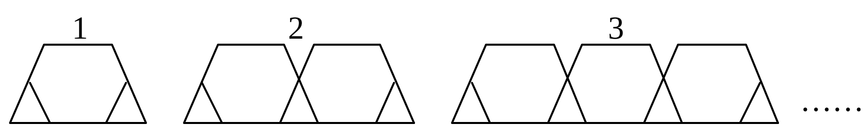


Design	Base x Height
1	2 x 1
2	2 x 2
3	2 x 3
4	2 x 4
5	2 x 5
8	
10	
20	
n	

Ans.
   a) 32. If $n = 16$, then $2n = 32$ □s
   b) Add 2 to each successive number, resulting in the even numbers or the multiples of 2.
   c) No. Only even numbers of blocks are used and 79 is odd.
   d) 134. If $n = 67$, then $2n = 134$ □s.
   e) 42nd. If $2n = 84$ □s, then $n = 42$.
   f) 2 x 8; 2 x 10; 2 x 20; 2 x n.

(E-4.2.1, E-4.3.2)

36. Pattern blocks have been used to construct the pattern below.



   a) How many hexagons will be needed for the 34th design? How many triangles?
   b) Describe the number pattern formed along the hexagons, the triangles, and the total.
   c) Which design can be built using a total of 97 blocks?
   d) Study the block designs and explain one way the arrangements show the generalization for the total number of blocks.

e) How many hexagons and how many triangles will be needed for the 87th design?
f) By how much do the numbers grow when looking at the totals? How does that constant difference relate to a generalization for design n?

Ans. a) 34, 35. For the 34th design, $n = 34$ & $n + 1 = 35$.
b) Hexagons: The numbers are consecutive (counting) numbers, starting with 1. Triangles: The numbers are consecutive numbers, starting with 2. Total: The numbers increase by 2, starting with 3 (the odd numbers).
c) 48th. If $2n + 1 = 97$, then $n = 48$.
d) There are always n hexagons in each row and $n + 1$ triangles in each row for a total of $n + (n + 1) = 2n + 1$ pattern blocks.
e) 87, 88. For design 87, $n = 87$ and $n + 1 = 88$.
f) The numbers grow by 2; 2 is the number in front of n in the generalization $2n + 1$.

(M-1.2.1) Division of Whole Numbers (Dividing Money)

37. Look at the dessert portion of the menu below. Three friends, Chris, Ward, and Phillip have decided that they will each buy a dessert and share the cost evenly. What is the least that each will pay if each orders a different dessert? The most?

Peaches 'n' More Desserts	
Apple Pie	1.89
Peach Cobbler	2.04
Ice Cream	0.84
Frozen Yogurt	0.96
Pecan Pie	2.10

Ans. (1.89 + 0.96 + 0.84) / 3 = $1.23
(2.10 + 2.04 + 1.89) / 3 = $2.01

(E-1.2.2) Divide Whole Numbers

38. List the first three whole numbers that will have a remainder of 3 when you divide by 7. Describe the pattern and explain how you found these numbers and how you would solve any similar problem.

Ans. 10, 17, 24. Explanation varies, but should mention that the pattern in the solution differs by the divisor and that it is found by adding the remainder to each of the multiples.

(E-1.3.2, M-1.3.2) Number Properties; Count Backward

39. Woe is me! My calculator does not have a 3 key that works! How can I use this broken calculator to do this problem? Explain your reasoning carefully and clearly.

$$\begin{array}{r} 23 \\ \underline{\times \quad 45} \end{array}$$

Ans. Varies, for example:
Using distributive property -- (45x25) - (45x2) or, (45x22) + 45

(M-1.2.5) Order of Operations

40. Adam, Betty, Charles, and Darlene were using their calculators to work on their math homework. The first problem they had to do was the following:

3 + 4 x 6 / 2 - 5

Adam used his Casio and got an answer of 16. Betty used a Math Explorer and got an answer of 10. Charles then used his Hewlett Packard and also got an answer of 10. Finally, Darlene used her Sharp calculator and the display showed an answer of 16.

They knew they pushed the right keys and they thought that each calculator worked the problem correctly, but the displays were different. How did each calculator solve the problem? How could you enter the problem on the Casio so that the answer would be 10?

Ans. The Casio and Sharp calculators worked the problem from left to right (arithmetic logic). The Math Explorer and Hewlett Packard calculators used the order of operations (algebraic logic) and did the multiplication and division first. Key entries on the Casio could vary, but would stress that the multiplication and division would have to be done first.

(E-1.2.2, E-1.3.3)
Divide Whole Numbers/ Place Value

41. You have a three-digit dividend and a one-digit divisor. Can you predict how many digits will be in the quotient? Give examples to support your answer.

    Ans. The quotient can have either 2 or 3 digits. It will always have 3 digits if the digit in the hundreds' place in the dividend is divisible by the divisor, e.g., 569/3 = 189 with a remainder where 5 is divisible by 3. If the digit in the hundreds' place in the dividend is not divisible by the divisor, then the quotient will have 2 digits, for example, 589/6 = 98 with a remainder where 5 is not divisible by 6.

(E-1.2.2, E-2.2.6)
Divide Whole Numbers/ Time

42. A jewelry store sold 108 pieces of jewelry in 13 days. How many pieces of jewelry did the store sell each week?

    Ans. If 13 days is over a 2-week period, then the store sold 54 pieces of jewelry per week. If 13 days is over a 3-week period, then the store sold 36 pieces per week.

(E-1.1.3, E-1.2.7)
Prime Numbers

43. A famous mathematician, Christian Goldbach, stated that every even number, except 2, is equal to the sum of two prime numbers. Goldbach's Conjecture has never been proven. Test his conjecture for the even numbers between 50 and 59.

    Ans. Varies, for example: 52 = 5,47; 54 = 7,47; 56 = 3,53

(E-1.1.3, E-4.2.1)
Odd/Even Multiples Find Rules for Patterns

44. Examine these number patterns:

15 = 7 + 8	15 = 4 + 5 + 6	15 = 1 + 2 + 3 + 4 + 5
9 = 4 + 5	9 = 2 + 3 + 4	

a) What kind of numbers can be written as a sum of 2, or 3, or 4 consecutive numbers? Give examples to verify your response.
b) What kind of numbers cannot be written as a sum of either 2, 3, or 4 consecutive numbers? Give examples of such numbers.
c) Describe the patterns you noticed.

Ans. a) All odd numbers can be written as the sum of 2 consecutive numbers. Numbers which are multiples of 3 can be written as the sum of 3 consecutive numbers. Numbers that can be written as the sum of 4 consecutive numbers are those in which twice the product of the sum of the middle two numbers is equal to the sum of all four numbers. Example (4): 26 = 5 + 6 + 7 + 8, or 2(6 + 7).

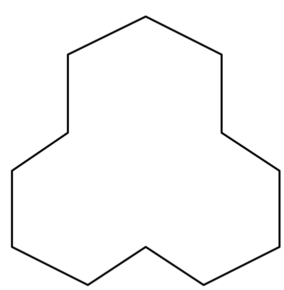
b) Even numbers cannot be written as the sum of two consecutive numbers. Numbers which are not multiples of 3 cannot be written as the sum of 3 consecutive numbers. No odd numbers can be written as the sum of 4 consecutive numbers, not can an even number be written as the sum of 4 consecutive numbers if it is divisible by 4. Example (4): 38 is an even number, but not divisible by 4. Dividing it by 2 gives 19 and the two consecutive numbers that add to 19 are 9 and 10. Thus the two middle numbers are 9 and 10 and the remaining numbers are 8 and 11, i.e., 8 + 9 + 10 + 11 = 38. 40 is divisible by 4 and thus is cannot be written as the sum of 4 consecutive numbers.

c) Various patterns can be observed such as those discussed above.

(E-4.3.2)
Sequences

45. a) Cover the figure below using 10 pattern blocks of 4 different colors.

b) Describe the strategy you used to solve the problem.



(Note for teacher: to make this pattern, trace around 3 yellow hexagon pattern blocks)

Ans. Varies, but will likely include beginning with some blocks that cover the shape and then substituting pieces one by one until the right number of total pieces is found.

(E-1.2.2, E-3.3.1) Divide Whole Numbers/ Organize Data

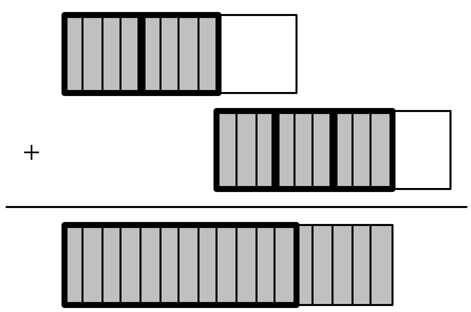
46. Vanessa is reading a book for a book report. The book is 132 pages long. Vanessa has already read 87 pages. She normally reads 12 pages a night. At this rate, how many more nights will it take her to finish the book? How many pages will Vanessa read the last night? Show how you solved this problem in at least two ways.

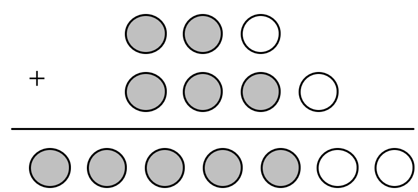
    Ans. 4 nights, including reading 9 pages the last night. Strategies could include showing arithmetic steps, making a table/chart, making a pictograph/bar graph, etc.

(E-1.2.3) Add Fractions

47. In last night's homework, Mrs. Jones asked that the students use manipulatives to find $\frac{2}{3}+\frac{3}{4}$. Johnny used fraction pieces and got a sum of $1\frac{5}{12}$. Kay used counters and got a sum of $\frac{5}{7}$. Could both of them have been right? Draw pictures to illustrate how each may have got their answer. What could the teacher have said to make it more clear how to do the problem?

    Ans. Fraction pieces: Counters:





The teacher should have been more specific in the instructions, e.g., state that the fractions were part of a whole (fraction pieces) or parts of a set (counters).

(E-1.2.2, E-1.2.6) 2-digit Division Estimation

48. Often there can be more than one way to estimate a quotient. Give several methods that you can use for the estimate for the quotient for 486 ÷ 72. Compare each of your methods and choose the one you think you would use.

    Ans. Varies, but should include not only rounding, but also at least one other method, such as front-end, compatible numbers, or truncating.

(E-1.2.2)
2-Digit
Division

49. Michael began working on his division homework and his work on the first problem looked like this:

$$26\overline{)182}\ \ \text{(quotient } 8\text{)}$$
$$208$$

Explain where Michael got the 208. What does 208 tell you about the quotient, 8? What number should Michael try next? Explain why you selected this number.

Ans. Varies, but should state something on the order of: "Michael estimated that 182 could be divided by 26 8 times, and when he tried it 8 x 26 was 208. This number is too large so he should try a smaller number. I would try 7 because it is the next closest number (or I know that 6 x 7 = 42 and the dividend 182 also ends in a 2)."

(E-2.1.3,
E-2.1.1,
E-2.2.4)
3-D Solids

50. a) Beth picked up a solid figure and described one of its characteristics by saying that it had a rectangular face. Identify the shape she may have picked up.
b) Gwen then picked up a solid and said it had a flat face. Identify the shape she may have picked up.

Ans. a) A rectangular prism, a triangular prism, or a cube.
b) A rectangular prism, a triangular prism, a cube, a square pyramid, a rectangular pyramid, a triangular pyramid, a cone, or a cylinder.

(E-2.1.3,
E-2.3.2)
3-D Solids

51. a) How many cubes of different sizes could you make if you have 16 multilink cubes?
b) How many rectangular prisms could you make with 8 multilink cubes? Describe any that might be the same if you changed their orientation, that is any that would be the same if you moved them around.

Ans. a) 4 cubes: 1x1, 2x2, 3x3, and 4x4
b) 12 different rectangular prisms: 1x1x1, 1x1x2, 1x1x3, 1x1x4, 1x1x5, 1x1x6, 1x1x7, 1x1x8, 2x2x1, 2x3x1, 2x4x1, 2x2x2. Any others are different orientations of these 12.

(E-2.1.5, E-1.2.3) Unit Pricing

52. Tabitha's mother asked her to buy a quart of orange juice for a recipe her mother was preparing. She knew that the cost of a pint of orange juice was \$0.79. How much would the price for a quart be to cost less? Explain how you figured this out.

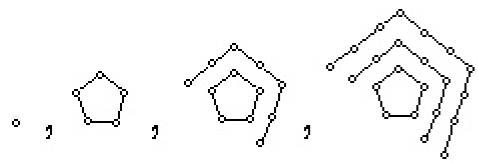
    Ans. Since there are 2 pints to a quart, the quart of O.J. would have to be less than twice \$0.79, or less than \$1.58.

GRADE 8 OPEN-ENDED QUESTIONS

(M-4.2.4, M- 4.2.5; H-1.2.4) *Variable in Sequences*

1. a) Draw the next two figures to continue this sequence of dot patterns.

   b) List the sequence of numbers that corresponds to the sequence of part (a). These are called pentagonal numbers.



   c) Complete this list of equations suggested by parts (a) and (b).

						1	=	1
				1	+	4	=	5
		1	+	4	+	7	=	12
1	+	4	+	7	+	10	=	22
______							=	____
______							=	____

   Observe that each pentagonal number is the sum of an arithmetic progression.

   d) Compute the 10^{th} term in the arithmetic progression 1, 4, 7, 10, ….

   e) Compute the 10^{th} pentagonal number.

   f) Determine the nth term in the arithmetic progression 1, 4, 7, 10, ….

   g) Compute the nth pentagonal number.

(M-3.2.1)

2. Yam, Bam, Uam, Iam, and Gam are aliens on a space ship.
   a) Yam is younger than Uam.
   b) Yam is not the youngest in the group.
   c) Only one alien is older than Gam.
   d) Gam is younger than Bam.

   Arrange Yam, Uam, Bam, Iam, and Gam in order of increasing age.

(H-3.1.4)
Permutations

3. Joe, Moe, and Hiram are brothers. One day, in some haste, they left home with each wearing the hat and coat of one of the others. Joe was wearing Moe's coat and Hiram's hat. (a) Whose hat and coat was each one wearing? (b) How many total possible clothing combinations were there?

(M-1.3.1, M-1.2.4)
Number Relationships

4. a) I'm thinking of a number. The sum of its digits is divisible by 2. The number is a multiple of 11. It is greater than 4x5. It is a multiple of 3. It is less than 7x8+23. What is the number? Is more than one answer possible?
   b) I'm thinking of a number. The number is even. It is not divisible by 3. It is not divisible by 4. It is not greater than 9^2. It is not less than 8^2. What is the number? Is more than one answer possible?

(M-1.3.1)
Number Relationships

5. a) Kathy Konrad chose one of the numbers 1, 2, 3, …, 1024 and challenged Sherrie Sherrill to determine the number by asking no more than 10 questions to which Kathy would respond truthfully either 'yes' or 'no.' Determine the number she chose if the questions and answers are as follows:

Questions	Answers
Is the number greater than 512?	No
Is the number greater than 256?	No
Is the number greater than 128?	Yes
Is the number greater than 192?	Yes
Is the number greater than 224?	No
Is the number greater than 208?	No
Is the number greater than 200?	Yes
Is the number greater than 204?	No
Is the number greater than 202?	No
Is the number 202?	No

   b) How many questions would Sherrie have to ask to determine Kathy's number if it is one of 1, 2, 3, …, 8192?
   c) How many possibilities might be disposed of with 20 questions?

(M-1.3.1)
Number Relationships

6. a) In any collection of seven natural numbers, show that there must be two whose sum or difference is divisible by ten.
   b) Find six numbers for which the conclusion of part (a) is false.

(M-1.3.1)
Number Relationships

7. When Joyce joined the Army, she decided to give her boom box to Ron and Jim, her two high-school-age brothers. After she heard Ron and Jim quarrel over using her boom box, Joyce told them they had better figure out a fair method of division so that one of them would get the stereo and the other would get a fair payment. Determine a method that is fair to Ron and Jim.

(M-1.1.4)
Place Value

8. a) Using each of 1, 2, 3, 4, 5, 6, 7, 8, and 9 once and only once fill in the circles in this diagram so that the sum of the three digit numbers formed is 999.

$$\begin{array}{ccc} \square & \square & \square \\ \square & \square & \square \\ \underline{\square} & \underline{\square} & \underline{\square} \\ 9 & 9 & 9 \end{array}$$

   b) Is there more than one solution to this problem? Explain briefly.

   c) Is there a solution to this problem with the digit 1 not in the hundreds column? Explain briefly.

(M-1.1.5, M-4.2.4; H-1.1.2)
Exponents/ Sequences

9. a) Write down the next three rows to continue this sequence of equations.

$$\begin{array}{rcrcrcrcrcrcl} & & & & & & 1 & = & 1 & = & 1^3 \\ & & & & 3 & + & 5 & = & 8 & = & 2^3 \\ & & 7 & + & 9 & + & 11 & = & 27 & = & 3^3 \\ 13 & + & 15 & + & 17 & + & 19 & = & 64 & = & 4^3 \end{array}$$

   b) Consider the sequence 1, 3, 7, 13, … of the first terms in the sums of part (a). Write the first ten terms of this sequence.

   c) Write out the tenth row in the pattern established in part (a).

(M-1.2.1)
Computation

10. Alice, Bill, and Carl have jointly inherited a piano, a car, a boat, and $20,000 in cash. The lawyer has asked each of them to bid on the value of the four items. Their bids are shown in the table. Decide which person receives each item, and what cash payments are made so that Alice, Bill, and Carl all feel they have received at least what each considers to be his or her fair share of the inheritance.

	Alice	Bill	Carl
Piano	\$1900	\$1500	\$2000
Car	\$5000	\$5200	\$6000
Boat	\$2500	\$1800	\$2000
Cash	\$20000	\$20000	\$20000
Total	\$29400	\$28500	\$30000
Fair Share	\$9800	\$9500	\$10000

(M-3.2.1)

11. Because of the high cost of living, Kimberly, Terry, and Otis each holds down two jobs, but no two have the same occupation. The occupations are doctor, engineer, teacher, lawyer, writer, and painter. Given the following information, determine the occupations of each individual.
    - The doctor had lunch with the teacher.
    - The teacher and the writer went fishing with Kimberly.
    - The painter is related to the engineer.
    - The doctor hired the painter to do a job.
    - Terry lives next door to the writer.
    - Otis beat Terry and the painter at tennis.
    - Otis is not the doctor.

(M-1.1.5, M-4.2.4; H-1.1.2) *Exponents/ Sequences*

12. a) Write down the next three rows to continue this sequence of equations.

$$2 = 1^3 + 1$$
$$4 + 6 = 2^3 + 2$$
$$8 + 10 + 12 = 3^3 + 3$$

b) Write down the tenth row in the sequence in part (a).

(M-2.1.1, M-4.2.4) *Basic Geometry Elements/ Variables in Numerical Patterns*

13. Consider a circle divided by n chords in such a way that every chord intersects every other chord interior to the circle and no three chords intersect in a common point. Complete this table and answer these questions.

a) Into how many regions is the circle divided by the chords?
b) How many points of intersection are there?
c) Into how many segments do the chords divide one another?

Number of chords	Number of Regions	Number of Intersections	Number of Segments
0	1	0	0
1	2	0	1
2	4	1	4
3			
4			
5			
6			
…			
N			

(M-1.1.1, M-1.2.1, M-3.2.1) *Rational Numbers*

14. While three watchmen were guarding an orchard, a thief slipped in a stole some apples. On his way out, he met the three watchmen one after another, and to each in turn he gave half the apples he had and two besides. In this way he managed to escape with one apple. How many had he stolen originally?

(M-4.1.2, M-4.2.5) *Functions*

15. In windy cold weather, the increased rate of heat loss makes the temperature feel colder than the actual temperature. To describe an equivalent temperature that more closely matches how it "feels," weather reports often give a windchill index, WCI. The WCI is a function of both the temperature F (in degrees Fahrenheit) and the wind speed v (in miles per hour). For wind speeds v between 4 and 45 miles per hour, the WCI is given by the formula

$$WCI = 91.4 - \frac{(10.45 + 6.69\sqrt{v} - 0.447v)(91.4 - F)}{22}$$

a) What is the WCI for a temperature of 10°F in a wind of 20 miles per hour?
b) A weather forecaster claims that a wind of 36 miles per hour has resulted in a WCI of –50°F. What is the actual temperature to the nearest degree?

(M-3.2.1, M-3.2.2) *Organize/ Represent Data*

16. In a college mathematics class all the students are also taking anthropology, history, or psychology and some of the students are taking two or even all three of these courses. If (i) forty students are taking anthropology, (ii) eleven students are taking history, (iii) twelve students are taking psychology, (iv) three students are taking all three courses, (v) six students are taking anthropology and history, and (vi) six students are taking psychology and anthropology.

a) How many students are taking only anthropology?
b) How many students are taking anthropology or history?
c) How many students are taking history and anthropology, but not psychology?

(M-4.1.2, M-4.2.4; H-1.1.2)
Generate Sequences

17. (Portfolio) Materials Needed: A calculator and a Fibonacci Sum Record Sheet for each student.

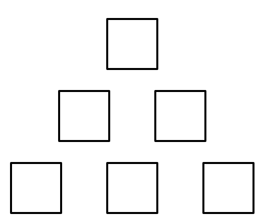
Directions

1. Start the investigation by placing any two natural numbers in the first two rows of column one of the record sheet and then complete the column by adding the consecutive entries to obtain the next entry in the Fibonacci manner. Finally, add the ten entries obtained and divide the sum by 11.
2. Repeat the process to complete all but the last column of the record sheet. Then look for a pattern and make a conjecture. Lastly, prove that your conjecture is correct by placing *a* and *b* in the first and second positions in the last column and then repeating the process as before.

	1	2	3	4	5	General Case
1						A
2						B
3						
4						
5						
6						
7						
8						
9						
10						
Sum						
Sum/11						

(M-1.2.1)
Add Fractions

18. Use the fractions $\frac{2}{3}, \frac{1}{3}, \frac{1}{6}, \frac{5}{6}, \frac{1}{2}$ and the whole number 1 to fill in the spaces below so that each side of the triangle will have the same sum. Use each number exactly once.

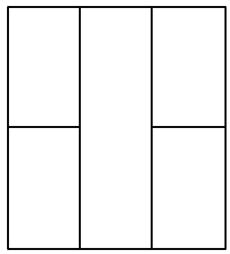


(M-2.1.2, M-2.2.1)
2-D Shapes

19. Arrange four points on a plane so the distance between any two points is one of two possible lengths; for example, one possible solution would be to have the four points be the vertices of a square. The side lengths of the square would be one length. The diagonals would be the second length. There are other possible solutions to this question. Draw diagrams/pictures to show the other solutions.

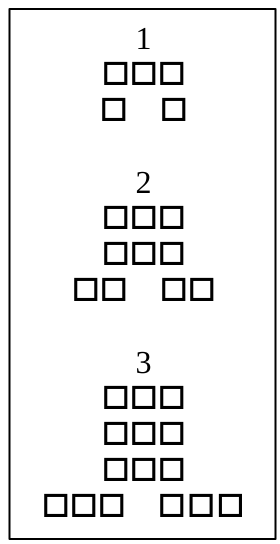
(M-2.3.2)
3-D Shapes

20. A solid is viewed as shown. This view is the same for both top and front. Draw the side view.



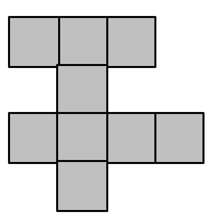
(M-4.2.4, M-4.2.5)
Variables in Numerical Patterns

21. Look at the pattern on the right.
    a) How many □s will be needed for the 15th design?
    b) Explain one way you can find the generalization for *n* by studying the block arrangements.
    c) How many □s are added to each successive design? How is this number used in your generalization for *n* □s?
    d) When exactly 95 □s are used, which design can be built?
    e) The 80th design will use how many □s?
    f) 585 □s will be needed to build which design?



(M-2.3.1)
Perimeter & Area

22. Nine square tiles are laid out on a table so that they make a solid pattern. Each tile must touch at least one other tile along an entire edge, e.g.



a) What are the possible perimeters of the figures that can be formed?
b) Which figure has the least perimeter?

(M-4.3.2)
Change in Variable

23. A homeowner has a 9x12 yard rectangular pool. She wants to build a concrete walkway around the pool. Complete the following table to show the area of the walkway.

Width of Walkway	Area of Walkway
0.5 yard	
1.0 yard	
1.5 yard	
2.0 yard	
N yard	

(M-4.2.4)
Variables in Numerical Patterns

24. The following figure shows the first five rows of Pascal's Triangle.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

a) Compute the sum of the elements in each of the first five rows.
b) Look for a pattern in the results and develop a general rule.
c) Use your rule to predict the next 3 rows of Pascal's Triangle.

(M-3.2.1, M-3.2.2)
Organize/ Represent Data

25. There are 75 students in the Travel Club. They discovered that 27 members have visited Mexico, 34 have visited Canada, 12 have been to England, 18 have visited both Mexico and Canada, 6 have been only to England, and 8 have been only to Mexico. Some club members have not been to any of the three foreign countries and, curiously, and equal number have been to all three countries.

a) How many students have been to all three countries?
b) How many students have been only to Canada?

(M-1.1.4)
Place Value

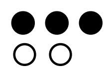
26. a) Place the digits 1, 3, 5, 7, and 9 in the proper boxes so that when multiplied they will produce the maximum product.

□□□
x □□

b) Generate a rule(s) that will enable you to properly place any five digits in a problem of a 3-digit number multiplied by a 2-digit number that will result in the maximum product.
c) Generate a similar rule that will enable you to property place any five digits in a problem of a 3-digit number multiplied by a 2-digit number that will result in the minimum product.
d) Give an example of a problem that follows your rule that will produce the smallest (minimum) product.

(M-1.1.1)
Integers

27. a) What 2-color counters need to be added to this array in order to represent –3? (Note: ○ = -1 and ● = +1)



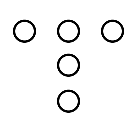
b) Could the question in part (a) be answered in more than one way?
c) How many different representations of –3 can be made with a total of 20 or fewer counters?

(M-1.2.1)
Subtract Integers

28. Solve each of these problems and indicate the subtraction facts illustrated by each. (Note: bringing a check adds a positive number; bringing a bill adds a negative number; taking away a check subtracts a positive number, and; taking away a bill subtracts a negative number.)
a) The mail carrier b rings you a check for $10 and takes away a check for $3. Are you richer or poorer and by how much?
b) The mail carrier brings you a bill for $10 and takes away a bill for $3. Are you richer or poorer and by how much?
c) The mail carrier brings you a check for $10 and takes away a bill for $3. Are you richer or poorer and by how much?
d) The mail carrier brings you a bill for $10 and takes away a check for $3. Are you richer or poorer and by how much?

(M-1.2.1)
Add Integers

29. Place the numbers –2, -1, 0, 1, 2 in the circles in the diagram so that the sum of the numbers in each direction is the same.



a) Can this be done with 0 in the middle of the top row? If so, show how. If not, why not?

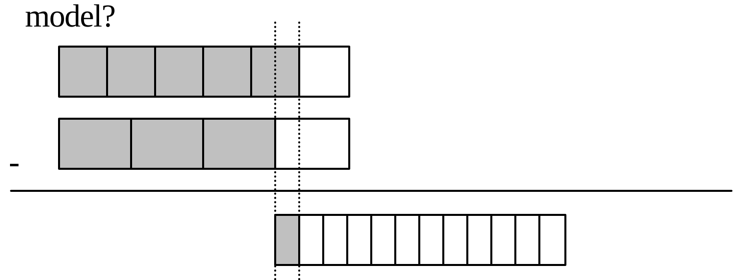
b) Can this be done with 2 in the middle of the top row? If so, show how. If not, why not?
c) Can this be done with –2 in the middle of the top row? If so, show how. If not, why not?
d) Can this be done with 1 or –1 in the middle of the top row? If so, show how. If not, why not?

(M-1.1.1, M-1.2.1)
Add Fractions

30. Depict the fraction $\frac{2}{3}$ with the following models.
a) colored region model
b) set model
c) fraction strip model
d) number line model
e) Choose one model to depict the sum of $\frac{2}{3}$ and $\frac{1}{6}$.

(M-1.1.1, M-1.2.1)
Subtract Fractions

31. a) What subtraction fact is illustrated by this fraction strip model?



b) Use the fraction strip (missing addend model), colored region (take-away model), and number line (measurement model) to illustrate $\frac{3}{4} - \frac{1}{3}$.

(M-1.2.3, M-4.2.5)
Solve Proportions/ Represent Functions with Tables

32. The eighth grade class at Washington Middle School is raffling off a turkey as a moneymaking project.
a) If the turkey cost $22 and raffle tickets are sold for $1.50 each, how many tickets will have to be sold for the class to break even?
b) Create a table to show how many tickets will have to be sold if the class is to make a profit of $20, $50, $100, $200, $*n*.

(M-1.2.3)
Solve Proportions

33. If ***a*** is to ***b*** as ***c*** is to ***d***; that is, if $\frac{a}{b} = \frac{c}{d}$
a) show that ***b*** is to ***a*** as ***d*** is to ***c***.
b) show that ***a-b*** is to ***b*** as ***c-d*** is to ***d***.
c) show that ***a*** is to ***a+b*** as ***c*** is to ***c+d***.
d) show that ***a+b*** is to ***c+d*** as ***b*** is to ***d***.

(M-1.2.3)
Solve Proportions

34. On an atlas map, the scale shows that one centimeter represents 250 miles. If the straight line distance from Chicago to St. Louis measures 1.1 cm on the map, what is the approximate airline distance from Chicago to St. Louis? Explain how you find the airline distance from any two cities in the continental United States using this map.

(M-3.2.3, M-3.1.2)
Line Plot Mean, Median, Mode, Outliers

35. The scores on the last math exam for Mrs. Thompson's class are in the following table.
    a) Depict the results in a line lot; compute the mean, median, and mode; and identify any outliers, if any.
    b) Present the data in a box-and-whiskers plot.
    c) Display the data in a stem-and-leaf plot.

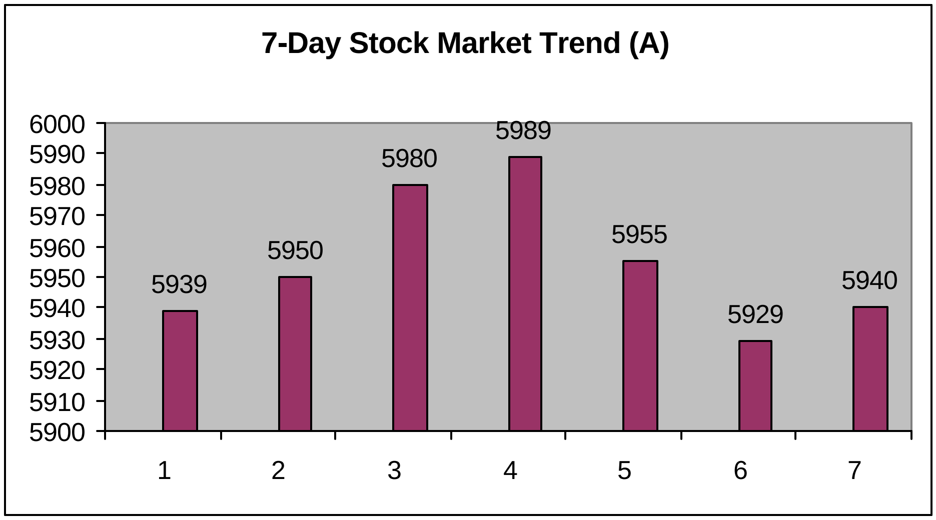
79	78	79	65	95	77
91	63	58	78	96	74
71	86	91	94	79	69
62	78	77	88	67	88
69	53	79	75	64	89

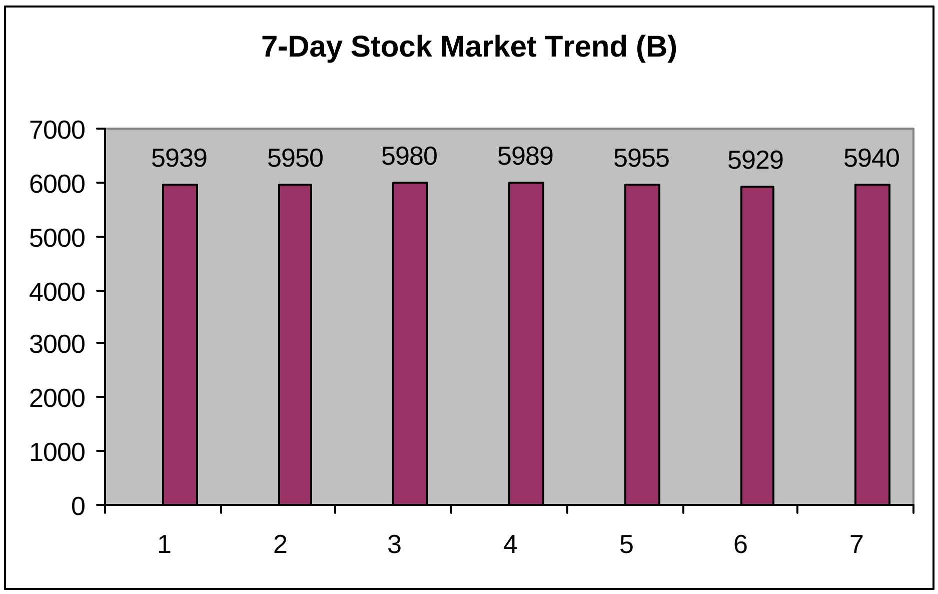
(M-3.1.3, M-3.3.3, M-3.2.2)
Misleading Graphs/ Interpreting Data

36. a) Discuss briefly why the television evening news might show histogram (A) below rather than (B) in reporting stock market activity for the last seven days. Is one of these histograms misleading? Why or why not?

    b) What was the percentage drop in the Dow Jones average from the fourth to the fifth day as shown in the following histograms? As an investor should I worry very much about this 34 point drop in the market?

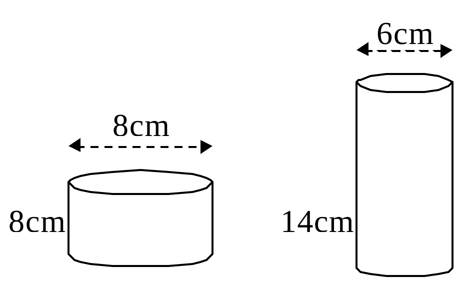
    c) Was the Dow Jones average on day five approximately half what it was on day four as suggested by histogram (A)?





(M-3.3.3, H-2.2.6) *Misleading Statistics/ Interpreting Data (Volume of Cylinder)*

37. Some merchandisers take advantage of optical illusions just as some pollsters, advertisers, and others do.



a) Which of the cans depicted here seems to have the greater volume?
b) Actually compute the volumes of the cans.
c) Which shape of can do you see more often in the grocery store? Why do you suppose this is so?

(M-3.2.2, M-3.2.3)
Interpret Data/ Find Mean, Median, Mode

38. a) Determine the mean, median, and mode for each section of the two pre-algebra classes from the data displayed in the stem-and-leaf plot below.

Section 1		Section 2
3	4	9
9 8 7 5	5	3 8
8 8 5 5 5 3 1	6	2 3 4 5 7 9 9
5 5 4 3 0	7	1 4 5 7 7 7 7 8 9
9 7 6 4 4 2 0	8	4 6 6 8 8
6 5 5 1	9	1 1 4 6 7
0 0 0	10	

b) Does the mode seem to be a good "typical" value of the scores in this case?

(M-3.1.1)
Meaning of Central Tendency

38. Produce sets of data that satisfy these conditions.
   a) mean = median < mode
   b) mean = mode < median
   c) median = mode < mean

(M-3.3.3)
Randomness/ Sampling

39. a) Suppose you generate a sequence of 0s and 1s by repeatedly tossing a die and recording a 0 each time an even number comes up and a 1 each time an odd number comes up. Is this a random sequence of 0s and 1s? Explain.

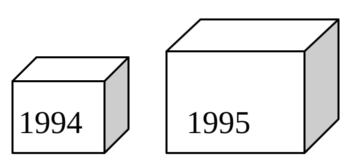
b) If you repeatedly tossed a pair of dice would this produce a random sequence of the numbers 2, 3, 4, …, 12? Why or why not?

(M-3.3.3)
Sampling

40. A television commercial states that eight out of ten dentists surveyed prefer Sparklin' White toothpaste. How could they make such a claim if, in fact, only 1 dentist in 10 actually prefers Sparklin' White?

(M-3.3.3)
Compare Data/ Misleading Statistics

41. a) Criticize this pictograph designed to suggest that the expenses for 1995 appear to be less than triple the 1994 expenses even though, in fact, the expenses did triple.



b) If the company officials are challenged by the stockholders can they honestly defend the pictographs?

(M-1.3.2)
Properties of Number Systems

43. a) Mark the cells in the following table to indicate if the property holds under the specified number system.

b) Give an example of those cells for which the property does not hold, e.g. the commutative property for subtraction does not hold for natural numbers since $5 - 7 \neq 7 - 5$.

Properties	**Natural numbers**	**Whole numbers**	**Integers**	**Rational numbers**	**Real numbers**
Closure Property for addition					
Closure Property for multiplication					
Closure Property for subtraction					
Closure Property for division except for division by zero					
Commutative Property for addition					
Commutative Property for multiplication					
Commutative Property for subtraction					
Commutative Property for division					
Associative Property for addition					
Associative Property for multiplication					
Associative Property for subtraction					
Associative Property for division					
Distributive Property for multiplication over addition					
Distributive Property for multiplication over subtraction					
Contains the additive identity					
Contains the multiplicative identity					
The multiplication property for 0 holds					
Each element possesses an additive inverse					
Each element except 0 posses a multiplicative inverse					

(M-3.2.6,
M-3.3.2,
M-3.2.4)
M-3.2.5)
Experimental & Theoretical Probability

44. a) Use simulation to determine the experimental probability that a family with three children contains at least one boy and at least one girl (If you use two-color counters, for example, red may represent a boy and yellow may represent a girl).

b) Calculate the theoretical probability that a family with three children contains at least one boy and at least one girl.

c) Compare the results of the experimental probability to the theoretical probability and discuss.

(M-3.3.2,
M-3.2.4)
Justify differences in Experimental & Theoretical Probabilities

45. Prepare a 3" by 5" card by writing the numbers 1, 2, 3, and 4 on it as shown. Show the card to 20 students and ask them to pick a number and tell you their choice. Record the results on the back of the card and then compute the experimental probability that a person chooses 3. Compare the results to the theoretical probability. Explain the difference in your findings.

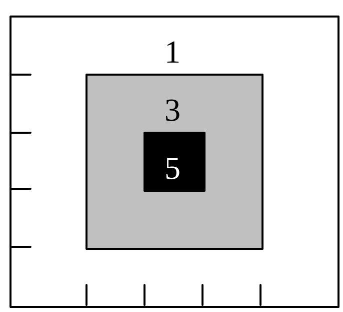
1	2	3	4

(M-3.2.6)
Probability (Counting Principle)

46. a) How many three digit natural numbers can be named using the digits 1, 2, 3, 4, or 5 at most once?
b) How many of the numbers in part a) begin with an odd digit?
c) How many of the numbers in part b) end with an odd digit?

(M-3.2.6,
M-3.2.7)
Predict Probability

47. A dartboard is marked as shown.



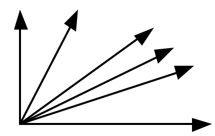
Josetta is good enough that she always hits the dartboard with her darts but beyond that, the darts hit in random locations. If a single dart is thrown, compute these probabilities.
a) The probability of scoring a 1.
b) The probability of scoring a 3.
c) The probability of scoring a 5.

(H-3.2.6, H-1.2.5))
Randomness (Combinations)

48. An urn contains eight red, five white, and six blue balls. Four balls are drawn at random. Compute the probability that all four are red.

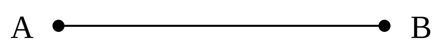
(M-2.1.1)
Geometric Elements/ Combinations

49. How many pairs of adjacent angles occur in this configuration of 6 rays? Explain your results.



(M-2.1.2, M-2.2.1)
Properties of 2-D shapes

50. The segment $\overline{AB}$ is to be completed to become a side of a triangle ABC.

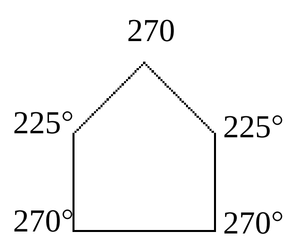


Describe in words and sketches, the set of points C so that:

a) ΔABC is a right triangle and $\overline{AB}$ is a leg.
b) ΔABC is a right triangle and $\overline{AB}$ is a hypotenuse.
c) ΔABC is an acute triangle.
d) ΔABC is an obtuse triangle.

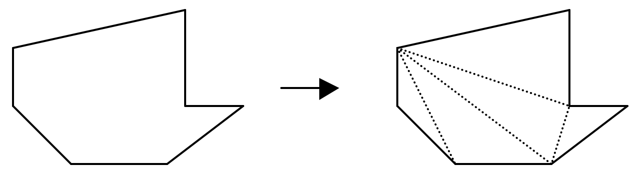
(M-4.2.6)
Use Formulas to Find Measurements

51. If the interior angle of a polygon has measure m, then 360° - m is called the measure of the conjugate angle at that vertex. Find a formula that gives the sum of the measures of the conjugate angles of an n-gon, and give a justification for your formula. As an example, the measures of the conjugate angles in this pentagon add up to 1260°.



(M-4.2.6)
Use Formulas to Find Measurements

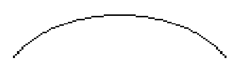
52. The heptagonal region shown on the left below has been broken into five triangular regions by drawing four nonintersecting diagonals across the interior of the polygon, as shown below on the right. In this way, we can say that the polygon is triangulated by diagonals.



a) Investigate how many diagonals are required to triangulate any *n*-gon.
b) How many triangles are in any triangulation of an *n*-gon by diagonals?
c) Explain how a triangulation by diagonals can give a new derivation of the formula $(n\text{-}2)\cdot 180°$ for the sum of the measures of the interior angles of any *n*-gon.

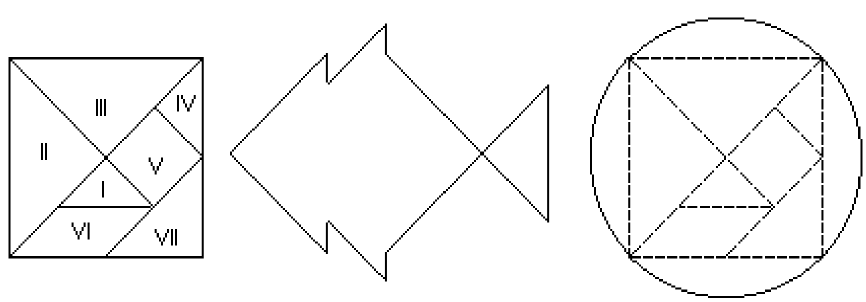
(M-2.2.2)
Use Appropriate Tools & Strategies

53. Explain how to construct the center of the circle whose arc is shown below.



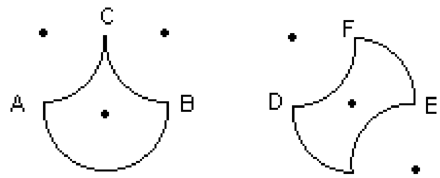
(M-2.2.2, M-2.2.4)
Use Appropriate Tools/ Estimate Measurements

54. Use the seven tangram pieces in which the unit of one tangram area is the small isosceles triangle to measure:
a) the area of each of the tangram pieces,
b) the area of the "fish,"
c) the area of the circle that circumscribes the square. Are your measurements exact, or approximate?



(M-2.3.1)
Relationship of Measurements

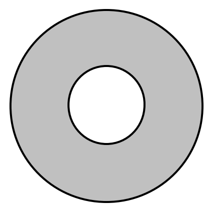
54. Leonardo da Vinci once became interested in showing how the areas of certain curvilinear regions could be determined and compared among themselves and to rectangular regions. The pendulum and the ax are two of the examples he worked out. The dots show the centers of the circular arcs that form the boundary of the region.



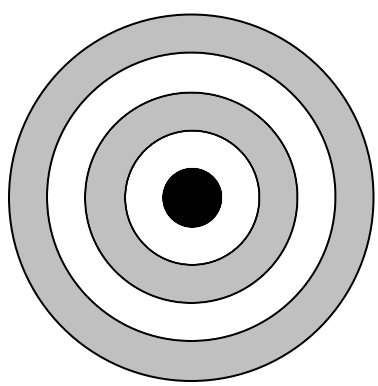
If the arcs forming the pendulum and the ax have radius 1, show that the areas of both figures are equal to that of a 1 by 2 rectangle.

(M-2.2.5)
Use Formulas to Find Measurements

56. An annulus is the region bounded by two concentric circles.

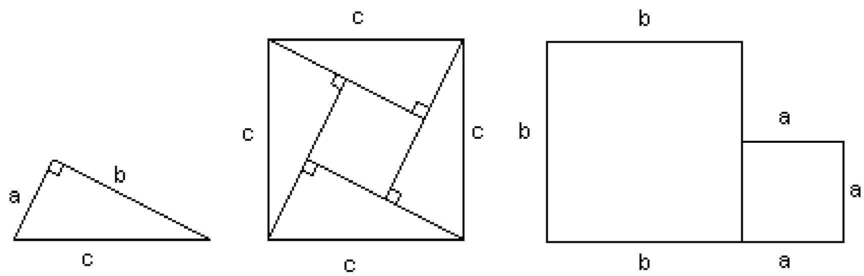


a) If the radius of the small circle is 1 and the radius of the larger circle is 2, what is the area of the annulus? Explain how you solved the problem.
b) A dartboard has four annular rings surrounding a bull's eye. The circles have radii 1, 2, 3, 4, and 5. Suppose a dart is equally likely to hit any point of the board. Is the dart more likely to hit in the outermost ring or inside the region consisting of the bull's eye and the two innermost rings? Show how you determined your answer.



(M-2.3.1, M-2.2.7)
How Measurements Are Related (Pythagorean Theorem)

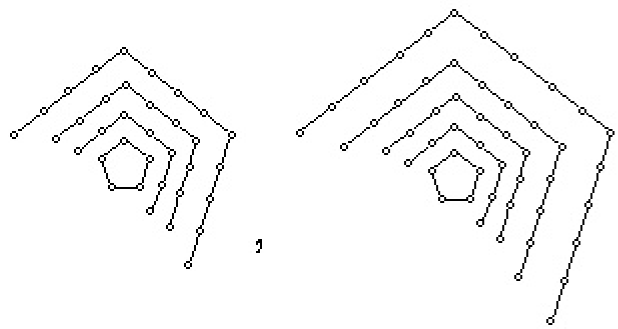
57. The twelfth century Hindu mathematician Bhaskara arranged four copies of a right triangle of side lengths *a, b, c* into a *c* by *c* square, filling in the remaining region with a small square.



a) Show how the five pieces in the *c* by *c* square can be arranged to fill the "double" square region at the right.
b) Explain how the Pythagorean Theorem follows from part a).

GRADE 8 SOLUTIONS

1. a)



   b) 1, 5, 12, 22, 35, 51
   c) 1 + 4 + 7 + 10 + 13 = 35   1 + 4 + 7 + 10 + 13 + 16 = 51
   d) 28
   e) 145
   f) $3n - 2$
   g) $[n(3n - 1)] / 2$

2. Iam, Yam, Uam, Gam, Bam

3. a) Moe was wearing Hiram's coat and Joe's hat; Hiram was wearing Moe's hat and Joe's coat.
   b) 9 hat-coat combinations for the three brothers.

4. a) 33, 66, or 99
   b) 70 or 74

5. a) 201
   b) 13 questions
   c) 1,048,576 or 2^{20}

6. a) Varies
   b) Varies, for example 2, 3, 4, 9, 12, and 13

7. The following method allows for the possibility that, while both Ron and Jim want the boom box, they have different assessments of its worth. Independently Ron and Jim are asked to write down what they think the stereo is worth. Let's say that Jim thinks it's worth \$90, but Ron thinks it's worth \$120. Since Ron places a higher value on the stereo, it should be his, and the question now is -- how much should Ron pay Jim?

The answer is \$52.50. To see why this is fair, recall that each brother feels he should get half of the value of Joyce's gift. Thus, Ron will be happy with a \$60 value, and Jim will feel he is fairly treated if he gets \$45. The average of the two values is \$52.50. For \$52.50 Ron gets \$60 in stereo value. At the same time Jim, who expects his half of the \$90 gift should be \$45, is also getting \$52.50. Each receives \$7.50 more value than expected. The method is a legally accepted practice in many states for settling estate claims.

8. a) Answers will vary. One solution is:

$$\begin{array}{r} 179 \\ 368 \\ \underline{452} \\ 999 \end{array}$$

   b) Yes. The digits in any column can be arranged in any order.
   c) No. The hundreds column digits must add up to 8 to allow for a carry from the tens column. If the digit 1 is not in the hundreds column, the smallest this sum can be is 2 + 3 + 4 = 9. Thus, the digit 1 must be in the hundreds column.

9. a) $21 + 23 + 25 + 27 + 29 = 5^3$
   $31 + 33 + 35 + 37 + 39 + 41 = 6^3$
   $43 + 45 + 47 + 49 + 51 + 53 + 55 = 7^3$

   b) 1, 3, 7, 13, 21, 31, 43, 57, 73, 91

   c) $91 + 93 + 95 + 97 + 99 + 101 + 103 + 105 + 107 + 109 = 10^3$

10. Each person expects to receive one-third of what they assess as the total value of the estate. Let's first award each item to its highest bidder, and then award enough of the cash to bring each person up to his or her "fair share" amount as shown in this table.

	Alice	**Bill**	**Carl**
"Fair Share"	\$9,800	\$9,500	\$10,000
Assigned items	Boat	….	Piano, Car
Value of items	\$2,500	0	\$8,000
Cash to bring to "fair share"	\$7,300	\$9,500	\$2,000

This uses \$7,300 + \$9,500 + \$2,000 = \$18,000 of the \$20,000 in cash, so there is a \$1,200 cash yet to be distributed. Dividing this equally among the three heirs, we can give each an additional \$400 more than they expect for a fair settlement. Thus, the final settlement will be:

Alice: Boat + \$7,700
Bill: \$9,900
Carl: Piano + Car + \$2,400

11. Kimberly is the lawyer and painter; Terry is the engineer and the doctor; Otis is the teacher and writer.

12. a) $14 + 16 + 18 + 20 = 4^3 + 4$
$22 + 24 + 26 + 28 + 30 = 5^3 + 5$
$32 + 34 + 36 + 38 + 40 + 42 = 6^3 + 6$

b) $92 + 94 + 96 + 98 + 100 + 102 + 104 + 106 + 108 + 110 = 10^3 + 10$

13. a) $\dfrac{n(n+1)}{2}+1$

b) $\dfrac{n(n-1)}{2}$

c) n^2

14. Working backwards he gave the third watchman five apples plus one more (he had 6 at the time). He gave the second watchman 10 apples (half of the 16 he had plus 2 more). He gave the first watchman 20 apples (half of the 36 he had plus 2 more). He had stolen 36 apples.

15. a) -25°F
b) 1°

16. a) 31
b) 45
c) 3

17.

1	2	3	4	5	6	7	8	9	10	Sum	Sum ÷ 11
a	b	a + b	a + 2b	2a + 3b	3a + 5b	5a + 8b	8a + 13b	13a + 21b	21a + 34b	55a + 88b	5a + 8b

18. Sum = $1\frac{1}{2}$

$\frac{1}{6}$
$1 \quad \frac{5}{6}$
$\frac{1}{3} \quad \frac{2}{3} \quad \frac{1}{2}$

Sum = $1\frac{5}{6}$

$\frac{1}{3}$
$\frac{1}{2} \quad \frac{5}{6}$
$1 \quad \frac{1}{6} \quad \frac{2}{3}$

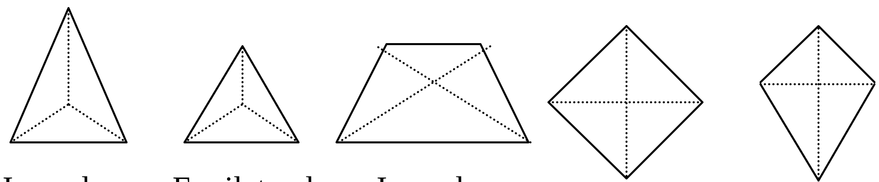
Sum = $1\frac{2}{3}$

$\frac{1}{2}$

$1 \quad \frac{1}{3}$

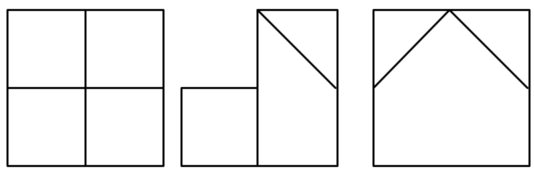
$\frac{1}{6} \quad \frac{2}{3} \quad \frac{5}{6}$

19.



Isosceles Triangle | Equilateral Triangle | Isosceles Trapezoid | Rhombus | Kite

20. There are multiple possible solutions, including:



21. a) 75. If $n = 15$, then $5n = 75$ □s.
    b) There are always 5 groups of blocks: three columns of n and 2 "wings" of n each at the bottom: $3n + 2n = 5n$.
    c) No; 42 is not a multiple of 5.
    d) 5. The 5 is the number in front of the n in the generalization of $5n$.
    e) 19th. If $5n = 95$ □s, then $n = 19$.
    f) 400. If $n = 80$, then $5n = 400$ □s.
    g) 117th. If $5n = 585$ □s, then $n = 117$.

22. a) Perimeters range from 12 units to 20 units. The perimeters must be even.
    b) The figure with the least perimeter is a 3x3 square.

23.

Width of Walkway	**Area of Walkway**
0.5 yd.	22 sq. yd.
1.0 yd.	46 sq. yd.
1.5 yd.	72 sq. yd.
2.0 yd.	100 sq. yd.
n yd.	$4n^2 + 42n$

24. a) $\text{Sum}_{\text{Row }0} = 1$; $\text{Sum}_{\text{Row }1} = 2$; $\text{Sum}_{\text{Row }2} = 4$; $\text{Sum}_{\text{Row }3} = 8$; $\text{Sum}_{\text{Row }4} = 16$
    b) The sum of each row after the first is twice the previous row. $\text{Sum}_{\text{Row }n} = 2^n$
    c) $\text{Sum}_{\text{Row }5} = 2^5 = 32$; $\text{Sum}_{\text{Row }6} = 2^6 = 64$; $\text{Sum}_{\text{Row }7} = 2^7 = 128$

25. a) 2 students have been to all three countries.
    b) 14 students have been only to Canada.

26\. a)

$$\begin{array}{r} 751 \\ \underline{\times \quad 93} \\ 69{,}843 \end{array}$$

b) The largest digit will be located in the tens place of the multiplier. The next largest digit will be located in the hundreds place of the multiplicand. The third largest digit will be in the tens place of the multiplicand and the fourth largest digit will be in the units place of the multiplier. Finally, the fifth and smallest digit will be in the units place of the multiplicand.

c) Nearly the opposite of the maximum value, i.e., place the smallest digit in the tens place of the multiplier, the next to smallest digit in the hundreds place of the multiplicand, *then the middle digit in units place of the multiplier. The next digit will be in the tens place of the multiplicand* and the largest digit in the units place of the multiplicand.

d) Using the digits 1, 2, 3, 5, and 6, the smallest possible product will be:

$$\begin{array}{r} 256 \\ \underline{\times \quad 13} \\ 3{,}328 \end{array}$$

27\. a) Varies, e.g. add 4 negatives such as: ● ● ●
○ ○ ○ ○ ○ ○

b) Yes, there are an infinite number of solutions. Part c shows some.

c)

●	0	1	2	3	4	5	6	7	8
○	3	4	5	6	7	8	9	10	11

28\. a) You are \$7 richer. This illustrates 10 - 3 = 7.
b) You are -\$7 poorer. This illustrates (-10) - (-3) = -7.
c) You are \$13 richer. This illustrates 10 - (-3) = 13.
d) You are -\$13 poorer. This illustrates (-10) - 3 = -13.

29\. a)

-1 0 +1
-2
2

b) -1 2 0

-2

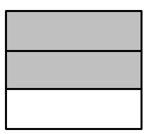
+1

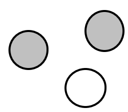
c) -1 -2 2

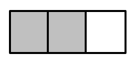
+1

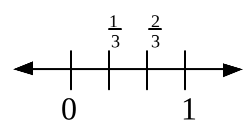
0

d) It can not be done with 1 or -1 in the middle of the top row since these are odd numbers. An odd number in the middle leaves only one odd number in either the row or column. Thus, one direction would be even and the other direction odd.

30. a) 

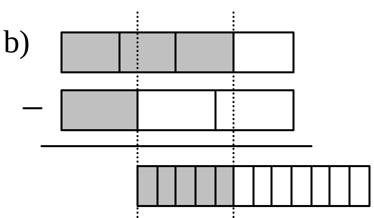
b) 

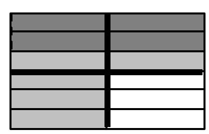
c) 

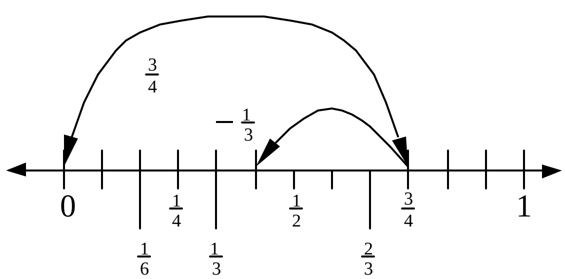
d) 

e) Any of the four models to show the sum is $\frac{5}{6}$.

31. a) $\frac{5}{6} - \frac{3}{4} = \frac{1}{12}$

b) 





32. a) $22 = 1.50\,x, x = 14{,}66\ldots$ To break even the class will have to sell at least 15 tickets.

b)

# tickets	28	48	82	148	$(n+22)/1.50$
profit	$20	$50	$100	$200	$\$n$

33. a) $ad = cb$

$da = cb$

$$\frac{d}{c} = \frac{b}{a}$$

b) $ad = cb$

$ad - bc = cb - bd$

$d\,(a - b) = b\,(c - d)$

$(a - b)\,d = (c - d)\,b$

$$\frac{a-b}{b} = \frac{c-d}{d}$$

c) $ad = cb$

$ac + ad = ac + cb$

$a\,(c + d) = c\,(a + b)$

$$\frac{a}{a+b} = \frac{c}{c+d}$$

d) $ad = cb$

$ad + bd = cb + bd$

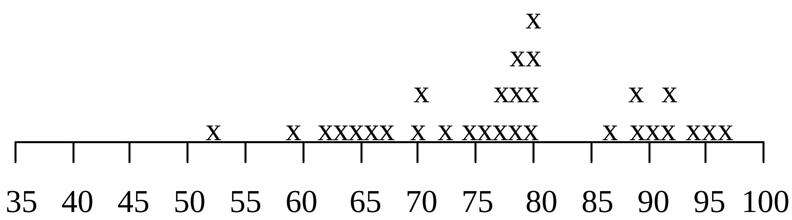
$d\,(a + b) = b\,(c + d)$

$(a + b)\,d = (c + d)\,b$

$$\frac{a+b}{c+d} = \frac{b}{d}$$

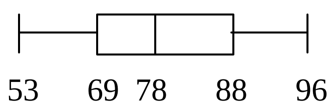
34. The airline distance is approximately 275 miles. The explanation will vary but should describe setting up a proportion using the map scale.

35. a)



Mean = 77.07; Median = 78; Mode = 79; there is no outlier. One method to determine an outlier is to use the five number summary and compute another value, the interquartile range (IQR). Minimum = 53; Lower Quartile = 69; Median = 78; Upper Quartile = 88; Maximum = 96. The IQR is the difference between the lower quartile and the upper quartile, i.e., 19 points (88 - 69). An outlier is a data value that is less than the lower quartile - (1.5 x IQR), i.e., 69 - 1.5x19, or 40.5 or greater than the upper quartile + (1.5 x IQR), i.e., 88 + 1.5x19, or 116.5. Since neither the minimum nor maximum falls outside these values, there are no outliers in this problem.

b)



Note: The box-and-whiskers plot may be drawn vertically and the stem-and-leaf plot may be drawn in inverse order.

c)

9	11456
8	6889
7	145778889999
6	2345799
5	38

36. a) Histogram (A) emphasizes the changes in the DJA by its choice of vertical scale. The changes appear to be large in (A) thus exaggerating the report of stock activity on the evening news. (B) makes it clear that the changes are minimal.

b) $34/5989 \cdot 100\% = 0.57\%$ which an investor should not worry about.

c) No

37. a) The taller can.

b) First can: $V = \pi\ (4\text{cm})^2(8\text{ cm}) \sim 402\text{ cm}^3$
Second can: $V = \pi\ (3\text{ cm})^2(14\text{ cm}) \sim 396\text{ cm}^3$

c) The tall cylinder is found most often in the stores because it creates an optical illusion that makes it appear to contain more.

38. a) Section 1: Median = 75; Mode = 77; Mean = 76.0
    Section 2: Median = 75; Mode = 65; Mean = 76.2
    b) The mode is not a good "typical" score.

39. Answers vary for each, e.g.:
    a) 5, 7, 10, 14, 14; median = 10, mode = 14, mean = 10.
    b) 0, 0.1, 0.1, 0.11, 0.12, 0.13, 0.14; median = 0.11, mode = 0.1, mean = 0.1.
    c) 9, 10, 10, 11, 12; median = 10, mode = 10, mean = 10.4.

40. a) Yes, since all sides of the die are equally likely to come up, all sequences of 0s and 1s are equally likely to appear.
    b) No, since, for example, a sum of 7 is more likely than a sum of 2 - 7 can be produced from 1 + 6, 2 + 5, and 3 + 4, whereas 2 can be found only with 1 + 1.

41. There is sampling error -- only those were surveyed that the pollster reasoned would be most likely to endorse Sparklin' White.

42. a) The volume of the larger box is about three times that of the smaller box. But the length of a side of the larger box is less than three times the length of a side of the smaller box, suggesting that the change was less than tripling.
    b) Since the volume of the larger box is about three times that of the smaller, then the pictograph is accurate.

43. a)

Properties	Natural numbers	Whole numbers	Integers	Rational numbers	Real numbers
Closure Property for addition	X	X	X	X	X
Closure Property for multiplication	X	X	X	X	X
Closure Property for subtraction			X	X	X
Closure Property for division except for division by zero				X	X
Commutative Property for addition	X	X	X	X	X
Commutative Property for multiplication	X	X	X	X	X
Commutative Property for subtraction					
Commutative Property for division					
Associative Property for addition	X	X	X	X	X
Associative Property for multiplication	X	X	X	X	X
Associative Property for subtraction					
Associative Property for division					
Distributive Property for multiplication over addition	X	X	X	X	X
Distributive Property for multiplication over subtraction	X	X	X	X	X
Contains the additive identity		X	X	X	X
Contains the multiplicative identity	X	X	X	X	X
The multiplication property for 0 holds		X	X	X	X
Each element possesses an additive inverse			X	X	X
Each element except 0 posses a multiplicative inverse				X	X

b) Examples vary.

44. a) Results will vary, but should be in range of 0.7 to 0.85.
    b) All possible outcomes include BBB, BGB, GBB, GGB, GBG, BGG and GGG. Thus, the theoretical probability would be 6 out of 8 or 0.75.
    c) Varies

45. The theoretical probability is 0.25, but the experimental probability will be much higher because people tend to choose a number in the middle (2 or 3) and not one on the end.

46. a) $5 \cdot 4 \cdot 3 = 60$
    b) $3 \cdot 4 \cdot 3 = 36$
    c) $3 \cdot 2 \cdot 3 = 18$

47. a) 16/25 or 0.64
    b) 9/25 or 0.36
    c) 1/25 or 0.04

48.
$$\frac{C(8,4)}{C(19,4)} = \frac{\;\frac{8\cdot 7\cdot 6\cdot 5}{1\cdot 2\cdot 3\cdot 4}\;}{\frac{19\cdot 18\cdot 17\cdot 16}{1\cdot 2\cdot 3\cdot 4}} = \sim 0.02$$

49. Notice that adjacent angles are not all composed from adjacent rays. But a pair of adjacent angles is determined by three rays and each collection of three rays determines a different pair of adjacent angels. So an equivalent question is: How many ways can a subset of three rays be chosen from six rays?

$$C(6,3) = \frac{6\cdot 5\cdot 4}{3\cdot 2} = 20$$

50. a) Varies
    b) Varies
    c) Varies
    d) Varies

51. At a vertex, the interior angle and the conjugate angle add up to 360º. For an n-gon, the sum of all interior angles and all conjugate angles is $n\cdot 360º$. All the interior angles add up to $(n-2)\cdot 180º$, so all the conjugate angles add up to $360º n-(n-2)\cdot 180º = 360º n - 180º n + 360º = 180º n + 360º = (n+2)\cdot 180º$.

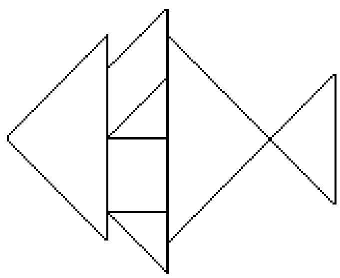
52. a) Sketches will vary.
    b) $n - 2$ triangles
    c) Since the sum of the interior angles of a triangle is 180º and there are $n - 2$ triangles in a triangulated n-gon, then the sum of the interior angles of an n-gon is $(n-2)\cdot 180º$.

53. Draw two different chords within the arc. Construct the perpendicular bisectors of each chord. The point of intersection of the perpendicular bisectors is the center of the circle and the radius is the measure from the center to the arc.

54. a)

Piece	Area
I, IV	1 tga
II, III	4 tga
V, VI, VII	2 tga

b) $4 + 2 + 1 + 2 + 1 + 4 + 2 = 16$ tga

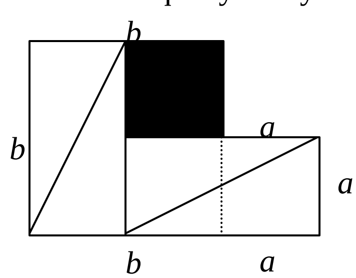


c) The circle can be covered by the seven tangram pieces, together with 12 additional copies of the unit triangle. This shows that the circle's area is between 16 tga and 28 tga. Thus, we might estimate the area at about 25 tga.

55. Draw line segments $\overline{AB}$ and $\overline{DE}$. Draw the perpendiculars from C to $\overline{AB}$ and from F to $\overline{DE}$. Cut along the line segments to divide each figure into three parts. Rearrange the parts into rectangles.

56. a) $\pi(2)^2 - \pi(1)^2 = 3\pi$ square units, or ~9.4 square units.
b) The area of the outermost ring = $\pi(5)^2 - \pi(4)^2 = 9\pi$ square units. The area of the bull's eye and the two innermost rings is $\pi(3)^2 = 9\pi$ square units. Thus, the possibilities are equally likely.

57. a)



b) Since the square and "double" square and tiled by the same five shapes, their areas are equal. The respective areas are c^2 and $a^2 + b^2$, so $c^2 = a^2 + b^2$.

ALGEBRA I / PROBABILITY / STATISTICS OPEN-RESPONSE QUESTIONS

To the teacher:

This set of questions is intended to be used with students for practice, i.e., questions can be given for students to work with in small groups or for homework. After completion of individual questions, using the answering model that has been established in the classroom, each solution and solution strategy should be discussed among members of the entire class. Many of the questions are more difficult than students will see on the KCCT, but the strategies they learn in answering these questions should be similar.

It is not intended that the questions be given as a block, but that there be ample opportunities for discussion between questions. It is not necessary that all questions be asked or that they are asked in any particular order. It should be your decision as to which questions to use with your students.

The solution key includes a cross-reference to the Core Content for Assessment descriptor that is being addressed for each question. It is suggested that, even if you don't use all questions, you select questions from a variety of different descriptors.

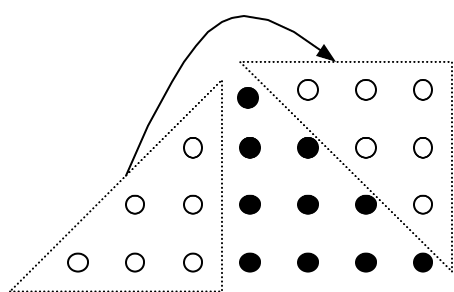
1. Alice received a check and she decided to cash it at her local bank. For some reason, the bank teller was confused and switched the dollars and cents; that is, what was written as cents on the check he gave to her in dollars, and what was written in dollars on the check he gave to her in cents. It was not until after she bought a piece of candy for five cents that she noticed the teller's error. At that point, she actually had twice the amount of money that was written on the check. How much money was the check made out for? Explain your thinking and your answer(s). Also, explain why you believe there is only one possible answer, or why you believe there is more than one possible answer.

2. Kim said her grocery items cost $25.86. Her mom said that twenty years ago the same items would have cost $20. At this rate, how much would it cost to purchase groceries now which would have cost $50 twenty years ago?

3. Kris lined cans in two rows on a shelf and had one left over; she tried again using three rows, and then four rows. Each time she had one can left over. Finally, she tried five rows, and had none left over! What possible numbers of cans could she have been working with?

4. Show that there are more possible seating arrangements in a classroom with 30 desks than there are drops of water in all the earth's oceans? (Note: The Earth's radius is about 6370 km.)

5. What famous summation formula is being illustrated in this informal pictorial proof?



6. To win the prize of $2000, Kris had to solve the following problem: The sum of two numbers is 2, the product of the same two numbers is 3. What is the sum of the reciprocals of the two numbers?

7. Stephen added the squares of 6 consecutive integers and got 1111. What integers did he square and then add?

8. Diana and Andrew each roll a standard die obtaining a number at random from 1 to 6. What is the probability that Diana's number is larger than Andrew's number?

9. What is half of 2^{40}?

10. 2^{4^2} means $(2^4)^2$ according to Max. 2^{4^2} means $2^{(4^2)}$ according to Phoebe. Find a positive integer that satisfies:

    $3n^{3^2} = 81n^{3^2}$ if Max wrote the right side and Phoebe wrote the left side.

11. Determine all pairs of numbers x and y such that xy, x/y, and $x - y$ are all equal.

12. Tolliver was asked how many miniature cars he had collected. He replied, "If I had as many more, half as many more, and $1\frac{1}{2}$ more, I would have two dozen." How many miniatures did Tolliver have?

13. Find the value of t and the digit symbolized by a such that $[3(230 + t)]^2$ will equal 492, a04.

14. Prove that for any integer n, the number $N = n^5 - 5n^3 + 4n$ is divisible by 120. This can be done without having to use mathematical induction.

15. Find all integral solutions of the equation

    $x^2 + y^2 + x + y = 1997.$

16. Find three integers in geometric progression whose sum is 21 and such that the sum of their reciprocals is 7/12.

17. Solve the system of equations:

    $6751x + 3249y = 26751$

    $3249x + 6751y = 23249$

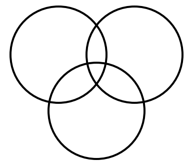
18. If you are given the sum of the squares of three consecutive integers, what five digits can never occur in the ones place?

19. You are given a nine-inch diameter pie plate and you conduct an experiment in which the plate is dropped, upside down from about waist height, onto a floor of nine-inch square tiles. Predict the number of tosses out of ten in which a corner is covered. What is the exact probability?

20. Each Necco roll contains forty flavored wafers. These forty wafers are randomly selected from eight available flavors. What is the probability that I will get a roll **without** my favorite color, chocolate? What is the probability that I get exactly seven flavors? Exactly six?

21. Three straws are chosen from a set of nine straws whose lengths are 1 inch, 2 inches, 3 inches, …, 9 inches. What is the probability that the three straws placed end-to-end will form a triangle?

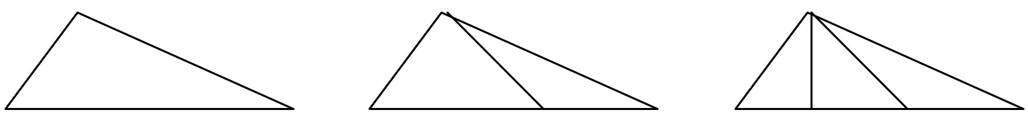
22. One circle divides a plane into two regions, two circles into four regions, three circles into ___ regions, four circles into ___ regions, and n circles into ___ regions.



23. Find positive integers x and y such that $\frac{1}{x}+\frac{1}{y}=\frac{1}{12}$.

24. Triangular Paper Folding Directions:
- Cut out a triangular piece of paper.
- Fold the triangle through one of its vertices.
- Fold the triangle through the same vertex many times.
- Each time you fold the paper, record the number of triangles formed by the folds.

Let N = number of folds; T = number of triangles formed by the folds.



0 folds
1 triangle

1 fold
3 triangles

2 folds
6 triangles

a) What pattern did you discover?
b) Find a formula in terms of N that will yield the Nth value of T.

25. You already know how to find the sum of the first n triangular numbers:

1	=	1
1 + 2	=	3
1 + 2 + 3	=	6
1 + 2 + 3 + 4	=	10, or $S_n = \frac{n(n+1)}{2}$.

Similarly, you know how to find the sum of the first n square numbers:

1^2	=	1
$1^2 + 2^2$	=	5
$1^2 + 2^2 + 3^2$	=	14
$1^2 + 2^2 + 3^2 + 4^2$	=	30, or $S_n = \frac{n(n+1)(2n+1)}{6}$.

a) Make a sequence of numbers using consecutive cubes.
b) Write a formula for the sum of n consecutive cubes.

ALGEBRA I / PROBABILITY / STATISTICS SOLUTIONS

1. *Solution of two variable linear equations; write and solve linear equations given real-life situations.* (H-4.2.1)

   Method #1: At first glance, this seems like a lot of guessing and checking. With 100 choices for dollars and 100 choices for cents, I would have to check out 10,000 possibilities. With random guessing it soon becomes apparent that the amount of cents must be more than double the number of dollars, and the number of dollars must be less than 50. After more tries, I realized that the only way the units digit can possibly work out is if the units digit of the dollars is 1, and the units digits of the cents is 3. (11 - 5 = 6, and 3 doubled is 6.) Thus, the only possibilities would be \$11.23, \$21.43, \$31.63, or \$41.83. The only one that works out is \$31.63.

   Method #2: Let d = # of dollars, c = # of cents. The problem stated that $100c + d - 5 = 2(100d + c)$, which simplifies to $98c - 5 = 199d$. The first observation in method #1 gives us that $c = 2d + 1$. Solving simultaneously, we get $d = 31$, $c = 63$.

2. *Concept of ratio and proportion in solving equations.* (H-1.3.4)
   25.86 " 20 :: x : 50. X = \$64.65

3. *Sequences and series*. 25, 85, 145, … (H-1.1.2)

4. *Factorials; combinations / permutations.* (H-1.2.2, H-1.2.5)
   Seating arrangements = 30! = 2.65×10^{32}.

   Earth's volume = $\dfrac{10 drops}{1\cancel{cm}^3} \times \dfrac{10^5 \cancel{cm}^3}{1\cancel{m}^3} \times \dfrac{10^9 \cancel{m}^3}{1\cancel{km}^3} \times \dfrac{\frac{4}{3} p (6370)^3 \cancel{km}^3}{1 earth} = 1.08 \times 10^{28}$
   drops - and that's if the whole world were water.

5. *Sequences and series.* (H-1.2.4) Any square number is the sum of the corresponding triangular number and the preceding triangular number.

   $\sum_{k=1}^{n}(2k-1)$ = 1 + 3 + 5 + … + (2n - 1) = n^2

6. *Non-linear equations.* (H-4.3.5) $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{x+y}{xy} = \dfrac{2}{3}$.

7. *Solution of one-variable equations.* (H-4.2.1) There are two possibilities: ± (11, 12, 13, 14, 15, 16).

8. *Probability*. (H-3.2.4) $\dfrac{5}{12}$

9. *Geometric sequences*. (H-1.2.4) 2^{39}

10. *Use operations on real numbers such as…power*. (H-1.2.2) $3n^9 + 81m^6$, then n=3.

11. *Systems of equations*. (H-4.2.3) $x = -\frac{1}{2}$ and $y = -1$

12. *Variables and constants in equations; solve a variety of equations*. (H-4.2.1)
$x + x + x/2 + 1\frac{1}{2} = 24$ $\qquad$ $x = 9$

13 *Variables and constants in equations; quadratic equations; solve a variety of equations; use operations on real numbers.* (H-4.2.6)
The expression $[3(230 + t)]^2$ is divisible by 9 and hence the number 492, a04 must also be divisible by 9. By the divisibility test for 9, this implies that a is 8. Substituting and solving the equation yields the value of 4 for t.

14. *Use operations on real numbers (divisibility); prime/composite numbers (grade8).* (M-1.2.4, H-4.2.7)
Rewrite the expression in factored form:

$$n(n^4 - 5n^2 + 4) = n(n + 1)(n - 1)(n + 2)(n - 2)$$

This is the product of five consecutive integers and hence it is divisible by five as well as three. It is also divisible by eight since one of the integers is divisible by four and a different one is divisible by two. Since 3, 5, and 8 are relatively prime, we know that N is divisible by 3x5x8, or 120.

15. *Variables and constants in expressions and equations; the complex number system which includes counting numbers, whole numbers, integers.* (H-4.2.7)
Rewrite the left side of the equation as $x(x + 1) + y(y + 1)$. Since the product of two consecutive integers is even, we have the sum of two even numbers. This can never be equal to the odd number, 1997, and thus the solution set is empty.

16. *Generate sequences and series; geometric means; variables and constants in equations.* (H-1.1.2, H-2.1.4, H-4.2.6)
Let the middle term of the three be x and let the common ratio be r. Then we have:

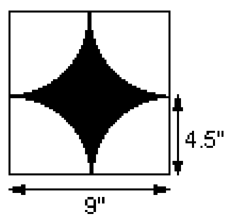
$$x\left(\frac{1}{r}+1+r\right)=21 \quad \text{and} \quad \frac{1}{x}\left(\frac{1}{r}+1+r\right)=\frac{7}{12}$$

Dividing the first equation by the second, we obtain $x^2 = 36$, and thus $x = 6$ or -6. Using $x = 6$, we obtain $r = 2$ or $r = \frac{1}{2}$. This in turn implies that the three integers are 3, 6, 12, or 12, 6, 3. If we let $x = -6$, we arrive at irrational values of r.

17. *Systems of equations.* (H-4.2.3)
    Add and then subtract the two equations and we arrive at the equivalent system:
    $10000x + 10000y = 50000$
    $3502x - 3502y = 3502$
    Divide through by 10000 and by 3502, respectively, and arrive at
    $x + y = 5$
    $x - y = 1$
    Solve to find that $x = 3$ and $y = 2$.

18. *The complex number system which includes counting numbers, whole numbers, integers, …; variables and constants in expressions.* (H-1.2.2)
    Call the integers $x - 1$, x, and $x + 1$. The squares will then be $x^2 - 2x + 1$, x^2, and $x^2 + 2x + 1$, yielding a sum of $3x^2 + 2$. X^2 must end in 0, 1, 4, 5, 6, or 9. This means that the sum of the squares could end in 0, 2, 5, 6, or 7 and will never end in 1, 3, 4, 8, or 9.

19. *Statistical models and probability simulations; design and conduct experiments and interpret the results; representing probability in different ways, including area models.* (H-3.2.5, H-3.2.4)



The above square represents a nine-inch piece of tile. The dark area represents the area that the center of the pie plate (with a nine-inch diameter) must lie within so that none of the plate will cover a corner. The rest of the square represents the area that the center of the plate must be within to cover a corner.

Light area = $4\left[\frac{1}{4}p(4.5)^2\right] = \frac{81}{4}p \approx 63.617$

Square area = $9^2 = 81$

Probability that a corner is covered when the plate is dropped is:
63.617 ÷ 81, or approximately 0.785. Thus, the experiment should yield 7 or 8 occurrences, out of ten, in which a corner is covered by the pie plate.

The exact value for the probability is, of course, $\frac{p}{4}$.

20. *Statistical models and probability simulations; design and conduct experiments and interpret the results.* (H-1.2.5, H-3.2.6)

a) $\begin{pmatrix}7\\8\end{pmatrix}^{40} \approx 0.00479$. The probability of choosing other than chocolate for any wafer is 7/8. This must happen forty consecutive times.

b) $\begin{pmatrix}8\\7\end{pmatrix} \bullet \begin{pmatrix}7\\8\end{pmatrix}^{40} - \begin{pmatrix}8\\6\end{pmatrix} \bullet \begin{pmatrix}6\\8\end{pmatrix}^{40} \approx 0.038$

c) 28 x (6/8) $\approx$ 0.000282. There are 28 different forty combinations of six flavors and each has a probability of 0.75. (Note: 0.038 … = 8(0.00479) - 0.000282).

21. *Statistical models and probability simulations; counting techniques; how properties of geometric shapes relate to each other.* (H-1.2.5, H-2.3.1, H-3.2.5)

Consider the triangle inequalities:

$a + b > c$
$b + c > a$
$a + c > b$

The total = $\begin{pmatrix}9\\3\end{pmatrix}$ = 84. (9 straws are taken three at a time.) Since there are only integral lengths and no two straws have the same length, no triangle will have a side of 1. Consider all of the combinations of sides:

One side measures	Other sides measure
2	(9,8), (8,7), (7,6), (6,5), (5,4), (4,3)
3	(9,8), (8,7), (7,6), (6,5), (5,4), (9,7), (8,6), (7,5), (6,4)
4	(9,8), (8,7), (7,6), (6,5), (9,7), (8,6), (7,5), (9,6), (8,5)
5	(9,8), (8,7), (7,6), (9,7), (8,6), (9,6)
6	(9,8), (8,7), (9,7)
7	(9,8)

A total of 34 combinations.
Then the probability is 34/84 $\approx$ 0.4.

22. *Represent patterns using functions.* (H-4.2.4)

Three circles divide a plane into eight regions, four circles divide a plane into 14 regions, five circles divide a plane into 22 regions, and n circles divide a plane into $n^2 - n + 2$ regions.

1	2	3	4	5	n
2	4	8	14	22	$n^2 - n + 2$

23. *Variables and constants in expressions and equations; solve a variety of equations.* (H-4.2.4) $(x,y) \in$ {(13,156), (156,13), (14,84), (84,14), (15,60), (60,15), (16,48), (48,16), (18,36), (36,18), (20,30), (30,20), (21,28), (28,21), (24,24)}

24. *Represent patterns using functions; how formulas, tables, graphs, and equations of functions relate to each other, generate sequences and series.* (H-1.1.2, H-1.2.4)
    a. The triangular numbers, i.e., $T = 1, 3, 6, 10, 15, 21, \ldots$
    b. $T = \dfrac{N(N+1)}{2}$

25. *Represent patterns using functions; how formulas, tables, graphs, and equations of functions relate to each other, generate sequences and series.* (H-1.2.4)
    a.

1^3	=	1	=	1^2
$1^3 + 2^3$	=	9	=	3^2
$1^3 + 2^3 + 3^3$	=	36	=	6^2
$1^3 + 2^3 + 3^3 + 4^3$	=	100	=	10^2

    b. $S_n = \dfrac{n^2(n+1)^2}{4}$

GEOMETRY OPEN-RESPONSE QUESTIONS

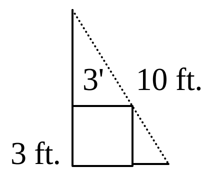
To the Teacher:

This set of questions is intended to be used with students for practice, i.e., questions can be given for students to work with in small groups or for homework. After completion of individual questions, using the answering model that has been established in the classroom, each solution and solution strategy should be discussed among members of the entire class. Many of the questions are more difficult geometry questions than students will see on the KCCT, but the strategies they learn in answering these questions should be similar.

It is not intended that the questions be given as a block, but that there be ample opportunities for discussion between questions. It is not necessary that all questions be asked or that they are asked in any particular order. It should be your decision as to which questions to use with your students.

The solution key includes a cross-reference to the Core Content for Assessment descriptor that is being addressed for each question. It is suggested that, even if you don't use all questions, you select questions from a variety of different descriptors.

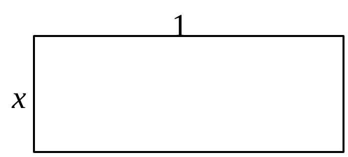
1. A 10-foot ladder leaning against a vertical wall just touches the corner of the box shown. How far up the wall does the ladder reach?



2. Liz said that she would take Jill to lunch if she could find the only two rectangles whose dimensions are integers and whose area and perimeter equal the same number. What should Jill's answer be?

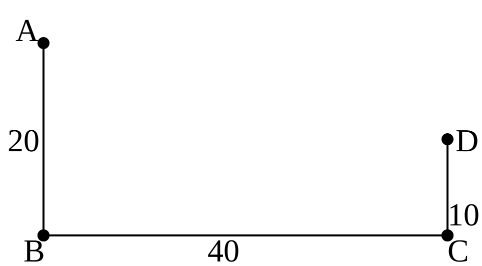
3. What percent of the area of a circle is enclosed by an isosceles triangle one of whose sides is the diameter of the circle?

4. Find x so that a vertical line that cuts the rectangle into two shapes yields (1) a square and (2) a rectangle that is similar to the original one.



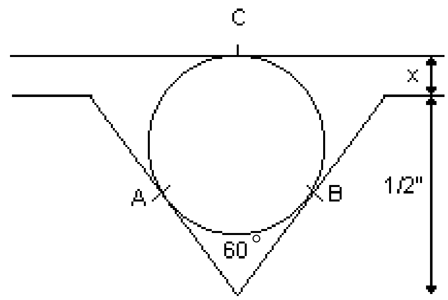
5. Given two equilateral triangles, find a third one whose area is the sum of the other two.

6. Tim must get from point A to point D and must touch some point between B and C along the way. (For example, he could go from A to B to D; or A to the midpoint between B and C and then to D.) What is the shortest length of such a journey satisfying these conditions?

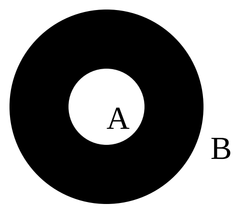


7. The end points of a diameter of a circle are (3, 9) and (11, 3). A triangle inscribed in this circle has two of its vertices at the given points. Find the coordinates of all points at which the third vertex of the triangle can be located for this triangle to have its maximum possible area.

8. If A, B, and C are points of tangency and the diameter of the circle is 3/8 inch, find x.



9. Segment AB, which is one inch long, is tangent to the inner of two concentric circles at A and intersects the outer circle at B. What is the area of the annular region (the ring) between the circles?



10. A running track is made up of two concentric circles. If the track is three meters wide, what is the difference in the two circumferences, i.e., the difference in running on the outside lane versus the inside lane?

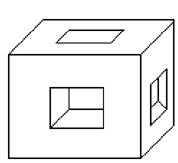
11. A recent advertisement read:

> **Waterfront Property**
> This triangular piece of property has an 895 foot river frontage and a 772 foot road frontage. The remaining 925 foot side is shared with a lumber yard.

What is the acreage of this property? (1 acre = 43,560 sq. ft.) Round to the nearest tenth acre.

12. A ball was floating in a lake when the lake froze. Someone removed it later (without breaking the ice), leaving a hole 24 cm across at the top and 8 cm deep. What was the diameter of the ball?

13. A wooden cube has edges 3 meters in length. Square holes with sides one meter in length and centered in each face are cut through to the opposite face with the edges of the holes parallel to the edges of the cube. What is the entire surface area, including the inside?



14. A semi-elliptical tunnel whose base has a width of 6 meters has to be of such a height that a truck with a height of 4 meters and width of 2 meters will just fit through it. What is the height of the tunnel?

15. Suppose you have a rectangular piece of paper whose dimensions are m x n, where $m>n$. The paper may be rolled to form a cylinder in two simple ways:
    - The edges of length m may be brought together and taped, or
    - The edges of length n may be brought together and taped.

    What is the ratio of the volume of the two cylinders? Which cylinder will have the greatest volume?

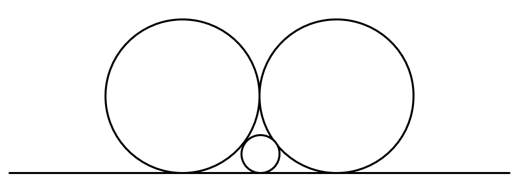
16. A cylindrical can of constant volume has its height increasing at a constant rate of 25% per day. At what rate will the radius be changing?

17. Find the distance between the parallel sides of a regular octagon of side s.

18. A horse is tethered to a rope at one corner of a square corral (outside of the corral) that is 10 feet on each side. The horse can graze at a distance of 18 feet from the corner of the corral where the rope is tied. What is the total grazing area for the horse?

19. Triangle ABC is isosceles with base AC. Points P and Q are respectively on CB and AB such that $AC = AP = PQ = QB$. What is the measure of angle B?

20. Two logs sit side by side so that they are tangent to the ground. Obviously there is enough room between the two logs to place another small log, also tangent to the ground. If the two larger logs are eight inches in radius, what is the radius of the smaller log? Generalize so that if the radius is x for the two larger logs, what is the radius of the smaller log?



GEOMETRY SOLUTIONS

1. *Pythagorean theorem.* (H-2.2.4) $(x + 3)^2 + (y + 3)^2 = 100$

   Similar triangles. (H-2.2.3) $\frac{x}{3} = \frac{3}{y} \Rightarrow y = \frac{9}{x}$

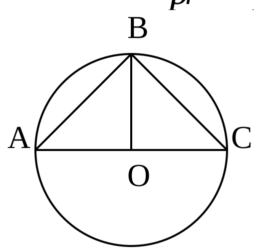
   \ $x^4 + 6x^3 - 82x^2 + 54x + 81 = 0 \Rightarrow x = 5.92$ or 1.52. Then the ladder reaches 8.92 or 4.52 feet up the wall.

2. *Perimeter and area.* (H-2.3.1) $ab = 2a + 2b = 2(a + b)$

a	*b*	*ab*	*2a + 2b*
1	1	1	4
2	2	4	4 + 4 = 8
4	4	16	8 + 8 = 16
6	3	18	12 + 6 = 18

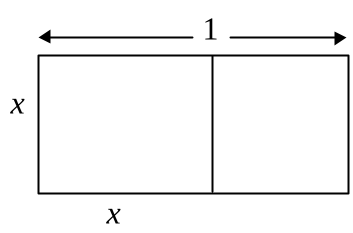
Thus, the only rectangles are a 4 x 4 and a 6 x 3.

3. *Area.* (H-2.3.1) Area of circle 0 is πr^2. Area of ΔABC is $\frac{1}{2}(2r)\ r = r^2$. The percent of the area of the circle is $\frac{r^2}{pr^2} = \frac{1}{p} \approx 0.318 = 31.8\%$.



4. *Proportions including geometric mean.* (H-2.1.4)

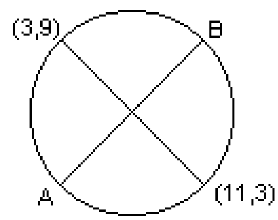
   $\frac{1}{x} = \frac{x}{1-x} \Rightarrow x = \frac{-1+\sqrt{5}}{2} \approx 0.618$ (A golden rectangle.)



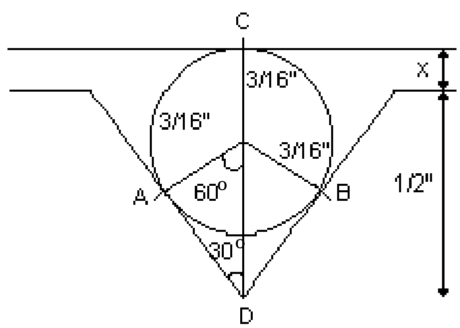
5. *Pythagorean theorem.* (H-2.2.4) The third equilateral triangle can be found by using a side of each given triangle as a leg in a right triangle. The result will be the triangle on the hypotenuse.

6. *Reflections; Pythagorean theorem.* (H-2.1.1, H-2.2.1, H-2.2.4) Reflect point D through line BC down to point D'. Going from point A to the line BC and then to the point D is the same as going from A to BC to D'. Since the shortest distance from A to D' is a straight line, we can use the Pythagorean theorem to find AD' = 50.

7. *Ratio measures such as slope; how algebraic procedures and geometric concepts are related.* (H-2.1.4, H-2.3.2) There are two locations for the third vertex — the points of intersection of the perpendicular bisector of the diameter and the circle, i.e., points A and B on the following diagram. The center of the circle is $\left[\left(\frac{3+11}{2}\right),\left(\frac{9+3}{2}\right)\right]$ = (7,6), the slope of the diameter is $\frac{9-3}{3-11} = -\frac{3}{4}$. So the slope of the perpendicular bisector is $\frac{4}{3}$. The two locations are A = (4, 2) and B = (10, 10).



8. *Use Pythagorean theorem (30/60/90 right triangle); how ratios relate to right triangles.* (H-2.2.4, H-2.1.4) $\frac{1}{16}$"



$\angle$ AOD = 60°, $\angle$ ADO = 30°, CO = 3/16" = AO = BO; DO = 6/16"

9. *Use Pythagorean theorem; how algebraic procedures and geometric concepts are related.* (H-2.2.4, H-2.3.1)
   The center of the circle and points A and B form a right triangle with the radius of the inner circle being the length of one side and the radius of the outer circle the hypotenuse. Let r be the radius of the inner circle and R be the radius of the outer circle. Then $R^2 = r^2 + 1$, or $R^2 - r^2 = 1$. The area of the ring is
   $\pi R^2 - \pi r^2 = \pi\ (R^2 - r^2) = \pi \bullet 1 = \pi$.

10. *How properties of geometric shapes relate to each other.* (H-2.3.1)
    6π

11. *How algebraic procedures and geometric concepts are related.* (H-2.3.1)

Using Heron's formulas, A = $\sqrt{s(s-a)(s-b)(s-c)}$ where s = $\dfrac{a+b+c}{2}$ = 1296 feet. Then the area is 317,854 square feet. Converting this to acres, we arrive at 7.3 acres.

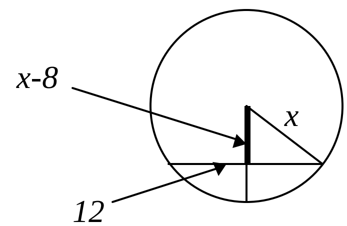
12. *Using Pythagorean theorem.* (H-2.2.4)

Assuming the ball floated with its center above the water (since it was removed without breaking the ice):

$(x - 8)^2 + 12^2 = x^2$

$x = 13$

\ Diameter = 26 cm



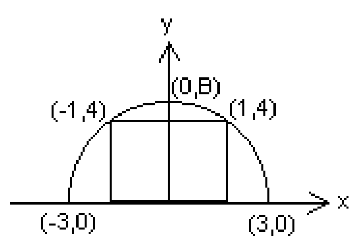
13. *Properties of geometric shapes.* (H-2.2.6)

Each exterior unit square that is removed exposes 4 interior unit squares, so the entire surface area is:

$6 \times 3^2 - 6 \times 1 + 6 \times 4 = 72\ m^2$

14. *How equations, lines, and curves are models of the relationship between two real world quantities; how algebraic procedures and geometric concepts are related; how position in the plane can be represented using rectangular coordinates.* (H-2.3.2)

Using a Cartesian coordinate system, the equation for the ellipse is $\dfrac{x^2}{3^2} + \dfrac{y^2}{B^2} = 1$



Substituting the value (1, 4): $\dfrac{1}{9} + \dfrac{16}{B^2} = 1 \Rightarrow B = 3\sqrt{2} \approx 4.24 \textit{meters}$

15. *How algebraic procedures and geometric concepts are related; construct geometric figures.* (H-2.2.6)

For cylinder 1: Height = m and circumference = n. Since $2pr = n$, $r = \frac{n}{2p}$.

Thus, $v_1 = p\left(\frac{n}{2p}\right)^2 m = \frac{mn^2}{4p}$.

For cylinder 2: Height = n and circumference = m. Since $2pr = m$, $r = \frac{m}{2p}$.

So $v_2 = p\left(\frac{m}{2p}\right)^2 n = \frac{m^2 n}{4p}$.

Then, $\frac{v_1}{v_2} = \frac{n}{m}$ and cylinder 2 has the greater volume.

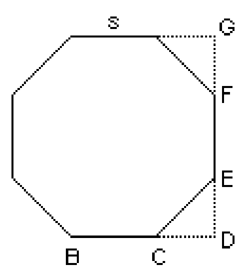
16. *Describe elements which change and elements which do not change under transformations; ratio measures such as rates.* (H-2.3.3)

-10.6%

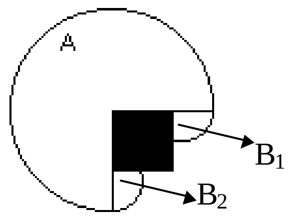
17. *How properties of geometric shapes relate to each other; use Pythagorean theorem.* H-2.2.4, H-2.2.2)

Extend sides $\overline{FE}$ and $\overline{BC}$ to D. ΔCDE is isosceles. So if $\overline{CE} = s$ and using the Pythagorean theorem, $\overline{CD} = \overline{ED} = \frac{1}{2}s\sqrt{2}$ and

$\overline{DG} = \overline{DE} + \overline{EF} + \overline{FG} = \frac{1}{2}s\sqrt{2} + s + \frac{1}{2}s\sqrt{2} = s + s\sqrt{2}$.



18. *How properties of geometric shapes relate to each other.* (H-2.3.1)
    Area of A $= \frac{3}{4}p\,(18)^2$
    Area of $B_1 = \frac{1}{4}p\ (8)^2$
    Area of $B_2 = \frac{1}{4}p\ (8)^2$
    Total area $\approx$ 863.6 sq. ft.



19. *How algebraic procedures and geometric concepts are related.* (H-2.1.3)
    ΔABC is *m Ð B + 3m Ð B + 3m Ð B = 180°*

    So, $m\angle B = \left(\frac{180}{7}\right)^{\circ}$

20. (H-2.1.4)
    Radius of smaller log is 2. In general, the radius of the smaller log is 1/4 the radius of the larger logs.