Mean-Value Estimates for the Derivative of the Riemann Zeta-Function

by

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To Qin for her love and support.

Curriculum Vitae

The author was born on February 14, 1979 in rural southwest Wisconsin. He attended high school at River Valley High School in Spring Green, Wisconsin and graduated in 1997 as the class valedictorian. He then enrolled at the University of Rochester where he graduated Summa cum Laude with a B.A. in Mathematics in 2001 and with an M.A. in Mathematics in 2002.

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Abstract

Let $\zeta(s)$ denote the Riemann zeta-function. This thesis is concerned with estimating discrete moments of the form

$$J_k(T) = \frac{1}{N(T)} \sum_{0<\gamma\leq T} \left|\zeta'(\rho)\right|^{2k},$$

where $k \in \mathbb{N}$ and the sum runs over the non-trivial (complex) zeros $\rho = \beta + i\gamma$ of $\zeta(s)$. Here the function $N(T) \sim \frac{T}{2\pi} \log T$ denotes the number of zeros of $\zeta(s)$ up to a height T counted with multiplicity. It is an open problem to determine the behavior of $J_k(T)$ as k varies. The main result of this thesis is to establish upper and lower bounds for $J_k(T)$ for each positive integer k that are very nearly of the same order of magnitude. In particular, assuming the Riemann Hypothesis we show that, for any $k \in \mathbb{N}$ and $\varepsilon > 0$ arbitrary, there exist positive constants $C_1 = C_1(k)$ and $C_2 = C_2(k, \varepsilon)$ such that the inequalities

$$C_1(\log T)^{k(k+2)} \leq J_k(T) \leq C_2(\log T)^{k(k+2)+\varepsilon}$$

hold when T is sufficiently large. The lower bound for $J_k(T)$ was proved jointly with Nathan Ng.

Two related problems are also considered. Assuming the Riemann Hypothesis S. M. Gonek has shown that $J_1(T) \sim \frac{1}{12}(\log T)^3$ as $T \to \infty$. As an application

of the L-functions Ratios Conjectures, J.B. Conrey and N. Snaith made a precise conjecture for the lower-order terms in the asymptotic expression for $J_1(T)$. By carefully following Gonek's original proof, we establish their conjecture.

The other problem is related to the average of the mean square of the reciprocal of $\zeta'(\rho)$. It is believed that the zeros of $\zeta(s)$ are all simple. If this is the case, then the sum $J_k(T)$ is defined when $k < 0$ and, for certain small values of k, conjectures exist about its behavior. Assuming the Riemann Hypothesis and that the zeros of $\zeta(s)$ are simple, we establish a lower bound for $J_{-1}(T)$ that differs from the conjectured value by a factor of 2.

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1 Introduction

Let $s=\sigma+it$. The Riemann zeta-function is the function of a complex variable s defined in the half-plane $\sigma>1$ by either the Dirichlet series or the Euler product

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{\substack{p \\ p \text{ prime}}} \left(1 - \frac{1}{p^s}\right)^{-1}. \tag{1.1}$$

It is defined by analytic continuation elsewhere in the complex plane except for a simple pole at $s=1$. If we let $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$, then it can be shown that the function $\xi(s)$ is entire and satisfies the functional equation $\xi(s) = \xi(1-s)$.

In his classic 1859 paper, Riemann [28] showed there is a deep connection between the behavior of the function $\zeta(s)$ and the distribution of the prime numbers. In fact, he sketched a proof of a formula that expresses the number of primes less than a number x explicitly in terms of the zeros of the zeta-function.

Every zero of $\zeta(s)$ corresponds to either a zero of $\xi(s)$ or a pole of $\Gamma(s)$. From well known properties of the gamma function, one can deduce that $\zeta(-2k) = 0$ for each positive integer k and, from (1.1) and the functional equation, that these are the only zeros of $\zeta(s)$ on the real-axis. We call these zeros the trivial zeros.

It is also known that $\zeta(s)$ has a countably infinite number of non-trivial (complex) zeros. We write these zeros as $\rho = \beta + i\gamma$. It can be shown that the non-trivial zeros of $\zeta(s)$ satisfy $0 \le \beta \le 1$ and that they are symmetric about the real-axis and about the line $\sigma = \frac{1}{2}$. Moreover, it is known that the number of non-trivial zeros with $0 < \gamma \le T$ is about $\frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi}$ as $T \to \infty$. Riemann famously conjectured that all the non-trivial zeros have $\beta = \frac{1}{2}$. This conjecture, known as the "Riemann Hypothesis", is widely regarded as one of the most important open problems in pure mathematics.

For an overview of the theory of the Riemann zeta-function, its connection to number theory, and related topics we refer the reader to the books by Ingham [18], Davenport [3], Titchmarsh [37], Edwards [4], Ivič [19], Iwaniec & Kowalski [20], and Montgomery & Vaughan [23] as well as the references contained within these sources.

1.1 Moments of $|\zeta'(\rho)|$.

This thesis is concerned with estimating discrete moments of the form

$$J_k(T) = \frac{1}{N(T)} \sum_{0<\gamma\le T} \left|\zeta'(\rho)\right|^{2k}, \tag{1.2}$$

where k is a real number and the sum runs over the non-trivial (complex) zeros $\rho{=}\beta{+}i\gamma$ of the Riemann zeta-function. Here, the function

$$N(T) = \sum_{0<\gamma\le T} 1 = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T)$$

denotes the number of zeros of $\zeta(s)$ up to a height T counted with multiplicity.

Initially, $J_k(T)$ is only defined for positive values of k. However, it is believed that the zeros of the Riemann zeta-function are all simple and, if this is the case, then $J_k(T)$ is defined for $k < 0$, as well.

It is an open problem to determine the behavior of $J_k(T)$ as k varies. Independently, Gonek [10] and Hejhal [14] have made the following conjecture.

Conjecture 1: *There exist positive constants A_k and B_k such that*

$$A_k(\log T)^{k(k+2)} \leq J_k(T) \leq B_k(\log T)^{k(k+2)} \tag{1.3}$$

for each fixed $k \in \mathbb{R}$ and T sufficiently large.

Though widely believed for positive values of k, there is evidence to suggest that this conjecture is false for $k \leq -3/2$. In fact, it is expected that there exist infinitely many zeros ρ of $\zeta(s)$ such that $|\zeta'(\rho)|^{-1} \geq |\gamma|^{1/3-\varepsilon}$ for each $\varepsilon > 0$. If this is the case, then there are arbitrarily large numbers T_ε such that

$$J_{-k}(T_\varepsilon) \geq (T_\varepsilon)^{2k/3-1-\varepsilon}.$$

When $k < -3/2$, this is not consistent with the upper bound in (1.3).

Until very recently, for $k > 0$, estimates in agreement with Conjecture 1 were only known in a few cases where k is small. Assuming the Riemann Hypothesis, which asserts that $\beta = \frac{1}{2}$ for each non-trivial zero $\rho = \beta + i\gamma$ of $\zeta(s)$, Gonek [8] has proved that

$$J_1(T) \sim \frac{1}{12}(\log T)^3 \quad \text{as } T \to \infty.$$

No asymptotic formula is known for $J_k(T)$ for any other value of k. Also assuming the Riemann Hypothesis, Ng [25] was able to exhibit positive constants D_1 and D_2 such that, for T sufficiently large,

$$D_1(\log T)^8 \leq J_2(T) \leq D_2(\log T)^8.$$

This establishes Conjecture 1 for the case $k = 2$.

There are a few related unconditional results where the sum in (1.2) is restricted only to the zeros of $\zeta(s)$ with $\beta = \frac{1}{2}$. In this case, Garaev [5] implicitly (see also Laurinčikas, Šlezevičiene, and Steuding [21; 34]) has shown that

$$\frac{1}{N(T)} \sum_{\substack{0<\gamma\le T \\ \beta=1/2}} |\zeta'(\rho)| \le C(\log T)^{5/4}$$

for a positive constant C. This result relies on a deep mean-value estimate for Dirichlet series due to Ramachandra [27]. Also, on this restricted set of zeros, Garunkštis and Steuding [6] have proved upper bounds of the conjectured order of magnitude for the mean-square and for the mean-fourth power of $\zeta'(\rho)$. In fact, by modifying a recent method of R. R. Hall [13], they give upper bounds for the more general quantity

$$J_k(T;\lambda) = \frac{1}{N(T)} \sum_{\substack{0<\gamma\le T,\ \beta=\frac{1}{2} \\ (\gamma^+-\gamma)\log T\le\lambda}} |\zeta'(\rho)|^{2k}$$

for any fixed $\lambda > 0$ when $k = 1$ or 2 where γ^+ is defined by

$$\gamma^+ := \min\{\gamma' : \zeta(\tfrac{1}{2} + i\gamma') = 0 \text{ and } \gamma' > \gamma\}.$$

If the Riemann Hypothesis is assumed, then these unconditional results can be applied to the sum $J_k(T)$ and, in that case, the value of the constant in the upper bound for $J_2(T)$ obtained by Garunkštis and Steuding improves slightly the value of the constant in Ng's upper bound for $J_2(T)$ mentioned above.

Very little is known about the moments $J_k(T)$ when $k > 2$. However, assuming the Riemann Hypothesis it can be deduced from well-known results of Littlewood (Theorems 14.14A-B of Titchmarsh [37]) that for $\sigma \ge 1/2$ and $t \ge 10$ the inequality

$$\left|\zeta'(\sigma+it)\right| \le \exp\left(\tfrac{A\log t}{\log\log t}\right)$$

holds for some constant $A > 0$. Assuming the Riemann Hypothesis, for $k > 0$, it immediately follows that

$$J_k(T) \le \exp\left(\tfrac{2kA\log T}{\log\log T}\right). \tag{1.4}$$

Non-trivial lower bounds for $J_k(T)$ can be obtained from the results mentioned above. For instance, if we apply Hölder's inequality to Ng's lower bound for $J_2(T)$ we can show that, for $k \ge 2$,

$$J_k(T) \ge B(\log T)^{4k} \tag{1.5}$$

for some positive constant $B = B(k)$. The main goal of this thesis is to improve the estimates in (1.4) and (1.5) by obtaining conditional upper and lower bounds for $J_k(T)$ that are very near the conjectured order of magnitude. In particular, we prove the following result.

Theorem 1.1.1. *Assume the Riemann Hypothesis. Let k be a positive integer and let $\varepsilon > 0$ be arbitrary. Then, there exist positive constants $C_1 = C_1(k)$ and $C_2 = C_2(k, \varepsilon)$ such that inequalities*

$$C_1(\log T)^{k(k+2)} \le \frac{1}{N(T)} \sum_{0<\gamma\le T} \left|\zeta'(\rho)\right|^{2k} \le C_2(\log T)^{k(k+2)+\varepsilon} \tag{1.6}$$

hold when T is sufficiently large.

This result lends strong support for Conjecture 1 in the case when $k \in \mathbb{N}$ and, up to the factor $(\log T)^{\varepsilon}$ in the upper bound, establish the appropriate size of these moments.

1.2 Negative Moments of $|\zeta'(\rho)|$

It is generally believed that the zeros of $\zeta(s)$ are all simple. If this is the case, then the moments

$$J_{-k}(T) = \frac{1}{N(T)} \sum_{0<\gamma\le T} \frac{1}{|\zeta'(\rho)|^{2k}}$$

are defined for $k > 0$. Through the work of Titchmarsh (Chapter 14 of [37]), Gonek (unpublished), and Ng [24], it is known that the behavior of these sums are intimately connected with the distribution of the summatory function

$$M(x) = \sum_{n\le x} \mu(n)$$

where $\mu(\cdot)$, the Möbius function, is defined by $\mu(1) = 1$, $\mu(n) = (-1)^k$ is n is divisible by k distinct primes, and $\mu(n) = 0$ if $n > 1$ is not square-free.

Using a heuristic method that is similar to Montgomery's study [22] of the pair-correlation of the imaginary parts of the zeros of $\zeta(s)$, Gonek [12] conjectured that

$$\sum_{0<\gamma\le T} \frac{1}{|\zeta'(\rho)|^2} \sim \frac{3T}{\pi^3} \quad \text{as } T \to \infty. \tag{1.7}$$

A completely different heuristic method due to Hughes, Keating, and O'Connell (discussed in the next section) leads to the same conjecture. As evidence for his conjecture, assuming the Riemann Hypothesis and that the zeros of $\zeta(s)$ are all simple, Gonek [10] has proven that

$$\sum_{0<\gamma\le T} \frac{1}{|\zeta'(\rho)|^2} \ge CT$$

for some constant $C > 0$ and T sufficiently large. In Chapter 4 of this thesis, we quantify this result by proving the following theorem.

Theorem 1.2.1. *Assume the Riemann Hypothesis and that the zeros of $\zeta(s)$ are all simple. Then the inequality*

$$\sum_{0<\gamma\le T} \frac{1}{|\zeta'(\rho)|^2} \ge \left(\frac{3}{2\pi^3} - \varepsilon\right) T$$

holds for $\varepsilon > 0$ arbitrary and T sufficiently large.

Our result provides a lower bound for $J_{-1}(T)$ that differs from Gonek's conjecture by a factor of 2. Combining the method used to prove Theorem 1.2.1 with the method used to prove the lower bound in Theorem 1.1.1, we could likely show that, for each positive integer k,

$$\frac{1}{N(T)} \sum_{0<\gamma\le T} \frac{1}{|\zeta'(\rho)|^{2k}} \ge C_k (\log T)^{k(k-2)}$$

where $C_k > 0$ is a constant depending on k. However, as mentioned above, this may not be the correct size of these moments when $k \ge 2$.

1.3 More Precise Conjectures for Moments of $|\zeta'(\rho)|$.

By using a random matrix model to study the behavior of the Riemann zeta-function and its derivative on the critical line, Hughes, Keating, and O'Connell [16] have refined Conjecture 1 above and formulated the following conjecture.

Conjecture 2: *For fixed $k > -3/2$,*

$$J_k(T) = \frac{1}{N(T)} \sum_{0<\gamma\le T} |\zeta'(\rho)|^{2k} \sim \frac{G^2(k+2)}{G(2k+3)} \cdot a(k) \cdot (\log \tfrac{T}{2\pi})^{k(k+2)}$$

as $T \to \infty$ where $G(\cdot)$ is the Barne's G-function and

$$a(k) = \prod_{p \text{ prime}} \left(1 - \frac{1}{p}\right)^{k^2} \sum_{m=0}^{\infty} \left(\frac{\Gamma(m+k)}{m!\, \Gamma(k)}\right)^2 p^{-m}.$$

It can be shown that $a(0) = a(1) = 1$ and that $a(-1) = a(2) = \frac{6}{\pi^2}$. Using these values of $a(k)$, we can check that Conjecture 2 agrees with the previously known and conjectured results. In particular, it agrees with the observation that $J_0(T) = 1$, which is trivial, the calculation that $J_1(T) \sim \frac{1}{12}(\log T)^3$, which is a result of Gonek mentioned previously, and it also agrees with the conjecture of Gonek for $J_{-1}(T)$ given in equation (1.7) above.

Using the Ratios Conjectures of Conrey, Farmer, and Zirnbauer [1] it is possible to further refine the above conjecture for $J_k(T)$ when k is a positive integer. This is done in Section 7 of a recent paper by Conrey and Snaith [2].

Conjecture 3: *For each positive integer k, there exists a polynomials $\mathcal{P}_k(\cdot)$ of degree $k(k+2)$ and leading coefficient $\frac{G^2(k+2)}{G(2k+3)} \cdot a(k)$ such that*

$$\frac{1}{N(T)} \sum_{0<\gamma\leq T} |\zeta'(\rho)|^{2k} = \mathcal{P}_k\left(\log \tfrac{T}{2\pi}\right) + O\left(T^{-1/2+\varepsilon}\right)$$

for $\varepsilon > 0$ arbitrary and T sufficiently large.

Conrey and Snaith only explicitly work out the coefficients of the polynomials $\mathcal{P}_k(\cdot)$ in the cases $k=1$ and $k=2$, but their method should work for any positive integer k. As evidence for Conjecture 3, in Chapter 5 we prove the following theorem.

Theorem 1.3.1. *Assume the Riemann Hypothesis. Then Conjecture 3 is true in the case $k=1$.*

1.4 Notation

Interchangeably, we use Landau's big-O notation, $f(T)=O\big(g(T)\big)$, and Vinogradov's $\ll$ notation, $f(T) \ll g(T)$, to mean that there exists a positive constant C such that the inequality

$$\big|f(T)\big| \leq C \cdot g(T)$$

holds for T sufficiently large. All constants implied by the big-O or the $\ll$ notations are absolute, unless otherwise stated, in which case we write $f(T) \ll_A g(T)$ to indicate that the implied constant depends on a parameter A. Similarly, we use the expressions $f(T) \gg g(T)$ (respectively, $f(T) \gg_A g(T)$) to mean that

$$|f(T)| \geq C \cdot g(T) \quad \text{ as } T \to \infty,$$

where the implied constant is absolute (respectively, depends on A). In addition, we use Landau's little-o notation, $f(T) = o(g(T))$, to mean that

$$\lim_{T\to\infty} \frac{|f(T)|}{|g(T)|} = 0.$$

Equivalently, $f(T) = o(g(T))$ means that the inequality $|f(T)| \leq \varepsilon|g(T)|$ holds for $\varepsilon > 0$ arbitrary and T sufficiently large (depending on ε).

2 The Proof of the Lower Bound in Theorem 1.1.1

In this chapter we establish the lower bound in Theorem 1.1.1. In particular, we show for each positive integer k we have

$$\sum_{0<\gamma\leq T} \left|\zeta'(\rho)\right|^{2k} \gg T(\log T)^{(k+1)^2} \tag{2.1}$$

where the implied constant depends on k. This estimate was proved jointly with Nathan Ng. The proof of (2.1) relies on a recent method developed by Rudnick and Soundararajan [29; 30] to compute lower bounds for moments of central values of L-functions in families. The main tools used in our proof are a mean-value estimate of Ng (our Lemma 2.2.1) and a well-known lemma of Gonek (our Lemma 2.3.1). It is likely that the proof of (2.1) can be adapted to prove a lower bound for $J_k(T)$ of the conjectured order of magnitude for any rational value of k with $k \geq 1$.

Let $k \in \mathbb{N}$ and define, for $\xi \geq 1$, the function $\mathcal{A}_\xi(s) = \sum_{n\leq\xi} n^{-s}$. When $\xi \ll |\Im s|$, this is a "short" Dirichlet polynomial approximation to $\zeta(s)$. Assuming the Riemann Hypothesis, we will estimate

$$\Sigma_1 = \sum_{0<\gamma\leq T} \zeta'(\rho)\mathcal{A}_\xi(\rho)^{k-1}\overline{\mathcal{A}_\xi(\rho)}^{k} \quad \text{and} \quad \Sigma_2 = \sum_{0<\gamma\leq T} \left|\mathcal{A}_\xi(\rho)\right|^{2k}, \tag{2.2}$$

where the sums run over the non-trivial zeros $\rho = \frac{1}{2} + i\gamma$ of $\zeta(s)$. Hölder's inequality implies that

$$\sum_{0<\gamma\leq T} \left|\zeta'(\rho)\right|^{2k} \geq \frac{\left|\Sigma_1\right|^{2k}}{\left(\Sigma_2\right)^{2k-1}},$$

and so to prove (2.1) it suffices to establish the estimates

$$\Sigma_1 \gg_k T(\log T)^{k^2+2} \quad \text{and} \quad \Sigma_2 \ll_k T(\log T)^{k^2+1} \tag{2.3}$$

for a particular choice of ξ.

For computational purposes, it is convenient to express Σ_1 and Σ_2 slightly differently. Assuming the Riemann Hypothesis, $1-\rho = \bar{\rho}$ for any non-trivial zero ρ of $\zeta(s)$, so $\overline{\mathcal{A}_\xi(\rho)} = \mathcal{A}_\xi(1-\rho)$. This allows us to re-write the sums in (2.2) as

$$\Sigma_1 = \sum_{0<\gamma\leq T} \zeta'(\rho)\mathcal{A}_\xi(\rho)^{k-1}\mathcal{A}_\xi(1-\rho)^k \tag{2.4}$$

and

$$\Sigma_2 = \sum_{0<\gamma\leq T} \mathcal{A}_\xi(\rho)^k\mathcal{A}_\xi(1-\rho)^k. \tag{2.5}$$

It is with these representations of Σ_1 and Σ_2 that we establish the bounds in (2.3), and thus the estimate in (2.1).

2.1 Some Estimates for Sums of Divisor Functions

For each real number $\xi \geq 1$ and each $k \in \mathbb{N}$, we define the arithmetic sequence of real numbers $\tau_k(n;\xi)$ by

$$\sum_{n\leq\xi^k} \frac{\tau_k(n;\xi)}{n^s} = \Big(\sum_{n\leq\xi} \frac{1}{n^s}\Big)^k = \mathcal{A}_\xi(s)^k. \tag{2.6}$$

The function $\tau_k(n;\xi)$ is a truncated approximation to the arithmetic function $\tau_k(n)$ (the k-th iterated divisor function) which is defined by

$$\zeta^k(s) = \Big(\sum_{n=1}^{\infty} \frac{1}{n^s}\Big)^k = \sum_{n=1}^{\infty} \frac{\tau_k(n)}{n^s} \tag{2.7}$$

for $\Re s > 1$. We require a few estimates for sums involving the functions $\tau_k(n)$ and $\tau_k(n;\xi)$ in order to establish the bounds for Σ_1 and Σ_2 in (2.3). First, we require a simple estimate for the size of $\zeta(s)$ to the right of the critical line.

Lemma 2.1.1. *Let $s = \sigma + it$ and set $\tau = |t| + 3$. Then*

$$|\zeta(\sigma+it)| = O\big(\tau^{1-\sigma}\big)$$

uniformly for $1/2 \le \sigma \le 3$.

Proof. By Theorem 4.11 of Titchmarsh [37], we have

$$\zeta(s) = \sum_{n\le x} \frac{1}{n^s} + \frac{x^{1-s}}{1-s} + O\big(x^{-\sigma}\big)$$

uniformly for $\sigma \ge \sigma_0 > 0, |t| < 2\pi x/C$, where C is a given constant greater than 1. Now choose $x = \tau$. Then, for $1/2 \le \sigma \le 3$,

$$|\zeta(\sigma+it)| \le \sum_{n\le \tau} \frac{1}{n^\sigma} + O\big(\tau^{-\sigma}\big) = O\big(\tau^{1-\sigma}\big).$$

This proves the lemma. □

Lemma 2.1.2. *For $x \ge 3$ and $k, \ell \in \mathbb{N}$, we have*

$$\sum_{n\le x} \frac{\tau_k(n)\tau_\ell(n)}{n} \asymp (\log x)^{k\ell} \tag{2.8}$$

where the implied constants depend on k and ℓ.

Proof. While proving the lemma, we allow any implied constants to depend on k and ℓ. Let $s = \sigma + it$ and set $\tau = |t| + 3$. Also, we put $m = \lfloor k\ell/4 \rfloor + 2$ where, as usual, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. We begin by noticing that

$$\sum_{n\le x} \frac{\tau_k(n)\tau_\ell(n)}{n}\Big(1-\frac{\log n}{\log x}\Big)^m \le \sum_{n\le x}\frac{\tau_k(n)\tau_\ell(n)}{n} \le 2^m \sum_{n\le x^2}\frac{\tau_k(n)\tau_\ell(n)}{n}\Big(1-\frac{\log n}{2\log x}\Big)^m,$$

so the lemma will follow if we can show that

$$S_{k,\ell,m}(x) := \sum_{n\le x}\frac{\tau_k(n)\tau_\ell(n)}{n}\Big(1-\frac{\log n}{\log x}\Big)^m \asymp (\log x)^{k\ell}. \tag{2.9}$$

Our proof of (2.9) relies on the identity

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^s}{s^{m+1}}ds = \begin{cases} \frac{1}{m!}(\log x)^m, & \text{if } x\ge 1,\\ 0, & \text{if } 0\le x<1,\end{cases} \tag{2.10}$$

which is valid for $c > 0$ when m is a positive integer (which is certainly the case for our choice of m). To use (2.10), we let

$$F_{k,\ell}(s) = \sum_{n=1}^{\infty}\frac{\tau_k(n)\tau_\ell(n)}{n^s} = \prod_p\Big(1+\sum_{j=1}^{\infty}\tbinom{k+j-1}{j}\tbinom{\ell+j-1}{j}p^{-js}\Big); \tag{2.11}$$

the second equality follows by using standard properties of $\tau_k(n)$. This series and product converge absolutely for $\sigma > 1$. Thus, by (2.10),

$$\begin{aligned}\frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty}F_{k,\ell}(s+1)\frac{x^s}{s^{m+1}}ds &= \sum_{n=1}^{\infty}\frac{\tau_k(n)\tau_\ell(n)}{n}\left(\frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty}\Big(\frac{x}{n}\Big)^s\frac{ds}{s^{m+1}}\right)\\ &= \frac{1}{m!}\sum_{n\le x}\frac{\tau_k(n)\tau_\ell(n)}{n}(\log x/n)^m\\ &= \frac{(\log x)^m}{m!}S_{k,\ell,m}(x).\end{aligned} \tag{2.12}$$

An elementary algebraic manipulation of the infinite product in (2.11) shows that

$$F_{k,\ell}(s) = G_{k,\ell}(s)\zeta(s)^{k\ell} \tag{2.13}$$

where

$$G_{k,\ell}(s) = \prod_p \Big(1 - \frac{1}{p^s}\Big)^{k\ell} \Big(1 + \sum_{j=1}^{\infty} \tbinom{k+j-1}{j}\tbinom{\ell+j-1}{j} p^{-js}\Big).$$

Since this product converges absolutely for $\sigma > 1/2$, the formula in (2.13) provides a meromorphic continuation of $F_{k,\ell}(s)$ to the half-plane $\sigma > 1/2$. Now, by (2.12), (2.13), and the calculus of residues, we see that

$$\begin{aligned} \frac{(\log x)^m}{m!} S_{k,\ell,m}(x) &= \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} G_{k,\ell}(s+1)\zeta(s+1)^{k\ell}\frac{x^s}{s^{m+1}}ds \\ &= \operatorname*{Res}_{s=0}\Big\{G_{k,\ell}(s+1)\zeta(s+1)^{k\ell}\frac{x^s}{s^{m+1}}\Big\} \\ &\quad + \frac{1}{2\pi i}\int_{-1/4-i\infty}^{-1/4+i\infty} G_{k,\ell}(s+1)\zeta(s+1)^{k\ell}\frac{x^s}{s^{m+1}}ds; \end{aligned} \tag{2.14}$$

the contour shift is justified by the absolute convergence of the $G_{k,\ell}(s)$ for $\sigma > 1/2$, the estimate in Lemma 2.1.1, and our choice of m. A straightforward calculation shows that

$$\operatorname*{Res}_{s=0}\Big\{G_{k,\ell}(s+1)\zeta(s+1)^{k\ell}\frac{x^s}{s^{m+1}}\Big\} = \mathcal{P}_{k,\ell,m}(\log x) \tag{2.15}$$

where $\mathcal{P}_{k,\ell,m}(\cdot)$ is a polynomial of degree $k\ell + m$ with a positive leading coefficient. Also, by Lemma 2.1.1,

$$\int_{-1/4-i\infty}^{-1/4+i\infty} G_{k,\ell}(s+1)\zeta(s+1)^{k\ell}\frac{x^s}{s^{m+1}}ds \ll x^{-1/4}\int_{-\infty}^{\infty}\frac{|\zeta(\frac{3}{4}+it)|^{k\ell}}{\tau^{m+1}}dt \ll x^{-1/4}. \tag{2.16}$$

Inserting estimates (2.15) and (2.16) into (2.14) establishes (2.9). By our initial observation, this proves the lemma. □

Lemma 2.1.3. *For $\xi \geq 3$ and $k \in \mathbb{N}$ we have*

$$(\log\xi)^{k^2} \ll \sum_{n\leq\xi}\frac{\tau_k(n)^2}{n} \leq \sum_{n\leq\xi^k}\frac{\tau_k(n;\xi)^2}{n} \leq \sum_{n\leq\xi^k}\frac{\tau_k(n)^2}{n} \ll (\log\xi)^{k^2} \tag{2.17}$$

where the implied constants depend on k.

Proof. From equations (2.6) and (2.7) we notice that $\tau_k(n;\xi)$ is non-negative and $\tau_k(n;\xi) \leq \tau_k(n)$ with equality holding when $n \leq \xi$. The lemma now follows by choosing $k = \ell$ in (2.8). □

2.2 A Lower Bound for Σ_1

In order to establish a lower bound for Σ_1, we require a mean-value estimate for sums of the form

$$S(X,Y;T) = \sum_{0<\gamma\leq T} \zeta'(\rho)X(\rho)Y(1-\rho)$$

where

$$X(s) = \sum_{n\leq N} \frac{x_n}{n^s} \qquad \text{and} \qquad Y(s) = \sum_{n\leq N} \frac{y_n}{n^s}$$

are Dirichlet polynomials. For $X(s)$ and $Y(s)$ satisfying certain reasonable conditions, a general formula for $S(X,Y;T)$ has been established by Nathan Ng [26]. Before stating the formula, we first introduce some notation. For T large, we let $\mathscr{L} = \log \frac{T}{2\pi}$ and $N = T^{\vartheta}$ for some fixed $\vartheta \geq 0$. As usual, the arithmetic functions $\mu(\cdot)$ and $\Lambda(\cdot)$ are defined as the coefficients in the Dirichlet series

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \qquad \text{and} \qquad -\frac{\zeta'}{\zeta}(s) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}.$$

It can be checked that $\mu(1) = 1$, $\mu(n) = (-1)^k$ if n is the product of k distinct prime factors, and $\mu(n) = 0$ if any prime divides n multiple times and that $\Lambda(n) = \log p$ if $n = p^k$ for some prime p and $\Lambda(n) = 0$ otherwise. In addition, we define the arithmetic function $\Lambda_2(\cdot)$ by $\Lambda_2(n) = (\mu * \log^2)(n)$ for each $n \in \mathbb{N}$. Here $*$ denotes Dirichlet convolution, that is $(a*b)(n) = \sum_{k\ell=n} a(k)b(\ell)$. Alternatively, with a little work, we can show that $\Lambda_2(n) = \Lambda(n)\log(n) + (\Lambda * \Lambda)(n)$.

Lemma 2.2.1. *Let x_n and y_n satisfy $|x_n|, |y_n| \ll \tau_\ell(n)$ for some $\ell \in \mathbb{N}$ and assume that $0 < \vartheta < 1/2$. Then for any $A > 0$, any $\varepsilon > 0$, and sufficiently large T we have*

$$S(X, Y; T) = S_1 + S_2 + S_3 + O_A\big(T(\log T)^{-A} + T^{3/4+\vartheta/2+\varepsilon}\big)$$

where

$$\begin{aligned} S_1 &= \frac{T}{2\pi} \sum_{mn \le N} \frac{x_m y_{mn}}{mn} \Big(\mathcal{P}_2(\mathscr{L}) - 2\mathcal{P}_1(\mathscr{L}) \log n + (\Lambda * \log)(n)\Big), \\ S_2 &= -\frac{T}{4\pi} \sum_{mn \le N} \frac{y_m x_{mn}}{mn} \mathcal{Q}_2(\mathscr{L} - \log n), \end{aligned}$$

and

$$S_3 = \frac{T}{2\pi} \sum_{\substack{a,b \le N \\ (a,b)=1}} \frac{r(a;b)}{ab} \sum_{g \le \min\left(\frac{N}{a}, \frac{N}{b}\right)} \frac{y_{ag} x_{bg}}{g}.$$

Here $\mathcal{P}_1, \mathcal{P}_2$, and $\mathcal{Q}_2$ are monic polynomials of degrees 1,2, and 2, respectively, and for $a, b \in \mathbb{N}$ the function $r(a;b)$ satisfies the bound

$$|r(a;b)| \ll \Lambda_2(a) + (\log T)\Lambda(a) \ . \tag{2.18}$$

Proof. This is a special case of Theorem 1.3 of Ng [26]. □

Using Lemma 2.2.1, we can now deduce a lower bound for Σ_1 from the divisor sum estimates given in Lemmas 2.1.2 and 2.1.3 of the previous section. Letting $\xi = T^{1/(4k)}$, we find that the choices $X(s) = \mathcal{A}_\xi(s)^{k-1}$ and $Y(s) = \mathcal{A}_\xi(s)^k$ satisfy the conditions of Lemma 2.2.1 with $\vartheta = 1/4$, $N = \xi^k$, $x_n = \tau_{k-1}(n;\xi)$, and $y_n = \tau_k(n;\xi)$. Consequently, for this choice of ξ,

$$\Sigma_1 = \mathcal{S}_{11} + \mathcal{S}_{12} + \mathcal{S}_{13} + O(T)$$

where

$$\mathcal{S}_{11} = \frac{T}{2\pi} \sum_{\substack{mn \le \xi^k \\ m \le \xi^{k-1}}} \frac{\tau_{k-1}(m;\xi)\tau_k(mn;\xi)}{mn} \Big(\mathcal{P}_2(\mathscr{L}) - 2\mathcal{P}_1(\mathscr{L}) \log n + (\Lambda * \log)(n) \Big),$$

$$\mathcal{S}_{12} = -\frac{T}{4\pi} \sum_{mn \le \xi^{k-1}} \frac{\tau_k(m;\xi)\tau_{k-1}(mn;\xi)}{mn} \mathcal{Q}_2(\mathscr{L} - \log n),$$

and

$$\mathcal{S}_{13} = \frac{T}{2\pi} \sum_{\substack{a,b \le \xi^k \\ (a,b)=1}} \frac{r(a;b)}{ab} \sum_{g \le \min\left(\frac{N}{a}, \frac{N}{b}\right)} \frac{\tau_k(ag;\xi)\tau_{k-1}(bg;\xi)}{g}.$$

We show that $\Sigma_1 \gg_k T(\log T)^{k^2+2}$ by establishing the estimates $S_{11} \gg_k T(\log T)^{k^2+2}$, $S_{12} \ll_k T(\log T)^{k^2+1}$, and $S_{13} \ll_k T(\log T)^{k^2+1}$. Throughout the remainder of this section, all implied constants are allowed to depend on k.

To estimate $\mathcal{S}_{11}$, we claim that $\big(\mathcal{P}_2(\mathscr{L}) - 2\mathcal{P}_1(\mathscr{L}) \log n + (\Lambda * \log)(n)\big) \gg \mathscr{L}^2$. To see why, notice that since $\mathcal{P}_2$ is monic we have

$$\mathcal{P}_2(\mathscr{L}) \ge \big(1 + o(1)\big)\mathscr{L}^2.$$

Also, since $n \le \xi^k = T^{1/4}$ and $\mathcal{P}_1$ is monic, we see that

$$\mathcal{P}_1(\mathscr{L}) \log n \le \Big(\frac{1}{4} + o(1)\Big)\mathscr{L}^2.$$

Therefore, since $(\Lambda * \log)(n) \ge 0$, these estimates imply that

$$\Big(\mathcal{P}_2(\mathscr{L}) - 2\mathcal{P}_1(\mathscr{L}) \log n + (\Lambda * \log)(n)\Big) \ge \Big(\frac{1}{2} + o(1)\Big)\mathscr{L}^2 \gg \mathscr{L}^2 \tag{2.19}$$

which proves the claim. Letting $mn = \ell$, we see that

$$\sum_{\substack{mn \le \xi^k \\ m \le \xi^{k-1}}} \frac{\tau_{k-1}(m;\xi)\tau_k(mn;\xi)}{mn} = \sum_{\ell \le \xi^k} \frac{\tau_k(\ell;\xi)}{\ell} \Bigg(\sum_{\substack{m \mid \ell \\ m \le \xi^{k-1}}} \tau_{k-1}(m;\xi) \Bigg).$$

But, noticing that

$$\sum_{\substack{m|\ell \\ m\le \xi^{k-1}}} \tau_{k-1}(m;\xi) \ge \tau_k(\ell;\xi),$$

we can conclude from (2.17) that

$$\sum_{\substack{mn\le \xi^k \\ m\le \xi^{k-1}}} \frac{\tau_{k-1}(m;\xi)\tau_k(mn;\xi)}{mn} \ge \sum_{\ell\le\xi} \frac{\tau_k(\ell;\xi)^2}{\ell} \gg (\log T)^{k^2}$$

which, when combined with (2.19), implies that $\mathcal{S}_{11} \gg T(\log T)^{k^2+2}$.

We can establish an upper bound for $\mathcal{S}_{12}$ by using the inequalities $\tau_k(n;\xi) \le \tau_k(n)$ and $\tau_k(mn) \le \tau_k(m)\tau_k(n)$. Since $\mathcal{Q}_2(\mathscr{L} - \log n) \ll \mathscr{L}^2$, by twice using (2.8), we find that

$$\begin{aligned}
\mathcal{S}_{12} &\ll T\mathscr{L}^2 \sum_{mn\le\xi^k} \frac{\tau_k(m)\tau_{k-1}(m)\tau_{k-1}(n)}{mn} \\
&\le T\mathscr{L}^2 \left(\sum_{m\le T} \frac{\tau_k(m)\tau_{k-1}(m)}{m}\right)\left(\sum_{n\le T}\frac{\tau_{k-1}(n)}{n}\right) \\
&\ll T(\log T)^{2+k(k-1)+k-1} \\
&\ll T(\log T)^{k^2+1}.
\end{aligned}$$

It remains to consider the contribution from $\mathcal{S}_{13}$. Again using the inequalities $\tau_k(n;\xi) \le \tau_k(n)$ and $\tau_k(mn) \le \tau_k(m)\tau_k(n)$ along with (2.18), it follows that

$$\begin{aligned}
\mathcal{S}_{13} &\ll \sum_{a,b\le\xi^k} \frac{(\Lambda_2(a) + (\log T)\Lambda(a))}{ab} \sum_{g\le\xi^k} \frac{\tau_k(a)\tau_k(g)\tau_{k-1}(b)\tau_{k-1}(g)}{g} \\
&\ll \sum_{a\le T} \frac{(\Lambda_2(a) + (\log T)\Lambda(a))\tau_k(a)}{a} \sum_{b\le T}\frac{\tau_{k-1}(b)}{b} \sum_{g\le T} \frac{\tau_k(g)\tau_{k-1}(g)}{g} \\
&\ll (\log T)^{k^2-1} \sum_{a\le T} \frac{\Lambda_2(a)\tau_k(a)}{a} + (\log T)^{k^2} \sum_{a\le T} \frac{\Lambda(a)\tau_k(a)}{a}.
\end{aligned}$$

Using the crude bound $\tau_k(n) \ll n^\varepsilon$ for $\varepsilon > 0$ arbitrary, it is easily shown that

$$\sum_{a\le T} \frac{\Lambda(a)\tau_k(a)}{a} = \sum_{p\le T} \frac{k\log p}{p} + O_k(1) = k\log T + O_k(1).$$

Using this estimate and, again, the inequality $\tau_k(mn) \le \tau_k(n)\tau_k(m)$, it follows that

$$\begin{aligned}\sum_{a\le T} \frac{\Lambda_2(a)\tau_k(a)}{a} &\ll (\log T)\sum_{a\le T}\frac{\Lambda(a)\tau_k(a)}{a} + \sum_{a\le T}\frac{(\Lambda*\Lambda)(a)\tau_k(a)}{a} \\ &\ll (\log T)^2 + \sum_{mn\le T}\frac{\tau_k(mn)\Lambda(m)\Lambda(n)}{mn} \\ &\ll (\log T)^2 + \left(\sum_{n\le T}\frac{\tau_k(n)\Lambda(n)}{n}\right)^2 \\ &\ll (\log T)^2.\end{aligned}$$

From the above estimates, we can now see that $\Sigma_3 \ll (\log T)^{k^2+1}$. Combining this with our estimates for $\mathcal{S}_{11}$ and $\mathcal{S}_{12}$, we conclude that $\Sigma_1 \gg T(\log T)^{k^2+2}$.

2.3 An Upper Bound for Σ_2

Assuming the Riemann Hypothesis, it follows from (2.5) that

$$\begin{aligned}\Sigma_2 &= \sum_{0<\gamma\le T}\sum_{m\le\xi^k}\frac{\tau_k(m;\xi)}{m^\rho}\sum_{n\le\xi^k}\frac{\tau_k(n;\xi)}{n^{1-\rho}} \\ &= N(T)\sum_{n\le\xi^k}\frac{\tau_k(n;\xi)^2}{n} + 2\Re\sum_{m\le\xi^k}\sum_{m<n\le\xi^k}\frac{\tau_k(m;\xi)\tau_k(n;\xi)}{n}\sum_{0<\gamma\le T}\left(\frac{n}{m}\right)^\rho\end{aligned} \tag{2.20}$$

where $N(T)$ denotes the number of non-trivial zeros of $\zeta(s)$ up to a height T. Recalling that $\xi = T^{1/(4k)}$ and using the estimate $N(T) \ll T\log T$ and Lemma 2.1.3, it follows that

$$N(T)\sum_{n\le\xi^k}\frac{\tau_k(n;\xi)^2}{n} \ll_k T(\log T)^{k^2+1}. \tag{2.21}$$

In order to bound the second sum on the right-hand side of (2.20), we require the following version of the Landau-Gonek explicit formula.

Lemma 2.3.1. *Let* $x, T > 1$ *and let* $\rho = \beta + i\gamma$ *denote a non-trivial zero of* $\zeta(s)$*. Then*

$$\sum_{0<\gamma\le T} x^{\rho} = -\frac{T}{2\pi}\Lambda(x) + O\big(x\log(2xT)\log\log(3x)\big) + O\Big(\log x \min\Big(T, \frac{x}{\langle x\rangle}\Big)\Big) + O\Big(\log(2T)\min\Big(T, \frac{1}{\log x}\Big)\Big),$$

where $\langle x\rangle$ *denotes the distance from* x *to the nearest prime power other than* x *itself,* $\Lambda(x) = \log p$ *if* x *is a positive integral power of a prime* p*, and* $\Lambda(x) = 0$ *otherwise.*

Proof. This is a result of Gonek [9; 11]. □

Applying the lemma, we find that

$$\begin{aligned}
\sum_{m\le\xi^k}\sum_{m<n\le\xi^k} \frac{\tau_k(m;\xi)\tau_k(n;\xi)}{n}\sum_{0<\gamma\le T}\Big(\frac{n}{m}\Big)^{\rho}
&= -\frac{T}{2\pi}\sum_{m\le\xi^k}\sum_{m<n\le\xi^k}\frac{\tau_k(m;\xi)\tau_k(n;\xi)\Lambda(\frac{n}{m})}{n} \\
&\quad + O\left(\mathscr{L}\log\mathscr{L}\sum_{m\le\xi^k}\sum_{m<n\le\xi^k}\frac{\tau_k(m;\xi)\tau_k(n;\xi)}{m}\right) \\
&\quad + O\left(\sum_{m\le\xi^k}\sum_{m<n\le\xi^k}\frac{\tau_k(m;\xi)\tau_k(n;\xi)}{m}\frac{\log\frac{n}{m}}{\langle\frac{n}{m}\rangle}\right) \\
&\quad + O\left(\log T\sum_{m\le\xi^k}\sum_{m<n\le\xi^k}\frac{\tau_k(m;\xi)\tau_k(n;\xi)}{n\log\frac{n}{m}}\right) \\
&= \mathcal{S}_{21} + \mathcal{S}_{22} + \mathcal{S}_{23} + \mathcal{S}_{24},
\end{aligned}$$

say. Since we only require an upper bound for Σ_2 (which, by definition, is clearly positive), we can ignore the contribution from $\mathcal{S}_{21}$ because $\tau_k(m;\xi)\tau_k(n;\xi)\Lambda(\frac{n}{m}) \geq 0$ for each $m, n \in \mathbb{N}$ so the sum is clearly negative.

In estimating the remaining terms, we use ε to denote a small positive constant, which may be different at each occurrence, and we allow any implied constants to depend on k. In order to estimate $\mathcal{S}_{22}$, we note that $\tau_k(n;\xi) \leq \tau_k(n) \ll_\varepsilon n^\varepsilon$. Since $\xi^k = T^{1/4}$, this implies that $\mathcal{S}_{22} \ll T^{1/4+\varepsilon}$. Turning to $\mathcal{S}_{23}$, we write n as $qm+\ell$ with $-\frac{m}{2} < \ell \leq \frac{m}{2}$ and find that

$$\mathcal{S}_{23} \ll T^\varepsilon \sum_{m \leq \xi^k} \frac{1}{m} \sum_{q \leq \lfloor \frac{\xi^k}{m} \rfloor + 1} \sum_{-\frac{m}{2} < \ell \leq \frac{m}{2}} \frac{1}{\langle q + \frac{\ell}{m} \rangle}$$

where $\lfloor x \rfloor$ denotes the greatest integer $\leq x$. Notice that $\langle q + \frac{\ell}{m} \rangle = \frac{|\ell|}{m}$ if q is a prime power and $\ell \neq 0$, otherwise $\langle q + \frac{\ell}{m} \rangle$ is $\geq \frac{1}{2}$. Hence,

$$\begin{aligned}
\mathcal{S}_{23} &\ll T^\varepsilon \Big(\sum_{m \leq \xi^k} \frac{1}{m} \sum_{\substack{q \leq \lfloor \frac{\xi^k}{m} \rfloor + 1 \\ \Lambda(q) \neq 0}} \sum_{1 \leq \ell \leq \frac{m}{2}} \frac{m}{\ell} + \sum_{m \leq \xi^k} \frac{1}{m} \sum_{q \leq \lfloor \frac{\xi^k}{m} \rfloor + 1} \sum_{1 \leq \ell \leq \frac{m}{2}} 1 \Big) \\
&\ll T^\varepsilon \Big(\sum_{m \leq \xi^k} \sum_{q \leq \lfloor \frac{\xi^k}{m} \rfloor + 1} 1 \Big) \ll T^{1/4+\varepsilon} .
\end{aligned}$$

It remains to consider $\mathcal{S}_{24}$. For integers $1 \leq m < n \leq \xi^k$, let $n = m + \ell$. Then

$$\log \frac{n}{m} = -\log \big(1 - \frac{\ell}{m}\big) > \frac{\ell}{m}.$$

Consequently,

$$\mathcal{S}_{24} \ll T^\varepsilon \sum_{m \leq \xi^k} \sum_{1 \leq \ell \leq \xi^k} \frac{1}{(m+\ell)\frac{\ell}{m}} \ll T^\varepsilon \xi^k = T^{1/4+\varepsilon}. \tag{2.22}$$

We can now establish the lower bound in Theorem 1.1.1. Combining (2.21) with our estimates for $\mathcal{S}_{22}$, $\mathcal{S}_{23}$, and $\mathcal{S}_{24}$ we deduce that $\Sigma_2 \ll T(\log T)^{k^2+1}$ which, when combined with our estimate for Σ_1, completes the proof of (2.1).

3 The Proof of the Upper Bound in Theorem 1.1.1

In this chapter, we prove the upper bound in Theorem 1.1.1. In particular, assuming the Riemann Hypothesis we show that

$$\sum_{0<\gamma\le T} \left|\zeta'(\rho)\right|^{2k} \ll_{k,\varepsilon} T(\log T)^{(k+1)^2+\varepsilon} \tag{3.1}$$

for $k \in \mathbb{N}$ and $\varepsilon > 0$ is arbitrary. Our proof is based upon a recent method of Soundararajan [35] that provides upper bounds for the frequency of large values of $|\zeta(\frac{1}{2}+it)|$. His method relies on obtaining an inequality for $\log|\zeta(\frac{1}{2}+it)|$ involving a "short" Dirichlet polynomial which is a smoothed approximation to the Dirichlet series for $\log\zeta(s)$. Using mean-value estimates for high powers of this Dirichlet polynomial, he deduces upper bounds for the measure of the set (as a function of V)

$$\{t \in [0,T] : \log|\zeta(\tfrac{1}{2}+it)| \ge V\}$$

and from this is able conclude that, for arbitrary positive values of k and ε,

$$\int_0^T \left|\zeta(\tfrac{1}{2}+it)\right|^{2k} \ll_{k,\varepsilon} T(\log T)^{k^2+\varepsilon}. \tag{3.2}$$

Soundararajan's techniques build upon the previous work of Selberg [31; 32; 33] who studied the distribution of values of $\log\zeta(\frac{1}{2}+it)$ in the complex plane.

Since $\log \zeta'(s)$ does not have a Dirichlet series representation, it is not clear that $\log|\zeta'(\frac{1}{2}+it)|$ can be approximated by a Dirichlet polynomial.[1] For this reason, we do not study the distribution of the values of $\zeta'(\rho)$ directly, but instead examine the frequency of large values of $|\zeta(\rho+\alpha)|$, where $\alpha \in \mathbb{C}$ is a small shift away from a zero of $\zeta(s)$. We proceed analogously to Soundararajan and use the functional equation for the zeta-function. This requires deriving an inequality for $\log|\zeta(\sigma+it)|$ involving a short Dirichlet polynomial that holds uniformly for values of σ in a small interval to the right of, and including, $\sigma = \frac{1}{2}$. By estimating high power moments of this Dirichlet polynomial averaged over the zeros of $\zeta(s)$, we are then able to derive upper bounds for the frequency of large values of $|\zeta(\rho+\alpha)|$ and use this information to prove the following theorem.

Theorem 3.0.2. *Assume the Riemann Hypothesis. Let $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$ and $|\Re\alpha - \frac{1}{2}| \leq (\log T)^{-1}$. Let $k \in \mathbb{R}$ with $k > 0$ and let $\varepsilon > 0$ be arbitrary. Then for sufficiently large T the inequality*

$$\sum_{0<\gamma\leq T} \left|\zeta(\rho+\alpha)\right|^{2k} \ll_{k,\varepsilon} T(\log T)^{k^2+\varepsilon} \tag{3.3}$$

holds uniformly in α.

Comparing the result of Theorem 3.0.2 with (3.2), we see that our theorem provides essentially the same upper bound (up to the implied constant) for discrete averages of the Riemann zeta-function near its zeros as can be obtained for continuos moments of $|\zeta(\frac{1}{2}+it)|$. Discrete moments similar to those in Theorem 3.0.2 have been studied previously. For instance, see the articles by Gonek [8] and Hughes [15].

[1] Hejhal [14] studied the distribution of $\log|\zeta'(\frac{1}{2}+it)|$ by a method that does not directly involve the use of Dirichlet polynomials.

We deduce (3.1) from Theorem 3.0.2 since, by Cauchy's integral formula, we can use our bounds for $\zeta(s)$ near its zeros to recover bounds for the values of $\zeta'(\rho)$. For a precise statement of this, see Lemma 3.6.1 below. Our proof allows us only to establish (3.1) when $k \in \mathbb{N}$ despite the fact that Theorem 3.0.2 holds for all $k \geq 2$.

3.1 An Inequality for $\log|\zeta(\sigma+it)|$ when $\sigma \geq \frac{1}{2}$.

Throughout the remainder of this chapter, we use $s = \sigma + it$ to denote a complex variable and use p to denote a prime number. We let $\lambda_0 = .5671...$ be the unique positive real number satisfying $e^{-\lambda_0} = \lambda_0$. Also, we put $\sigma_\lambda = \sigma_{\lambda,x} = \frac{1}{2} + \frac{\lambda}{\log x}$ and let

$$\log^+|x| = \begin{cases} 0, & \text{if } |x| < 1, \\ \log|x|, & \text{if } |x| \geq 1. \end{cases}$$

As usual, we denote by $\Lambda(\cdot)$ the arithmetic function defined by $\Lambda(n) = \log p$ when $n = p^k$ and $\Lambda(n) = 0$ when $n \neq p^k$. The main result of this section is the following lemma.

Lemma 3.1.1. *Assume the Riemann Hypothesis. Let* $\tau = |t| + 3$ *and* $2 \leq x \leq \tau^2$. *Then, for any* λ *with* $\lambda_0 \leq \lambda \leq \frac{\log x}{4}$, *the estimate*

$$\log^+\left|\zeta(\sigma+it)\right| \leq \left|\sum_{n\leq x} \frac{\Lambda(n)}{n^{\sigma_\lambda+it}\log n}\frac{\log x/n}{\log x}\right| + \frac{(1+\lambda)}{2}\frac{\log\tau}{\log x} + O(1) \tag{3.4}$$

holds uniformly for $1/2 \leq \sigma \leq \sigma_\lambda$.

In [35], Soundararajan proved an inequality similar to Lemma 3.1.1 for the function $\log\left|\zeta(\frac{1}{2}+it)\right|$. In his case, when $\zeta(\frac{1}{2}+it) \neq 0$, an inequality slightly stronger than (3.4) holds with the constant λ_0 replaced by $\delta_0 = .4912...$ where δ_0 is the unique positive real number satisfying $e^{-\delta_0} = \delta_0 + \frac{1}{2}\delta_0^2$. Our proof of the lemma is a modification of his argument.

Proof of Lemma 3.1.1. We assume that $|\zeta(\sigma+it)| \geq 1$, as otherwise the lemma holds for a trivial reason. In particular, we are assuming that $\zeta(\sigma+it) \neq 0$. Assuming the Riemann Hypothesis, we denote a non-trivial zeros of $\zeta(s)$ as $\rho = \frac{1}{2} + i\gamma$ and define the function

$$F(s) = \Re \sum_{\rho} \frac{1}{s-\rho} = \sum_{\rho} \frac{\sigma - \frac{1}{2}}{(\sigma-\frac{1}{2})^2 + (t-\gamma)^2}.$$

Notice that $F(s) \geq 0$ whenever $\sigma \geq \frac{1}{2}$ and $s \neq \rho$. The partial fraction decomposition of $\zeta'(s)/\zeta(s)$ (equation (2.12.7) of Titchmarsh [37]) says that for $s \neq 1$ and s not coinciding with a zero of $\zeta(s)$, we have

$$\frac{\zeta'}{\zeta}(s) = \sum_{\rho} \left(\frac{1}{s-\rho} + \frac{1}{\rho} \right) - \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\tfrac{1}{2}s+1\right) - \frac{1}{s-1} + B \tag{3.5}$$

where the constant $B = \log 2\pi - 1 - 2\gamma_0$; γ_0 denotes Euler's constant. Taking the real part of each term in (3.5), we find that

$$-\Re \frac{\zeta'}{\zeta}(s) = -\Re \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\tfrac{1}{2}s+1\right) - F(s) + O(1). \tag{3.6}$$

Stirling's asymptotic formula for the gamma function implies that

$$\frac{\Gamma'}{\Gamma}(s) = \log s - \frac{1}{2s} + O\big(|s|^{-2}\big) \tag{3.7}$$

for $\delta > 0$ fixed, $|\arg s| < \pi - \delta$, and $|s| > \delta$ (see Appendix A.7 of Ivič [19]). By combining (3.6) and (3.7) with the observation that $F(s) \geq 0$, we find that

$$\begin{aligned} -\Re \frac{\zeta'}{\zeta}(s) &= \tfrac{1}{2}\log\tau - F(s) + O(1) \\ &\leq \tfrac{1}{2}\log\tau + O(1). \end{aligned} \tag{3.8}$$

uniformly for $\frac{1}{2} \le \sigma \le 1$. Consequently, the inequality

$$\begin{aligned} \log|\zeta(\sigma+it)| - \log|\zeta(\sigma_\lambda+it)| &= \Re \int_{\sigma}^{\sigma_\lambda} \Big[-\frac{\zeta'}{\zeta}(u+it) \Big] du \\ &\le \big(\sigma_\lambda - \sigma\big)\big(\tfrac{1}{2}\log\tau + O(1)\big) \\ &\le \big(\sigma_\lambda - \tfrac{1}{2}\big)\big(\tfrac{1}{2}\log\tau + O(1)\big) \end{aligned} \tag{3.9}$$

holds uniformly for $\frac{1}{2} \le \sigma \le \sigma_\lambda$.

To complete the proof of the lemma, we require an upper bound for $\log|\zeta(\sigma_\lambda+it)|$ which, in turn, requires an additional identity for $\zeta'(s)/\zeta(s)$. Specifically, for $s \ne 1$ and s not coinciding with a zero of $\zeta(s)$, we have

$$\begin{aligned} -\frac{\zeta'}{\zeta}(s) = \sum_{n \le x} \frac{\Lambda(n)}{n^s} \frac{\log(x/n)}{\log x} + \frac{1}{\log x}\Big(\frac{\zeta'}{\zeta}(s)\Big)' + \frac{1}{\log x}\sum_{\rho} \frac{x^{\rho-s}}{(\rho-s)^2} \\ - \frac{1}{\log x}\frac{x^{1-s}}{(1-s)^2} + \frac{1}{\log x}\sum_{k=1}^{\infty} \frac{x^{-2k-s}}{(2k+s)^2}. \end{aligned} \tag{3.10}$$

This identity is due to Soundararajan (Lemma 1 of [35]). Integrating over σ from σ_λ to ∞, we deduce from the above identity that

$$\begin{aligned} \log|\zeta(\sigma_\lambda+it)| = \Re \sum_{n \le x} \frac{\Lambda(n)}{n^{\sigma_\lambda+it}\log n} \frac{\log x/n}{\log x} - \frac{1}{\log x}\Re \frac{\zeta'}{\zeta}(\sigma_\lambda+it) \\ + \frac{1}{\log x}\sum_{\rho} \Re \int_{\sigma_\lambda}^{\infty} \frac{x^{\rho-s}}{(\rho-s)^2} d\sigma + O\Big(\frac{1}{\log x}\Big). \end{aligned} \tag{3.11}$$

We now estimate the second and third terms on the right-hand side of this expression. Arguing as above, using (3.5) and (3.7), we find that

$$\Re \frac{\zeta'}{\zeta}(\sigma_\lambda+it) = \tfrac{1}{2}\log\tau - F(\sigma_\lambda+it) + O(1). \tag{3.12}$$

Also, observing that

$$\begin{aligned} \sum_{\rho} \Big| \int_{\sigma_\lambda}^{\infty} \frac{x^{\rho-s}}{(\rho-s)^2} d\sigma \Big| &\le \sum_{\rho} \int_{\sigma_\lambda}^{\infty} \frac{x^{1/2-\sigma}}{|\rho-s|^2} d\sigma \\ &= \sum_{\rho} \frac{x^{1/2-\sigma_\lambda}}{|\rho-\sigma_\lambda-it|^2 \log x} = \frac{x^{1/2-\sigma_\lambda}F(\sigma_\lambda+it)}{(\sigma_\lambda+it)\log x}, \end{aligned} \tag{3.13}$$

and combining (3.12) and (3.13) with (3.11), we see that

$$\begin{aligned}\log|\zeta(\sigma_\lambda+it)| \le \Re\sum_{n\le x}\frac{\Lambda(n)}{n^{\sigma_\lambda+it}\log n}\frac{\log x/n}{\log x} + \frac{1}{2}\frac{\log\tau}{\log x} \\ + \frac{F(\sigma_\lambda+it)}{\log x}\Big(\frac{x^{1/2-\sigma_\lambda}}{(\sigma_\lambda-\frac{1}{2})\log x} - 1\Big) + O\Big(\frac{1}{\log x}\Big).\end{aligned}$$

If $\lambda \ge \lambda_0$, then the term on the right-hand side involving $F(\sigma_\lambda+it)$ is less than or equal to zero, so omitting it does not change the inequality. Thus,

$$\log|\zeta(\sigma_\lambda+it)| \le \Re\sum_{n\le x}\frac{\Lambda(n)}{n^{\sigma_\lambda+it}\log n}\frac{\log x/n}{\log x} + \frac{1}{2}\frac{\log\tau}{\log x} + O\Big(\frac{1}{\log x}\Big). \tag{3.14}$$

Since we have assumed that $|\zeta(\sigma+it)| \ge 1$, the lemma now follows by combining the inequalities in (3.9) and (3.14) and then taking absolute values.

3.2 A Variation of Lemma 3.1.1

In this section, we prove a version of Lemma 3.1.1 in which the sum on the right-hand side of the inequality is restricted just to the primes. A sketch of the proof of the lemma appearing below has been given previously by Soundararajan (see [35], Lemma 2). Our proof is different and the details are provided for completeness.

Lemma 3.2.1. *Assume the Riemann Hypothesis. Put $\tau = |t| + e^{30}$. Then, for $\sigma \ge \frac{1}{2}$ and $2 \le x \le \tau^2$, we have*

$$\Bigg|\sum_{n\le x}\frac{\Lambda(n)}{n^{\sigma+it}\log n}\frac{\log x/n}{\log x} - \sum_{p\le x}\frac{1}{p^{\sigma+it}}\frac{\log x/n}{\log x}\Bigg| = O\big(\log\log\log\tau\big).$$

As a consequence, for any λ with $\lambda_0 \le \lambda \le \frac{\log x}{4}$, the estimate

$$\log^+\big|\zeta(\sigma+it)\big| \le \Bigg|\sum_{p\le x}\frac{1}{p^{\sigma_\lambda+it}}\frac{\log x/p}{\log x}\Bigg| + \frac{(1+\lambda)}{2}\frac{\log\tau}{\log x} + O\big(\log\log\log\tau\big)$$

holds uniformly for $\frac{1}{2} \le \sigma \le \sigma_\lambda$ and $2 \le x \le \tau^2$.

Proof. First we observe that, for $\sigma \geq \frac{1}{2}$,

$$\begin{aligned}\sum_{n\leq x}\frac{\Lambda(n)}{n^s\log n}\frac{\log x/n}{\log x}-\sum_{p\leq x}\frac{1}{p^s}\frac{\log x/p}{\log x} &= \frac{1}{2}\sum_{p\leq\sqrt{x}}\frac{1}{p^{2s}}\frac{\log\sqrt{x}/n}{\log\sqrt{x}}+O(1). \\ &= \frac{1}{2}\sum_{n\leq\sqrt{x}}\frac{\Lambda(n)}{n^{2s}\log n}\frac{\log\sqrt{x}/n}{\log\sqrt{x}}+O(1).\end{aligned}$$

Thus, if we let $w = u + iv$ and $\nu = |v| + e^{30}$, the lemma will follow if we can show that

$$\sum_{n\leq z}\frac{\Lambda(n)}{n^w\log n}\frac{\log z/n}{\log z} = O\big(\log\log\log\nu\big) \tag{3.15}$$

uniformly for $u \geq 1$ and $2 \leq z \leq \nu$. In what follows, we can assume that $z \geq (\log\nu)^2$ as otherwise

$$\sum_{n\leq z}\frac{\Lambda(n)}{n^w\log n}\frac{\log z/n}{\log z} \ll \sum_{p<\log^2\nu}\frac{1}{p} \ll \log\log\log\nu.$$

Let $c = \max(2, 1+u)$. Then, by expressing $\frac{\zeta'}{\zeta}(s+w)$ as a Dirichlet series and interchanging the order of summation and integration (which is justified by absolute convergence), it follows that

$$\begin{aligned}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Big[-\frac{\zeta'}{\zeta}(s+w)\Big]z^s\frac{ds}{s^2} &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Big[\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{s+w}}\Big]z^s\frac{ds}{s^2} \\ &= \frac{1}{2\pi i}\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^w}\int_{c-i\infty}^{c+i\infty}\Big(\frac{z}{n}\Big)^s\frac{ds}{s^2} \\ &= \sum_{n\leq z}\frac{\Lambda(n)}{n^w}\log(z/n).\end{aligned}$$

Here we have made use of the standard identity

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}x^s\frac{ds}{s^2} = \begin{cases}\log x, & \text{if } x\geq 1,\\ 0, & \text{if } 0\leq x<1,\end{cases}$$

which is valid for $c > 0$. By moving the line of integration in the integral left to $\Re s = \sigma = \frac{3}{4} - u$, we find by the calculus of residues that

$$\sum_{n \le z} \frac{\Lambda(n)}{n^w} \log(z/n) = -(\log z)\frac{\zeta'}{\zeta}(w) - \Big(\frac{\zeta'}{\zeta}(w)\Big)' + \frac{z^{1-w}}{(w-1)^2} + \frac{1}{2\pi i}\int_{\frac{3}{4}-u-i\infty}^{\frac{3}{4}-u+i\infty} \Big[-\frac{\zeta'}{\zeta}(s+w)\Big] z^s \frac{ds}{s^2}. \tag{3.16}$$

That there are no residues obtained from poles of the integrand at the non-trivial zeros of $\zeta(s)$ follows from the Riemann Hypothesis. To estimate the integral on the right-hand side of the above expression, we use Theorem 14.5 of Titchmarsh [37], namely, that if the Riemann Hypothesis is true, then

$$\Big|\frac{\zeta'}{\zeta}(\sigma+it)\Big| \ll (\log\tau)^{2-2\sigma}$$

uniformly for $\frac{5}{8} \le \sigma \le \frac{7}{8}$, say. Using this estimate, it immediately follows that

$$\int_{\frac{3}{4}-u-i\infty}^{\frac{3}{4}-u+i\infty} \Big[-\frac{\zeta'}{\zeta}(s+w)\Big] z^s \frac{ds}{s^2} \ll z^{3/4-u}\sqrt{\log\nu}.$$

Inserting this estimate into equation (3.16) and dividing by $\log z$, it follows that

$$\sum_{n \le z} \frac{\Lambda(n)}{n^w}\frac{\log(z/n)}{\log z} = -\frac{\zeta'}{\zeta}(w) - \frac{1}{\log z}\Big(\frac{\zeta'}{\zeta}(w)\Big)' + \frac{z^{1-w}}{(w-1)^2\log z} + O\Big(\frac{z^{3/4-u}}{\log z}\sqrt{\log\nu}\Big). \tag{3.17}$$

Integrating the expression in (3.17) from ∞ to u (along the line $\sigma + i\nu$, $u \le \sigma < \infty$), we find that

$$\sum_{n \le z} \frac{\Lambda(n)}{n^w \log n}\frac{\log(z/n)}{\log z} = \log\zeta(w) + \frac{1}{\log z}\frac{\zeta'}{\zeta}(w) + O\Big(\frac{z^{1-u}}{\nu^2(\log z)^2} + \frac{z^{3/4-u}}{(\log z)^2}\sqrt{\log\nu}\Big).$$

Assuming the Riemann Hypothesis, we can estimate the terms on the right-hand side of of the above expression by invoking the bounds

$$|\log\zeta(\sigma+it)| \ll \log\log\log\tau \qquad \text{and} \qquad \left|\frac{\zeta'}{\zeta}(\sigma+it)\right| \ll \log\log\tau \tag{3.18}$$

which hold uniformly for $\sigma \geq 1$ and $|t| \geq 1$. (For a discussion of such estimates see Heath-Brown's notes following Chapter 14 in Titchmarsh [37].) Using the estimates in (3.18) and recalling that we are assuming that $u \geq 1$ and $z \geq (\log\nu)^2$, we find that

$$\begin{aligned}\sum_{n\leq z}\frac{\Lambda(n)}{n^w\log n}\frac{\log(z/n)}{\log z} &\ll \log\log\log\nu + \frac{\log\log\nu}{\log z} + \frac{z^{1-u}}{\nu^2(\log z)^2} + z^{-1/4}\frac{\sqrt{\log\nu}}{(\log z)^2}\\ &\ll \log\log\log\nu.\end{aligned}$$

This establishes (3.15) and, thus, the lemma. □

3.3 A Sum over the Zeros of $\zeta(s)$

In this section we prove an estimate for the mean-square of a Dirichlet polynomial averaged over the zeros of $\zeta(s)$. Our estimate follows from the the Landau-Gonek explicit formula (Lemma 2.3.1 above).

Lemma 3.3.1. *Assume the Riemann Hypothesis and let $\rho = \frac{1}{2} + i\gamma$ denote a non-trivial zero of $\zeta(s)$. For any sequence of complex numbers $\mathscr{A} = \{a_n\}_{n=1}^{\infty}$ define, for $\xi \geq 1$,*

$$m_\xi = m_\xi(\mathscr{A}) = \max_{1\leq n\leq\xi}\big(1, |a_n|\big).$$

Then for $3 \leq \xi \leq T(\log T)^{-1}$ and any complex number α with $\Re\alpha \geq 0$ we have

$$\sum_{0<\gamma\leq T}\left|\sum_{n\leq\xi}\frac{a_n}{n^{\rho+\alpha}}\right|^2 \ll m_\xi T\log T\sum_{n\leq\xi}\frac{|a_n|}{n}, \tag{3.19}$$

where the implied constant is absolute (and independent of α).

Proof. Assuming the Riemann Hypothesis, we note that $1-\rho=\bar{\rho}$ for any non-trivial zero $\rho=\frac{1}{2}+i\gamma$ of $\zeta(s)$. This implies that

$$\left|\sum_{n\le\xi}\frac{a_n}{n^{\rho+\alpha}}\right|^2=\sum_{m\le\xi}\sum_{n\le\xi}\frac{a_m}{m^{\rho+\alpha}}\frac{\overline{a_n}}{n^{1-\rho+\bar{\alpha}}},$$

and, moreover, that

$$\sum_{0<\gamma\le T}\left|\sum_{n\le\xi}\frac{a_n}{n^{\rho+\alpha}}\right|^2=N(T)\sum_{n\le\xi}\frac{|a_n|^2}{n^{1+2\Re\alpha}}+2\Re\sum_{m\le\xi}\frac{a_m}{m^{\alpha}}\sum_{m<n\le\xi}\frac{\overline{a_n}}{n^{1+\bar{\alpha}}}\sum_{0<\gamma\le T}\left(\frac{n}{m}\right)^{\rho}$$

where $N(T)\sim\frac{T}{2\pi}\log T$ denotes the number of zeros ρ with $0<\gamma\le T$. Since $\Re\alpha\ge 0$, it follows that

$$N(T)\sum_{n\le\xi}\frac{|a_n|^2}{n^{1+2\Re\alpha}}\ll T\log T\sum_{n\le\xi}\frac{|a_n|^2}{n}\ll m_\xi T\log T\sum_{n\le\xi}\frac{|a_n|}{n}.$$

Appealing to Lemma 2.3.1, we find that

$$\sum_{m\le\xi}\frac{a_m}{m^{\alpha}}\sum_{n<m}\frac{\overline{a_n}}{n^{1+\bar{\alpha}}}\sum_{0<\gamma\le T}\left(\frac{n}{m}\right)^{\rho}=\Sigma_1+\Sigma_2+\Sigma_3+\Sigma_4,$$

where

$$\Sigma_1=-\frac{T}{2\pi}\sum_{m\le\xi}\frac{a_m}{m^{\alpha}}\sum_{m<n\le\xi}\frac{\overline{a_n}}{n^{1+\bar{\alpha}}}\Lambda\Big(\frac{n}{m}\Big),$$

$$\Sigma_2=O\left(\log T\log\log T\sum_{m\le\xi}\frac{|a_m|}{m^{1+\Re\alpha}}\sum_{m<n\le\xi}\frac{|a_n|}{n^{\Re\alpha}}\right),$$

$$\Sigma_3=O\left(\sum_{m\le\xi}\frac{|a_m|}{m^{1+\Re\alpha}}\sum_{m<n\le\xi}\frac{|a_n|}{n^{\Re\alpha}}\frac{\log\frac{m}{n}}{<\frac{m}{n}>}\right),$$

and

$$\Sigma_4=O\left(\log T\sum_{m\le\xi}\frac{|a_m|}{m^{\Re\alpha}}\sum_{m<n\le\xi}\frac{|a_n|}{n^{1+\Re\alpha}\log\frac{n}{m}}\right).$$

We estimate Σ_1 first. Making the substitution $n = mk$, we re-write our expression for Σ_1 as

$$-\frac{T}{2\pi}\sum_{m\le\xi}\frac{a_m}{m^\alpha}\sum_{k\le\frac{\xi}{m}}\frac{\overline{a_{mk}}\Lambda(k)}{(mk)^{1+\bar{\alpha}}} = -\frac{T}{2\pi}\sum_{m\le\xi}\frac{a_m}{m^{1+2\Re\alpha}}\sum_{k\le\frac{\xi}{m}}\frac{\overline{a_{mk}}\Lambda(k)}{k^{1+\bar{\alpha}}}.$$

Again using the assumption that $\Re\alpha \ge 0$, we find that

$$\Sigma_1 \ll m_\xi T\sum_{n\le\xi}\frac{|a_n|}{n}\sum_{m\le\frac{\xi}{n}}\frac{\Lambda(m)}{m} \ll m_\xi T\log T\sum_{n\le\xi}\frac{|a_n|}{n}.$$

Here we have made use of the standard estimate $\sum_{m\le\xi}\frac{\Lambda(m)}{m} \ll \log\xi$. We can replace $\Re\alpha$ by 0 in each of the sums Σ_i (for $i = 2, 3$, or 4), as doing so will only make the corresponding estimates larger. Thus, using the assumption that $3 \le \xi \le T/\log T$, it follows that

$$\Sigma_2 \ll m_\xi \log T\log\log T\sum_{n\le\xi}\frac{|a_n|}{n}\sum_{m<n\le\xi}1 \ll m_\xi T\log T\sum_{n\le\xi}\frac{|a_n|}{n}.$$

Next, turning to Σ_3, we find that

$$\Sigma_3 \ll m_\xi\sum_{m\le\xi}\frac{|a_m|}{m}\sum_{m<n\le\xi}\frac{\log\frac{n}{m}}{<\frac{n}{m}>}.$$

Writing n as $qm+\ell$ with $-\frac{m}{2} < \ell \le \frac{m}{2}$, we have

$$\Sigma_3 \ll m_\xi\sum_{m\le\xi}\frac{|a_m|}{m}\sum_{q\le\lfloor\frac{\xi}{m}\rfloor+1}\sum_{-\frac{m}{2}<\ell\le\frac{m}{2}}\frac{\log\left(q+\frac{\ell}{m}\right)}{<q+\frac{\ell}{m}>},$$

where, as usual, $\lfloor x\rfloor$ denotes the greatest integer less than or equal to x. Now $<q+\frac{\ell}{m}>=\frac{|\ell|}{m}$ if q is a prime power and $\ell \ne 0$, otherwise $<q+\frac{\ell}{m}>$ is $\ge \frac{1}{2}$. Using the estimate $\sum_{n\le\xi}\Lambda(n)\ll\xi$, we now find that

$$\begin{aligned}
\Sigma_3 &\ll m_\xi \sum_{m\le\xi}\frac{|a_m|}{m}\sum_{q\le\lfloor\frac{\xi}{m}\rfloor+1}\Lambda(q)\sum_{1\le\ell\le\frac{m}{2}}\frac{m}{\ell}\\
&\qquad + m_\xi \sum_{m\le\xi}\frac{|a_m|}{m}\sum_{q\le\lfloor\frac{\xi}{m}\rfloor+1}\log(q+1)\sum_{1\le\ell\le\frac{m}{2}}1\\
&\ll m_\xi \sum_{m\le\xi}|a_m|\log m\sum_{q\le\lfloor\frac{\xi}{m}\rfloor+1}\Lambda(q)\\
&\qquad + m_\xi \sum_{m\le\xi}|a_m|\sum_{q\le\lfloor\frac{\xi}{m}\rfloor+1}\log(q+1)\\
&\ll m_\xi(\xi\log\xi)\sum_{m\le\xi}\frac{|a_m|}{m}\\
&\ll m_\xi T\log T\sum_{m\le\xi}\frac{|a_m|}{m}.
\end{aligned}$$

It remains to consider the contribution from Σ_4 which is

$$\ll m_\xi\log T\sum_{m\le\xi}|a_m|\sum_{m<n\le\xi}\frac{1}{n\log\frac{n}{m}}\ll m_\xi\log T\sum_{m\le\xi}\frac{|a_m|}{m}\sum_{m<n\le\xi}\frac{1}{\log\frac{n}{m}}\ ,$$

since $\frac{1}{m}>\frac{1}{n}$ if $n>m$. Writing $n=m+\ell$, we see that

$$\sum_{m<n\le\xi}\frac{1}{\log\frac{n}{m}}=\sum_{1\le\ell\le\xi-m}\frac{1}{\log\left(1+\frac{\ell}{m}\right)}\ll\sum_{1\le\ell\le\xi-m}\frac{m}{\ell}\ll m\log\xi\ll\xi\log\xi.$$

Consequently,

$$\Sigma_4\ll m_\xi T\log T\sum_{m\le\xi}\frac{|a_m|}{m}.$$

Now, by combining estimates, we obtain the lemma. □

3.4 The Frequency of Large Values of $|\zeta(\rho+\alpha)|$

Our proof of Theorem 3.0.2 requires the following lemma concerning the distribution of values of $|\zeta(\rho+\alpha)|$ where ρ is a zero of $\zeta(s)$ and $\alpha\in\mathbb{C}$ is a small shift. In what follows, $\log_3(\cdot)$ stands for $\log\log\log(\cdot)$.

Lemma 3.4.1. *Assume the Riemann Hypothesis. Let T be large, $V \geq 3$ a real number, and $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$ and $0 \leq \Re\alpha - \frac{1}{2} \leq (\log T)^{-1}$. Consider the set*

$$\mathcal{S}_\alpha\big(T;V\big) = \big\{\gamma \in (0,T] : \log|\zeta(\rho{+}\alpha)| \geq V\big\}$$

where $\rho = \frac{1}{2} + i\gamma$ denotes a non-trivial zero of $\zeta(s)$. Then, the following inequalities for $\#\mathcal{S}_\alpha\big(T;V\big)$, the cardinality of $\mathcal{S}_\alpha\big(T;V\big)$, hold.

(i) When $\sqrt{\log\log T} \leq V \leq \log\log T$, we have

$$\#\mathcal{S}_\alpha\big(T;V\big) \ll N(T)\frac{V}{\sqrt{\log\log T}}\exp\Big(-\tfrac{V^2}{\log\log T}\Big(1-\tfrac{4}{\log_3 T}\Big)\Big).$$

(ii) When $\log\log T \leq V \leq \frac{1}{2}(\log\log T)\log_3 T$, we have

$$\#\mathcal{S}_\alpha\big(T;V\big) \ll N(T)\frac{V}{\sqrt{\log\log T}}\exp\Big(-\tfrac{V^2}{\log\log T}\Big(1-\tfrac{4V}{(\log\log T)\log_3 T}\Big)\Big).$$

(iii) Finally, when $V > \frac{1}{2}(\log\log T)\log_3 T$, we have

$$\#\mathcal{S}_\alpha\big(T;V\big) \ll N(T)\exp\big(-\tfrac{V}{201}\log V\big).$$

Here, as usual, the function $N(T) \sim \frac{T}{2\pi}\log T$ denotes the number of zeros ρ of $\zeta(s)$ with $0 < \gamma \leq T$.

Proof. Since $\lambda_0 < \frac{3}{5}$, by taking $x = (\log\tau)^{2-\varepsilon}$ in Lemma 3.2.1 and estimating the sum over primes trivially, we find that

$$\log^+|\zeta(\sigma+i\tau)| \leq \Big(\frac{1+\lambda_0}{4} + o(1)\Big)\frac{\log\tau}{\log\log\tau} \leq \frac{2}{5}\frac{\log\tau}{\log\log\tau}$$

for $|\tau|$ sufficiently large. Therefore, we may suppose that $V \leq \frac{2}{5}\frac{\log T}{\log\log T}$, for otherwise the set $\mathcal{S}_\alpha(T;V)$ is empty.

We define a parameter

$$A = A(T,V) = \begin{cases} \frac{1}{2}\log_3(T), & \text{if } \ V \le \log\log T, \\ \frac{\log\log T}{2V}\log_3(T), & \text{if } \ \log\log T < V \le \frac{1}{2}(\log\log T)\log_3 T, \\ 1, & \text{if } \ V > \frac{1}{2}(\log\log T)\log_3 T, \end{cases}$$

and set $x = \min\big(T^{1/2}, T^{A/V}\big)$ and put $z = x^{1/\log\log T}$. Further, we let

$$S_1(s) = \sum_{p\le z} \frac{1}{p^{s+\frac{\lambda_0}{\log x}}}\frac{\log(x/p)}{\log x} \qquad \text{and} \qquad S_2(s) = \sum_{z<p\le x} \frac{1}{p^{s+\frac{\lambda_0}{\log x}}}\frac{\log(x/p)}{\log x}.$$

Then Lemma 3.2.1 implies that

$$\log^+|\zeta(\rho+\alpha)| \le |S_1(\rho)| + |S_2(\rho)| + \frac{(1+\lambda_0)}{2A}V + O\big(\log_3 T\big) \tag{3.20}$$

for any non-trivial zero $\rho = \frac{1}{2}+i\gamma$ of $\zeta(s)$ with $0<\gamma\le T$. Here we have used that $\lambda_0 \ge 1/2$, $x \le T^{1/2}$, and $0 \le \Re\alpha - \frac{1}{2} \le (\log T)^{-1}$ which together imply that

$$\frac{1}{2} \le \Re(\rho+\alpha) \le \frac{1}{2} + \frac{1}{\log T} \le \frac{1}{2} + \frac{\lambda_0}{\log x}.$$

Since $\lambda_0 < 3/5$, it follows from the inequality in (3.20) that

$$\log^+|\zeta(\rho+\alpha)| \le |S_1(\rho)| + |S_2(\rho)| + \tfrac{4}{5}\tfrac{V}{A} + O\big(\log_3 T\big).$$

It follows that if $\rho \in \mathcal{S}_\alpha(T;V)$, then either

$$|S_1(\rho)| \ge V\big(1-\tfrac{9}{10A}\big) \qquad \text{or} \qquad |S_2(\rho)| \ge \tfrac{V}{10A}.$$

For simplicity, we put $V_1 = V\big(1-\frac{9}{10A}\big)$ and $V_2 = \frac{V}{10A}$.

Let $N_1(T;V)$ be the number of ρ with $0<\gamma\le T$ such that $|S_1(\rho)| \ge V_1$ and let $N_2(T;V)$ be the number of ρ with $0<\gamma\le T$ such that $|S_2(\rho)| \ge V_2$. We prove the lemma by obtaining upper bounds for the size of the sets $N_i(T;V)$ for $i=1$ and 2 using the inequality

$$N_i(T;V)\cdot V_i^{2k} \le \sum_{0<\gamma\le T} |S_i(\rho)|^{2k}, \tag{3.21}$$

which holds for any positive integer k. With some restrictions on the size of k, we can use Lemma 3.3.1 to estimate the sums appearing on the right-hand side of this inequality.

We first turn our attention to estimating $N_1(T;V)$. If we define the sequence $\alpha_k(n) = \alpha_k(n, x, z)$ by

$$\sum_{n \le z^k} \frac{\alpha_k(n)}{n^s} = \left(\sum_{p \le z} \frac{1}{p^s} \frac{\log x/p}{\log x} \right)^k ,$$

then it is easily seen that $|\alpha_k(n)| \le k!$. Thus, Lemma 3.3.1 implies that the estimate

$$\begin{aligned} \sum_{0<\gamma\le T} |S_1(\rho)|^{2k} &\ll N(T)\ k! \left(\sum_{p \le z} \frac{1}{p} \frac{\log(x/p)}{\log x} \right)^k \\ &\ll N(T)\ k! \left(\sum_{p \le z} \frac{1}{p} \right)^k \\ &\ll N(T)\sqrt{k} \left(\frac{k \log\log T}{e} \right)^k \end{aligned}$$

holds for any positive integer k with $z^k \le T(\log T)^{-1}$ and T sufficiently large. Using (3.21), we deduce from this estimate that

$$N_1(T;V) \ll N(T)\sqrt{k} \left(\frac{k \log\log T}{eV_1^2} \right)^k . \tag{3.22}$$

It is now convenient to consider separately the case when $V \le (\log\log T)^2$ and the case $V > (\log\log T)^2$. When $V \le (\log\log T)^2$ we choose $k = \lfloor V_1^2 / \log\log T \rfloor$ where, as before, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. To see that this choice of k satisfies $z^k \le T(\log T)^{-1}$, we notice from the definition of A that

$$VA \le \max\left(V,\ \tfrac{1}{2}(\log\log T)\log_3 T\right) .$$

Therefore, we find that

$$\begin{aligned} z^k \le z^{V_1^2/\log\log T} &= \exp\Big(\tfrac{VA\log T}{(\log\log T)^2}\big(1-\tfrac{9}{10A}\big)^2\Big) \\ &\le \exp\Big(\log T\big(1-\tfrac{9}{10A}\big)^2\Big) \\ &\le T/\log T. \end{aligned}$$

Thus, by (3.22), we see that for $V \le (\log\log T)^2$ and T large we have

$$N_1(T;V) \ll N(T)\frac{V}{\sqrt{\log\log T}}\exp\Big(-\frac{V_1^2}{\log\log T}\Big). \tag{3.23}$$

When $V > (\log\log T)^2$ we choose $k = \lfloor 10V \rfloor$. This choice of k satisfies $z^k \le T(\log T)^{-1}$ since $z^{10V} = T^{10/\log\log T} \le T(\log T)^{-1}$ for large T. With this choice of k, we conclude from (3.22) that

$$\begin{aligned} N_1(T;V) &\ll N(T)\exp\Big(\tfrac{1}{2}\log V - 10V\log\Big(\tfrac{eV}{1000\log\log T}\Big)\Big) \\ &\ll N(T)\exp\big(-10V\log V + 11V\log_3(T)\big) \end{aligned} \tag{3.24}$$

for T sufficiently large. Since $V > (\log\log T)^2$, we have that $\log V \ge 2\log_3(T)$ and thus it follows from (3.24) that

$$N_1(T;V) \ll N(T)\exp\big(-4V\log V\big). \tag{3.25}$$

By combining (3.23) and (3.25), we have shown that, for any choice of V,

$$N_1(T;V) \ll N(T)\frac{V}{\sqrt{\log\log T}}\exp\Big(-\frac{V_1^2}{\log\log T}\Big) + N(T)\exp\big(-4V\log V\big). \tag{3.26}$$

We now turn our attention to estimating $N_2(T;V)$. If we define the sequence $\beta_k(n) = \beta_k(n,x,z)$ by

$$\sum_{n\le x^k}\frac{\beta_k(n)}{n^s} = \Bigg(\sum_{z<p\le x}\frac{1}{p^s}\frac{\log x/p}{\log x}\Bigg)^k,$$

then it can be seen that $|\beta_k(n)| \le k!$. Thus, Lemma 3.3.1 implies that

$$\begin{aligned}\sum_{0<\gamma\le T} |S_2(\rho)|^{2k} &\ll N(T)\ k! \Big(\sum_{z<p\le x} \frac{1}{p}\frac{\log(x/p)}{\log x}\Big)^k \\ &\ll N(T)\ k! \Big(\sum_{z<p\le x}\frac{1}{p}\Big)^k \\ &\ll N(T)\ k! \Big(\log_3(T) + O(1)\Big)^k \\ &\ll N(T)\ k! \big(2\log_3(T)\big)^k \\ &\ll N(T)\big(2k\log_3(T)\big)^k \end{aligned} \tag{3.27}$$

for any natural number k with $x^k \le T/\log T$ and T sufficiently large. The choice of $k = \lfloor \frac{V}{A} - 1 \rfloor$ satisfies $x^k \le T/\log T$ when T is large. To see why, recall that $A \ge 1, x = T^{A/V}$, and $V \le \frac{2}{5}\frac{\log T}{\log\log T}$. Therefore,

$$x^k \le x^{(V/A-1)} \le T^{1-A/V} \le T^{1-1/V} = T(\log T)^{-5/2} \le T(\log T)^{-1}.$$

Also, observing that $A \le \frac{1}{2}\log_3(T)$ and recalling that $V \ge \sqrt{\log\log T}$, with this choice of k and T large it follows from (3.21) that

$$\begin{aligned} N_2(T;V) &\ll N(T)\Big(\frac{10A}{V}\Big)^{2k}\big(2k\log_3(T)\big)^k \\ &\ll N(T)\exp\Big(-2k\log(\tfrac{V}{10A}) + k\log(2k\log_3(T))\Big) \\ &\ll N(T)\exp\Big(-2\tfrac{V}{A}\log(\tfrac{V}{10A}) + 2\log\tfrac{V}{10A} + \tfrac{V}{A}\log\big(\tfrac{2V}{A}\log_3(T)\big)\Big) \\ &\ll N(T)\exp\big(-\tfrac{V}{2A}\log V\big). \end{aligned} \tag{3.28}$$

Using our estimates for $N_1(T;V)$ and $N_2(T;V)$ we can now complete the proof of the lemma by checking the various ranges of V. By combining (3.26) and (3.28), we see that

$$\begin{aligned} \#\mathcal{S}_\alpha(T;V) \ll N(T)&\frac{V}{\sqrt{\log\log T}}\exp\big(-\tfrac{V_1^2}{\log\log T}\big) + N(T)\exp\big(-4V\log V\big) \\ &+ N(T)\exp\big(-\tfrac{V}{2A}\log V\big). \end{aligned} \tag{3.29}$$

If $\sqrt{\log\log T} \le V \le \log\log T$, then $A = \frac{1}{2}\log_3(T)$ and (3.29) implies that, for T sufficiently large,

$$\begin{aligned}\#\mathcal{S}_\alpha(T;V) &\ll N(T)\frac{V}{\sqrt{\log\log T}}\exp\left(-\tfrac{V^2}{\log\log T}\left(1-\tfrac{9}{5\log_3 T}\right)^2\right)\\ &\ll N(T)\frac{V}{\sqrt{\log\log T}}\exp\left(-\tfrac{V^2}{\log\log T}\left(1-\tfrac{4}{\log_3 T}\right)\right).\end{aligned} \tag{3.30}$$

If $\log\log T < V \le \frac{1}{2}(\log\log T)\log_3(T)$, then $A = \frac{\log\log T}{2V}\log_3(T)$ and we deduce from (3.29) that

$$\begin{aligned}\#\mathcal{S}_\alpha(T;V) \ll N(T)&\frac{V}{\sqrt{\log\log T}}\exp\left(-\tfrac{V^2}{\log\log T}\left(1-\tfrac{9}{5(\log\log T)\log_3 T}\right)^2\right)\\ &+N(T)\exp\left(-\tfrac{V^2\log V}{(\log\log T)\log_3 T}\right)+N(T)\exp\big(-4V\log V\big).\end{aligned} \tag{3.31}$$

For V in this range, $\frac{\log V}{(\log\log T)\log_3 T} > \frac{1}{\log\log T}$ and $\frac{V}{\log V} < \log\log T$, so (3.31) implies that

$$\begin{aligned}\#\mathcal{S}_\alpha(T;V) &\ll N(T)\frac{V}{\sqrt{\log\log T}}\exp\left(-\tfrac{V^2}{\log\log T}\left(1-\tfrac{9}{5(\log\log T)\log_3 T}\right)^2\right)\\ &\ll N(T)\frac{V}{\sqrt{\log\log T}}\exp\left(-\tfrac{V^2}{\log\log T}\left(1-\tfrac{4}{(\log\log T)\log_3 T}\right)\right).\end{aligned} \tag{3.32}$$

Finally, if $V \ge \frac{1}{2}(\log\log T)\log_3 T$, then $A = 1$ and we deduce from (3.29) that

$$\#\mathcal{S}_\alpha(T;V) \ll N(T)\exp\left(\log V - \tfrac{V^2}{100\log\log T}\right) + N(T)\exp\left(-\tfrac{V}{2}\log V\right). \tag{3.33}$$

Certainly, if $V \ge \frac{1}{2}(\log\log T)\log_3 T$ then we have that $\frac{V^2}{100\log\log T} - \log V > \frac{1}{201}V\log V$ for T sufficiently large and so it follows from (3.33) that

$$\#\mathcal{S}_\alpha(T;V) \ll N(T)\exp\left(-\tfrac{V}{201}\log V\right). \tag{3.34}$$

The lemma now follows from the estimates in (3.30), (3.32), and (3.34). $\square$

3.5 Proof of Theorem 3.0.2

Using Lemma 3.4.1, we first prove Theorem 3.0.2 in the case where $|\alpha| \le 1$ and $0 \le \Re\alpha \le (\log T)^{-1}$. Then, from this result, the case when $-(\log T)^{-1} \le \Re\alpha < 0$ can be deduced from the functional equation for $\zeta(s)$ and Stirling's formula for the gamma function. In what follows, $k \in \mathbb{R}$ is fixed and we let $\varepsilon > 0$ be an arbitrarily small positive constant which may not be the same at each occurrence.

First, we partition the real axis into the intervals $I_1 = (-\infty, 3], I_2 = (3, 4k \log\log T]$, and $I_3 = (4k \log\log T, \infty)$ and set

$$\Sigma_i = \sum_{\nu \in I_i \cap \mathbb{Z}} e^{2k\nu} \cdot \#\mathcal{S}_\alpha(T, \nu)$$

for $i = 1, 2$, and 3. Then we observe that

$$\sum_{0<\gamma\le T} \left|\zeta(\rho+\alpha)\right|^{2k} \le \sum_{\nu\in\mathbb{Z}} e^{2k\nu} \Big[\#\mathcal{S}_\alpha(T,\nu) - \#\mathcal{S}_\alpha(T,\nu-1)\Big] \le \Sigma_1 + \Sigma_2 + \Sigma_3. \quad (3.35)$$

Using the trivial bound $\#\mathcal{S}_\alpha(T,\nu) \le N(T)$, which holds for every $\nu \in \mathbb{Z}$, we find that $\Sigma_1 \le e^{6k} N(T)$. To estimate Σ_2, we use the bound

$$\#\mathcal{S}_\alpha(T,\nu) \ll N(T)(\log T)^\varepsilon \exp\left(\frac{-\nu^2}{\log\log T}\right)$$

which follows from the first two cases of Lemma 3.4.1 when $\nu \in I_2 \cap \mathbb{Z}$. From this, it follows that

$$\begin{aligned}
\Sigma_2 &\ll N(T)(\log T)^\varepsilon \int_3^{4k\log\log T} \exp\left(2ku - \tfrac{u^2}{\log\log T}\right) du \\
&\ll N(T)(\log T)^\varepsilon \int_0^{4k} (\log T)^{u(2k-u)}\, du \\
&\ll N(T)(\log T)^{k^2+\varepsilon}
\end{aligned}$$

When $\nu \in I_3 \cap \mathbb{Z}$, the second two cases of Lemma 3.4.1 imply that

$$\#\mathcal{S}_\alpha(T,\nu) \ll N(T)(\log T)^\varepsilon e^{-4k\nu}.$$

Thus,

$$\Sigma_3 \ll N(T)(\log T)^{\varepsilon} \int_{4k\log\log T}^{\infty} e^{-2ku}\, du$$
$$\ll N(T)(\log T)^{-8k^2+\varepsilon}.$$

In light of (3.35), by collecting estimates, we see that

$$\sum_{0<\gamma\le T} \left|\zeta(\rho+\alpha)\right|^{2k} \ll N(T)(\log T)^{k^2+\varepsilon} \tag{3.36}$$

for every $k > 0$ when $|\alpha| \le 1$ and $0 \le \Re\alpha \le (\log T)^{-1}$.

The functional equation for the zeta-function states that $\zeta(s) = \chi(s)\zeta(1-s)$ where $\chi(s) = 2^s\pi^{s-1}\Gamma(1-s)\sin\left(\frac{\pi s}{2}\right)$. Stirling's asymptotic formula for the gamma function (see Appendix A.7 of Ivič [19]) can be used to show that

$$\left|\chi(\sigma+it)\right| = \left(\frac{|t|}{2\pi}\right)^{1/2-\sigma}\left(1+O\left(\frac{1}{|t|}\right)\right)$$

uniformly for $-1 \le \sigma \le 2$ and $|t| \ge 1$. Using the Riemann Hypothesis, we see that

$$\begin{aligned}\left|\zeta(\rho+\alpha)\right| &= \left|\chi(\rho+\alpha)\zeta(1-\rho-\alpha)\right| \\ &= \left|\chi(\rho+\alpha)\zeta(\bar\rho-\alpha)\right| \\ &= \left|\chi(\rho+\alpha)\zeta(\rho-\bar\alpha)\right| \\ &\le C\left|\zeta(\rho-\bar\alpha)\right|\end{aligned}$$

for some absolute constant $C > 0$ when $|\alpha| \le 1$, $|\Re\alpha - \frac{1}{2}| \le (\log T)^{-1}$, and $0 < \gamma \le T$. Consequently, for $\Re\alpha < 0$,

$$\sum_{0<\gamma\le T} \left|\zeta(\rho+\alpha)\right|^{2k} \le C^{2k} \cdot \sum_{0<\gamma\le T} \left|\zeta(\rho-\bar\alpha)\right|^{2k}. \tag{3.37}$$

Applying the inequality in (3.36) to the right-hand side of (3.37) we see that

$$\sum_{0<\gamma\le T} \left|\zeta(\rho+\alpha)\right|^{2k} \ll N(T)(\log T)^{k^2+\varepsilon} \tag{3.38}$$

for every $k > 0$ when $|\alpha| \le 1$ and $-(\log T)^{-1} \le \Re\alpha < 0$. The theorem now follows from the estimates in (3.36) and (3.38).

3.6 Proof of the Upper Bound in Theorem 1.1.1

The upper bound in Theorem 1.1.1 can now be established as a simple consequence of Theorem 3.0.2 and the following lemma.

Lemma 3.6.1. *Assume the Riemann Hypothesis. Let $k, \ell \in \mathbb{N}$ and let $R > 0$ be arbitrary. Then we have*

$$\sum_{0<\gamma\leq T} \left|\zeta^{(\ell)}(\rho)\right|^{2k} \leq \left(\frac{\ell!}{R^\ell}\right)^{2k} \cdot \left[\max_{|\alpha|\leq R} \sum_{0<\gamma\leq T} \left|\zeta(\rho+\alpha)\right|^{2k}\right]. \tag{3.39}$$

Proof. Since the function $\zeta^{(\ell)}(s)$ is real when $s \in \mathbb{R}$, $\zeta^{(\ell)}(\bar{s}) = \overline{\zeta^{(\ell)}(s)}$. Hence, assuming the Riemann Hypothesis, the identity

$$\left|\zeta^{(\ell)}(1-\rho+\alpha)\right| = \left|\zeta^{(\ell)}(\bar{\rho}+\alpha)\right| = \left|\zeta^{(\ell)}(\rho+\overline{\alpha})\right| \tag{3.40}$$

holds for any non-trivial zero ρ of $\zeta(s)$ and any $\alpha \in \mathbb{C}$. For each positive integer k, let $\hat{\alpha}_k = (\alpha_1, \alpha_2, \ldots, \alpha_{2k})$ and define

$$\mathcal{Z}\big(s; \hat{\alpha}_k\big) = \prod_{i=1}^{k} \zeta(s+\alpha_i)\zeta(1-s+\alpha_{i+k}).$$

If we suppose that each $|\alpha_i| \leq R$ for $i = 1, \ldots, 2k$ and apply Hölder's inequality in the form

$$\left|\sum_{n=1}^{N} \left(\prod_{i=1}^{2k} f_i(s_n)\right)\right| \leq \prod_{i=1}^{2k} \Big(\sum_{n=1}^{N} |f_i(s_n)|^{2k}\Big)^{\frac{1}{2k}},$$

we see that (3.40) implies that

$$\begin{aligned} \left|\sum_{0<\gamma\leq T} \mathcal{Z}\big(\rho; \hat{\alpha}_k\big)\right| &\leq \prod_{i=1}^{k} \left(\sum_{0<\gamma\leq T} \left|\zeta(\rho+\alpha_i)\right|^{2k}\right)^{\frac{1}{2k}} \left(\sum_{0<\gamma\leq T} \left|\zeta(\rho+\overline{\alpha_{k+i}})\right|^{2k}\right)^{\frac{1}{2k}} \\ &\leq \max_{|\alpha|\leq R} \sum_{0<\gamma\leq T} \left|\zeta(\rho+\alpha)\right|^{2k} \end{aligned} \tag{3.41}$$

In order to prove the lemma, we first rewrite the left-hand side of equation (3.39) using the function $\mathcal{Z}(s;\hat{\alpha}_k)$ and then apply the inequality in (3.41). By Cauchy's integral formula and another application of (3.40), we see that

$$\begin{aligned}\sum_{0<\gamma\le T}\left|\zeta^{(\ell)}(\rho)\right|^{2k} &= \sum_{0<\gamma\le T}\left(\prod_{i=1}^{k}\zeta^{(\ell)}(\rho)\zeta^{(\ell)}(1-\rho)\right)\\ &= \frac{(\ell!)^{2k}}{(2\pi i)^{2k}}\int_{\mathscr{C}_1}\cdots\int_{\mathscr{C}_{2k}}\left(\sum_{0<\gamma\le T}\mathcal{Z}(\rho;\hat{\alpha}_k)\right)\prod_{i=1}^{2k}\frac{d\alpha_i}{\alpha_i^{\ell+1}}\end{aligned} \tag{3.42}$$

where, for each $i=1,\ldots,2k$, the contour $\mathscr{C}_i$ denotes the positively oriented circle in the complex plane centered at 0 with radius R. Now, combining (3.41) and (3.42) we find that

$$\begin{aligned}\sum_{0<\gamma\le T}\left|\zeta^{(\ell)}(\rho)\right|^{2k} &\le \left(\frac{\ell!}{2\pi}\right)^{2k}\cdot\left[\max_{|\alpha|\le R}\sum_{0<\gamma\le T}\left|\zeta(\rho+\alpha)\right|^{2k}\right]\cdot\int_{\mathscr{C}_1}\cdots\int_{\mathscr{C}_{2k}}\prod_{i=1}^{2k}\frac{d\alpha_i}{|\alpha_i|^{\ell+1}}\\ &\le \left(\frac{\ell!}{2\pi}\right)^{2k}\cdot\left[\max_{|\alpha|\le R}\sum_{0<\gamma\le T}\left|\zeta(\rho+\alpha)\right|^{2k}\right]\cdot\left(\frac{2\pi}{R^{\ell}}\right)^{2k}\\ &\le \left(\frac{\ell!}{R^{\ell}}\right)^{2k}\cdot\left[\max_{|\alpha|\le R}\sum_{0<\gamma\le T}\left|\zeta(\rho+\alpha)\right|^{2k}\right],\end{aligned}$$

as claimed. □

Proof of the upper bound in Theorem 1.1.1. Let $k\in\mathbb{N}$ and set $R=(\log T)^{-1}$. Then, it follows from Theorem 3.0.2 and Lemma 3.6.1 that

$$\sum_{0<\gamma\le T}\left|\zeta^{(\ell)}(\rho)\right|^{2k}\ll_{k,\ell,\varepsilon} N(T)(\log T)^{k(k+2\ell)+\varepsilon} \tag{3.43}$$

for any $\ell\in\mathbb{N}$ and for $\varepsilon>0$ arbitrary. By setting $\ell=1$ and using the estimate $N(T)\ll T\log T$, we establish (3.1) and, thus, the upper bound in Theorem 1.1.1.

4 The Proof of Theorem 1.2.1

In this chapter, we prove Theorem 4. The theorem says that assuming the Riemann Hypothesis and that the zeros of $\zeta(s)$ are simple we have

$$\sum_{0<\gamma\leq T} \frac{1}{|\zeta'(\rho)|^2} \geq \Big(\frac{3}{2\pi^3} - \varepsilon\Big)T \tag{4.1}$$

for $\varepsilon > 0$ arbitrary and T sufficiently large. Under the same assumptions, Gonek has previously shown that

$$\sum_{0<\gamma\leq T} \frac{1}{|\zeta'(\rho)|^2} \gg T \tag{4.2}$$

and, as mentioned in the introduction, he conjectured that in fact

$$\sum_{0<\gamma\leq T} \frac{1}{|\zeta'(\rho)|^2} \sim \frac{3}{\pi^3}T.$$

Gonek conjectured this formula using a heuristic similar to Montgomery's study of the pair correlation of the ordinates of the zeros of $\zeta(s)$. Independently, using a heuristic based upon random matrix theory, Hughes, Keating, and O'Connell have conjectured the same asymptotic formula [16]. We see that our lower bound for $J_{-1}(T)$ given in (4.1) differs from the conjectured value by a factor of 2 and quantifies the constant implicit in Gonek's estimate given in (4.2).

The proof of the inequality in (4.1) is similar to the method used in the proof the lower bound in Theorem 1.1.1 given in Chapter 2. Let $\mathcal{M}_\xi(s) = \sum_{n\le\xi} \mu(n)n^{-s}$ where the $\mu(\cdot)$ is the Möbius function. Assuming the Riemann Hypothesis, for any non-trivial zero $\rho = \frac{1}{2} + i\gamma$, we have $\overline{\mathcal{M}_\xi(\rho)} = \mathcal{M}_\xi(1-\rho)$. Using this fact and letting

$$M_1 = \sum_{0<\gamma\le T} \frac{1}{\zeta'(\rho)}\mathcal{M}_\xi(1-\rho) \qquad \text{and} \qquad M_2 = \sum_{0<\gamma\le T} \left|\mathcal{M}_\xi(\rho)\right|^2,$$

Cauchy's inequality implies that

$$\sum_{0<\gamma\le T} \frac{1}{|\zeta'(\rho)|^2} \ge \frac{\left|M_1\right|^2}{M_2}.$$

Here we are implicitly assuming the zeros of $\zeta(s)$ are simple so that the sums involved in the above argument are well defined. The estimate in (4.1) now follows if we can show that $M_1 \sim \frac{3}{\pi^3}T\log T$ and that $M_2 \sim \frac{6}{\pi^3}T(\log T)^2$ for a particular choice of ξ.

4.1 The Calculation of M_1

We first estimate the sum M_1. Assuming the Riemann Hypothesis there exist a sequence $\mathcal{T} = \{\tau_n\}_{n=3}^\infty$, $n < \tau_n \le n+1$, such that

$$\left|\zeta(\sigma + i\tau_n)\right|^{-1} \ll \exp\left(\frac{A\log\tau_n}{\log\log\tau_n}\right) \tag{4.3}$$

for some constant $A > 0$ and uniformly for $\frac{1}{2} \le \sigma \le 2$. For a proof of this fact, see Theorem 14.16 of Titchmarsh [37]. In order to prove (4.1), we can assume, with no loss of generality, that T is in the sequence $\mathcal{T}$. In what follows, we will also assume that $\xi = o(T)$ and choose ξ as a function of T after we have evaluated M_1 and M_2.

Let $c = 1 + (\log T)^{-1}$ and recall that $|\gamma| > 14$ for every non-trivial zero $\rho = \frac{1}{2} + i\gamma$ of $\zeta(s)$. By assuming that all the zeros of $\zeta(s)$ are simple, Cauchy's integral theorem then implies that

$$\begin{aligned} M_1 &= \frac{1}{2\pi i}\left(\int_{c+i}^{c+iT} + \int_{c+iT}^{1-c+iT} + \int_{1-c+iT}^{1-c+1} + \int_{1-c+i}^{c+i}\right) \frac{1}{\zeta(s)}\mathcal{M}_\xi(1-s)\, ds \\ &= \frac{1}{2\pi i}\Big(I_1 + I_2 + I_3 + I_4\Big), \end{aligned}$$

say. Here we have used the fact that the residue of the function $1/\zeta(s)$ at $s = \rho$ equals $1/\zeta'(\rho)$ if ρ is a simple zero of $\zeta(s)$.

The main contribution to M_1 comes from the integral I_1; the rest of the integrals contribute an error term. To handle the integral I_1, we write $1/\zeta(s)$ as a Dirichlet series and interchange the sums and the integral. After a change of variables, we find that

$$2\pi I_1 = \sum_{m=1}^{\infty}\sum_{n\le\xi} \frac{\mu(m)\mu(n)}{n^c m^{1-c}} \int_1^T \left(\frac{m}{n}\right)^{it} dt.$$

Integrating and using the fact that $|\mu(n)| \le 1$, it follows that

$$2\pi I_1 = (T-1)\sum_{n\le\xi}\frac{\mu(n)^2}{n} + O\left(\sum_{m=1}^{\infty}\sum_{\substack{n\le\xi \\ n\ne m}} \frac{1}{m^c n^{1-c}|\log m/n|}\right).$$

The big O-term is

$$\begin{aligned} &\ll \sum_{m=1}^{\infty}\frac{1}{m^c}\sum_{\substack{n\le\xi \\ n\ne m}}\frac{1}{|\log m/n|} \\ &\ll \xi\log\xi\sum_{m=1}^{\infty}\frac{1}{m^c} \\ &\ll \zeta(c)\ \xi\log\xi \\ &\ll \xi\ (\log T)^2, \end{aligned}$$

since $\zeta(c) \ll \log T$ for our choice of c. From the standard estimate

$$\sum_{n \le \xi} \frac{\mu(n)^2}{n} = \frac{6}{\pi^2} \log \xi + O(1) \tag{4.4}$$

we deduce that

$$I_1 = \frac{3}{\pi^3} T \log \xi + O\big(\xi (\log T)^2\big) + O(T).$$

To estimate the contribution from the integral I_2, we recall that the functional equation for the Riemann zeta-function says that

$$\zeta(s) = \chi(s)\zeta(1-s) \quad \text{ where } \quad \chi(s) = 2^s \pi^{s-1} \Gamma(1-s) \sin\left(\frac{\pi s}{2}\right).$$

Stirling's asymptotic formula for the gamma function can be used to show that

$$\big|\chi(\sigma + it)\big| = \left(\frac{|t|}{2\pi}\right)^{1/2-\sigma} \Big(1 + O\big(|t|^{-1}\big)\Big) \tag{4.5}$$

uniformly for $-1 \le \sigma \le 2$ and $|t| \ge 1$. Combining this estimate and (4.3), it follows that, for $T \in \mathfrak{T}$,

$$\big|\zeta(\sigma + it)\big|^{-1} \ll T^{\min(\sigma - 1/2), 0)} \exp\left(\frac{A \log T}{\log\log T}\right)$$

uniformly for $-1 \le \sigma \le 2$ for some constant $A > 0$. As a consequence, we can estimate the integral I_2 trivially to find that

$$I_2 \ll \exp\left(\frac{A \log T}{\log\log T}\right) \sum_{n \le \xi} n^{c-1} \ll \xi \exp\left(\frac{A \log T}{\log\log T}\right)$$

since $\xi^{c-1} \ll T^{c-1} = e$.

To handle the integral I_3, we notice that the functional equation for $\zeta(s)$ combined with the estimate in (4.5) implies that, for $1 \le |t| \le T$,

$$\big|\zeta(\sigma + it)\big|^{-1} \ll |t|^{1/2-c} \big|\zeta(c - it)\big|^{-1} \ll |t|^{1/2-c} \zeta(c) \ll |t|^{-1/2} \log T,$$

and thus that

$$I_3 \ll \log T \Big(\sum_{n \le \xi} \frac{|\mu(n)|}{n^c} \Big) \int_1^T t^{-1/2} \, dt.$$

Since $|\mu(n)| \le 1$ and

$$\sum_{n \le \xi} \frac{1}{n^c} \le \sum_{n \le \xi} \frac{1}{n} \ll \log \xi \ll \log T,$$

integrating we find that $I_3 \ll \sqrt{T} (\log T)^2$.

Finally, since $1/\zeta(s)$ and $\mathcal{M}_\xi(1-s)$ are bounded on the interval $[1-c+i, c+i]$, we find that $I_4 \ll 1$. Thus, our combined estimates for I_1, I_2, I_3, and I_4 imply that

$$M_1 = \frac{3}{\pi^3} T \log \xi + O\Big(\xi \exp \Big(\frac{A \log T}{\log \log T} \Big) \Big) + O(T)$$

for a fixed constant $A > 0$.

4.2 The Calculation of M_2

We now turn our attention to M_2. Assuming the Riemann Hypothesis, we notice that

$$\begin{aligned} M_2 &= \sum_{0 < \gamma \le T} \mathcal{M}_\xi(\rho) \mathcal{M}_\xi(1-\rho) \\ &= N(T) \sum_{n \le \xi} \frac{\mu(n)^2}{n} + \; 2\Re \sum_{m \le \xi} \mu(m) \sum_{m < n \le \xi} \frac{\mu(n)}{n} \sum_{0 < \gamma \le T} \Big(\frac{n}{m} \Big)^\rho \end{aligned} \tag{4.6}$$

where $N(T) = \frac{T}{2\pi} \log T + O(T)$ denotes the number of non-trivial zeros of $\zeta(s)$ up to height T. Combining this estimate for $N(T)$ with (4.4), we see that

$$N(T) \sum_{n \le \xi} \frac{\mu(n)^2}{n} = \frac{3}{\pi^3} T \log T \log \xi + O(T \log T).$$

To handle the triple sum on the second line of (4.6), we use the Landau-Gonek Formula (Lemma 2.3.1 above) and see that

$$\begin{aligned}\sum_{m\le\xi}\sum_{n<m}\frac{\mu(m)\mu(n)}{n}\sum_{0<\gamma\le T}\Big(\frac{n}{m}\Big)^{\rho} &= -\frac{T}{2\pi}\sum_{m\le\xi}\sum_{m<n\le\xi}\frac{\mu(m)\mu(n)}{n}\Lambda\Big(\frac{n}{m}\Big)\\ &\quad + O\left(\log T\log\log T\sum_{m\le\xi}\sum_{m<n\le\xi}\frac{1}{m}\right)\\ &\quad + O\left(\sum_{m\le\xi}\frac{1}{m}\sum_{m<n\le\xi}\frac{\log\frac{m}{n}}{<\frac{m}{n}>}\right)\\ &\quad + O\left(\log T\sum_{m\le\xi}\sum_{m<n\le\xi}\frac{1}{n\log\frac{n}{m}}\right)\\ &= \Sigma_1+\Sigma_2+\Sigma_3+\Sigma_4,\end{aligned}$$

say. To estimate Σ_1, let $m = p^k n$ for primes p and then we find that

$$\begin{aligned}\Sigma_1 &= -\frac{T}{2\pi}\sum_{m\le\xi}\frac{\mu(m)}{m}\sum_{p^k\le\xi/m}\frac{\mu(p^k m)\log p}{p^k}\\ &= -\frac{T}{2\pi}\sum_{m\le\xi}\frac{\mu(m)}{m}\sum_{p\le\xi/m}\frac{\mu(pm)\log p}{p}\\ &= \frac{T}{2\pi}\sum_{m\le\xi}\frac{\mu(m)^2}{m}\sum_{\substack{p\le\xi/m\\(p,m)=1}}\frac{\log p}{p} + O\Big(T\log T\max_{m\le\xi}\sum_{p\mid m}\frac{\log p}{p}\Big)\\ &= \frac{T}{2\pi}\sum_{m\le\xi}\frac{\mu(m)^2}{m}\sum_{p\le\xi/m}\frac{\log p}{p} + O\Big(T\log T\max_{m\le\xi}\sum_{p\mid m}\frac{\log p}{p}\Big)\\ &= \frac{T}{2\pi}\sum_{m\le\xi}\frac{\mu(m)^2}{m}\log\tfrac{\xi}{m} + O\Big(T\log T\max_{m\le\xi}\sum_{p\mid m}\frac{\log p}{p}\Big)\end{aligned}$$

where we have used Merten's well-known estimate

$$\sum_{p\le\xi}\frac{\log p}{p} = \log\xi + O(1). \tag{4.7}$$

By (4.4) and partial summation, we see that

$$\sum_{m\le\xi}\frac{\mu(m)^2}{m}\log\tfrac{\xi}{m}=\frac{3}{\pi^2}(\log\xi)^2+O(\log\xi).$$

To finish our estimate for Σ_1, we need to bound the sum over primes in the O-term. Let $p_1, p_2, p_3, \dots$ denote the sequence of primes (in order) and, for each integer $m \ge 3$, let k be the unique number such that

$$p_1p_2\cdots p_{k-1}\le m<p_1p_2\cdots p_k.$$

The Prime Number Theorem then implies that

$$p_{k-1}\ll\sum_{j=1}^{k-1}\log p_j\le\log m$$

and hence, by (4.7),

$$\sum_{p|m}\frac{\log p}{p}\ll\sum_{p\le p_k}\frac{\log p}{p}\ll\log p_k\ll\log\log m.$$

Therefore, collecting estimates, we deduce that

$$\Sigma_1=\frac{3T}{2\pi^2}\log\xi+O(T\log T\log\log T).$$

Also, it is easily shown that $\Sigma_2\ll\xi(\log T)^2\log\log T\ll\xi(\log T)^3$. Turning to Σ_3, we write n as $qm+\ell$ with $-\frac{m}{2}<\ell\le\frac{m}{2}$ and find that

$$\Sigma_3\ll\log\xi\sum_{m\le\xi}\frac{1}{m}\sum_{q\le\lfloor\frac{\xi}{m}\rfloor+1}\;\sum_{-\frac{m}{2}<\ell\le\frac{m}{2}}\frac{1}{\langle q+\frac{\ell}{m}\rangle}$$

where $\lfloor x\rfloor$ denotes the greatest integer $\le x$. Notice that $\langle q+\frac{\ell}{m}\rangle=\frac{|\ell|}{m}$ if q is a prime power and $\ell\ne0$, otherwise $\langle q+\frac{\ell}{m}\rangle$ is $\ge\frac{1}{2}$. Hence,

$$\Sigma_3\ll\log\xi\Big(\sum_{m\le\xi}\frac{1}{m}\sum_{\substack{q\le\lfloor\frac{\xi}{m}\rfloor+1\\ \Lambda(q)\ne0}}\;\sum_{1\le\ell\le\frac{m}{2}}\frac{m}{\ell}+\sum_{m\le\xi}\frac{1}{m}\sum_{q\le\lfloor\frac{\xi}{m}\rfloor+1}\;\sum_{1\le\ell\le\frac{m}{2}}1\Big)$$

$$\ll\log\xi\Big(\sum_{m\le\xi}\;\sum_{q\le\lfloor\frac{\xi}{m}\rfloor+1}1\Big)\ll\xi(\log\xi)^2\ll\xi(\log T)^2.$$

It remains to consider Σ_4. For integers $1 \le m < n \le \xi$, let $n = m + \ell$. Then

$$\log \frac{n}{m} = -\log\left(1 - \frac{\ell}{m}\right) > \frac{\ell}{m}.$$

Consequently,

$$\Sigma_4 \ll \sum_{m \le \xi} \sum_{1 \le \ell \le \xi} \frac{1}{(m+\ell)\frac{\ell}{m}} \ll \sum_{m \le \xi} \sum_{1 \le \ell \le \xi} \frac{1}{\ell} \ll \xi(\log \xi) \ll \xi(\log T)^2.$$

Collecting all our estimates, we have shown that

$$M_2 = \frac{3}{\pi^3} T \log T \log \xi + \frac{3}{\pi^3} T(\log \xi)^2 + O\big(\xi(\log \xi)^3\big) + O(T \log T)$$

and that

$$M_1 = \frac{3}{\pi^3} T \log \xi + O\Big(\xi \exp\Big(\frac{A \log T}{\log \log T}\Big)\Big) + O(T)$$

for a fixed constant $A > 0$. To complete to the proof of (4.1), it remains to choose a value of ξ with $\xi = o(T)$ so that $M_1 \sim \frac{3}{\pi^3} T \log T$ and $M_2 \sim \frac{6}{\pi^3} T(\log T)^2$. To accomplish this, we simply choose a constant $B > A$ and let $\xi = T \exp\big(\frac{-B \log T}{\log \log T}\big)$.

4.3 Some Remarks

Let $\mathcal{P}$ be a polynomial. If we repeat the above calculation with the function

$$\mathcal{M}_\xi\big(s; \mathcal{P}\big) = \sum_{n \le \xi} \frac{\mu(n)}{n^s} \mathcal{P}\big(\tfrac{\log n}{\log \xi}\big)$$

in place of $\mathcal{M}_\xi(s)$, it can be shown that

$$\sum_{0 < \gamma < T} \frac{1}{|\zeta'(\rho)|^2} \ge \Big(\frac{3T}{\pi^3} + o(T)\Big) \cdot I(P)$$

where

$$I(P) = \frac{\Big(\int_0^1 P(x) dx\Big)^2}{\int_0^1 P(x)^2 dx + 2\int_0^1 \int_0^{1-y} P(x)P(x+y) dx dy}\ .$$

The choice of $P = 1$ recovers our estimate in (4.1). Though I have not been able to prove it, numerical calculations seem to imply that $I(P) \leq 1/2$. If that is the case, the estimate in (4.1) is the best that can be attained by this method when $\xi \leq T$. It would be interesting to see if a better lower bound for $J_{-1}(T)$ can be obtained using mean-value theorems of long Dirichlet polynomials (i.e. the case when $\xi \gg T$) along with the assumption of certain correlation sum estimates for the Möbius function of the form

$$\sum_{n \leq \xi} \mu(n)\mu(kn + h) \ll \xi^{1/2+\varepsilon}$$

with a certain amount of uniformity in k and h. In this case, the analogue of the sum M_1 could be handled using a result of Goldston and Gonek [7]. To handle the analogue of the sum M_2, we would need a formula for the mean-square of long Dirichlet polynomials summed over the zeros of $\zeta(s)$. Jim Bian, Steve Gonek, Heekyoung Hahn, Nathan Ng, and I have begun work on such a formula.

5 The Proof of Theorem 1.3.1

In this chapter we prove a conjecture of Conrey and Snaith (see the remark at the end of this chapter) concerning the lower-order terms of the second moment of $|\zeta'(\rho)|$. Throughout this chapter we let $T \geq 1$, $\mathscr{L} = \log \frac{T}{2\pi}$, and $\varepsilon > 0$ be arbitrary. Then, assuming the Riemann Hypothesis, we show that

$$\sum_{0<\gamma\leq T} |\zeta'(\rho)|^2 = \frac{T}{24\pi}\mathscr{L}^4 + \frac{(2\gamma_0 - 1)}{6\pi}T\mathscr{L}^3 + B_2 T\mathscr{L}^2 + B_1 T\mathscr{L} + B_0 T + O(T^{1/2+\varepsilon}) \tag{5.1}$$

where γ_0 is Euler's constant and B_2, B_1, and B_0 are other, computable, constants. Our method allows us to express the B_i (for $i = 0, 1, 2$) in terms of the the coefficients γ_k in the Laurent series expansion

$$\zeta(s) = \frac{1}{s-1} + \sum_{k=0}^{\infty} \frac{(-1)^k \gamma_k}{k!}(s-1)^k \tag{5.2}$$

around $s = 1$. This will be done at the end of the chapter.

From (5.1), we recover a well-known result of Gonek [8] which, assuming the

Riemann Hypothesis, says that

$$\sum_{0<\gamma\le T} |\zeta'(\rho)|^2 = \frac{T}{24\pi}\log^4 T + O\big(T\log^3 T\big). \tag{5.3}$$

In fact, our proof of (5.1) is a modification of Gonek's method taking care to keep track of all lower-order terms.

5.1 Some Preliminary Results

In this section we collect a few results that are necessary in order to prove the estimate in (5.1). We begin by recalling a classical result of Ingham [17].

Lemma 5.1.1. *Let $\nu \ge 0$. Then for $T \ge 1$ we have*

$$\int_1^T |\zeta^{(\nu)}(\tfrac{1}{2}+it)|^2 dt = \frac{T}{2\nu+1}\sum_{n=0}^{2\nu} C_n \binom{2\nu+1}{n}\Big(\log\frac{T}{2\pi}\Big)^{2\nu+1-n} + \Big(C_{2\nu} + \frac{2C_{2\nu+1}}{2\nu+1}\Big)T + O(T^{1/2}\mathscr{L}^{2\nu+1}),$$

where the coefficients C_n are defined by $\frac{s\zeta(1+s)}{1+s} = \sum_{n=0}^{\infty}\frac{C_n}{n!}s^n$ for $|s|<1$.

In the case $\nu = 1$, which is the only case we use, Lemma 5.1.1 and equation (5.2) imply that, for $T \ge 1$,

$$\int_1^T |\zeta'(\tfrac{1}{2}+it)|^2 = T\ \mathcal{P}_1\big(\mathscr{L}\big) + O(T^{1/2}\mathscr{L}^3),$$

where the polyinomial $\mathcal{P}_1$ is defined as

$$\mathcal{P}_1(x) = \frac{x^3}{3} - (1-\gamma_0)x^2 + 2(1-\gamma_0-\gamma_1)x - 2(1-\gamma_0-\gamma_1-\gamma_2). \tag{5.4}$$

The functional equation for $\zeta(s)$ can be written

$$\zeta(s) = \chi(s)\zeta(1-s), \quad \text{where} \quad \chi(s) = \pi^{s-1/2}\frac{\Gamma\big(\frac{1-s}{2}\big)}{\Gamma\big(\frac{s}{2}\big)}. \tag{5.5}$$

Using Stirling's asymptotic formula for the gamma function, we can deduce that the estimates

$$\bigl|\chi(\sigma+it)\bigr| = |t|^{\sigma-1/2}\bigl(1+O\bigl(|t|^{-1}\bigr)\bigr) \tag{5.6}$$

and

$$\frac{\chi'}{\chi}(\sigma+it) = -\log\tfrac{|t|}{2\pi} + O\bigl(|t|^{-1}\bigr) \tag{5.7}$$

hold uniformily for $-1 \le \sigma \le 2$ and $|t| \ge 1$. By combining these estimates, it follows that

$$\begin{aligned}\chi'(1-s) &= \frac{\chi'}{\chi}(1-s)\chi(1-s)\\ &= -\chi(1-s)\log\tfrac{|t|}{2\pi} + O\bigl(|t|^{\sigma-1/2}\bigr)\end{aligned} \tag{5.8}$$

uniformily for $-1 \le \sigma \le 2$ and $t \ge 1$, as well.

We also require a few additional lemmas.

Lemma 5.1.2. *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of complex numbers satisfying $a_n \ll n^{\varepsilon}$ for $\varepsilon > 0$ arbitrary. Let $c > 1$ be fixed and let m be a non-negative integer, then for $T \ge 1$ we have*

$$\begin{aligned}\frac{1}{2\pi}\int_1^T \Bigl(\sum_{n=1}^{\infty}\frac{a_n}{n^{c+it}}\Bigr)\chi(1-c-it)\bigl(\log\frac{t}{2\pi}\bigr)^m dt\\ = \sum_{1\le n\le \frac{T}{2\pi}} a_n(\log n)^m + O\bigl(T^{c-1/2}(\log T)^m\bigr).\end{aligned} \tag{5.9}$$

Proof. This is Lemma 5 of Gonek [8]. □

Lemma 5.1.3. *Let $\mathcal{F}(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ be a Dirichlet series with a finite abscissa of convergence σ_a and satisfying $\sum_{n=1}^{\infty}|a_n|n^{-\sigma} \ll (\sigma-\sigma_a)^{-\alpha}$ for $\sigma > \sigma_a$ and some* *real number $\alpha > 0$. Suppose there exists a non-decreasing function $\varphi(x)$ such that $|a_n| \le \varphi(n)$ for $n \ge 1$. Let $x \ge 2, U \ge 2$, $\sigma \le \sigma_a$, and $\kappa = \sigma_a - \sigma + 1/\log x$. Then*

$$\sum_{n\le x} \frac{a_n}{n^s} = \frac{1}{2\pi i}\int_{\kappa-iU}^{\kappa+iU} \mathcal{F}(s+w)x^w \frac{dw}{w} + O\Big(x^{\sigma_a-\sigma}\frac{(\log x)^\alpha}{U} + \frac{\varphi(2x)}{x^\sigma}\big(1+x\frac{\log U}{U}\big)\Big).$$

Proof. This is Corollary 2.1, p. 133 of Tenenbaum [36]. □

Lemma 5.1.4. *Assume the Riemann Hypothesis. Let $s = \sigma + it$ and put $\tau = |t| + 3$. Then, for $\sigma > \frac{1}{2}$ and $s \ne 1$, we have*

$$\frac{\zeta'}{\zeta}(s) \ll \left(\frac{\log \tau}{\sigma - \frac{1}{2}} + \frac{1}{|\sigma-1|}\right),$$

where the implied constant is absolute.

Proof. Let $\rho = \frac{1}{2} + i\gamma$ denote a zero of $\zeta(s)$. Then Theorem 9.6A of Titchmarsh [37] implies that, for $s \ne 1$ and $s \ne \rho$,

$$\frac{\zeta'}{\zeta}(s) = \sum_{|s-\rho|\le 1} \frac{1}{s-\rho} + O(\log \tau) + O\left(\frac{1}{|s-1|}\right) \tag{5.10}$$

uniformly for $-1 \le \sigma \le 2$. From this, the fact that the number of terms in the sum is $O(\log \tau)$, and the estimate $|\zeta'(s)/\zeta(s)| \ll 1$ for $\sigma \ge 2$, the lemma is immediate. □

5.2 Proof of Theorem 1.3.1

The Riemann Hypothesis (RH) will be assumed during the course of the proof. If we let $\rho = \frac{1}{2} + i\gamma$ be a non-trivial zero of $\zeta(s)$, then the RH implies that γ is real and that $\bar{\rho} = 1 - \rho$. Moreover, this implies that

$$\overline{\zeta'(\rho)} = \zeta'(1-\rho)$$

as $\zeta'(s)$ is real-valued for real s. An upper bound for the maximum size of $\zeta(s)$ and its derivatives is provided by the Lindelöf Hypothesis, which follows from the RH, and asserts that for any $\varepsilon > 0$,

$$|\zeta(\sigma+it)| \ll t^{\varepsilon} \quad \text{and} \quad |\zeta'(\sigma+it)| \ll t^{\varepsilon} \tag{5.11}$$

for $\sigma \geq \frac{1}{2}$ and $t \geq 1$. We choose a number $\mathfrak{T}$ with $T \leq \mathfrak{T} \leq T+1$ such that

$$\left|\frac{\zeta'}{\zeta}(\sigma+i\mathfrak{T})\right| \ll \log^2 T \tag{5.12}$$

uniformily for $-1 \leq \sigma \leq 2$ and $T \geq 2$. The existence of such a $\mathfrak{T}$ can be deduced from equation (5.10) above. It is known that no zero $\rho = \frac{1}{2} + i\gamma$ of $\zeta(s)$ satisfies $0 < \gamma \leq 1$ and that the number of zeros of $\zeta(s)$ with ordinates between T and $T+1$ is $O(\log T)$. Thus, it follows from Cauchy's Theorem that, for $\varepsilon > 0$ arbitirary,

$$\begin{aligned} \sum_{0<\gamma\leq T} |\zeta'(\rho)|^2 &= \sum_{1<\gamma\leq \mathfrak{T}} \zeta'(\rho)\zeta'(1-\rho) + O(T^{\varepsilon}) \\ &= \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{\zeta'}{\zeta}(s)\zeta'(s)\zeta'(1-s)\, ds + O(T^{\varepsilon}) \end{aligned} \tag{5.13}$$

where $\mathcal{C}$ is the positively oriented rectangular contour with vertices at the points

$$c+i, \quad c+i\mathfrak{T}, \quad 1-c+i\mathfrak{T}, \quad \text{and} \quad 1-c+i$$

for a fixed constant c (to be chosen at the end of the proof) satisfying $1 < c \leq 2$.

We first estimate the contribution from the horizontal portions of the contour to the integral on the right-hand side of (5.13). By combining the estimates in (5.5), (5.6), (5.8), and (5.11), we see that $|\zeta'(\sigma+it)| \ll t^{1/2-\sigma+\varepsilon}$ for $\sigma \leq \frac{1}{2}$ and $t \geq 1$. From this, (5.8), and (5.12) it can be seen that the horizontal portions of the integral around $\mathcal{C}$ contribute an amount that is $O\big(T^{c-1/2+\varepsilon}\big)$. In other words, we have that

$$\sum_{0<\gamma\leq T} |\zeta'(\rho)|^2 = I_1 - I_2 + O(T^{c-1/2+\varepsilon}) \tag{5.14}$$

where

$$I_1 = \frac{1}{2\pi i}\int_{c+i}^{c+iT} \frac{\zeta'}{\zeta}(s)\zeta'(s)\zeta'(1-s)\ ds$$

and

$$I_2 = \frac{1}{2\pi i}\int_{1-c+i}^{1-c+iT} \frac{\zeta'}{\zeta}(s)\zeta'(s)\zeta'(1-s)\ ds.$$

The integral I_2 can be expressed in a more convenient manner. Logarithmically differentiating the functional equation for $\zeta(s)$, we see that

$$\frac{\zeta'}{\zeta}(1-s) = \frac{\chi'}{\chi}(1-s) - \frac{\zeta'}{\zeta}(s),$$

from which it follows that

$$\begin{aligned} I_2 &= \frac{1}{2\pi}\int_1^T \frac{\zeta'}{\zeta}(1-c+it)\zeta'(1-c+it)\zeta'(c-it)dt \\ &= \frac{1}{2\pi}\int_1^T \left(\frac{\chi'}{\chi}(1-c+it) - \frac{\zeta'}{\zeta}(c-it)\right)\zeta'(1-c+it)\zeta'(c-it)dt \\ &= I_3 - \overline{I_1} \end{aligned}$$

where

$$I_3 = \frac{1}{2\pi}\int_1^T \frac{\chi'}{\chi}(1-c+it)\zeta'(1-c+it)\zeta'(c-it)dt.$$

In particular, we see that for any $c > 1$,

$$\sum_{0<\gamma\le T} |\zeta'(\rho)|^2 = -I_3 - 2\ \mathrm{Re}\ I_1 + O(T^{c-1/2+\varepsilon}). \tag{5.15}$$

The integral I_3 can be computed using Lemma 5.1.1. Using Cauchy's Theorem and estimating as above, we can shift the line of integration in I_3 from $1-c$ to $1/2$ with an error of $O(T^{c-1/2+\varepsilon})$. Thus, it follows that

$$I_3 = \frac{1}{2\pi}\int_1^T \frac{\chi'}{\chi}(\tfrac{1}{2}+it)|\zeta'(\tfrac{1}{2}+it)|^2dt + O(T^{c-1/2+\varepsilon}).$$

Using estimate (5.7), we see that

$$\begin{aligned} I_3 &= -\frac{1}{2\pi}\int_1^T |\zeta'(\tfrac{1}{2}+it)|^2 \log \tfrac{t}{2\pi} dt + O(T^{c-1/2+\varepsilon}) \\ &= \frac{-\mathscr{L}}{2\pi}\int_1^T |\zeta'(\tfrac{1}{2}+it)|^2 dt + \frac{1}{2\pi}\int_1^T \frac{1}{t}\int_1^t |\zeta'(\tfrac{1}{2}+iu)|^2 du\ dt + O(T^{c-\frac{1}{2}+\varepsilon}). \end{aligned}$$

The last equality follows from integration by parts. Lemma 5.1.1 now implies that

$$I_3 = -\frac{T\mathscr{L}}{2\pi}\mathcal{P}_1(\mathscr{L}) + \frac{1}{2\pi}\int_1^T \mathcal{P}_1(\log \frac{t}{2\pi})\ dt + O(T^{c-1/2+\varepsilon}) \tag{5.16}$$

where $\mathcal{P}_1(x)$ is the polynomial defined in equation (5.4).

The integral I_1 can be calculated as follows. Differentiating the functional equation for $\zeta(s)$ and using estimates (5.8) and (5.11) we have, for $\sigma \geq \frac{1}{2}$ and $t \geq 1$, that

$$\begin{aligned} \zeta'(1-s) &= \chi'(1-s)\zeta(s) - \chi(1-s)\zeta'(s) \\ &= -\chi(1-s)\big(\zeta(s)\log \tfrac{t}{2\pi} + \zeta'(s)\big) + O\big(|t|^{\sigma-3/2+\varepsilon}\big). \end{aligned}$$

From this and Lemma 5.1.2, it follows that

$$\begin{aligned} I_1 &= \frac{1}{2\pi}\int_1^T \frac{\zeta'}{\zeta}(c+it)\zeta'(1-c-it)\zeta'(c+it)dt \\ &= -\frac{1}{2\pi}\int_1^T \frac{\zeta'}{\zeta}(c+it)\zeta'(c+it)\chi(1-c-it)\Big(\zeta(c+it)\log \tfrac{t}{2\pi} + \zeta'(c+it)\Big)dt \\ &\quad + O(T^{c-1/2+\varepsilon}) \\ &= -\frac{1}{2\pi}\int_1^T \zeta'(c+it)^2\chi(1-c-it)\log \tfrac{t}{2\pi} dt \\ &\quad - \frac{1}{2\pi}\int_1^T \frac{\zeta'}{\zeta}(c+it)\zeta'(c+it)^2\chi(1-c-it)dt + O(T^{c-1/2+\varepsilon}). \\ &= \sum_{1\leq \ell mn \leq \frac{T}{2\pi}} \Lambda(\ell)\log m \log n - \sum_{1\leq mn\leq \frac{T}{2\pi}} \log m \log n \log mn + O(T^{c-1/2+\varepsilon}) \\ &= \sum_{1\leq \ell mn \leq \frac{T}{2\pi}} \Lambda(\ell)\log m \log n - 2\sum_{1\leq mn\leq \frac{T}{2\pi}} \log^2 m \log n + O(T^{c-1/2+\varepsilon}). \end{aligned} \tag{5.17}$$

Here $\Lambda(n)$, the von Mangoldt function, denotes the coefficients of the Dirichlet series for $-\zeta'(s)/\zeta(s)$. If we define

$$\mathcal{R}_1(x) = \operatorname*{Res}_{s=1} \frac{\zeta'}{\zeta}(s)\zeta'(s)^2\Big(\frac{x^s}{s}\Big) \quad \text{and} \quad \mathcal{R}_2(x) = \operatorname*{Res}_{s=1} \zeta'(s)\zeta''(s)\Big(\frac{x^s}{s}\Big),$$

then using Lemma 5.1.3 we can show that

$$\sum_{1\le \ell mn\le \frac{T}{2\pi}} \Lambda(\ell)\log m \log n = -\mathcal{R}_1\big(\tfrac{T}{2\pi}\big) + O(T^{1/2+\varepsilon}) \tag{5.18}$$

and also that

$$\sum_{1\le mn\le \frac{T}{2\pi}} \log^2 m \log n = -\mathcal{R}_2\big(\tfrac{T}{2\pi}\big) + O(T^{1/2+\varepsilon}). \tag{5.19}$$

We will only prove (5.18) as formula (5.19) can be established in an analogous manner. Since

$$\sum_{k\ell m=n} \Lambda(k)\log \ell \log m \le \log^2 n \sum_{k|m} \Lambda(k) \le \log^3 n,$$

from Lemma 5.1.3 we see that, for $a = 1 + 1/\log x$ and $x \ge 2$,

$$\sum_{1\le \ell mn\le x} \Lambda(\ell)\log m \log n = -\frac{1}{2\pi i}\int_{a-iU}^{a+iU} \frac{\zeta'}{\zeta}(s)\zeta'(s)^2\big(\frac{x^s}{s}\big)ds$$
$$+ O(x^a U^{-1}\log^5 x) + O(\log^3 x).$$

Now we shift the line of integration from a to $\frac{1}{2} + (\log x)^{-1}$. In doing so we pass over the poles of the integrand at $s = 1$ which contribute a residue of $\mathcal{R}_1(x)$. Using Lemma 5.1.4, the Lindelöf Hypothesis, and equation (5.11), the resulting integrals are easily shown to be $O(x^{1/2+\varepsilon}U^{\varepsilon} + x^{1+\varepsilon}U^{-1+\varepsilon})$. If we choose $U = x^{1/2}$, it follows that

$$\begin{aligned}\sum_{1\le \ell mn\le x} \Lambda(\ell)\log m \log n &= -\mathcal{R}_1(x) + O(x^{1/2+\varepsilon}U^{\varepsilon}) + O(x^{1+\varepsilon}U^{-1+\varepsilon}) \\ &\quad + O(x^a U^{-1}\log^5 x) + O(\log^3 x) \\ &= -\mathcal{R}_1(x) + O(x^{1/2+\varepsilon}).\end{aligned}$$

Finally, letting $x = \frac{T}{2\pi}$, we obtain (5.18).

Combining (5.15)-(5.19), we find that

$$\sum_{1<\gamma\le T} |\zeta'(\rho)|^2 = \frac{TL}{2\pi}\mathcal{P}_1(L) - \frac{1}{2\pi}\int_1^T \mathcal{P}_1(\log\frac{t}{2\pi})dt \\ - 2 \text{ Re } \mathcal{R}_1(\frac{T}{2\pi}) + 4 \text{ Re } \mathcal{R}_2(\frac{T}{2\pi}) + O(T^{c-1/2+\varepsilon}). \tag{5.20}$$

It now remains only to calculate the residues $\mathcal{R}_1(\frac{T}{2\pi})$ and $\mathcal{R}_2(\frac{T}{2\pi})$. In order to do this, we need the Laurent series coefficients for $\zeta'(s)/\zeta(s)$ around $s=1$. Writing

$$\frac{\zeta'}{\zeta}(s) = \frac{-1}{s-1} + \sum_{k=0}^{\infty}\frac{(-1)^k\xi_k}{k!}(s-1)^k \tag{5.21}$$

and using (5.2), we can solve for the ξ_k recursively in terms of the γ_k by setting

$$\frac{\zeta'}{\zeta}(s)\zeta(s) = \zeta'(s)$$

and comparing coefficients. The first few values of ξ_k are:

k	ξ_k
0	γ_0
1	$2\gamma_1 + \gamma_0^2$
2	$3\gamma_2 + 6\gamma_1\gamma_0 + 2\gamma_0^3$
3	$4\gamma_3 + 12\gamma_2\gamma_0 + 24\gamma_1\gamma_0^2 + 12\gamma_1^2 + 6\gamma_0^4$

We are now in a position to finish the proof of (5.1). By choosing $c = 1+\varepsilon$, using the above values of ξ_k and (5.4), we deduce from (5.20) that

$$\sum_{0<\gamma\le T} |\zeta'(\rho)|^2 = \frac{T}{24\pi}\mathscr{L}^4 + \frac{(2\gamma_0 - 1)}{6\pi}T\mathscr{L}^3 \\ + B_2T\mathscr{L}^2 + B_1T\mathscr{L} + B_0T + O(T^{1/2+2\varepsilon}), \tag{5.22}$$

where the constants are:

$$
\begin{aligned}
B_2 &= \frac{1}{2\pi}(1 - 2\gamma_0 + \gamma_0^2 - 2\gamma_1), \\
B_1 &= -\frac{1}{2\pi}(2 - 4\gamma_0 + 2\gamma_0^2 + 2\gamma_0^3 + 10\gamma_0\gamma_1 - 4\gamma_1 + \gamma_2),
\end{aligned}
$$

and

$$
B_0 = \frac{1}{6\pi}(6+6\gamma_0(5\gamma_1+4\gamma_2-2) +6\gamma_0^2(\gamma_0+\gamma_0^2+6\gamma_1+1)-12\gamma_1+42\gamma_1^2+3\gamma_2+10\gamma_3).
$$

By replacing 2ε by ε (which is possible since $\varepsilon > 0$ was arbitrary), we complete the proof of (5.1).

Remark: *In Section 7.1 of [2], Conrey and Snaith conjectured that, for $\varepsilon > 0$ arbitrary,*

$$
\begin{aligned}
\sum_{0<\gamma\le T} |\zeta'(\rho)|^2 = \int_1^T \Bigg(& \frac{1}{24\pi}\log^4\frac{t}{2\pi} + \frac{\gamma_0}{3\pi}\log^3\frac{t}{2\pi} + \left(\frac{\gamma_0^2}{2\pi} - \frac{\gamma_1}{\pi}\right)\log^2\frac{t}{2\pi} \\
& - \left(\frac{\gamma_0^3}{\pi} + \frac{\gamma_0\gamma_1}{\pi} + \frac{\gamma_2}{2\pi}\right)\log\frac{t}{2\pi} + \left(\frac{\gamma_0^4}{\pi} + \frac{\gamma_0^2\gamma_1}{\pi} + \frac{7\gamma_1^2}{\pi} + \frac{4\gamma_0\gamma_2}{\pi} + \frac{5\gamma_3}{3\pi}\right)\Bigg) dt \\
& + O\left(T^{1/2+\varepsilon}\right),
\end{aligned}
$$

assuming the Riemann Hypothesis. That our expression in (5.22) is equivalent to their conjecture follows by performing the integration on the right-hand side of the above expression and comparing terms.

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