The Riemann Hypothesis and the Riemann Zeta Function

Khanh Nguyen

Colgate University

1 May 2017

Introduction

- The Riemann Hypothesis - one of the most important problems in mathematics.
- Introduced by Bernhard Riemann in his article 'On the Number of Primes Less Than a Given Magnitude' in 1859.
- A solution of the Riemann Hypothesis gives a better understanding of the Riemann zeta function and many other important consequences in mathematics.
- The Clay Mathematics Institute listed the Riemann Hypothesis as one of the seven Millennium Problems in 2000.

The Riemann Zeta Function

- Studied extensively by Euler in the first half of the eighteenth century as a real variable function.
- Riemann extended Euler's definition to a function of a complex variable, and established the functional equation form.
- Generalizations of the function appear frequently in modern mathematics
- Most common definition: The Riemann Zeta Function is a function of a complex variable that analytically continues the sum of the Dirichlet series:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

for Re(s) > 1

Euler's Progresses

- Euler has made important progresses in the study of the real analytic version of the zeta function.
- He resolved the Basel's problem in 1734 by establishing its convergence to $\pi^2/6$.
- Basel's Problem:

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + ... = \frac{\pi^2}{6} \quad (2)$$

- Apery's constant:

$$1 + \frac{1}{2^3} + \frac{1}{3^3} + ... \approx 1.202 \quad (3)$$

- Other values of the series have important roles in formulating various laws of physics.

How Euler gave us a cool T-shirt idea

Figure 1: Euler Formula

Euler's Product Formula

- Often called the 'Golden Key', the Euler Product Formula is a fascinating way to relate the Riemann Zeta Function with the prime numbers
- Introduced and proved by Euler at the St Petersburg Academy in 1737
- The proof utilized ideas from the sieve of Eratosthenes
- Complete form of the formula:

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p} \frac{1}{1 - \frac{1}{p^s}} \tag{4}$$

Convergence Properties of the Zeta Function in the defined domain

- Since $n^s = e^{sln(n)}$, we have $|n^s| = |e^{sln(n)}| = e^{Re(s)\ ln(n)} = n^{Re(s)}$.
- Therefore:
$$\sum_{n=1}^{\infty} \left| \frac{1}{n^s} \right| = \sum_{n=1}^{\infty} \frac{1}{n^{Re(s)}} \tag{5}$$
- The series then converges with $Re(s) > 1$
- The zeta function thus converges absolutely in the domain of definition.
- Very useful in establishing the analyticity of the function in this domain.

Analytic Continuation of the Riemann Zeta Function when $s < 1$

- A general idea could be shown using the eta function, which is an alternating version of the zeta function
- Eta function:
$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} \tag{6}$$
- Closed form relationship between the zeta and the eta function:
$$\zeta(s) = \eta(s) \div \left(1 - \frac{1}{2^{s-1}}\right) \tag{7}$$
- Values of the zeta function can be obtained based on values of the eta function in the region $0 < Re(s) < 1$.
- Riemann was able to extend the zeta function to an analytic function in all of C except for a simple pole at 1.

The Functional Equation Form of the Riemann Zeta Function

- The functional equation form of the Riemann Zeta Function is a very useful way to calculate the values of the Riemann Zeta Function and understand its analytic property.
- Relates the Riemann Zeta Function with the Gamma Function, another important function in mathematics
- The functional equation:

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s) \tag{8}$$

Zeros of the Riemann Zeta Function

- Zeros which lie on the left half of the complex plane are called 'trivial zeros' of the Riemann Zeta Function
- Zeros in this region occur at negative even integers
- It is known that all the other zeros (the nontrivial zeros) only occur on the critical strip where $0 < \mathrm{Re}(s) < 1$

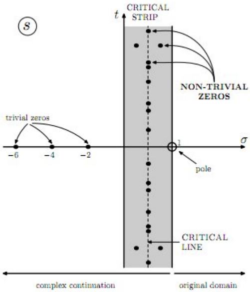

Figure 2: The Critical Strip

The Statement of the Riemann Hypothesis

- The official Riemann Hypothesis reads: 'The Riemann Zeta Function has its zeros only at the negative even integers and the complex numbers with real part $\frac{1}{2}$.
- No zero can occur outside the line of symmetry of the critical strip
- This property has been verified so far for the first 10,000,000,000,000 solutions

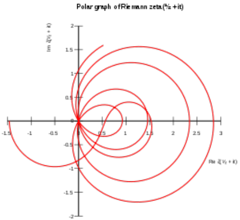

Figure 3: The Value Plane on the Critical Line of Re(s) $= \frac{1}{2}$

Generalized Riemann Hypothesis

- The Generalized Riemann Hypothesis can be stated for Dirichlet L-functions, which are formally similar to the Riemann Zeta Function.
- A Dirichlet character is a completely multiplicative arithmetic function χ such that there exists a positive integer k with $\chi(n+k) = \chi(n)$ for all n and $\chi(n) = 0$ whenever gcd$(n, k) > 1$.
- The corresponding Dirichlet L-function is defined by:

$$L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \tag{9}$$

for every complex number s with Re(s) > 1

Generalized Riemann Hypothesis (cont.)

- The function can be extended to a meromorphic function defined on the entire complex plane.
- The generalized Riemann Hypothesis asserts that for every Dirichlet character χ and every complex number s with $L(\chi, s) = 0$, then it is actually $\frac{1}{2}$.

The Riemann Hypothesis and the Prime Number Theorem

- The Prime Number Theorem states a way to estimate $\pi(x)$, the number of primes between 1 and x.
- A strong form of the Prime Number Theorem uses the Logarithmic Integral, or $Li(x)$, as a way to estimate $\pi(x)$.

$$Li(x) = \int_2^x \frac{1}{logt} dt \tag{10}$$

- Von Koch and Schoenfeld proved that the Riemann hypothesis is equivalent to the following statement:

$$|\pi(x) - Li(x)| < \sqrt{x} log(x) \tag{11}$$

for all x $\geq$ 3

Estimates using the Prime Number Theorem

x	$\pi(x)$	$\frac{x}{ln(x)}$	$\frac{x}{ln(x)-1}$	$\int_2^x \frac{1}{ln(t)} dt$
10^2	25	22	28	29
10^3	168	145	169	177
10^4	1229	1086	1218	1245
10^5	9592	8686	9512	9629
10^6	78498	72382	78030	78627
10^7	664579	620421	661459	664917
10^8	5761455	5428681	5740304	5762208
10^9	50847534	48254942	50701542	50849234
10^{10}	455052511	434294482	454011971	455055614

Figure 4: Table of estimates of $\pi(x)$

Importance of the Riemann Hypothesis

- A proof of the Riemann hypothesis will provide accurate estimates for the distribution of prime numbers.
- It also implies strong bounds on the growth of many arithmetic functions, besides the prime counting function.
- Many other important problems in number theory, such as the prime gap conjecture or the Goldbach conjecture, can be attacked using implications from the Riemann Hypothesis.

Progress on the Riemann Hypothesis

- Partial progress on the hypothesis (in the forms of zero-free regions for the zeta function) have been made.
- It is known that there are no zeroes of the zeta function on the line $Re(s) = 1$.
- Numerical evidence and research indicate the validity of the conjecture, but it remains unproven until this day.
- Progress towards a proof has been made using different approaches (complex analysis, algebraic geometry, random matrix theory, etc.)

Bernhard Riemann - The man behind the hypothesis

- Bernhard Riemann was a German mathematician in the 19th century.
- He was described as an extremely pious and shy man.
- Always tried to look at mathematics in a larger philosophical context.
- A man of great brilliance and staggering boldness underneath the diffident appearance.

Figure 5: Bernhard Riemann

Quotes on the Riemann Hypothesis

- "Right now, when we tackle problems without knowing the truth of the Riemann hypothesis, it's as if we have a screwdriver. But when we have it, it'll be more like a bulldozer." - Peter Sarnak
- "The consequences [of the Riemann Hypothesis] are fantastic: the distribution of primes, these elementary objects of arithmetic. And to have tools to study the distribution of these of objects." - Henryk Iwaniec
- "If I were to awaken after having slept for a thousand years, my first question would be: Has the Riemann hypothesis been proven?" - David Hilbert

Figure 6: Henryk Iwaniec

Figure 7: Peter Sarnak

Figure 8: David Hilbert

References

- Derbyshire, John. *The Prime Obsession: Bernhard Riemann and the Greatest Unsolved Problem in Mathematics*. Washington, D.C: Joseph Henry Press, 2003.
- Gowers, Timothy et al. "The Prime Number Theorem and the Riemann Hypothesis." *The Princeton Companion to Mathematics*. Princeton: Princeton University Press, 2008. 714-715. Print
- Watkins, Matthew. "The Riemann Hypothesis: FAQs and Resources". 1 May 2017. Web. 1 May 2017. http://empslocal.ex.ac.uk/people/staff/mrwatkin/zeta/riemannhyp.htmquotes.
- "Consequences of the Riemann Hypothesis". 1 May 2017. Web. 1 May 2017. https://mathoverflow.net/questions/17209/consequences-of-the-riemann-hypothesis/17232.
- Image courtesy of en.wikipedia.org and images.google.com