

Graph-Theoretic Algorithms Project Report

Trapezoid Graphs

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1 Introduction

Trapezoid graphs are intersection graphs of trapezoids between two horizontal lines. They generalize both interval graphs and permutation graphs. Interval graphs are intersection graphs of intervals on the real line. Permutation graphs are intersection graphs of straight lines between two horizontal lines. They can be represented by a permutation diagram consisting of a pair of horizontal lines (a channel) where each line has n points. The points of one line are numbered from 1 to n and the points of the other line are numbered by a permutation of $(1, 2, \dots, n)$. The graph has n vertices v_1, v_2, \dots, v_n and vertices v_i and v_j are adjacent if and only if the line joining two i points intersect (cross) the line joining two j points. Figure 1 shows a permutation graph and a permutation diagram for it.

Trapezoid graphs generalize these two graph classes by replacing points with intervals in the permutation diagram. The vertices v_i and v_j are adjacent if and only if the trapezoid joining the two intervals of i intersect the trapezoid joining the two intervals of j . In next section we define trapezoid graphs more precisely, present some equivalent characterization for them, and show an

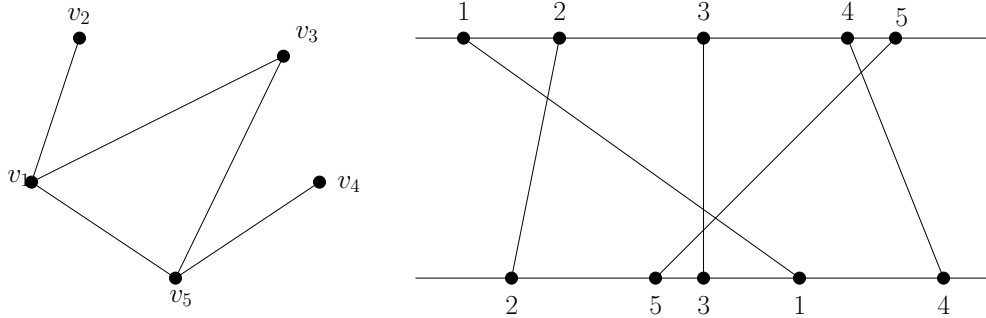


Figure 1: A permutation graph and a permutation diagram for it.

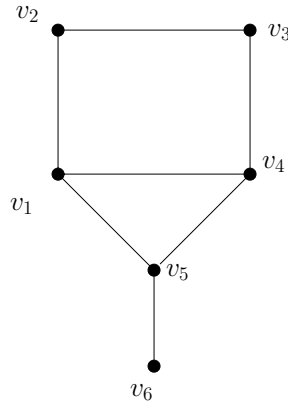


Figure 2: A trapezoid graph G .

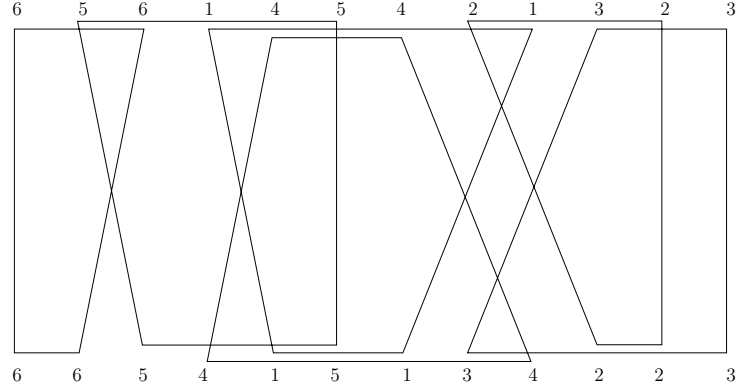


Figure 3: A trapezoidal representation for graph G shown in figure 2.

application of them in channel routing problems of VLSI design. Relationship of trapezoid graphs to other graph classes are studied in section 3. In section 4 we discuss algorithms for trapezoid graph recognition. Section 5 contains information about some well-known problems that have efficient algorithms on trapezoid graphs.

2 Trapezoid graphs

In this section we introduce three equivalent characterizations of trapezoid graphs and show an application of these graphs. Informally, they are intersection graphs of a finite set of trapezoids between two parallel lines.

Definition 1 *A graph G is a trapezoid graph if there exists a set of trapezoids between a pair of horizontal lines such that for each vertex v_i of G there is a trapezoid T_i and there is an edge (v_i, v_j) if and only if $T_i \cap T_j \neq \emptyset$. We call this family of trapezoids a trapezoidal representation for G . Each trapezoid T_i has an interval $I_i^u = [a_i, b_i]$ on the upper line and one interval $I_i^l = [c_i, d_i]$ on the lower line.*

A trapezoid graph G is shown in figure 2. Figure 3 shows a trapezoidal representation for G .

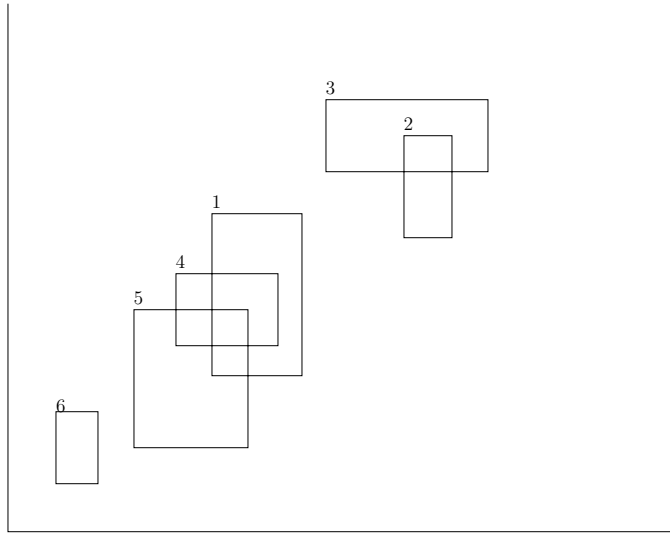


Figure 4: A box representation for trapezoid graph G shown in figure 2.

Trapezoid graphs were introduced by Dagan et al. [DGP88]. Corneil and Kamula [CK87] independently introduced the same class but they refer to them as *II* (interval-interval) graphs. Felsner, Muller, and Wernisch [FMW94] introduced an equivalent characterization (representation) for trapezoid graphs. For two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ in plane we say P_1 is *dominated* by P_2 , denoted by $P_1 < P_2$, if $x_1 < x_2$ and $y_1 < y_2$. A *box* b in plane is the set of points $\{(x, y) \in \mathbb{R}^2 : l_1 \leq x \leq u_1, l_2 \leq y \leq u_2\}$ where $L = (l_1, l_2)$ is the lower corner and $U = (u_1, u_2)$ is the upper corner of b . A box b_1 dominates a box b_2 if the lower corner of b_1 dominates the upper corner of b_2 . Two boxes are *comparable* if one of them dominates the other. Otherwise they are *incomparable*. Now we have the following definition for trapezoid graphs.

Definition 2 A graph G is a trapezoid graph if there exists a set of boxes in plane such that for each vertex v_i of G there is a box b_i and there is an edge (v_i, v_j) if and only if b_i and b_j are incomparable. We call this family of boxes a box representation for G .

A box representation of graph G is shown in figure 4.

To observe that these two definitions are equivalent, map lower and upper lines in the trapezoidal representation to the x-axis and the y-axis in the box representation, respectively. Now for each vertex v_i , $I_i^l = [c_i, d_i]$ lies on the x-axis and $I_i^u = [a_i, b_i]$ lies on the y-axis. These two intervals define an axis-parallel box b_i whose projection on the x-axis and the y-axis coincides with I_i^l and I_i^u , respectively. Two boxes b_i and b_j are incomparable if and only if $T_i \cap T_j \neq \emptyset$. So these two definitions are equivalent.

Next we show another equivalent characterization for trapezoid graphs, but first we need some definitions.

Definition 3 A partially ordered set (poset) $P = (A, <)$ is an interval order if we can map each element $a \in A$ to an interval I_a of the real line such that $x < y$ if and only if $I_x \ll I_y$, i.e. I_x is totally to the left of I_y . The interval order dimension of a poset P , denoted by $\text{iodimm}(P)$ is the minimum number k of interval orders P_1, P_2, \dots, P_k such that $P = P_1 \cap P_2 \cap \dots \cap P_k$, i.e. $x < y$ if and only if $x < y$ in all interval orders P_1, \dots, P_k .

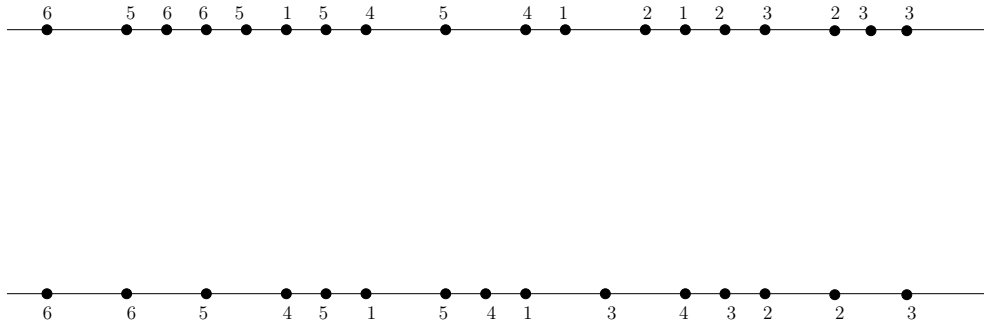


Figure 5: A channel and its labeled terminals.

Definition 4 *The co-comparability graph of a poset $P = (A, <)$ is a graph G whose vertices correspond to elements of P and $(a, b) \in E(G)$ if and only if a and b are incomparable in P , i.e. $x \not\leq y$ and $y \not\leq x$.*

Note that interval graphs are co-comparability graphs of interval orders. Now we are ready to describe another equivalent characterization for trapezoid graphs.

Theorem 1 [DGP88] *A graph G is a trapezoid graph if and only if it is the co-comparability graph of a poset P with $\text{iodim}(P) \leq 2$.*

Proof: The proof of this theorem is straightforward. We present the proof idea for one direction, the proof of other direction is similar. Assume that G is a trapezoid graph. Consider an arbitrary trapezoidal representation of G . Define interval orders P_1 and P_2 corresponding to the intervals on the upper and lower line, respectively. Finally consider poset $P = P_1 \cap P_2$ and show that $T_i \cap T_j = \emptyset$ if and only if $(I_i^u \ll I_j^u \text{ and } I_i^l \ll I_j^l)$ or $(I_j^u \ll I_i^u \text{ and } I_j^l \ll I_i^l)$. The complete proof can be found in [DGP88]. \square

Now we describe an application of trapezoid graphs in VLSI circuits design. We have a pair of horizontal lines called channel and some points along them called terminals or ports. Each terminal has an integral number between 1 and n as its label. All terminals with the same label constitute a net. A channel and its terminals are shown in figure 5. In the channel routing problem we want to connect all terminals of each net so that no two nets intersect. Routing all nets in a single layer is not always possible. In the multi-wire model we can use several layers but each net should use only one layer, i.e. no layer changes is allowed along the wires connecting a net. The problem is to find the minimum number of layers needed to connect all nets. We can model this problem using trapezoid graphs. Each net can be modeled by a trapezoid that connects rightmost and leftmost terminals of that net on two horizontal lines. The channel of figure 5 can be modeled by the trapezoid graph shown in figure 2. One can show [DGP88] that two nets can be routed without intersection in the same layer if and only if their corresponding trapezoids do not intersect. Thus the minimum number of required layers equals to minimum number of colors needed to color the corresponding trapezoid graph.

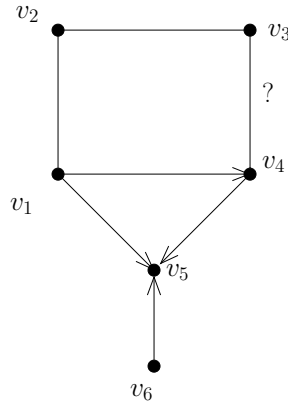


Figure 6: An incomplete orientation for graph G shown in figure 2.

3 Relationship to other graph classes

In this section we first study the relationship of trapezoid graphs to some other graph classes and then introduce some other graphs that are very similar to trapezoid graphs.

3.1 Relationship to other graph classes

We introduced trapezoid graphs as generalization of both interval and permutation graphs. Now we show that trapezoid graphs strictly include these two graph classes. For permutation graphs consider the case where on both horizontal lines no two intervals intersect. In this case each interval can be considered as a point and each trapezoid as a straight line. Thus each permutation graph is a trapezoid graph. Interval graphs can be obtained in the case where $a_i = c_i$ and $b_i = d_i$ for each $v_i \in V(G)$, i.e. one horizontal line is the mirror image of the other. In this case two trapezoids intersect if and only if their corresponding intervals on the upper line (lower line) intersect. Therefore each interval graph is a trapezoid graph. Figure 2 shows a trapezoid graph G that is neither an interval graph nor a permutation graph. A trapezoidal representation for G is shown in figure 3. Therefore G is a trapezoidal graph. G is not an interval graph because it contains C_4 as an induced subgraph and we know that interval graphs are chordal. We use the following lemma to show that G is not a permutation graph.

Lemma 1 [PLE71] *A graph G is a permutation graph if and only if G and \overline{G} are comparability graphs.*

So it suffices to show that G is not a comparability graph. To prove this, assume for contradiction that it has a transitive orientation. (v_6, v_5) can be oriented either $v_6 \rightarrow v_5$ or $v_5 \rightarrow v_6$. We consider the case that it is oriented $v_6 \rightarrow v_5$, the other case is similar. Now (v_5, v_4) ((v_5, v_1)) should be oriented $v_4 \rightarrow v_5$ ($v_1 \rightarrow v_5$), for if it were oriented $v_5 \rightarrow v_4$ ($v_5 \rightarrow v_1$), then by transitivity we would have to have edge $v_6 \rightarrow v_4$ ($v_6 \rightarrow v_1$) but there is no such edge. Then by symmetry we may assume that (v_1, v_4) is oriented $v_1 \rightarrow v_4$. Figure 6 shows orientations we have done so far. But now we cannot orient (v_3, v_4) . For if we have $v_3 \rightarrow v_4$ then transitivity implies an edge $v_3 \rightarrow v_5$, which does not exist, and if we have $v_4 \rightarrow v_3$ then transitivity implies an edge $v_1 \rightarrow v_3$, which does not exist. Contradiction, so the graph is not a comparability graph. Thus we have the following result.

Theorem 2 *Trapezoid graphs strictly contain interval graphs and permutation graphs.*

Theorem 1 shows that trapezoid graphs are co-comparability graphs. We know that co-comparability graphs are perfect. So trapezoid graphs are perfect. Corneil and Kamula [CK87] showed that trapezoid graphs are contained in both the class of asteroidal triple free graphs and the class of weakly chordal graphs.

Definition 5 *An asteroidal triple in a graph G is a set of three pairwise non-adjacent vertices of G such that any two can be connected by a path that avoids the neighborhood of third. A graph that contains no asteroidal triple is called an asteroidal triple free graph.*

Theorem 3 [CK87] *Trapezoid graphs do not contain an asteroidal triple.*

Definition 6 *A graph G is weakly chordal if it has no induced subgraph isomorphic to C_k or $\overline{C_k}$ for any $k \geq 5$.*

Theorem 4 [CK87] *Trapezoid graphs are strictly contained in the weakly chordal graphs.*

If both G and \overline{G} are trapezoid graphs then theorem 1 implies that both of them are comparability graphs and so according to lemma 1 they are permutation graphs.

3.2 Friends of trapezoid graphs

There are several graph classes that are very similar to trapezoid graphs. We briefly introduce some of them. PI graphs and PI^* graphs were introduced by Corneil and Kamula [CK87]. They are intersection graphs of triangles between two horizontal lines. In PI (point-interval) graphs each triangle has a point on the upper line and an interval on the lower line while in PI^* graphs each triangle has a point on lower or upper line and an interval on the other line. So PI graphs are generalization of permutation graphs and interval graphs, PI^* graphs are generalization of PI graphs, and trapezoid graph are generalization of PI^* graphs. It is shown in [CK87] that all of these containments are strict. Simple trapezoid graphs are another class of similar graphs that were introduced by Lin [Lin02] as intersection graphs of straight lines (sticks) and rectangles (blocks) between two horizontal lines. In these graphs each vertex is represented by either a straight line or a rectangle and two vertices intersect if and only if their corresponding lines or rectangles intersect. Therefore simple trapezoid graphs are generalization of both permutation (only sticks) and interval (only blocks) graphs. Lin [Lin02] has shown that there is no relationship between PI graphs and simple trapezoid graphs. Parallelogram graphs are another subgraph of trapezoid graphs and are defined as intersection graphs of parallelograms between two horizontal lines. Bogart et al. [BFIL95] showed that parallelogram graphs are exactly bounded tolerance graphs.

We can also generalize the concept of trapezoid graphs.

Definition 7 [Flo95] *Let m be a positive integer. Consider $m + 1$ parallel lines g_1, g_2, \dots, g_{m+1} , where the g_i are indexed according to their ordering. On g_i there is given a closed interval $[a_i, b_i]$. These intervals determine an m -trapezoid on g_1, g_2, \dots, g_{m+1} , i.e. the interior of closed polygon $a_1, \dots, a_{m+1}, b_{m+1}, \dots, b_1$. A graph $G = (V, E)$ is called an m -trapezoid graph if and only if there exists a family $(T_v)_{v \in V}$ of m -trapezoids on g_1, \dots, g_{m+1} holding the following condition: $(u, v) \in E \iff T_u \cap T_v \neq \emptyset$. $(T_v)_{v \in V}$ is said to be an m -trapezoid representation.*

The graph is independent from the ordering of $m + 1$ parallel lines g_i [Flo95]. The class of 0-trapezoid graphs is exactly the class of interval graphs. The class of 1-trapezoid graphs is exactly the class of trapezoid graphs. Flotow [Flo95] has proved the following results about m -trapezoid graphs.

Theorem 5 *The class of m -trapezoid graphs is exactly the class of co-comparability graphs of an order P with interval dimension less than or equal to $m + 1$.*

Theorem 6 *Given a co-comparability graph G , there exists a positive integer m such that G is an m -trapezoid graph.*

Circle-trapezoid graphs were introduced by Felsner et al. [FMW94] as a generalization of trapezoid graphs, circle graphs, and circular-arc graphs.

Definition 8 *A circle-trapezoid is the region between two non-crossing chords of a circle. Alternatively, it is the convex hull of two disjoint arcs on the circle. Circle-trapezoid graphs are the intersection graphs of families of circle-trapezoids on a fixed circle.*

4 Trapezoid graph recognition

Cogis [Cog82] developed a polynomial time algorithm for the recognition of posets with interval dimension two. Ma and Spinrad [MS94] refined that algorithm and use theorem 1 to develop a $O(n^2)$ time recognition algorithm for trapezoid graphs. This algorithm is currently the fastest trapezoid graph recognition algorithm. Cheah and Corneil [CC96] developed a $O(n^3)$ recognition algorithm that also produce a trapezoidal representation for trapezoid graphs. They introduced an operation called vertex splitting that transforms trapezoid graphs to permutation graphs satisfying a specific condition. The recognition algorithm tests whether the resulting permutation graph satisfies the condition. Although this algorithm is slower, it is completely graph theoretical and can be easily implemented.

The problem of efficient recognition of PI graphs, PI^* graphs, and simple trapezoid graphs is open. Yannakakis [Yan82] proved that for a given poset P and $m \geq 3$, the problem of deciding whether $iodim(P) = m$ is NP-complete. Thus theorem 5 implies that problem of recognition m -trapezoid graphs is NP-complete for $m \geq 2$.

5 Problems on trapezoid graphs

Several important graph problems that are NP-hard in general case have polynomial time algorithms for trapezoid graphs. We describe outline of two algorithms for coloring problem and summarize the results for several other problems.

5.1 Coloring

We have studied an application for finding chromatic number of trapezoid graphs in section 2. Finding the chromatic number is NP-hard for arbitrary graphs and takes $O(n^{2.5})$ time for co-comparability graphs. It has more efficient algorithms for trapezoid graphs. Historically the first algorithm for trapezoid graphs was the coloring algorithm developed by Dagan et al. [DGP88]. This

algorithm receives a trapezoidal representation of a trapezoid graph G and finds the chromatic number and a valid coloring for G . The algorithm takes $O(nk)$ time where k is the chromatic number of G . It is a simple greedy algorithm that considers trapezoids from left to right according to their top left corner. First the set of used colors is empty. As the algorithm proceeds, it maintains a trapezoid $T_{max}(j)$ for each color j . $T_{max}(j)$ is a trapezoid with color j that has the rightmost bottom right corner among all previous trapezoids with color j . For each trapezoid T_i it first finds all the previous colors that can be used by T_i . Note that a color j can be used by T_i if and only if $T_{max}(j) \cap T_i = \emptyset$. If there is no such color it assigns a new color h to T_i and sets $T_{max}(h) = T_i$. Otherwise it selects an available color c for which $T_{max}(c)$ has the rightmost bottom right corner, colors T_i with c , and sets $T_{max}(c) = T_i$. Dagan et al. [DGP88] proved that this obtains the minimum coloring by showing that G have a clique of the same size.

Felsner et al. [FMW94] developed an $O(n \lg n)$ algorithm for finding the chromatic number. They use a box representation of the trapezoid graph G (see definition 2). Based on the given box representation we can define a *trapezoid order* whose elements are the given boxes and for two boxes b_1 and b_2 we have $b_1 < b_2$ if and only if b_1 is dominated by b_2 . A set of mutually comparable elements of an order is a *chain* and a set of mutually incomparable elements is an *antichain*. As we discussed in section 2 two boxes are incomparable if and only if the corresponding vertices of the trapezoid graph are joined. So a minimum coloring of G is a minimum chain partition of G . The algorithm uses a sweep line technique which is widely used in computational geometry problems. The sweep line is a vertical line L that goes from left to right and is initialized with a dummy point d such that d is below all points of all boxes. Event points are lower and upper corners of boxes in given box representation. The sweep line L maintains the maximal elements of the chains of this partition ordered by y-coordinate. For each event point p depending on whether it is a lower corner or an upper corner, some specific computations are done and L is updated appropriately. Felsner et al. [FMW94] proved that this algorithm finds a minimum chain partition of the given trapezoid order in time $O(n \lg n)$. Thus a minimum coloring of G can be found in this time.

5.2 Dominating set and related problems

In this subsection we consider the dominating set problem and its variants. They have many applications in areas like bus routing, radio broadcast, coding theory, and communication networks.

Definition 9 *A dominating set of a connected undirected graph $G = (V, E)$ is a subset S of V such that every vertex not in S is adjacent to at least one vertex in S . A dominating set S is connected or independent if the subgraph induced by S is connected or has no edge, respectively. An independent dominating set S is efficient if every vertex not in S is adjacent to exactly one vertex of S . A minimum cardinality dominating set is a dominating set with the minimum number of vertices. If graph G is weighted, i.e. each vertex $v \in V$ is associated with a real $w(v)$, we define the weight of S as $\sum_{v \in S} w(v)$.*

According to this definition we have several domination problems. Finding *minimum cardinality connected dominating set* is *NP*-hard in arbitrary graphs. Liang [Lia95] presented an $O(m + n)$ time algorithm for solving this problem in trapezoid graphs and Kohler [Koh00] improved this result by developing an $O(n)$ time algorithm. Both of these algorithms require that a trapezoidal representation for G is given as input. Finding *minimum cardinality dominating clique* is *NP*-hard in co-comparability graphs [KS93]. Kohler [Koh00] presented a $O(n + m)$ time algorithm that solves the problem in trapezoid graphs provided that a trapezoidal representation is given.

Weighted dominating set problem tries to find a *minimum weight dominating set* in a weighted graph. This problem is *NP*-hard for co-comparability graphs [Cha97]. Liang [Lia94] showed that it can be solved in $O(nm)$ time for trapezoid graphs. Lin [Lin98] developed two $O(n \lg n)$ time algorithms that find *minimum weighted independent dominating set* and *minimum weighted efficient dominating set* in trapezoid graphs.

5.3 Other problems

Consider the definition of trapezoid order in subsection 5.1. Assume that G is a trapezoid graph and P is its corresponding trapezoid order. The following results are implied by this definition. A *maximum weighted clique* in G is a maximum weighted antichain in P . A *maximum weighted independent set* in G is a maximum weighted chain in P . A *minimum clique cover* is a minimum antichain partition of P . Felsner et al. [FMW94] developed an algorithm that uses a sweep line and simultaneously computes a maximum weighted independent set and a minimum clique cover of a trapezoid graph G in $O(n \lg n)$ time provided that a box representation of G is given. They also developed a similar algorithm that computes a maximum weighted clique of G in $O(n \lg n)$ time.

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