Equivalent Viscous Damping

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September 24, 2009
Revised: 11-13-2013

1 Derivation of Equivalent Viscous Damping

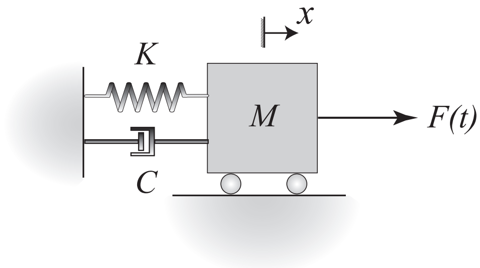

Figure 1. Forced mass-spring-damper system.

The energy lost per cycle in a damper in a harmonically forced system may be expressed as

$$W_d = \oint F_d dx \tag{1}$$

where F_d represents the damping force. The simplest case mathematically is that of viscous damping, where $F_d = C\dot{x}$. Letting the steady-state solution be expressed as

$$x = X\sin(\omega t - \phi) \tag{2}$$

we have

$$\dot{x} = \omega X\cos(\omega t - \phi) \tag{3}$$

Hence,

$$W_d = \oint C\dot{x}dx = \oint C\dot{x}^2 dt \tag{4}$$

where we recall that $dx = \dot{x}dt$. Substitution of (2) into (4) yields

$$\boxed{W_d = C\omega^2 X^2 \int_0^{\frac{2\pi}{\omega}} \cos^2(\omega t - \phi)dt = \pi C\omega X^2} \tag{5}$$

The relationship between damper force, displacement, and energy dissipated, is more easily assuming forcing at the resonance frequency.

At resonance, we have $\omega = \omega_n = \sqrt{\frac{K}{M}}$, and noting that $C = 2\zeta\sqrt{KM}$, we have from Equation (5)

$$W_d(\omega_n) = 2\zeta\pi K X^2 \tag{6}$$

We may recast (**??**) as

$$\dot{x} = \pm\omega X\sqrt{1 - \sin^2(\omega t - \phi)} = \pm\omega\sqrt{X^2 - x^2} \tag{7}$$

Thus, we may express the damping force as

$$F_d = C\dot{x} = \pm C\omega\sqrt{X^2 - x^2} \tag{8}$$

Rearranging (8), we have

$$\left(\frac{F_d}{C\omega X}\right)^2 + \left(\frac{x}{X}\right)^2 = 1 \tag{9}$$

The ellipse expressed by Equation (9) may be represented graphically, as shown in Figure 2. Other loss

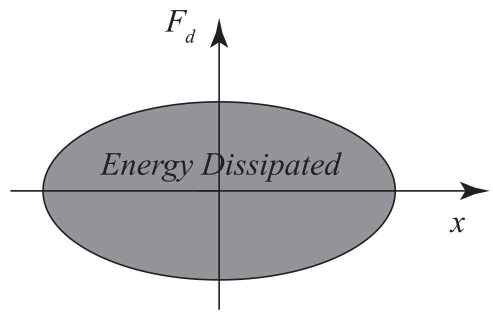

Figure 2. Viscous damper dissipated energy ellipse at resonace.

mechanisms may be modeled as equivalent viscous dissipation by equating the work done in one cycle to that done by a viscous damper

$$W_d = \pi C_{eq}\omega X^2 \tag{10}$$

Hence, the equivalent viscous damping constant is defined as

$$\boxed{C_{eq} = \frac{W_d}{\pi\omega X^2}} \tag{11}$$

1.1 Equivalent Viscous Damping for Coulomb Friction

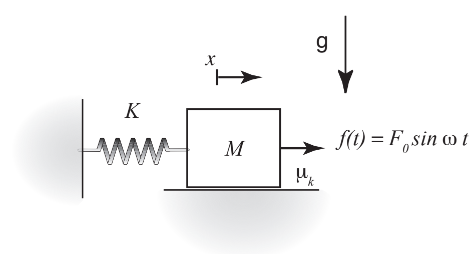

Figure 3. Simple Coulomb friction model.

The resistance force, F_c, in the case of Coulomb friction dissipates $W_c/4 = F_cX$ in energy over each quarter cycle as shown in Figure 4, hence, equating the total dissipative work per cycle to that done by a viscous damper, we have

$$W_c = 4F_cX = \pi C_c\omega X^2 \tag{12}$$

Hence, the equivalent viscous damping constant for Coulomb friction is given by

$$\boxed{C_c = \frac{4F_c}{\pi\omega X}} \tag{13}$$

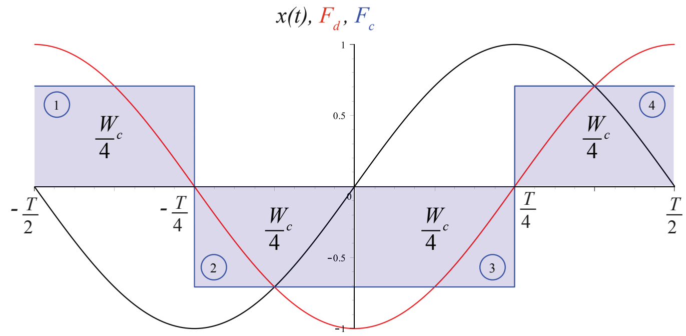

Figure 4. Normalized viscous and Coulomb friction resistance force, and displacement over one period.

The equation of motion for the equivalent viscously damped case is given by

$$M\ddot{x} + C_c\dot{x} + Kx = F_0 \sin \omega t \tag{14}$$

Hence, the steady-state magnitude may be written

$$|X| = \frac{F_0}{\sqrt{\left(K - M\omega^2\right)^2 + C_c^2\omega^2}} \tag{15}$$

Substitution of (13) into (15) yields

$$\boxed{|X| = \frac{\sqrt{F_0^2 - \left(\frac{4F_c}{\pi}\right)^2}}{K - M\omega^2} = \frac{F_0}{K}\frac{\sqrt{1 - \left(\frac{4F_c}{\pi F_0}\right)^2}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}} \tag{16}$$

Note that unlike the viscous damping case, the amplitude grows unbounded as $\omega \to \omega_n$. In addition, for real (physically meaningful) solutions, we must have

$$\frac{4F_c}{\pi F_0} \leq 1 \tag{17}$$

1.2 Quadratic Damping

Another form of damping that may be encountered in various shock absorbers yields a force proportional to the square of the difference in the velocities of the ends of the damper. The quadratic damping force, shown in Figure 5, may be modeled as

$$F_d = \alpha_q \,\mathrm{sgn}(v)\, v^2 \tag{18}$$

where

$$\mathrm{sgn}(v) = \begin{cases} -1 & \text{for } v > 0 \\ 0 & \text{for } v = 0 \\ 1 & \text{for } v < 0 \end{cases} \tag{19}$$

and where $v = \dot{x}$, and α_q is a constant. Hence, the energy dissipated in one cycle is given by

$$W_d = \oint \alpha_q \dot{x}^2 dx = \alpha_q \,\mathrm{sgn}(\dot{x}) \oint \dot{x}^3 dt \tag{20}$$

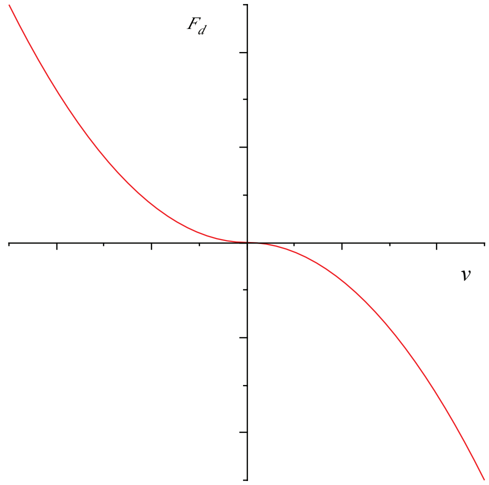

Figure 5. Force due to quadratic damping

Assuming a linear steady state response of the same form as that given by Equation (2), we obtain an equation analogous to Equation (5), but where we must use the symmetry of the force and eliminate the sgn function by integrating over a quarter of the period, and multiplying quadrupling the result

$$W_d = 4\alpha_q \omega^3 X^3 \int_0^{\frac{\pi}{2\omega}} \cos^3(\omega t - \phi) dt = \frac{8}{3}\alpha_q \omega^2 X^3 \tag{21}$$

Equating the dissipated quadratic damping energy to that dissipated by a linear viscous damper, as done for Coulomb damping in Equation (12), yields

$$\boxed{C_q = \frac{8}{3}\frac{\alpha_q \omega X}{\pi}} \tag{22}$$

The assumption of a linear response is very likely invalid for large displacements, but assuming reasonably linear behavior, the equation of motion may be written as

$$M\ddot{x} + C_q\dot{x} + Kx = F_0 \sin \omega t \tag{23}$$

and may be solved to yield an expression for the amplitude identical to Equation (16), with the exception that C_c is replace by C_q. However, when the expression for C_q, given by (22) is substituted, the equation for the amplitude becomes

$$|X| = \frac{F_0}{\sqrt{(K - M\omega^2)^2 + \frac{64\alpha_q^2\omega^4|X|^2}{9\pi^2}}} \tag{24}$$

Equation (24) reveals that the steady-state amplitude is a function of itself! Squaring both sides of Equation (24), and rearranging yields a quartic equation in $|X|$, for which only positive real-valued solutions are valid:

$$\frac{64\alpha_q^2\omega^4}{9\pi^2}|X|^4 + \left(K - M\omega^2\right)^2 |X|^2 - F_0^2 = 0 \tag{25}$$

Solving for $|X|$ yields:

$$\boxed{|X| = \frac{3\pi}{8\sqrt{2}\alpha_q\omega^2}\sqrt{\sqrt{\frac{256F_0^2\alpha_q^2}{9\pi^2}\omega^4 + (K - M\omega^2)^4} - (K - M\omega^2)^2}} \tag{26}$$

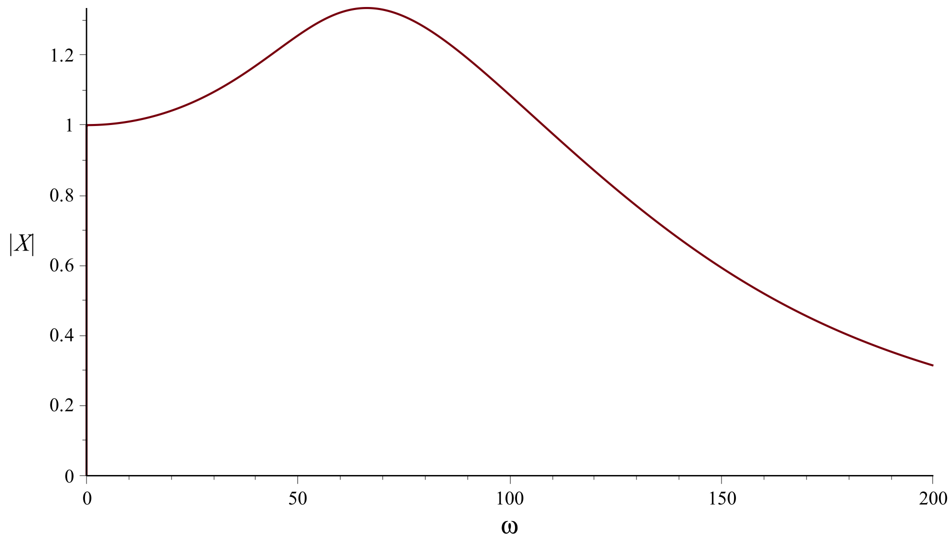

Figure 6. Amplitude of quadratically-damped harmonic response.

While not immediately apparent, $\lim_{\omega\to 0}|X| = F_0/K$, which is the same result one obtains in the linear viscous damping case given by Equation (15). The peak amplitude and its location is strongly dependent upon α_q and F_0, and of the four extrema, only two are positive, and only one of the positive extrema corresponds to a maximum value of $|X|$; the forcing frequency at the maximum value of $|X|$ is given by

$$\boxed{\omega\text{max} = \frac{1}{3M\pi}\sqrt{\frac{9\,K^2M^2\pi^2 + 32\,\alpha^2F^2 - 8\sqrt{9\,F^2K^2M^2\pi^2\alpha^2 + 16\,F^4\alpha^4}}{M\,K}}} \tag{27}$$

Plotting $|X|$ with $\alpha = 1$, $K = F = 10000$, and $M = 1$, as shown in Figure 6, reveals a maximum amplitude at $\omega \approx 66.2$ rad/s.

1.3 Material (Hysteretic) Damping

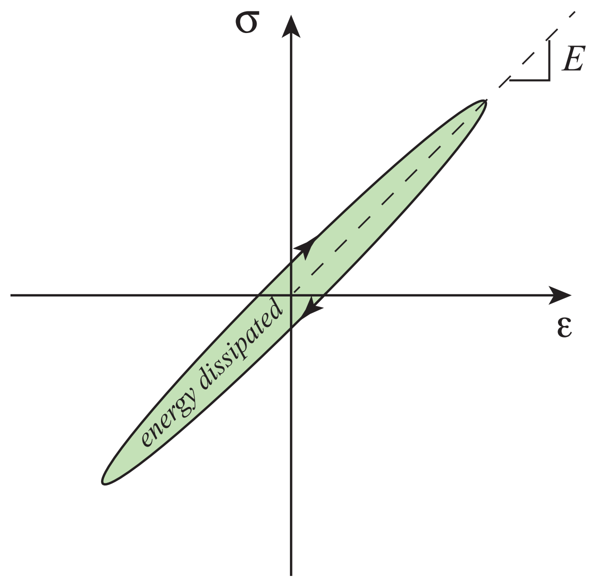

Figure 7. Stress-strain curve for hysteretic damping model.

The energy dissipated in metals over a cycle of deformation has been found to be independent of frequency over a wide range of frequencies, and proportional to the square of the amplitude of vibration. This behavior forms the basis of the so-called *material damping* or *hysteric damping* model; the resulting stress-strain curve forms a tilted ellipse with average slope equal to Young's modulus as shown in Figure 7. The energy dissipated over a cycle is given by

$$W_h = \alpha_h X^2 \tag{28}$$

Hence,

$$\boxed{C_{eq} = C_h = \frac{\alpha_h}{\pi\omega}} \tag{29}$$