Question 1: Spherical Pendulum

Consider a two-dimensional pendulum of length l with mass M at its end. It is easiest to use spherical coordinates centered at the pivot since the magnitude of the position vector is constant: $|\vec{r}| = \sqrt{(l\hat{r})\cdot(l\hat{r})} = \sqrt{l^2(\hat{r}\cdot\hat{r})} = l$. In other words, the mass is restricted to move along the surface of sphere of radius $r = l$ that is centered on the pivot. The motion of the pendulum can therefore be described by the polar angle θ, the azimuthal angle ϕ, and their rates of change.

(a) The Lagrangian for a spherical pendulum

Let's assume that the mass is on "bottom half" of the sphere, so that the mass has a Cartesian coordinate $z = -l\cos\theta$. Since gravity is the only external, non-constraint force acting on the mass, with potential energy $U = Mgz = -Mgl\cos\theta$, the Lagrangian ($\mathcal{L}$) can be first written as:

$$\mathcal{L} = T - U = \frac{1}{2}M|\vec{v}|^2 + Mgl\cos\theta$$

In spherical coordinates, $\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\dot{\phi}\sin\theta\hat{\phi}$, and for the problem under consideration we note that $\dot{r} = \dot{l} = 0$. So we can write $\mathcal{L}$ as an explicit function of the spherical coordinates,

$$\boxed{\mathcal{L} = \frac{1}{2}M\left(l^2\dot{\theta}^2 + l^2\dot{\phi}^2\sin^2\theta\right) + Mgl\cos\theta} \tag{1}$$

There are two equations of motion for the spherical pendulum, since $\mathcal{L}$in Equation 1 is a function of both θ and ϕ; we therefore use the Euler-Lagrange equation for both coordinates to obtain them. For θ and ϕ,

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) - \frac{\partial\mathcal{L}}{\partial\theta} = 0 \Longrightarrow \boxed{Ml^2\ddot{\theta} - Ml^2\dot{\theta}^2\sin\theta\cos\theta + Mgl\sin\theta = 0} \tag{2}$$

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\phi}}\right) - \frac{\partial\mathcal{L}}{\partial\phi} = 0 \Longrightarrow \boxed{\frac{d}{dt}\left(Ml^2\dot{\phi}\sin^2\theta\right) = 0} \tag{3}$$

Equation 3 suggests that the quantity $Ml^2\dot{\phi}\sin^2\theta$ is a constant in time. This quantity has the same form as the z-component of the angular momentum vector for the mass (L_z), so Equation 3 states that *angular momentum is conserved in the z direction for our spherical pendulum.*

(b) Energy in terms of θ and L_z

For Lagrangians that don't explicitly depend on time, and for potentials that only depend on the coordinates (and not explicitly in their rates of change), the total mechanical energy $E = T + U$ is another constant of the motion. Equation 1 satisfies both of these conditions, so E is constant for the spherical pendulum. We can eliminate the $\dot{\phi}$ term in T since, according to Equation 3, $L_z = Ml^2\dot{\phi}\sin^2\theta$ is constant in time. So the expression for E becomes

$$\begin{aligned} E &= \frac{1}{2}M\left(l^2\dot{\theta}^2 + l^2\dot{\phi}^2\sin^2\theta\right) - Mgl\cos\theta \\ &= \frac{1}{2}M\left[l^2\dot{\theta}^2 + l^2\left(\frac{L_z}{Ml^2\sin^2\theta}\right)^2\sin^2\theta\right] - Mgl\cos\theta \\ &= \boxed{\frac{1}{2}Ml^2\dot{\theta}^2 + \frac{L_z^2}{Ml^2\sin^2\theta} - Mgl\cos\theta = E} \end{aligned} \tag{4}$$

Notice that E was re-written as a function of only one coordinate (θ); one can therefore talk about an equivalent one-body problem, where $E = T_{\text{eff}} + U_{\text{eff}}$, $T_{\text{eff}} = Ml^2\dot{\theta}^2/2$, and

$$U_{\text{eff}} = \frac{L_z^2}{Ml^2 \sin^2\theta} - Mgl\cos\theta \tag{5}$$

Given the one-body form of Equation 4, we can technically solve for the solution for θ as a function of time (or vice versa). Noting that $\dot{\theta} = d\theta/dt$ and solving for it in Equation 4, we get

$$\frac{d\theta}{dt} = \sqrt{\frac{2[E - U_{\text{eff}}(\theta)]}{Ml^2}} \Longrightarrow dt = \sqrt{M}l \frac{d\theta}{\sqrt{2[E - U_{\text{eff}}(\theta)]}}$$

$$\boxed{t - t_0 = \int_{t_0}^{t} dt = l\sqrt{M}\int_{\theta_0}^{\theta} \frac{d\theta}{\sqrt{2[E - U_{\text{eff}}(\theta)]}}} \tag{6}$$

For the *full* solution of the spherical pendulum, we also need to find the solution for ϕ. This can be done using the integrated form of Equation 3, $L_z = Ml^2\dot{\phi}\sin^2\theta$, and using the chain rule of advanced calculus to put the eventual integral in terms of θ. In the form of equations, this gives

$$\begin{aligned} \frac{L_z}{Ml^2\sin^2\theta} &= \frac{d\phi}{dt} \\ &= \frac{d\phi}{d\theta}\frac{d\theta}{dt} \\ &= \frac{d\phi}{d\theta}\sqrt{\frac{2[E - U_{\text{eff}}(\theta)]}{Ml^2}} \end{aligned}$$

and, solving for $d\phi/d\theta$, we finally get

$$\frac{d\phi}{d\theta} = \frac{L_z}{l\sqrt{M}} \frac{1}{\sin^2\theta\sqrt{2[E - U_{\text{eff}}(\theta)]}} \Longrightarrow \boxed{\phi - \phi_0 = \int_{\phi_0}^{\phi} d\phi = \frac{L_z}{l\sqrt{2M}}\int_{\theta_0}^{\theta}\frac{d\theta}{\sin^2\theta\sqrt{E - U_{\text{eff}}(\theta)}}} \tag{7}$$

(c) Max and min values of θ

At the maximum and minimum values of θ, the mass has no (effective) kinetic energy (i.e. $\dot{\theta} = 0$) and so the total mechanical energy at those points is equal to the effective potential energy:

$$\begin{aligned} E &= U_{\text{eff}} \\ &= \frac{L_z^2}{2Ml^2\sin^2\theta} - mgl\cos\theta \\ &= \frac{L_z^2}{2Ml^2(1-\cos^2\theta)} - mgl\cos\theta \end{aligned} \tag{8}$$

We can then find an algebraic (cubic) equation for the maximum and minimum values of $\cos\theta$ by solving for it in Equation 8:

$$\boxed{Mgl(\cos^3\theta - \cos\theta) + E(\cos^2\theta - 1) + \frac{L_z^2}{2Ml^2} = 0} \tag{9}$$

Question 2: Radial Oscillations

In the classic two-body problem, the total mechanical energy for a pair of point-like particles with an interaction potential $V(r)$ can be re-written to reflect an effective one-body problem when using constants of the motion (i.e. conservation laws), as well as a suitable reference frame and coordinate system. Therefore, instead of describing the motion of two particles with masses m_1 and m_2 undergoing motion dictated by the potential $V(r)$, one can instead talk about a particle of reduced mass $\mu = m_1 m_2/(m_1 + m_2)$ moving in an "effective potential" V_{eff}, which has the form

$$V_{\text{eff}} = V(r) + \frac{L^2}{2\mu^2 r^2} \tag{10}$$

and r is the radial coordinate of the particle with mass μ measured relative to the center of field. Note that Equation 10 yields the *potential*, which is the potential energy per unit (reduced) mass, and that the potential energy can be computed to be $U_{\text{eff}} = \mu V_{\text{eff}}$.

For circular orbits, $r = r_0 =$ constant in time, and for general central-force problems in Newtonian mechanics the orbital angular momentum (L) is a conserved quantity[1] of the motion: $L = \mu r_0^2 \dot{\phi}^2$. The latter equation can be solved to find the period of orbital motion,

$$\dot{\phi} = \frac{d\phi}{dt} = \frac{L}{\mu r_0^2} \Longrightarrow \int_0^{2\pi} d\phi = 2\pi = \int_0^T \frac{L}{\mu r_0^2} dt = \frac{L}{\mu r_0^2} T$$

and finally can be put in terms of r_0 and L,

$$T = 2\pi \frac{\mu r_0^2}{L} \tag{11}$$

(a) Newtonian potential for inverse-square-law force

Consider the case when the interaction potential $V(r) = -V_0/r$. The effective potential can be explicitly written as a function of r,

$$V_{\text{eff}} = -\frac{V_0}{r} + \frac{L^2}{2\mu^2 r^2}. \tag{12}$$

A circular orbit corresponds to a system with total mechanical energy E that is equal to U_{eff} at all times. In other words, a circular orbit has a value of r_0 that corresponds to the minimum value of U_{eff} (or, equivalently, V_{eff}), which can be found to be

$$\left.\frac{dV_{\text{eff}}}{dr}\right|_{r=r_0} = \frac{V_0}{r_0^2} - \frac{L^2}{\mu^2 r_0^3} = 0 \Longrightarrow \boxed{r_0 = \frac{L^2}{\mu^2 V_0}} \tag{13}$$

We can find the orbital period for this potential by combining the general formula for T (Equation 11) with Equation 13 to eliminate L; this is most easily done by finding T^2:

$$\boxed{T^2 = 4\pi^2 \frac{\mu^2 r_0^4}{L^2} = 4\pi^2 \frac{\mu^2 r_0^4}{(r_0 \mu^2 V_0)} = 4\pi^2 \frac{r_0^3}{V_0}} \tag{14}$$

If $V_0 = GM$, where G is Newton's gravitational constant and $M = m_1 + m_2$ is the total (gravitational) mass of the system, then Equation 14 is *Kepler's third law of planetary motion.*

[1] These equations are written for an orbit with its plane embedded in the $x - y$ Cartesian plane, so that $\theta = \pi/2$ is constant in time.

For small perturbations in r relative to r_0, we can express V_{eff} as a Taylor expansion in r:

$$V_{\text{eff}}(r) \approx V_{\text{eff}}(r_0) + \left.\frac{dV_{\text{eff}}}{dr}\right|_{r=r_0} (r - r_0) + \frac{1}{2}\left.\frac{d^2V_{\text{eff}}}{dr^2}\right|_{r=r_0} (r - r_0)^2 + \dots \tag{15}$$

where the first derivative of V_{eff} is equal to 0 by definition. We are therefore left with a constant term and a quadratic term in r for the Taylor-expanded V_{eff}. This has an equivalent form to the potential for a simple harmonic oscillator in one dimension, since the constant term will vanish when we compute the Euler-Lagrange equation of motion for r. We can therefore find the frequency of oscillations in r by computing the second derivative in V_{eff},

$$\begin{aligned}\frac{d^2V_{\text{eff}}}{dr^2} &= -2\frac{V_0}{r_0^3} + 3\frac{L^2}{\mu^2 r_0^4} = -2\frac{V_0}{r_0^3} + 3\frac{(r_0\mu^2 V_0)}{\mu^2 r_0^4} = -2\frac{V_0}{r_0^3} + 3\frac{V_0}{r_0^3} \\ &= \frac{V_0}{r_0^3}\end{aligned} \tag{16}$$

where the frequency (ω) can be finally found to be:

$$\omega = \sqrt{\frac{V_0}{r_0^3}} \tag{17}$$

We can compare this to the frequency of orbital motion (Ω) by noting that $\Omega = 2\pi/T$, and from Equation 14 this can be found to be:

$$\boxed{\Omega = \sqrt{\frac{V_0}{r_0^3}} = \omega} \tag{18}$$

So, in short, the orbital and oscillation frequencies are equal for the case where the interaction potential corresponds to an inverse-square-law force.

(b) Two-dimensional harmonic potential

Now consider the case when the interaction potential $V(r) = kr^2/2$, where k is positive constant. The effective potential can be explicitly written as a function of r,

$$V_{\text{eff}} = \frac{1}{2}kr^2 + \frac{L^2}{2\mu^2 r^2}. \tag{19}$$

We can now apply the same procedures as done for part (a) of this problem to find the various quantities of interest. For starters, we can compute the radius of a circular orbit for this effective potential by setting the first derivative of Equation 19 to 0:

$$\left.\frac{dV_{\text{eff}}}{dr}\right|_{r=r_0} = kr_0 - \frac{L^2}{\mu^2 r_0^3} = 0 \Longrightarrow \boxed{r_0 = \left(\frac{L^2}{\mu^2 k}\right)^{1/4}} \tag{20}$$

We can combine the above result with the general relation for the orbital period T and orbital frequency Ω to find them in terms of r_0 and k:

$$T^2 = \frac{4\pi^2\mu^2 r_0^4}{L^2} = \frac{4\pi^2\mu^2 r_0^4}{k\mu^2 r_0^4} = \frac{4\pi^2}{k} \Longrightarrow \boxed{\Omega = \frac{2\pi}{T} = \sqrt{k}} \tag{21}$$

Using Equation 15, we can find the solution for small oscillations in r about r_0 by computing the second derivative of V_{eff} with respect to r:

$$\left.\frac{d^2V_{\text{eff}}}{dr^2}\right|_{r=r_0} = k + 3\frac{L^2}{\mu^2 r_0^4} = k + 3\frac{k\mu^2 r_0^4}{\mu^2 r_0^4} = 4k \tag{22}$$

and so the frequency of oscillations is found to be:

$$\boxed{\omega = 2\sqrt{k} = 2\Omega} \tag{23}$$