AN INTRODUCTION TO THE THEORY OF p-ADIC REPRESENTATIONS

by

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Abstract. — This informal article is an expanded version of two lectures given in Padova during the "Dwork Trimester" in June 2001. Their goal was to explain the proof of the p-adic monodromy theorem for de Rham representations and to give some background on p-adic representations.

Résumé. — Cet article informel est une version longue de deux exposés donnés à Padoue en Juin 2001 au "Trimestre Dwork". Leur objet était d'expliquer la démonstration du théorème de monodromie p-adique pour les représentations de de Rham et de donner des rappels sur les représentations p-adiques.

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2000 *Mathematics Subject Classification.* — 11F80, 11R23, 11S25, 12H25, 13K05, 14F30.

I. Introduction

I.1. Introduction

I.1.1. Motivation. — One of the aims of arithmetic geometry is to understand the structure of the Galois group $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, or at least to understand its action on representations coming from geometry. A good example is provided by the Tate module $T_\ell E$ of an elliptic curve E defined over $\mathbf{Q}$. The action of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ on $T_\ell E$ carries a lot of arithmetical information, including the nature of the reduction of E at various primes and the number of points in $E(\mathbf{F}_q)$.

Let $D_p \subset \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ be the decomposition group of a place above p; it is naturally isomorphic to $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$. The aim of the theory of p-adic representations is to extract information from the action of D_p, on $\mathbf{Q}_p$-vector spaces. This is in stark contrast to the theory of ℓ-adic representations, which endeavors to understand the action of D_p on $\mathbf{Q}_\ell$-vector spaces with $\ell \neq p$.

In this latter situation, the topology of D_p is mostly incompatible with that of an ℓ-adic vector space (essentially because the wild inertia of D_p is a pro-p-group), and the result is that the theory of ℓ-adic representations is of an algebraic nature. On the other hand, in the p-adic case, the topologies are compatible and as a result there are far too many representations. The first step is therefore to single out the interesting objects, and to come up with significant invariants attached to them. Unlike the ℓ-adic situation, the study of p-adic representations is therefore of a rather (p-adic) analytic nature.

For example, there exists a p-adically continuous family of characters of the group $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$, given by χ^s where χ is the cyclotomic character and s varies in weight space (essentially $p-1$ copies of $\mathbf{Z}_p$). Out of those characters, only those corresponding to integer values of s "come from geometry". This kind of phenomenon does not arise in the ℓ-adic case, where every character is "good".

The aim of this article is to introduce some of the objects and techniques which are used to study p-adic representations, and to provide explanations of recent developments.

I.1.2. Organization of the article. — This article is subdivided in chapters, each of which is subdivided in sections made up of paragraphs. At the end of most paragraphs, I have added references to the literature. This article's goal is to be a quick survey of some topics and a point of entry for the literature on those subjects. In general, I have tried to give my point of view on the material rather than complete detailed explanations.

References are indicated at the end of paragraphs. For each topic, I have tried to indicate a sufficient number of places where the reader can find all the necessary details.

I have not always tried to give references to original articles, but rather to more recent (and sometimes more readable) accounts.

I.1.3. Acknowledgments. — The basis for this article are the two lectures which I gave at the "Dwork Trimester" in Padova, and I thank the organizers, especially F. Baldassarri, P. Berthelot and B. Chiarellotto for the time and effort they spent to make this conference a success. After I wrote a first version of this article, M. Çiperiani, J-M. Fontaine and H. J. Zhu took the time to read it, pointed out several inaccuracies and made many suggestions for improvement. Any remaining inaccuracies are entirely my fault.

I.2. p-adic representations

I.2.1. Some notations. — The results described in this article are true in a rather general setting. Let k be any perfect field of characteristic p (*perfect* means that the map $x \mapsto x^p$ is an automorphism), and let $F = W(k)[1/p]$ be the fraction field of $\mathcal{O}_F = W(k)$, the ring of Witt vectors over k (for reminders on Witt vectors, see paragraph II.2.1). Let K be a finite totally ramified extension of F, and let $\mathbf{C} = \widehat{\overline{F}} = \widehat{\overline{K}}$ be the p-adic completion of the algebraic closure of F (not to be confused with the field $\mathbb{C}$ of complex numbers). If k is contained in the algebraic closure of $\mathbf{F}_p$, then $\mathbf{C} = \mathbf{C}_p$, the field of so-called p-adic complex numbers.

An important special case is when k is a finite extension of $\mathbf{F}_p$, so that K is a finite extension of $\mathbf{Q}_p$, and F is then the maximal unramified extension of $\mathbf{Q}_p$ contained in K. The reader can safely assume that we're in this situation throughout the article. Another important special case though is when k is algebraically closed.

Let μ_m denote the subset of $\overline{K}$ defined by $\mu_m = \{x \in \overline{K},\ x^m = 1\}$. We'll choose once and for all a compatible sequence of primitive p^n-th roots of unity, $\varepsilon^{(0)} = 1$, and $\varepsilon^{(n)} \in \mu_{p^n} \subset \overline{K}$, such that $\varepsilon^{(1)} \neq 1$ and $(\varepsilon^{(n+1)})^p = \varepsilon^{(n)}$. Let $K_n = K(\varepsilon^{(n)})$ and $K_\infty = \cup_{n=0}^{+\infty} K_n$. Making such a choice of $\varepsilon^{(n)}$ is like choosing an orientation in p-adic Hodge theory, in the same way that choosing one of $\pm i$ is like choosing an orientation in classical geometry. Here are the various fields that we are considering:

$$F \subset \overbrace{\underbrace{K \subset K_n \subset K_\infty}_{\Gamma_K} = \underbrace{K_\infty \subset \overline{F} = \overline{K}}_{H_K}}^{G_K} \subset \mathbf{C}$$

Let G_K be the Galois group $\mathrm{Gal}(\overline{K}/K)$. The *cyclotomic character* $\chi : G_K \to \mathbf{Z}_p^*$ is defined by $\sigma(\zeta) = \zeta^{\chi(\sigma)}$ for every $\sigma \in G_K$ and $\zeta \in \mu_{p^\infty}$. The kernel of the cyclotomic character is $H_K = \mathrm{Gal}(\overline{K}/K_\infty)$, and χ therefore identifies $\Gamma_K = \mathrm{Gal}(K_\infty/K) = G_K/H_K$ with an open subgroup of $\mathbf{Z}_p^*$.

I.2.2. Definitions. — A *p-adic representation* V of G_K is a finite dimensional $\mathbf{Q}_p$-vector space with a continuous linear action of G_K. The dimension of V as a $\mathbf{Q}_p$-vector space will always be denoted by d. Here are some examples of p-adic representations:

1. If $r \in \mathbf{Z}$, then $\mathbf{Q}_p(r) = \mathbf{Q}_p \cdot e_r$ where G_K acts on e_r by $\sigma(e_r) = \chi(\sigma)^r e_r$. This is the r-th *Tate twist* of $\mathbf{Q}_p$;
2. if E is an elliptic curve, then the Tate module of E, $V = \mathbf{Q}_p \otimes_{\mathbf{Z}_p} T_p E$ is a p-adic representation of dimension $d = 2$;
3. more generally, if X is a proper and smooth variety over K, then the étale cohomology $H^i_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p)$ is a p-adic representation of G_K.

This last example is really the most interesting (the first two being special cases), and it was the motivation for the systematic study of p-adic representations. Grothendieck had suggested in 1970 the existence of a "mysterious functor" (*le foncteur mystérieux*) directly linking the étale and crystalline cohomologies of a p-divisible group. Fontaine gave an algebraic construction of that functor which conjecturally allowed one to recover, for any i and any proper and smooth X/K, the de Rham cohomology of X/K (which is a filtered K-vector space) from the data of $H^i_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p)$ as a p-adic representation. His construction was shown to be the right one in general by Tsuji; we'll come back to that in II.5.1.

The above result is a p-adic analogue of the well-known fact that if X is a proper smooth variety over a number field L, then over the complex numbers $\mathbb{C}$ one has an isomorphism

$$\mathbb{C} \otimes_L H^i_{dR}(X/L) \simeq \mathbb{C} \otimes_{\mathbf{Z}} H^i(X, \mathbf{Z})$$

given by integrating differential forms on cycles.

I.2.3. Fontaine's strategy. — Fontaine's strategy for studying p-adic representations was to construct *rings of periods*, which are topological $\mathbf{Q}_p$-algebras B, with a continuous and linear action of G_K and some additional structures which are compatible with the action of G_K (for example: a Frobenius φ, a filtration Fil, a monodromy map N, a differential operator ∂), such that the B^{G_K}-module $D_B(V) = (B \otimes_{\mathbf{Q}_p} V)^{G_K}$, which inherits the additional structures, is an interesting invariant of V. For Fontaine's constructions to work, one needs to assume that B is *G_K-regular*, which means that if $b \in B$ is such that the line $\mathbf{Q}_p \cdot b$ is stable by G_K, then $b \in B^*$. In particular, B^{G_K} has to be a field.

In general, a simple computation shows that $\dim_{B^{G_K}} D_B(V) \leq d = \dim_{\mathbf{Q}_p} V$, and we say that V is *B-admissible* if equality holds, which is equivalent to having $B \otimes_{\mathbf{Q}_p} V \simeq B^d$ as $B[G_K]$-modules. In this case, $B \otimes_{B^{G_K}} D_B(V) \simeq B \otimes_{\mathbf{Q}_p} V$, and the coefficients of a matrix of this isomorphism in two bases of $D_B(V)$ and V are called the *periods* of V.

Let us briefly mention a cohomological version of this: a p-adic representation V determines a class $[V]$ in $H^1(G_K, \mathrm{GL}(d, \mathbf{Q}_p))$, and therefore a class $[V]_B$ in $H^1(G_K, \mathrm{GL}(d, B))$. The representation V is B-admissible if and only if $[V]_B$ is trivial. In this case, $[V]_B$ is a coboundary, given explicitly by writing down a G_K-invariant basis of $B \otimes_{\mathbf{Q}_p} V$.

Here are some examples of rings of periods:

1. If $B = \overline{K}$, then $B^{G_K} = K$ and V is $\overline{K}$-admissible if and only if the action of G_K on V factors through a finite quotient. This is essentially Hilbert 90;
2. If $B = \mathbf{C}$, then $B^{G_K} = K$ (the so-called theorem of Ax-Sen-Tate, first shown by Tate) and V is $\mathbf{C}$-admissible if and only if the action of the inertia I_K on V factors through a finite quotient. This was conjectured by Serre and proved by Sen. We will return to this in IV.4.2;
3. Let $B = \mathbf{B}_{\mathrm{dR}}$ be Fontaine's ring of p-adic periods (defined below in II.2.3). It is a field, equipped with a filtration, and $B^{G_K} = K$. If $V = H^i_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p)$, for a proper smooth X/K, then V is $\mathbf{B}_{\mathrm{dR}}$-admissible, and $\mathbf{D}_{\mathrm{dR}}(V) = \mathbf{D}_{\mathbf{B}_{\mathrm{dR}}}(V) \simeq H^i_{\mathrm{dR}}(X/K)$ as filtered K-vector spaces. This is one of the most important theorems of p-adic Hodge theory.

For rings of periods and Tannakian categories in a general setting, see Fontaine's [**Fo94b**].

I.3. Fontaine's classification

By constructing many rings of periods, Fontaine has defined several subcategories of the category of all p-adic representations, and in this paragraph, we shall list a number of them along with categories of invariants which one can attach to them. Many of the words used here will be defined later in the text, but the table below should serve as a guide to the world of p-adic representations.

p-adic representations	Some invariants attached to those representations	References in the text	ℓ-adic analogue
all of them	(φ, Γ)-modules	III.2.2	–
Hodge-Tate	Hodge-Tate weights	II.1.2	–
de Rham	1. p-adic differential equations 2. filtered K-vector spaces	II.2.3, IV.4.1	all ℓ-adic representations
potentially semi-stable	1. quasi-unipotent differential equations 2. admissible filtered $(\varphi, N, G_{L/K})$-modules	II.3.3, IV.3.3	quasi-unipotent representations
semi-stable	1. unipotent differential equations 2. admissible filtered (φ, N)-modules	II.3.3, IV.3.3	unipotent representations
crystalline	1. trivial differential equations 2. admissible filtered φ-modules	II.3.1, IV.3.3	representations with good reduction

Each category of representations is a subcategory of the one above it. One can associate to every p-adic representation a (φ, Γ)-module, which is an object defined on the boundary of the open unit disk. This object extends to a small annulus, and if V is de Rham, the action of the Lie algebra of Γ gives a p-adic differential equation. This equation is unipotent exactly when V (restricted to G_{K_n} for some n) is semi-stable. In this case, the kernel of the connection is a (φ, N)-module which coincides with the (φ, N)-module attached to V by p-adic Hodge theory (one loses the filtration, however).

All of this will be explained later in the body of the text.

II. p-adic Hodge theory

In this chapter, we'll define various rings of periods which are used in p-adic Hodge theory, and give some simple examples of Fontaine's construction for an explicit geometric object (an elliptic curve).

II.1. The field $\mathbf{C}$ and the theory of Sen

Before we define the rings of periods which are used in p-adic Hodge theory, we'll review some simple properties of the field $\mathbf{C}$ of p-adic complex numbers. As we have seen above, $\mathbf{C}$ is not a great ring of periods (since $\mathbf{C}$-admissible representations are potentially unramified while representations coming from arithmetic geometry are much more complicated than that), but one can still extract a lot of arithmetic information from the data of $\mathbf{C} \otimes_{\mathbf{Q}_p} V$: this is the content of *Sen's theory.*

II.1.1. The action of G_K on $\mathbf{C}$. — An important property of $\mathbf{C}$ that we will need is that we can explicitly describe $\mathbf{C}^H$ where H is a closed subgroup of G_K. Clearly, $\overline{K}^H \subset \mathbf{C}^H$

and therefore $\widehat{\overline{K}^H} \subset \mathbf{C}^H$. The Ax-Sen-Tate theorem says that the latter inclusion is actually an equality: $\widehat{\overline{K}^H} = \mathbf{C}^H$. This was first shown by Tate, and the proof was later improved and generalized by Sen and Ax. Following Sen, Ax gave a natural proof of that result, by showing that if an element of $\overline{K}$ is "almost invariant" by H, then it is "almost" in $\overline{K}^H$.

The first indication that $\mathbf{C}$ was not a good choice for a ring of periods was given by a theorem of Tate, which asserts that $\mathbf{C}$ does not contain periods for characters which are too ramified (for example: the cyclotomic character). More precisely, he showed that if $\psi : G_K \to \mathbf{Z}_p^*$ is a character which is trivial on H_K but which does not factor through a finite quotient of Γ_K, then

$$H^0(K, \mathbf{C}(\psi^{-1})) = \{x \in \mathbf{C},\ g(x) = \psi(g)x \text{ for all } g \in G_K\} = \{0\}.$$

In particular, there is no period in $\mathbf{C}$ for the cyclotomic character (a non-zero element of the above set is a period for ψ^{-1}). Let us explain the proof of Tate's result: by the Ax-Sen-Tate theorem, the invariants of $\mathbf{C}$ under the action of H_K are given by $\mathbf{C}^{H_K} = \widehat{K}_\infty$. The main argument in Tate's proof is the construction of generalized trace maps $\mathrm{pr}_{K_n} : \widehat{K}_\infty \to K_n$. The map pr_{K_n} is a continuous, K_n-linear, and Γ_K-equivariant section of the inclusion $K_n \subset \widehat{K}_\infty$. In addition, if $x \in \widehat{K}_\infty$, then $x = \lim_{n\to\infty} \mathrm{pr}_{K_n}(x)$. We see that we can and should set $\mathrm{pr}_{K_n}(x) = \lim_{m\to+\infty}[K_{n+m} : K_n]^{-1} \mathrm{Tr}_{K_{n+m}/K_n}(x)$. The proof of the convergence of the above limit depends essentially on a good understanding of the ramification of K_∞/K.

Using these maps, one can prove Tate's theorem. Let x be a period of ψ. Since $\psi_{|H_K} = 1$, one has $x \in \mathbf{C}^{H_K} = \widehat{K}_\infty$. We therefore have $x = \lim_{n\to\infty} x_n$ where $x_n = \mathrm{pr}_{K_n}(x)$, and since $g(x) - \psi(g)x = 0$ for all $g \in G_K$, and pr_{K_n} is Galois-equivariant, one also has $g(x_n) - \psi(g)x_n = 0$ for all $g \in G_K$. If $x_n \neq 0$, this would imply that ψ factors through $\mathrm{Gal}(K_n/K)$, a contradiction, so that $x_n = 0$ for every n. Since $x = \lim_{n\to\infty} x_n$, we also have $x = 0$.

General facts on $\mathbf{C}$ can be found in Koblitz's [**Kob84**], which is a good introduction to p-adic numbers. The beginning of [**DGS94**] is a wonderful introduction too. The proof of Ax-Sen-Tate's theorem that we referred to is in Ax's [**Ax70**], see also Colmez's [**Col02**, §4]. Tate's theorems on the cohomology of $\mathbf{C}$ are in [**Tat66**] or in Fontaine's [**Fon00**, §1].

II.1.2. Sen's theory. — The point of Sen's theory is to study the residual action of Γ_K on the $\widehat{K}_\infty$-vector space $(\mathbf{C} \otimes_{\mathbf{Q}_p} V)^{H_K}$, where V is a p-adic representation of G_K. We can summarize his main result as follows. If $d \geq 1$, then $H^1(H_K, \mathrm{GL}(d, \mathbf{C}))$ is trivial and the natural map: $H^1(\Gamma_K, \mathrm{GL}(d, K_\infty)) \to H^1(\Gamma_K, \mathrm{GL}(d, \widehat{K}_\infty))$ induced by the inclusion $K_\infty \subset \widehat{K}_\infty$ is a bijection.

One can show that this implies the following: given a p-adic representation V, the $\widehat{K}_\infty$-vector space $(\mathbf{C} \otimes_{\mathbf{Q}_p} V)^{H_K}$ has dimension $d = \dim_{\mathbf{Q}_p}(V)$, and the union of the finite dimensional K_∞-subspaces of $(\mathbf{C} \otimes_{\mathbf{Q}_p} V)^{H_K}$ stable under the action of Γ_K is a K_∞-vector space of dimension d. We shall call it $\mathbf{D}_{\mathrm{Sen}}(V)$, and the natural map $\widehat{K}_\infty \otimes_{K_\infty} \mathbf{D}_{\mathrm{Sen}}(V) \to (\mathbf{C} \otimes_{\mathbf{Q}_p} V)^{H_K}$ is then an isomorphism. The K_∞-vector space $\mathbf{D}_{\mathrm{Sen}}(V)$ is endowed with an action of Γ_K, and *Sen's invariant* is the linear map giving the action of $\mathrm{Lie}(\Gamma_K)$ on $\mathbf{D}_{\mathrm{Sen}}(V)$. It is the operator defined in $\mathrm{End}(\mathbf{D}_{\mathrm{Sen}}(V))$ by $\Theta_V = \log(\gamma)/\log_p(\chi(\gamma))$, where $\gamma \in \Gamma_K$ is close enough to 1 (the definition of Θ_V doesn't depend on the choice of γ).

More precisely, for any $k \geq 1$, $(1-\gamma)^k$ is a K-linear operator on $\mathbf{D}_{\mathrm{Sen}}(V)$ and one can show that if $\gamma \in \Gamma_K$ is close enough to 1, then the series of operators:

$$-\frac{1}{\log_p(\chi(\gamma))} \sum_{k \geq 1} \frac{(1-\gamma)^k}{k}$$

converges (in $\mathrm{End}(\mathbf{D}_{\mathrm{Sen}}(V))$) to an operator Θ_V which is K_∞-linear and does not depend on the choice of γ.

The operator Θ_V is then an invariant canonically attached to V. Let us give a few examples: we say that V is *Hodge-Tate*, with *Hodge-Tate weights* $h_1, \cdots, h_d \in \mathbf{Z}$, if there is a decomposition of $\mathbf{C}[G_K]$-modules: $\mathbf{C} \otimes_{\mathbf{Q}_p} V = \oplus_{j=1}^d \mathbf{C}(h_j)$. This is equivalent to Θ_V being diagonalizable with integer eigenvalues. For this reason, the eigenvalues of Θ_V are usually called the *generalized Hodge-Tate weights* of V. All representations coming from a proper smooth variety X/K (the subquotients of its étale cohomology groups) are Hodge-Tate, and the integers $-h_j$ are the jumps of the filtration on the de Rham cohomology of X. For example, the Hodge-Tate weights of $V = \mathbf{Q}_p \otimes_{\mathbf{Z}_p} T_p E$, where E is an elliptic curve, are 0 and 1. Here is a representation which is not Hodge-Tate: let V be a two dimensional $\mathbf{Q}_p$-vector space on which G_K acts by

$$\begin{pmatrix} 1 & \log_p(\chi(\overline{g})) \\ 0 & 1 \end{pmatrix} \qquad \text{so that} \qquad \Theta_V = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Relevant papers of Sen are [**Sen72, Sen73, Sen80**] and [**Sen93**] which deals with families of representations. Colmez has given a different construction more in the spirit of the "ring of periods" approach (by constructing a ring $\mathbf{B}_{\mathrm{Sen}}$), see [**Col94**]. For an interesting discussion of all this, see Fontaine's course [**Fon00**, §2].

II.2. The field $\mathbf{B}_{\mathrm{dR}}$

II.2.1. Reminder: Witt vectors. — Before we go any further, we'll briefly review the theory of Witt vectors. Let R be a perfect ring of characteristic p. For example, R could be a finite field or an algebraically closed field, or the ring of integers of an algebraically closed field (in characteristic p, of course). The aim of the theory of Witt vectors is to construct a ring A, in which p is not nilpotent, and such that A is separated and complete for the topology defined by the ideals p^nA. We say that A is a *strict p-ring* with residual ring R. The main result is that if R is a perfect ring of characteristic p, then there exists a unique (up to unique isomorphism) strict p-ring $A = W(R)$ with residual ring R. It is called the ring of *Witt vectors* over R. Furthermore, because of the unicity, if one has a map $\xi : R \to S$, then it lifts to a map $\xi : W(R) \to W(S)$. In particular, the map $x \to x^p$ lifts to a Frobenius automorphism $\varphi : W(R) \to W(R)$.

Let us give a few simple examples: if $R = \mathbf{F}_p$, then $W(R) = \mathbf{Z}_p$ and more generally, if R is a finite field, then $W(R)$ is the ring of integers of the unique unramified extension of $\mathbf{Q}_p$ whose residue field is R. If $R = \overline{\mathbf{F}}_p$, then $W(R) = \mathcal{O}_{\widehat{\mathbf{Q}_p^{\mathrm{unr}}}}$. In the following paragraphs, we will see more interesting examples.

If $x = x_0 \in R$, then for every $n \geq 0$, choose an element $\widetilde{x}_n$ in A whose image in R is $x^{p^{-n}}$. The sequence $\widetilde{x}_n^{p^n}$ then converges in A to an element $[x]$ which depends only on x. This defines a multiplicative map $x \mapsto [x]$ from $R \to A$, which is a section of the projection $x \mapsto \overline{x}$, called the *Teichmüller map*. The Teichmüller elements (the elements in the image of the Teichmüller map) are a distinguished set of representatives of the elements of R. One can write every element $x \in A$ in a unique way as $x = \sum_{n=0}^{+\infty} p^n[x_n]$ with $x_n \in R$. Given two elements $x, y \in A$, one can then write

$$x + y = \sum_{n=0}^{+\infty} p^n[S_n(x_i, y_i)] \quad \text{and} \quad xy = \sum_{n=0}^{+\infty} p^n[P_n(x_i, y_i)]$$

where S_n and $P_n \in \mathbf{Z}[X_i^{p^{-n}}, Y_i^{p^{-n}}]_{i=0 \cdots n}$ are universal homogeneous polynomials of degree one (if one decides that the degrees of the X_i and Y_i are 1). For example, $S_0(X_0, Y_0) = X_0 + Y_0$ and $S_1(X_0, X_1, Y_0, Y_1) = X_1 + Y_1 + p^{-1}((X_0^{1/p} + Y_0^{1/p})^p - X_0 - Y_0)$. The simplest way to construct $W(R)$ is then by setting $W(R) = \prod_{n=0}^{+\infty} R$ and by making it into a ring using the addition and multiplication defined by the P_n and S_n, which are given by (not so) simple functional equations.

Finally, let us mention that if R is not perfect, then there still exist strict p-rings A such that $A/pA = R$, but A is not unique anymore. Such a ring is called a *Cohen ring*. For example, if $R = \mathbf{F}_p[[X]]$, then one can take $A = \mathbf{Z}_p[[X]]$, but for all $\alpha \in p\mathbf{Z}_p$, the map $X \mapsto X + \alpha$ is a non-trivial isomorphism of A which induces the identity on R.

The above summary is inspired by a course given by P. Colmez. The best place to start further reading is Harder's survey [**Har97**]. The construction of Witt vectors is also explained by Serre in [**Ser68**] (or in English in [**Ser79**]).

II.2.2. The universal cover of $\mathbf{C}$. — Let $\widetilde{\mathbf{E}}^+$ be the set defined by

$$\widetilde{\mathbf{E}}^+ = \varprojlim_{x\mapsto x^p} \mathcal{O}_{\mathbf{C}} = \{(x^{(0)}, x^{(1)}, \cdots) \mid (x^{(i+1)})^p = x^{(i)}\}$$

which we make into a ring by deciding that if $x = (x^{(i)})$ and $y = (y^{(i)})$ are two elements of $\widetilde{\mathbf{E}}^+$, then their sum and their product are defined by:

$$(x+y)^{(i)} = \lim_{j\to\infty} (x^{(i+j)} + y^{(i+j)})^{p^j} \text{ and } (xy)^{(i)} = x^{(i)}y^{(i)}.$$

This makes $\widetilde{\mathbf{E}}^+$ into a perfect local ring of characteristic p. Let $\varepsilon = (\varepsilon^{(i)})$ where the $\varepsilon^{(i)}$ are the elements which have been chosen in I.2.1. It is easy to see that $\mathbf{F}_p((\varepsilon-1)) \subset \widetilde{\mathbf{E}} = \widetilde{\mathbf{E}}^+[(\varepsilon-1)^{-1}]$ and one can show that $\widetilde{\mathbf{E}}$ is a field which is the completion of the algebraic (non-separable!) closure of $\mathbf{F}_p((\varepsilon-1))$, so it is really a familiar object.

We define a valuation $v_{\mathbf{E}}$ on $\widetilde{\mathbf{E}}$ by $v_{\mathbf{E}}(x) = v_p(x^{(0)})$ so that $\widetilde{\mathbf{E}}^+$ is the integer ring of $\widetilde{\mathbf{E}}$ for $v_{\mathbf{E}}$. For example, $v_{\mathbf{E}}(\varepsilon-1) = \lim_{n\to\infty} v_p(\varepsilon^{(n)}-1)^{p^n} = p/(p-1)$.

Finally, let us point out that there is a natural map $\widetilde{\mathbf{E}}^+ \to \varprojlim_{x\mapsto x^p} \mathcal{O}_{\mathbf{C}}/p$ and it's not hard to show that this map is an isomorphism.

There is a natural map θ from $\widetilde{\mathbf{E}}^+$ to $\mathcal{O}_{\mathbf{C}}$, which sends $x = (x^{(i)})$ to $x^{(0)}$, and the map $\theta : \widetilde{\mathbf{E}}^+ \to \mathcal{O}_{\mathbf{C}}/p$ is a homomorphism. Let $\widetilde{\mathbf{A}}^+ = W(\widetilde{\mathbf{E}}^+)$ and

$$\widetilde{\mathbf{B}}^+ = \widetilde{\mathbf{A}}^+[1/p] = \{\sum_{k\gg-\infty} p^k[x_k],\ x_k \in \widetilde{\mathbf{E}}^+\}$$

where $[x] \in \widetilde{\mathbf{A}}^+$ denotes the Teichmüller lift of $x \in \widetilde{\mathbf{E}}^+$. The map θ then extends to a surjective homomorphism $\theta : \widetilde{\mathbf{B}}^+ \to \mathbf{C}$, which sends $\sum p^k[x_k]$ to $\sum p^k x_k^{(0)}$. Let $[\varepsilon_1] = [(\varepsilon^{(1)}, \cdots)]$ so that $\varepsilon_1^p = \varepsilon$, and let $\omega = ([\varepsilon]-1)/([\varepsilon_1]-1)$. Then $\theta(\omega) = 1 + \varepsilon^{(1)} + \cdots + (\varepsilon^{(1)})^{p-1} = 0$, and actually, the kernel of θ is the ideal generated by ω.

Here is a simple proof: obviously, the kernel of $\theta : \widetilde{\mathbf{E}}^+ \to \mathcal{O}_{\mathbf{C}}/p$ is the ideal of $x \in \widetilde{\mathbf{E}}^+$ such that $v_{\mathbf{E}}(x) \geq 1$. Let y be any element of $\widetilde{\mathbf{A}}^+$ killed by θ whose reduction modulo p satisfies $v_{\mathbf{E}}(\overline{y}) = 1$. The map $y\widetilde{\mathbf{A}}^+ \to \ker(\theta)$ is then injective, and surjective modulo p; since both sides are complete for the p-adic topology, it is an isomorphism. Now, we just need to observe that the element ω is killed by θ and that $v_{\mathbf{E}}(\overline{\omega}) = 1$.

These constructions are given in Fontaine's [**Fo94a**], but the reader should be warned that the notation is rather different; for example, $\widetilde{\mathbf{E}}^+$ is Fontaine's $\mathcal{R}$ and $\widetilde{\mathbf{A}}^+$ is his $\mathbf{A}_{\mathrm{inf}}$. In [**Fo94a**], the title of this paragraph is also explained (the pair $(\widetilde{\mathbf{B}}^+, \theta : \widetilde{\mathbf{B}}^+ \to \mathbf{C})$ is the solution of a universal problem). The most up-to-date place to read about these rings is Colmez' [**Col02**, §8].

II.2.3. Construction of $\mathbf{B}_{\mathrm{dR}}$. — Using this we can finally define $\mathbf{B}_{\mathrm{dR}}$; let $\mathbf{B}_{\mathrm{dR}}^+$ be the ring obtained by completing $\widetilde{\mathbf{B}}^+$ for the $\ker(\theta)$-adic topology, so that $\mathbf{B}_{\mathrm{dR}}^+ = \varprojlim_n \widetilde{\mathbf{B}}^+/(\ker(\theta))^n$. In particular, since $\ker(\theta) = (\omega)$, every element $x \in \mathbf{B}_{\mathrm{dR}}^+$ can be written (in many ways) as a sum $x = \sum_{n=0}^{+\infty} x_n \omega^n$ with $x_n \in \widetilde{\mathbf{B}}^+$. The ring $\mathbf{B}_{\mathrm{dR}}^+$ is then naturally a F-algebra. Let us construct an interesting element of this ring; since $\theta(1-[\varepsilon]) = 0$, the element $1-[\varepsilon]$ is "small" for the topology of $\mathbf{B}_{\mathrm{dR}}^+$ and the following series

$$-\sum_{n=1}^{+\infty} \frac{(1-[\varepsilon])^n}{n}$$

will converge in $\mathbf{B}_{\mathrm{dR}}^+$, to an element which we call t. Of course, one should think of t as $t = \log([\varepsilon])$. For instance, if $g \in G_F$, then

$$g(t) = g(\log([\varepsilon])) = \log([g(\varepsilon^{(0)}, \varepsilon^{(1)}, \cdots)]) = \log([\varepsilon^{\chi(g)}]) = \chi(g)t$$

so that t is a period for the cyclotomic character.

We now set $\mathbf{B}_{\mathrm{dR}} = \mathbf{B}_{\mathrm{dR}}^+[1/t]$, which is a field that we endow with the filtration defined by $\mathrm{Fil}^i \mathbf{B}_{\mathrm{dR}} = t^i \mathbf{B}_{\mathrm{dR}}^+$. This is the natural filtration on $\mathbf{B}_{\mathrm{dR}}$ coming from the fact that $\mathbf{B}_{\mathrm{dR}}^+$ is a complete discrete valuation ring. By functoriality, all the rings we have defined are equipped with a continuous linear action of G_K. One can show that $\mathbf{B}_{\mathrm{dR}}^{G_K} = K$, so that if V is a p-adic representation, then $\mathbf{D}_{\mathrm{dR}}(V) = (\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V)^{G_K}$ is naturally a filtered K-vector space. We say that V is *de Rham* if $\dim_K \mathbf{D}_{\mathrm{dR}}(V) = d$.

We see that $\mathrm{Gr}\, \mathbf{B}_{\mathrm{dR}} \simeq \oplus_{i \in \mathbf{Z}} \mathbf{C}(i)$, and therefore, if V is a de Rham representation (a $\mathbf{B}_{\mathrm{dR}}$-admissible representation), then there exist d integers $h_1, \cdots, h_d$ such that $\mathbf{C} \otimes_{\mathbf{Q}_p} V \simeq \oplus_{j=1}^d \mathbf{C}(h_j)$. A de Rham representation is therefore Hodge-Tate. Furthermore, one sees easily that the jumps of the filtration on $\mathbf{D}_{\mathrm{dR}}(V)$ are precisely the opposites of Hodge-Tate weights of V (that is, $\mathrm{Fil}^{-h_j}(D) \neq \mathrm{Fil}^{-h_j+1}(D)$). For example, if $V = \mathbf{Q}_p \otimes_{\mathbf{Z}_p} T_p E$, then the Hodge-Tate weights of V are 0 and 1.

References for this paragraph are Fontaine's [**Fo94a**] for the original construction of $\mathbf{B}_{\mathrm{dR}}$, and Colmez's [**Col02**, §8] for a more general presentation. For the behavior of $\mathbf{B}_{\mathrm{dR}}$ under the action of some closed subgroups of G_K, one can see Iovita-Zaharescu's [**IZ99a, IZ99b**].

II.2.4. Sen's theory for $\mathbf{B}_{\mathrm{dR}}^+$. — Fontaine has done the analogue of Sen's theory for $\mathbf{B}_{\mathrm{dR}}^+$, that is, he defined a $K_\infty[[t]]$-module $\mathbf{D}_{\mathrm{dif}}^+(V)$ which is the union of the finite dimensional $K_\infty[[t]]$-submodules of $(\mathbf{B}_{\mathrm{dR}}^+ \otimes_{\mathbf{Q}_p} V)^{H_K}$ which are stable by Γ_K. He then proved that $\mathbf{D}_{\mathrm{dif}}^+(V)$ is a d-dimensional $K_\infty[[t]]$-module endowed with a residual action of Γ_K. The action of $\mathrm{Lie}(\Gamma_K)$ gives rise to a differential operator ∇_V. The representation V is de Rham if and only if ∇_V is trivial on $K_\infty((t)) \otimes_{K_\infty[[t]]} \mathbf{D}_{\mathrm{dif}}^+(V)$. Furthermore, one recovers $(\mathbf{D}_{\mathrm{Sen}}(V), \Theta_V)$ from $(\mathbf{D}_{\mathrm{dif}}^+(V), \nabla_V)$ simply by applying the map $\theta : \mathbf{B}_{\mathrm{dR}}^+ \to \mathbf{C}$.

This construction is carried out in Fontaine's course [**Fon00**, §3,4], where $\mathbf{B}_{\mathrm{dR}}$-representations are classified.

II.3. The rings $\mathbf{B}_{\mathrm{cris}}$ and $\mathbf{B}_{\mathrm{st}}$

II.3.1. Construction of $\mathbf{B}_{\mathrm{cris}}$. — One unfortunate feature of $\mathbf{B}_{\mathrm{dR}}^+$ is that it is too coarse a ring: there is no natural extension of the natural Frobenius $\varphi : \widetilde{\mathbf{B}}^+ \to \widetilde{\mathbf{B}}^+$ to a continuous map $\varphi : \mathbf{B}_{\mathrm{dR}}^+ \to \mathbf{B}_{\mathrm{dR}}^+$. For example, $\theta([\widetilde{p}^{1/p}]-p) \neq 0$, so that $[\widetilde{p}^{1/p}]-p$ is invertible in $\mathbf{B}_{\mathrm{dR}}^+$, and so $1/([\widetilde{p}^{1/p}]-p) \in \mathbf{B}_{\mathrm{dR}}^+$. But if φ is a natural extension of $\varphi : \widetilde{\mathbf{B}}^+ \to \widetilde{\mathbf{B}}^+$, then one should have $\varphi(1/([\widetilde{p}^{1/p}]-p)) = 1/([\widetilde{p}]-p)$, and since $\theta([\widetilde{p}]-p)=0$, $1/([\widetilde{p}]-p) \notin \mathbf{B}_{\mathrm{dR}}^+$.

Another way to see this is that since $\mathbf{B}_{\mathrm{dR}}^{G_L} = L$ for every finite extension L/K, the existence of a canonical Frobenius map $\varphi : \mathbf{B}_{\mathrm{dR}} \to \mathbf{B}_{\mathrm{dR}}$ would imply the existence of a Frobenius map $\varphi : \overline{K} \to \overline{K}$, which is of course not the case. One would still like to have a Frobenius map, and there is a natural way to complete $\widetilde{\mathbf{B}}^+$ (where one avoids inverting elements like $[\widetilde{p}^{1/p}]-p$) such that the completion is still endowed with a Frobenius map.

The ring $\mathbf{B}_{\mathrm{cris}}^+$ is a subring of $\mathbf{B}_{\mathrm{dR}}^+$, consisting of the limits of sequences of elements of $\mathbf{B}_{\mathrm{dR}}^+$ which satisfy some growth condition. For example, $\sum_{n=0}^{+\infty} p^{-n^2} t^n$ converges in $\mathbf{B}_{\mathrm{dR}}^+$ but not in $\mathbf{B}_{\mathrm{cris}}^+$. The ring $\mathbf{B}_{\mathrm{cris}}^+$ is then equipped with a continuous Frobenius φ. More precisely, recall that every element $x \in \mathbf{B}_{\mathrm{dR}}^+$ can be written (in many ways) as $x = \sum_{n=0}^{+\infty} x_n \omega^n$ with $x_n \in \widetilde{\mathbf{B}}^+$. One then has:

$$\mathbf{B}_{\mathrm{cris}}^+ = \{x \in \mathbf{B}_{\mathrm{dR}}^+,\ x = \sum_{n=0}^{+\infty} x_n \frac{\omega^n}{n!},\ \text{where } x_n \to 0 \text{ in } \widetilde{\mathbf{B}}^+\}$$

Let $\mathbf{B}_{\mathrm{cris}} = \mathbf{B}_{\mathrm{cris}}^+[1/t]$ (note that $\mathbf{B}_{\mathrm{cris}}$ is not a field. For example, $\omega - p$ is in $\mathbf{B}_{\mathrm{cris}}$ but not its inverse); one can show that $(\mathbf{B}_{\mathrm{cris}})^{G_K} = F$, the maximal absolutely unramified subfield of K. Those representations V of G_K which are $\mathbf{B}_{\mathrm{cris}}$-admissible are called *crystalline*, and using Fontaine's construction one can therefore associate to every such V a filtered φ-module $\mathbf{D}_{\mathrm{cris}}(V) = (\mathbf{B}_{\mathrm{cris}} \otimes_{\mathbf{Q}_p} V)^{G_K}$ (a *filtered φ-module* D is an F-vector space with a decreasing, exhaustive and separated filtration indexed by $\mathbf{Z}$ on $K \otimes_F D$, and a σ_F-semi-linear map $\varphi : D \to D$. We do not impose any compatibility condition between φ and Fil). One can associate to a filtered φ-module D two polygons: its Hodge polygon $P_H(D)$, coming from the filtration, and its Newton polygon $P_N(D)$, coming from the slopes of φ. We say that D is *admissible* if for every subobject D' of D, the Hodge polygon of D' lies below the Newton polygon of D', and the endpoints of the Hodge and Newton polygons of D are the same. One can show that $\mathbf{D}_{\mathrm{cris}}(V)$ is always admissible.

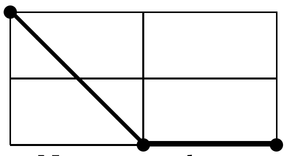
Furthermore, a theorem of Colmez and Fontaine shows that the functor $V \mapsto \mathbf{D}_{\mathrm{cris}}(V)$ is an equivalence of categories between the category of crystalline representations and the category of admissible filtered φ-modules [(1)].

The construction of $\mathbf{B}_{\mathrm{cris}}$ can be found in Fontaine's [**Fo94a**] or Colmez's [**Col02**, §8]. One should also look at Fontaine's [**Fo94b**] for information on filtered φ-modules. The theorem of Colmez-Fontaine is proved in Colmez-Fontaine's [**CF00**], as well as in Colmez's [**Col02**, §10] and it is reviewed in Fontaine's [**Fon00**, §5]. The ring $\mathbf{B}_{\mathrm{cris}}^+$ has an interpretation in crystalline cohomology, see Fontaine's [**Fon83**] and Fontaine-Messing's [**FM87**].

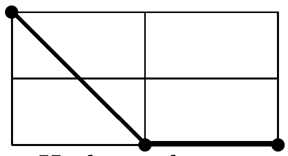
II.3.2. Example: elliptic curves. — If $V = \mathbf{Q}_p \otimes_{\mathbf{Z}_p} T_p E$, where E is an elliptic curve over F with good ordinary reduction, then $\mathbf{D}_{\mathrm{cris}}(V)$ is a 2-dimensional F-vector space with a basis x, y, and there exists $\lambda \in F$ and $\alpha_0, \beta_0 \in \mathcal{O}_F^*$ depending on E such that:

$$\begin{cases} \varphi(x) & = \alpha_0 p^{-1} x \\ \varphi(y) & = \beta_0 y \end{cases} \qquad \text{and} \qquad \mathrm{Fil}^i \mathbf{D}_{\mathrm{cris}}(V) = \begin{cases} \mathbf{D}_{\mathrm{cris}}(V) & \text{if } i \leq -1 \\ (y + \lambda x)F & \text{if } i = 0 \\ \{0\} & \text{if } i \geq 1 \end{cases}$$

The Newton and Hodge polygons of $\mathbf{D}_{\mathrm{cris}}(V)$ are then as follows:

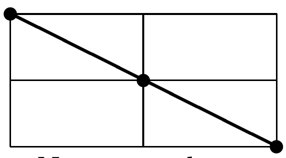


Newton polygon

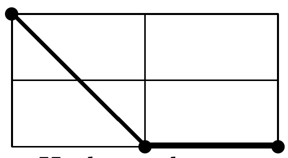


Hodge polygon

If on the other hand an elliptic curve E has good supersingular reduction, then the operator $\varphi : \mathbf{D}_{\mathrm{cris}}(V) \to \mathbf{D}_{\mathrm{cris}}(V)$ is irreducible and the Newton and Hodge polygons are as follows:



Newton polygon



Hodge polygon

In both cases, it is clear that $\mathbf{D}_{\mathrm{cris}}(V)$ is admissible.

For basic facts about elliptic curves, see for example Silverman's [**Sil86, Sil96**]. For basic facts on Newton polygons, see the first chapter of [**DGS94**] and for isocrystals, see [**Fon79, Kz79**].

(1) admissible modules were previously called *weakly admissible*, but since Colmez and Fontaine showed that being weakly admissible is the same as being admissible (previously, D was said to be admissible if there exists some V such that $D = \mathbf{D}_{\mathrm{cris}}(V)$), we can drop the "weakly" altogether.

II.3.3. Semi-stable representations. — If an elliptic curve E has bad semi-stable reduction, then V is not crystalline but it is *semi-stable*, that is, it is $\mathbf{B}_{\mathrm{st}}$-admissible where $\mathbf{B}_{\mathrm{st}} = \mathbf{B}_{\mathrm{cris}}[Y]$, where we have decided that $\varphi(Y) = Y^p$ and $g(Y) = Y + c(g)t$, where $c(g)$ is defined by the formula $g(p^{1/p^n}) = p^{1/p^n}(\varepsilon^{(n)})^{c(g)}$. Of course, the definition of Y depends on a number of choices, but two such $\mathbf{B}_{\mathrm{st}}$'s are isomorphic. In addition to a Frobenius, $\mathbf{B}_{\mathrm{st}}$ is equipped with the *monodromy map* $N = -d/dY$.

Let $\widetilde{p} \in \widetilde{\mathbf{E}}^+$ be an element such that $\widetilde{p}^{(0)} = p$, and let $\log[\widetilde{p}] \in \mathbf{B}_{\mathrm{dR}}^+$ be the element defined by

$$\log[\widetilde{p}] = \log_p(p) - \sum_{n=1}^{+\infty} \frac{(1 - [\widetilde{p}]/p)^{n-1}}{n}.$$

One can define a Galois equivariant and $\mathbf{B}_{\mathrm{cris}}$-linear embedding of $\mathbf{B}_{\mathrm{st}}$ into $\mathbf{B}_{\mathrm{dR}}$, by mapping Y to $\log[\widetilde{p}] \in \mathbf{B}_{\mathrm{dR}}^+$, but doing so requires us to make a choice of $\log_p(p)$. As a consequence, there is no canonically defined filtration on $\mathbf{D}_{\mathrm{st}}(V)$, only on $\mathbf{D}_{\mathrm{dR}}(V)$: one has to be a little careful about this. This in contrast to the fact that the inclusion $K \otimes_F \mathbf{D}_{\mathrm{cris}}(V) \subset \mathbf{D}_{\mathrm{dR}}(V)$ is canonical. It is customary to choose $\log_p(p) = 0$ which is what we'll always assume from now on.

One can then associate to every semi-stable representation V a filtered (φ, N)-module and Colmez and Fontaine showed that the functor $V \mapsto \mathbf{D}_{\mathrm{st}}(V)$ is an equivalence of categories between the category of semi-stable representations and the category of admissible filtered (φ, N)-modules.

See the references for paragraph II.3.1 on $\mathbf{B}_{\mathrm{cris}}$. See II.4.3 for Fontaine's original definition of $\mathbf{B}_{\mathrm{st}}$.

II.3.4. Frobenius and filtration. — Although the ring $\mathbf{B}_{\mathrm{cris}}$ is endowed with both a Frobenius map and the filtration induced by the inclusion $\mathbf{B}_{\mathrm{cris}} \subset \mathbf{B}_{\mathrm{dR}}$, these two structures have little compatibility. For example, here is an exercise: let $r = \{r_n\}_{n \geq 0}$ be a sequence with $r_n \in \mathbf{Z}$. Show that there exists an element $x_r \in \cap_{k \geq 0} \varphi^k \mathbf{B}_{\mathrm{cris}}$ such that for every $n \geq 0$, one has $\varphi^{-n}(x_r) \in \mathrm{Fil}^{r_n} \mathbf{B}_{\mathrm{dR}} \setminus \mathrm{Fil}^{r_n+1} \mathbf{B}_{\mathrm{dR}}$ (for a solution, see paragraph IV.3.2). The reader should also be warned that $\mathbf{B}_{\mathrm{cris}}^+ \subset \mathrm{Fil}^0 \mathbf{B}_{\mathrm{cris}} = \mathbf{B}_{\mathrm{cris}} \cap \mathbf{B}_{\mathrm{dR}}^+$ but that the latter space is much larger. It is true however that if $\mathbf{B}'_{\mathrm{cris}}$ is the set of elements $x \in \mathbf{B}_{\mathrm{cris}}$ such that for every $n \geq 0$, one has $\varphi^n(x) \in \mathrm{Fil}^0 \mathbf{B}_{\mathrm{cris}}$, then $\varphi(\mathbf{B}'_{\mathrm{cris}}) \subset \mathbf{B}_{\mathrm{cris}}^+ \subset \mathbf{B}'_{\mathrm{cris}}$ ($\varphi^2(\mathbf{B}'_{\mathrm{cris}})$ if $p = 2$).

Given the above facts, it is rather surprising that there is a relation of some sort between φ and Fil. One can show that the natural map $\mathbf{B}_{\mathrm{cris}}^{\varphi=1} \to \mathbf{B}_{\mathrm{dR}}/\mathbf{B}_{\mathrm{dR}}^+$ is surjective, and that its kernel is $\mathbf{Q}_p$. This gives rise to an exact sequence

$$0 \to \mathbf{Q}_p \to \mathbf{B}_{\mathrm{cris}}^{\varphi=1} \to \mathbf{B}_{\mathrm{dR}}/\mathbf{B}_{\mathrm{dR}}^+ \to 0$$

called the *fundamental exact sequence*. It is used to define *Bloch-Kato's exponential.*

See Fontaine's [**Fo94a**] and [**Fo94b**] or Colmez's [**Col02**, §8]. For Bloch-Kato's exponential, see Bloch-Kato's [**BK91**, §3] and Kato's [**Kat93**].

II.3.5. Some remarks on topology. — We'll end this section with a few remarks on the topologies of the rings we just introduced. Although $\mathbf{B}_{\mathrm{dR}}^+$ is a discrete valuation ring, complete for that valuation, the natural topology on $\mathbf{B}_{\mathrm{dR}}^+$ is weaker than the topology coming from that valuation. It is actually the topology of the projective limit on $\mathbf{B}_{\mathrm{dR}}^+ = \varprojlim_n \widetilde{\mathbf{B}}^+/\ker(\theta)^n$, and the topology of $\widetilde{\mathbf{B}}^+ = \widetilde{\mathbf{A}}^+[1/p]$ combines the p-adic topology and the topology of the residue ring $\widetilde{\mathbf{A}}^+/p = \widetilde{\mathbf{E}}^+$ which is a valued ring. In particular, $\mathbf{B}_{\mathrm{dR}}^+/\ker(\theta)^n$ is p-adic Banach space, which makes $\mathbf{B}_{\mathrm{dR}}^+$ into a p-adic Fréchet space.

The topology on $\mathbf{B}_{\mathrm{cris}}$ is quite unpleasant, as Colmez points out: "By the construction of $\mathbf{B}_{\mathrm{cris}}^+$, the sequence $x_n = \omega^{p^n-1}/(p^n-1)!$ does not converge to 0 in $\mathbf{B}_{\mathrm{cris}}^+$, but the sequence ωx_n does; we deduce from this the fact that the sequence tx_n converges to 0 in $\mathbf{B}_{\mathrm{cris}}^+$, and therefore that $x_n \to 0$ in $\mathbf{B}_{\mathrm{cris}}$, so that the topology of $\mathbf{B}_{\mathrm{cris}}^+$ induced by that of $\mathbf{B}_{\mathrm{cris}}$ is not the natural topology of $\mathbf{B}_{\mathrm{cris}}^+$." The reason for this is that the sequence $n!$ converges to 0 in a pretty chaotic manner, and it is more convenient to use the ring

$$\mathbf{B}_{\max}^+ = \{x \in \mathbf{B}_{\mathrm{dR}}^+,\ x = \sum_{n=0}^{+\infty} x_n \frac{\omega^n}{p^n},\ \text{where } x_n \to 0 \text{ in } \widetilde{\mathbf{B}}^+\},$$

which is also endowed with a Frobenius map. In any case, the periods of crystalline representations live in

$$\widetilde{\mathbf{B}}_{\mathrm{rig}}^+[1/t] = \cap_{n=0}^{+\infty}\varphi^n\mathbf{B}_{\mathrm{cris}}^+[1/t] = \cap_{n=0}^{+\infty}\varphi^n\mathbf{B}_{\max}^+[1/t]$$

because they live in finite dimensional F-vector subspaces of $\mathbf{B}_{\mathrm{cris}}$ stable by φ.

Finally, let us mention an interesting result of Colmez, that has yet to be applied: $\overline{K}$ is naturally a subring of $\mathbf{B}_{\mathrm{dR}}^+$, and he showed that $\mathbf{B}_{\mathrm{dR}}^+$ is the completion of $\overline{K}$ for the induced topology, which is finer than the p-adic topology. This generalizes an earlier result of Fontaine, who showed that $\overline{K}$ is dense $\mathbf{B}_{\mathrm{dR}}^+/t^2$. The topology of $\overline{K}$ induced by $\mathbf{B}_{\mathrm{dR}}^+$ is a bit like the "uniform convergence of a function and all its derivatives", if one views $x \in \overline{K}$ as an algebraic function of p. For example, the series $\sum_{n=0}^{+\infty} p^n\varepsilon^{(n)}$ is not convergent in $\mathbf{B}_{\mathrm{dR}}^+$. A series which converges in $\mathbf{B}_{\mathrm{dR}}^+$ does so in $\mathbf{C}$, so we get a map $\theta : \mathbf{B}_{\mathrm{dR}}^+ \to \mathbf{C}$, which coincides with the one previously defined.

The remark on the topology of $\mathbf{B}_{\mathrm{cris}}$ can be found in Colmez's [**Col98a**, §III], and Colmez's theorem is proved in the appendix to Fontaine's [**Fo94a**]. Fontaine's earlier result was used by Fontaine and Messing in [**FM87**]. The ring $\widetilde{\mathbf{B}}_{\mathrm{rig}}^+$ has an interpretation in rigid cohomology, as was explained to me by Berthelot in [**Blt01**].

II.4. Application: Tate's elliptic curve

We will now explicitly show that if E is an elliptic curve with bad semi-stable reduction, then $V = \mathbf{Q}_p \otimes_{\mathbf{Z}_p} T_p E$ is $\mathbf{B}_{\mathrm{dR}}$-admissible. After that, we will show that V is actually semi-stable. We'll assume throughout this section that $K = F$, ie that K is absolutely unramified.

II.4.1. Tate's elliptic curve. — Let q be a formal parameter and define

$$s_k(q) = \sum_{n=1}^{+\infty} \frac{n^k q^n}{1-q^n} \qquad a_4(q) = -s_3(q) \qquad a_6(q) = -\frac{5s_3(q)+7s_5(q)}{12}$$

$$x(q,v) = \sum_{n=-\infty}^{+\infty} \frac{q^n v}{(1-q^n v)^2} - 2s_1(q) \qquad y(q,v) = \sum_{n=-\infty}^{+\infty} \frac{(q^n v)^2}{(1-q^n v)^3} + s_1(q).$$

All those series are convergent if $q \in p\mathcal{O}_F$ and $v \notin q^{\mathbf{Z}} = \langle q \rangle$ (the multiplicative subgroup of F^* generated by q). For such $q \neq 0$, let E_q be the elliptic curve defined by the equation $y^2 + xy = x^3 + a_4(q)x + a_6(q)$. The theorem of Tate is then: the elliptic curve E_q is defined over F, it has bad semi-stable reduction, and E_q is uniformized by $\overline{F}^*$, that is, there exists a map $\alpha : \overline{F}^* \to E_q(\overline{F})$, given by

$$v \mapsto \begin{cases} (x(q,v), y(q,v)) & \text{if } v \notin q^{\mathbf{Z}} \\ 0 & \text{if } v \in q^{\mathbf{Z}} \end{cases}$$

which induces an isomorphism of groups with G_F-action $\overline{F}^*/\langle q \rangle \to E_q(\overline{F})$.

Furthermore, if E is an elliptic curve over F with bad semi-stable reduction, then there exists q such that E is isomorphic to E_q over F.

For basic facts about Tate's elliptic curve, see Silverman's [**Sil96**, V.3] for example.

II.4.2. The p-adic representation attached to E_q. — Using Tate's theorem, we can give an explicit description of $T_p(E_q)$. Let $\varepsilon^{(i)}$ be the p^i-th roots of unity chosen in I.2.1 and let $q^{(i)}$ be elements defined by $q^{(0)} = q$ and the requirement that $(q^{(i+1)})^p = q^{(i)}$. Then α induces isomorphisms

$$\begin{array}{ccc} \overline{F}^*/\langle q \rangle & \longrightarrow & E_q(\overline{F}) \\ \{x \in \overline{F}^*/\langle q \rangle, x^{p^n} \in \langle q \rangle\} & \longrightarrow & E_q(\overline{F})[p^n] \end{array}$$

and one sees that $\{x \in \overline{F}^*/\langle q \rangle, x^{p^n} \in \langle q \rangle\} = \{(\varepsilon^{(n)})^i (q^{(n)})^j, 0 \leq i,j < p^n - 1\}$. The elements $\varepsilon^{(n)}$ and $q^{(n)}$ therefore form a basis of $E_q(\overline{F})[p^n]$, so that a basis of $T_p(E_q)$ is given by $e = \varprojlim_n \varepsilon^{(n)}$ and $f = \varprojlim_n q^{(n)}$. This makes it possible to compute explicitly the Galois action on $T_p(E_q)$. We have $g(e) = \varprojlim_n g(\varepsilon^{(n)}) = \chi(g)e$ and $g(f) = \varprojlim_n g(q^{(n)}) = \varprojlim_n q^{(n)}(\varepsilon^{(n)})^{c(g)} = f + c(g)e$ where $c(g)$ is some p-adic integer, determined by the fact that $g(q^{(n)}) = q^{(n)}(\varepsilon^{(n)})^{c(g)}$. Note that $[g \mapsto c(g)] \in H^1(F, \mathbf{Z}_p(1))$. The matrix of g in the basis (e,f) is therefore given by

$$\begin{pmatrix} \chi(g) & c(g) \\ 0 & 1 \end{pmatrix}$$

II.4.3. p-adic periods of E_q. — We are looking for p-adic periods of $V = \mathbf{Q}_p \otimes_{\mathbf{Z}_p} T_p(E_q)$ which live in $\mathbf{B}_{\mathrm{dR}}$, that is for elements of $(\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V)^{G_F}$. An obvious candidate is $t^{-1} \otimes e$ since $g(t) = \chi(g)t$ and $g(e) = \chi(g)e$. Let us look for a second element of $(\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V)^{G_F}$, of the form $a \otimes e + 1 \otimes f$. We see that this element will be fixed by G_F if and only if $g(a)\chi(g) + c(g) = a$.

Let $\widetilde{q}$ be the element of $\widetilde{\mathbf{E}}^+$ defined by $\widetilde{q} = (q^{(0)}, q^{(1)}, \cdots)$. Observe that we have

$$g(\widetilde{q}) = (g(q^{(0)}), g(q^{(1)}), \cdots) = \widetilde{q}\varepsilon^{c(g)},$$

and that $\theta([\widetilde{q}]/q^{(0)} - 1) = 0$, so that $[\widetilde{q}]/q^{(0)} - 1$ is small in the $\ker(\theta)$-adic topology. The series

$$\log_p(q^{(0)}) - \sum_{n=1}^{+\infty} \frac{(1 - [\widetilde{q}]/q^{(0)})^n}{n}$$

therefore converges in $\mathbf{B}_{\mathrm{dR}}^+$ to an element which we call u. One should think of u as being $u = \log([\widetilde{q}])$. In particular, $g(u) = g(\log([\widetilde{q}])) = \log([g(\widetilde{q})]) = \log([\widetilde{q}]) + c(g)\log([\varepsilon]) = u + c(g)t$, and we readily see that $a = -u/t$ satisfies the equation $g(a)\chi(g) + c(g) = a$. A basis of $\mathbf{D}_{\mathrm{dR}}(V) = (\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V)^{G_F}$ is therefore given by

$$\begin{cases} x = t^{-1} \otimes e \\ y = -ut^{-1} \otimes e + 1 \otimes f \end{cases}$$

and this shows that $T_p(E_q)$ is $\mathbf{B}_{\mathrm{dR}}$-admissible. Furthermore, one sees that $\theta(u - \log_p(q^{(0)})) = 0$, so that $u - \log_p(q^{(0)})$ is divisible by t and

$$\operatorname{Fil}^i \mathbf{D}_{\mathrm{dR}}(V) = \begin{cases} \mathbf{D}_{\mathrm{dR}}(V) & \text{if } i \leq -1 \\ (y + \log_p(q^{(0)})x)F & \text{if } i = 0 \\ \{0\} & \text{if } i \geq 1 \end{cases}$$

This gives us a description of $\mathbf{D}_{\mathrm{dR}}(V)$. We shall now prove that V is semi-stable. It's clearly enough to show that $t, u \in \mathbf{B}_{\mathrm{st}}^+$. The series which defines t converges in $\mathbf{B}_{\mathrm{cris}}^+$ (that is, the cyclotomic character is crystalline), and the series which defines $\log[\widetilde{q}/\widetilde{p}^{v_p(q)}]$ also does. As a consequence, one can write $u = v_p(q)Y + \log[\widetilde{q}/\widetilde{p}^{v_p(q)}] \in \mathbf{B}_{\mathrm{st}}^+$. This implies that V is semi-stable. Actually, Fontaine defined $\mathbf{B}_{\mathrm{st}}$ so that it would contain $\mathbf{B}_{\mathrm{cris}}$ and a period of E_q, so that the computation of this paragraph is a little circular.

Let us compute the action of Frobenius in the case of Tate's elliptic curve. On a ring of characteristic p, one expects Frobenius to be $x \mapsto x^p$, and therefore $\varphi([x])$ should be $[x^p]$ so that $\varphi(\log[x]) = p\log[x]$. In particular, one has $\varphi(t) = pt$ and $\varphi(u) = pu$ and the action of Frobenius on $\mathbf{D}_{\mathrm{st}}(V)$ is therefore given by $\varphi(x) = p^{-1}x$ and $\varphi(y) = y$. Let us point out one more time that the filtration is defined on $\mathbf{D}_{\mathrm{dR}}(V)$, and that the identification $\mathbf{D}_{\mathrm{st}}(V) \simeq \mathbf{D}_{\mathrm{dR}}(V)$ depends on a choice of $\log_p(p)$. The p-adic number $\log_p(q^{(0)}/p^{v_p(q^{(0)})})$ is canonically attached to V and is called the *$\mathcal{L}$-invariant* of V.

II.4.4. Remark: Kummer theory. — What we have done for Tate's elliptic curve is really a consequence of the fact that $V = \mathbf{Q}_p \otimes_{\mathbf{Z}_p} T_p E_q$ is an extension of $\mathbf{Q}_p$ by $\mathbf{Q}_p(1)$, namely that there is an exact sequence $0 \to \mathbf{Q}_p(1) \to V \to \mathbf{Q}_p \to 0$. All of these extensions are classified by the cohomology group $H^1(K, \mathbf{Q}_p(1))$, which is described by Kummer theory. Recall that for every $n \geq 1$, there is an isomorphism $\delta_n : K^*/(K^*)^{p^n} \to H^1(K, \mu_{p^n})$. By taking the projective limit over n, we get a map $\delta : \widehat{K^*} \to H^1(K, \mathbf{Z}_p(1))$ because $\varprojlim_n \mu_{p^n} \simeq \mathbf{Z}_p(1)$ once we have chosen a compatible sequence of $\varepsilon^{(n)}$. By tensoring with $\mathbf{Q}_p$, we get an isomorphism $\delta : \mathbf{Q}_p \otimes_{\mathbf{Z}_p} \widehat{K^*} \to H^1(K, \mathbf{Q}_p(1))$ which is defined in the following way: if $q = q^{(0)} \in \mathbf{Q}_p \otimes_{\mathbf{Z}_p} \widehat{K^*}$, choose a sequence $q^{(n)}$ such that $(q^{(n)})^p = q^{(n-1)}$, and define $c = \delta(q)$ by $(\varepsilon^{(n)})^{c(g)} = g(q^{(n)})/(q^{(n)})$. Of course, this depends on the choice of $q^{(n)}$, but two different choices give cohomologous cocycles.

It is now easy to show that every extension of $\mathbf{Q}_p$ by $\mathbf{Q}_p(1)$ is semi-stable. This is because $tc(g) = g(\log[\widetilde{q}]) - \log[\widetilde{q}]$ with notations similar to those above, and $\widetilde{q} = (q^{(n)})$. If $q \in \mathbf{Q}_p \otimes_{\mathbf{Z}_p} \widehat{\mathcal{O}_K^*}$ then the series which defines $\log[\widetilde{q}]$ converges in $\mathbf{B}^+_{\mathrm{cris}}$ and the extension V is crystalline. In general, if $q \in \mathbf{Q}_p \otimes_{\mathbf{Z}_p} \widehat{K^*}$, then $\log[\widetilde{q}]$ will be in $\mathbf{B}^+_{\mathrm{cris}} + v_p(q)Y \subset \mathbf{B}^+_{\mathrm{st}}$. The F-vector space $\mathbf{D}_{\mathrm{st}}(V)$ will then have a basis $x = t^{-1} \otimes e$ and $y = -\log[\widetilde{q}]t^{-1} \otimes e + 1 \otimes f$ so that $\varphi(x) = p^{-1}x$ and $\varphi(y) = y$. If one chooses $\log_p(p) = 0$, then the filtration on $\mathbf{D}_{\mathrm{dR}}(V)$ is given by $\mathrm{Fil}^0 \mathbf{D}_{\mathrm{dR}}(V) = (y + \log_p(q)x)F$.

II.5. p-adic representations and Arithmetic Geometry

II.5.1. Comparison theorems. — If X/K is a proper smooth variety over K, then by a *comparison theorem*, we mean a theorem relating $H^i_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p)$ and $H^i_{\mathrm{dR}}(X/K)$.

It was shown early on by Fontaine that the Tate modules $V = \mathbf{Q}_p \otimes_{\mathbf{Z}_p} T_p A$ of all abelian varieties A are de Rham (he actually showed in a letter to Jannsen that they were potentially semi-stable), and that $\mathbf{D}_{\mathrm{dR}}(V)$ is isomorphic to the dual of the de Rham cohomology of A. Fontaine and Messing then found another proof, in which they explicitly construct a pairing between V (interpreted as a quotient of the étale $\pi_1(A)$) and $H^1_{\mathrm{dR}}(A/K)$ (interpreted as the group of isomorphism classes of vectorial extensions of A). One should remember that for an abelian variety A, we have $\mathrm{Hom}(T_p A, \mathbf{Z}_p(1)) \simeq H^1_{\text{ét}}(A_{\overline{K}}, \mathbf{Z}_p)$.

After that, Fontaine and Messing proved the comparison theorem for the $H^i_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p)$ of proper smooth X for $i \leq p-1$ and $K/\mathbf{Q}_p$ finite unramified. These results were then extended by Kato and his school (Hyodo, Tsuji). Finally, the general statement that for a variety X/K, one can recover $H^i_{\mathrm{dR}}(X/K)$ from the data of $V = H^i_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p)$ as a p-adic representation was shown by Tsuji. He showed that if X has semi-stable reduction, then $V = H^i_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p)$ is $\mathbf{B}_{\text{st}}$-admissible. A different proof was given by Niziol (in the good reduction case) and also by Faltings (who proved that V is crystalline if X has good reduction and that V is de Rham otherwise).

In the case of an abelian variety, the rings $\mathbf{B}_{\text{cris}}$ and $\mathbf{B}_{\text{st}}$ are exactly what it takes to decide, from the data of V alone, whether A has good or semi-stable reduction. Indeed, Coleman-Iovita and Breuil showed that A has good reduction if and only if V is crystalline, and that A has semi-stable reduction if and only if V is semi-stable. This can be seen as a p-adic analogue of the (ℓ-adic) Néron-Ogg-Shafarevich criterion.

In another direction, Fontaine and Mazur have conjectured the following: let V be a p-adic representation of $\text{Gal}(\overline{\mathbf{Q}}/L)$ where L is a finite extension of $\mathbf{Q}$. Then, V "comes from geometry" if and only if it is unramified at all but finitely many primes ℓ, and if its restriction to all decomposition groups above p are potentially semi-stable.

Note that among all potentially semi-stable representations V of G_K, where K is a p-adic field, there are many which do not come from geometry: indeed, if $V = H^i_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p)$ then the eigenvalues of φ on $\mathbf{D}_{\text{st}}(V)$ should at least be Weil numbers.

There were many partial results before Tsuji's theorem was proved in [**Tsu99**] (see [**Tsu02**] for a survey), and we refer the reader to the bibliography of that article. For a different approach (integrating forms on cycles), see Colmez's [**Col98b**].

The conjecture of Fontaine and Mazur was proposed by them in [**FM93**]. There is little known in that direction, but there are some partial results: see Taylor's [**Tay01**] and Kisin's [**Kis03**] for example.

Regarding the criteria for good or semi-stable reduction, see Coleman-Iovita's [**CI99**] and Breuil's [**Bre00**].

II.5.2. Weil-Deligne representations. — Let V be a potentially semi-stable representation of G_K, so that there exists L, a finite extension of K such that the restriction of V to G_L is semi-stable. One can then consider the F-vector space $\mathbf{D}^L_{\text{st}}(V) = (\mathbf{B}_{\text{st}} \otimes_{\mathbf{Q}_p} V)^{G_L}$. It is a finite dimensional (φ, N)-module with an action of $\text{Gal}(L/K)$. One can attach to such an object several interesting invariants: L-factors, ϵ-factors, and a representation of the Weil-Deligne group.

In particular, if E is an elliptic curve, one can recover from the p-adic representation T_pE pretty much the same information as from the ℓ-adic representation $T_\ell E$.

The action of the Weil-Deligne group on $\mathbf{D}^L_{\text{st}}(V)$ was defined by Fontaine in [**Fo94c**].

III. Fontaine's (φ,Γ)-modules

III.1. The characteristic p theory

A powerful tool for studying p-adic representations is Fontaine's theory of (φ,Γ)-modules. We will first define φ-modules for representations of the Galois group of a local field of characteristic p (namely $k((\pi))$) and then apply this to the characteristic zero case, making use of Fontaine-Wintenberger's theory of the field of norms.

III.1.1. Local fields of characteristic p. — Let π_K be a formal variable (for now), let F' be the maximal unramified extension of F in K_∞ and [(2)] let $\mathbf{A}_K$ be the ring

$$\mathbf{A}_K = \{ \sum_{k=-\infty}^{\infty} a_k \pi_K^k,\ a_k \in \mathcal{O}_{F'},\ a_{-k} \to 0\},$$

so that $\mathbf{A}_K/p = k_{F'}((\pi_K))$. The ring $\mathbf{A}_K$ (which is an example of a Cohen ring, as in II.2.1) is endowed with actions of φ and Γ_K, such that $\varphi(\pi_K) = \pi_K^p \mod p$. The exact formulas depend on K, but if $K = F$ then $\varphi(\pi_K) = (1+\pi_K)^p - 1$ and if $\gamma \in \Gamma_K$, then $\gamma(\pi_K) = (1+\pi_K)^{\chi(\gamma)} - 1$. We won't use the action of Γ_K in the "characteristic p case". Let $\mathbf{E}_K = k_{F'}((\pi_K)) = \mathbf{A}_K/p$, and let $\mathbf{E}$ be the separable closure of $\mathbf{E}_K$. Let G_{E_K} be the Galois group of $\mathbf{E}/\mathbf{E}_K$. In this paragraph, we will look at p-adic representations of G_{E_K}, that is, finite dimensional $\mathbf{Q}_p$-vector spaces V, endowed with a continuous linear action of G_{E_K}.

Let $\mathbf{B}_K$ be the fraction field of $\mathbf{A}_K$ (one only needs to invert p). A *φ-module* M is a finite dimensional $\mathbf{B}_K$-vector space with a semi-linear action of φ. We say that M is *étale* (or *slope* 0 or also *unit-root*) if M admits an $\mathbf{A}_K$-lattice M_A which is stable by φ and such that $\varphi^* M_A = M_A$ (which means that $\varphi(M_A)$ generates M_A over $\mathbf{A}_K$). This follows for example from $\varphi(M_A) \subset M_A$ and $p \nmid \det(\varphi)$. The first result is that there is an equivalence of categories

$$\{p\text{-adic representations of } G_{E_K}\} \qquad \longleftrightarrow \qquad \{\text{étale } \varphi\text{-modules}\}$$

Let us explain where this comes from. The correspondence $T \mapsto (\mathbf{E} \otimes_{\mathbf{F}_p} T)^{G_{E_K}}$ is (by Hilbert 90) an equivalence of categories between the category of $\mathbf{F}_p$-representations of G_{E_K}, and étale $\mathbf{E}_K$-modules. Let $\mathbf{A}$ be a Cohen ring over $\mathbf{E}$ (we will give a more precise definition of $\mathbf{A}$ below. Suffice it to say that $\mathbf{A}$ should be the ring of Witt vectors over $\mathbf{E}$, but $\mathbf{E}$ is not perfect, so that there are several possible choices for $\mathbf{A}$). The ring $\mathbf{A}$ is endowed with an action of G_{E_K} and $\mathbf{A}^{G_{E_K}} = \mathbf{A}_K$. Then by lifting things to characteristic 0 and inverting p, we get an equivalence of categories between the category of

[(2)]it is incorrectly assumed throughout [**Ber02**] that $F = F'$. The problem is that even if K/F is totally ramified, K_∞/F_∞ does not have to be. In general in [**Ber02**] one should take $e_K = e(K_\infty/F_\infty)$ and not $[K_\infty : F_\infty]$.

$\mathbf{Q}_p$-representations of G_{E_K}, and étale $\mathbf{B}_K$-modules with a Frobenius (these constructions were previously used, for example, by Bloch and Katz).

We will now give a construction of the ring of periods $\mathbf{A}$. Let $\widetilde{\mathbf{E}}^+$ be the ring introduced in II.2.2 and let $\widetilde{\mathbf{E}}$ be the field of fractions of $\widetilde{\mathbf{E}}^+$. Then $\mathbf{E}_K$ embeds in $\widetilde{\mathbf{E}}$. For example, if $K = F$, then $\mathbf{E}_F = k((\varepsilon - 1)) \subset \widetilde{\mathbf{E}}$. Let $\mathbf{E}$ be the completion of the separable closure of $\mathbf{E}_K$ in $\widetilde{\mathbf{E}}$.

One can show that $\widetilde{\mathbf{E}}$ is the completion of the algebraic closure of $\mathbf{E}_F$ so that $\widetilde{\mathbf{E}}$ is the completion of the perfection of $\mathbf{E}$. By a theorem of Ax, $\widetilde{\mathbf{E}}$ is also the completion of $\mathbf{E}$.

Let $\widetilde{\mathbf{A}} = W(\widetilde{\mathbf{E}})$ and $\widetilde{\mathbf{B}} = \widetilde{\mathbf{A}}[1/p]$. It is easy to see (at least when $K = F$) that $\mathbf{B}_K$ is a subfield of $\widetilde{\mathbf{B}}$, with $\pi_F = [\varepsilon] - 1$. If $K \neq F$, then one should take for π_K an element of $\mathbf{A}$ whose image modulo p is a uniformiser of $\mathbf{E}_K = \mathbf{E}^{H_K}$. Let $\mathbf{B}$ be the completion of the maximal unramified extension of $\mathbf{B}_K$ in $\widetilde{\mathbf{B}}$, and $\mathbf{A} = \mathbf{B} \cap \widetilde{\mathbf{A}}$. The field $\mathbf{B}$ is endowed with an action of G_{E_K}, and one indeed has $\mathbf{B}^{G_{E_K}} = \mathbf{B}_K$. The field $\mathbf{B}$ is also naturally endowed with a Frobenius map φ.

These ideas appear for example in Katz' [**Kz73**, chap 4]. We gave their local version, which is in Fontaine's [**Fon91**, A1].

III.1.2. Representations of G_{E_K} and differential equations. — Let us mention an application of the theory we just sketched. Let δ be the differential operator defined by $\delta(f(\pi)) = (1+\pi)df/d\pi$ on the field $\mathbf{B}_F$. This operator extends to $\mathbf{B}$ because it extends to the maximal unramified extension of $\mathbf{B}_F$, and then to its completion by continuity. One can use it to associate to every p-adic representation of G_{E_K} a $\mathbf{B}_F$-vector space with a Frobenius φ and a differential operator δ which satisfy $\delta \circ \varphi = p\varphi \circ \delta$. When the action of the inertia of G_{E_K} factors through a finite quotient on a representation V, then there exists a basis of $\mathbf{D}(E)$ in which δ is "overconvergent" (in the sense of III.3 below). One can use this fact to associate to every potentially unramified representation of G_{E_K} an overconvergent differential equation. This condition (φ and δ overconvergent) is much stronger than merely requiring φ to be overconvergent (which happens very often, see III.3).

There are many interesting parallels between the theory of finite Galois representations in characteristic p and differential equations: see Crew's [**Cre85, Cre00**] and Matsuda's [**Mat95**] for a starting point.

III.2. The characteristic zero theory

III.2.1. The field of norms. — The next step of the construction is the theory of the field of norms (of Fontaine and Wintenberger) which gives a canonical isomorphism between G_{E_K} and H_K. Let $\mathcal{N}_K$ be the set $\varprojlim_n K_n$ where the transition maps are given by $N_{K_n/K_{n-1}}$, so that $\mathcal{N}_K$ is the set of sequences $(x^{(0)}, x^{(1)}, \cdots)$ with $x^{(n)} \in K_n$ and $N_{K_n/K_{n-1}}(x^{(n)}) = x^{(n-1)}$. If we define a ring structure on $\mathcal{N}_K$ by

$$(xy)^{(n)} = x^{(n)}y^{(n)} \quad \text{and} \quad (x+y)^{(n)} = \lim_{m \to +\infty} N_{K_{n+m}/K_n}(x^{(n+m)} + y^{(n+m)}),$$

then $\mathcal{N}_K$ is actually a field, called the *field of norms* of K_∞/K. It is naturally endowed with an action of H_K. Furthermore, for every finite Galois extension L/K, $\mathcal{N}_L/\mathcal{N}_K$ is a finite Galois extension whose Galois group is $\mathrm{Gal}(L_\infty/K_\infty)$, and every finite Galois extension of $\mathcal{N}_K$ is of this kind so that the absolute Galois group of $\mathcal{N}_K$ is naturally isomorphic to H_K.

On the other hand, one can prove that $\mathcal{N}_K$ is a local field of characteristic p isomorphic to $\mathbf{E}_K \simeq k_{F'}((\pi_K))$. More precisely, by ramification theory, the map $N_{K_n/K_{n-1}}$ is "close" to the p-th power map and there is therefore a well-defined ring homomorphism from $\mathcal{N}_K$ to $\widetilde{\mathbf{E}}$ given by sending $(x^{(n)}) \in \mathcal{N}_K$ to $(y^{(n)}) \in \widetilde{\mathbf{E}}$ where $y^{(n)} = \lim_{m\to+\infty}(x^{(n+m)})^{p^m}$. This map then realizes an isomorphism between $\mathcal{N}_K$ and $\mathbf{E}_K$, so that the two Galois groups H_K and G_{E_K} are naturally isomorphic.

For the theory of the field of norms in a much more general setting, see Fontaine and Wintenberger's [**FW79**] and Wintenberger's [**Win83**]. For the construction of the isomorphism $\mathcal{N}_K \to \mathbf{E}_K$ and its relation to Coleman series, see Fontaine's appendix to [**Per94**] and Cherbonnier-Colmez's [**CC99**].

III.2.2. (φ, Γ)-modules. — By combining the construction of III.1.1 and the theory of the field of norms, we see that we have an equivalence of categories:

$$\{p\text{-adic representations of } H_K\} \quad \longleftrightarrow \quad \{\text{étale } \varphi\text{-modules}\}.$$

We immediately deduce from this the equivalence of categories we were looking for:

$$\{p\text{-adic representations of } G_K\} \quad \longleftrightarrow \quad \{\text{étale } (\varphi, \Gamma_K)\text{-modules}\}.$$

One associates to V the étale φ-module $\mathbf{D}(V) = (\mathbf{B} \otimes_{\mathbf{Q}_p} V)^{H_K}$, which is an étale φ-module endowed with the residual action of Γ_K: it is a (φ, Γ_K)*-module*. The inverse functor is then given by $D \mapsto (\mathbf{B} \otimes_{\mathbf{B}_K} D)^{\varphi=1}$.

In general, it is rather hard to write down the (φ, Γ)-module associated to a representation V. We can therefore only give a few trivial examples, such as $\mathbf{D}(\mathbf{Q}_p(r)) = \mathbf{B}_F(r)$. See also the examples in IV.5.4.

The original theory of (φ, Γ)-modules is the subject of Fontaine's [**Fon91**]. It has been modified a bit by Cherbonnier and Colmez in [**CC99**], whose constructions we have followed. For explicit families of (φ, Γ)-modules, see [**BLZ03**].

III.2.3. Computation of Galois cohomology. — Since the category of étale (i.e. slope 0) (φ, Γ)-modules is equivalent to that of p-adic representations, it should be possible to recover all properties of p-adic representations in terms of (φ, Γ)-modules. For example, Herr showed in his thesis how one could compute the Galois cohomology of V from $\mathbf{D}(V)$. Let $H^i(K, V)$ denote the groups of continuous cohomology of V. Herr's main result is that one can recover the $H^i(K, V)$'s from $\mathbf{D}(V)$.

Let Δ_K be the torsion subgroup of Γ_K; since Γ_K is an open subgroup of $\mathbf{Z}_p^*$, Δ_K is a finite subgroup whose order divides $p-1$ (or 2 if p=2). Let p_Δ be the idempotent defined by $p_\Delta = \frac{1}{|\Delta_K|}\sum_{\delta\in\Delta_K}\delta$ so that if M is a $\mathbf{Z}_p[[\Gamma_K]]$-module, then p_Δ is a projection map from M to M^{Δ_K} (at least if $p \neq 2$). Let γ be a topological generator of Γ_K/Δ_K.

Let $\mathbf{D}'(V) = \mathbf{D}(V)^{\Delta_K}$. If α is a map $\alpha : \mathbf{D}'(V) \to \mathbf{D}'(V)$ which commutes with Γ_K, let $C_{\alpha,\gamma}(K, V)$ be the following complex :

$$0 \to \mathbf{D}'(V) \xrightarrow{f} \mathbf{D}'(V) \oplus \mathbf{D}'(V) \xrightarrow{g} \mathbf{D}'(V) \to 0$$

where $f(x) = ((\alpha-1)x, (\gamma-1)x)$ and $g(x, y) = (\gamma-1)x - (\alpha-1)y$.

The cohomology of the complex $C_{\varphi,\gamma}(K, V)$ is then naturally isomorphic to the Galois cohomology of V. For example, we see immediately that $H^i(K, V) = 0$ if $i \geq 3$.

This was proved by Herr in [**Her98**]. For various applications, see Herr's [**Her98, Her01, Her00**], Benois' [**Ben00**], [**Ber01**, chap VI] and [**Ber03a, Ber03c**], Cherbonnier-Colmez's [**CC99**], and Colmez's [**Col99**].

III.3. Overconvergent (φ, Γ)-modules

Since the theory of (φ, Γ)-modules is so good at dealing with p-adic representations, we would like to be able to recover from $\mathbf{D}(V)$ the invariants associated to V by p-adic Hodge theory. This is the subject of the next chapter, on reciprocity formulas, but in this paragraph we will introduce the main technical tool, the ring of overconvergent elements.

By construction, the field $\mathbf{B}$ is a subfield of

$$\widetilde{\mathbf{B}} = W(\widetilde{\mathbf{E}})[1/p] = \{\sum_{k\gg-\infty} p^k[x_k],\ x_k \in \widetilde{\mathbf{E}}\}.$$

Let $\mathbf{B}^{\dagger,r}$ be the subring of $\mathbf{B}$ defined as follows:

$$\mathbf{B}^{\dagger,r} = \{x \in \mathbf{B},\ x = \sum_{k\gg-\infty} p^k[x_k],\ k + \frac{p-1}{pr}v_E(x_k) \to +\infty\}.$$

If $r_n = p^{n-1}(p-1)$ for some $n \geq 0$, then the definition of $\mathbf{B}^{\dagger,r_n}$ boils down to requiring that $\sum_{k\gg-\infty} p^k x_k^{(n)}$ converge in $\mathbf{C}$, which in turn is equivalent to requiring that $\sum_{k\gg-\infty} p^k[x_k^{p^{-n}}]$ converge in $\mathbf{B}_{\mathrm{dR}}^+$. If e_K denotes the ramification index of K_∞/F_∞, and F' is the maximal unramified extension of F contained in K_∞, then one can show that if $\pi_K \in \mathbf{A}_K$ is the "variable" introduced previously, (see the end of III.1.1) and $r \gg 0$, then the invariants of $\mathbf{B}^{\dagger,r}$ under the action of H_K are given by

$$(\mathbf{B}^{\dagger,r})^{H_K} = \mathbf{B}_K^{\dagger,r} = \{ \sum_{k=-\infty}^{+\infty} a_k \pi_K^k, \text{ where } a_k \in F' \text{ and } \sum_{k=-\infty}^{+\infty} a_k X^k \text{ is convergent and bounded on } p^{-1/e_K r} \leq |X| < 1 \}.$$

If $K = F$ (so that $e_K = 1$), then one can take $\pi_F = \pi$, and the above description is valid for all $r \geq p-1$.

A p-adic representation is said to be *overconvergent* if, for some $r \gg 0$, $\mathbf{D}(V)$ has a basis consisting of elements of $\mathbf{D}^{\dagger,r}(V) = (\mathbf{B}^{\dagger,r} \otimes_{\mathbf{Q}_p} V)^{H_K}$. This is equivalent to requiring that there exist a basis of $\mathbf{D}(V)$ in which $\mathrm{Mat}(\varphi) \in \mathrm{M}(d, \mathbf{B}_K^{\dagger,r})$ for some $r \gg 0$.

The main result on the (φ,Γ)-modules of p-adic representations (or, equivalently, on étale (φ,Γ)-modules) is a theorem of Colmez and Cherbonnier which shows that every p-adic representation of G_K (equivalently, every étale (φ,Γ)-module) is overconvergent. It is not true that every étale φ-module is overconvergent, and their proof uses the action of Γ_K in a crucial way. For instance, there is no such result in the characteristic p theory.

The above result is the main theorem of Cherbonnier-Colmez's [**CC98**]. Most applications of (φ,Γ)-modules to p-adic Hodge theory make use of it. If V is absolutely crystalline, then one can say more about the periods of $\mathbf{D}(V)$, see Colmez's [**Col99**], [**Ber02**, 3.3] and [**Ber03b**]. See also the next chapter.

IV. Reciprocity formulas for p-adic representations

IV.1. Overview

IV.1.1. Reciprocity laws in class field theory. — The aim of this chapter is to give constructions relating the theory of (φ,Γ)-modules to p-adic Hodge theory. The first thing we'll do is explain why we (and others) have chosen to call such constructions reciprocity formulas. Recall that, in its simplest form, the aim of class field theory is to provide a description of $\mathrm{Gal}(K^{\mathrm{ab}}/K)$, where K is a field. For example, if K is a local field, then one has for every finite extension L/K the norm residue symbol $(\cdot, L/K) : K^* \to \mathrm{Gal}(L/K)^{\mathrm{ab}}$, which is a surjective map whose kernel is $N_{L/K}(L^*)$. This is a form of the local reciprocity law, and the aim of explicit reciprocity laws is to describe (explicitly!) the map $(\cdot, L/K)$ (more precisely, the Hilbert symbol). For example, a theorem of Dwork shows that if ζ is a p^n-th root of unity, then one has $(u^{-1}, \mathbf{Q}_p(\zeta)/\mathbf{Q}_p) \cdot \zeta = \zeta^u$.

Let $V = \mathbf{Q}_p(1)$, which is the Tate module of the multiplicative group $\mathbf{G}_m$. The classical reciprocity map relates the tangent space $\mathbf{D}_{\mathrm{dR}}(V)$ of $\mathbf{G}_m$ to the Galois cohomology $H^1(G_K, V)$. This is why we call a *reciprocity map* those maps which relate Galois cohomology and p-adic Hodge theory. Since the Galois cohomology of V naturally occurs in (φ,Γ)-modules, it is natural to call "reciprocity map" those maps which relate (φ,Γ)-modules and p-adic Hodge theory.

This is the aim of this chapter: we will show how to recover $\mathbf{D}_{\mathrm{cris}}(V)$ or $\mathbf{D}_{\mathrm{st}}(V)$ from $\mathbf{D}(V)$ and how to characterize de Rham representations. As an application, we will explain the proof of Fontaine's monodromy conjecture.

The first important constructions relating (φ,Γ)-modules and p-adic Hodge theory were carried out in Cherbonnier-Colmez's [**CC99**], and are closely related to Perrin-Riou's exponential, as in her [**Per94**] and Colmez's [**Col98a**]. See also [**Ber03a**] for "explicit formulas" for Bloch-Kato's maps.

IV.2. A differential operator on (φ,Γ)-modules

In order to further relate the theory of (φ,Γ)-modules to p-adic Hodge theory, we will need to look at the action of the Lie algebra of Γ_K on $\mathbf{D}^\dagger(V)$. On $\mathbf{B}_K^\dagger$ it acts through a differential operator ∇, given by $\nabla = \log(\gamma)/\log_p(\chi(\gamma))$, and one can easily show that $\nabla(f(\pi)) = \log(1+\pi)(1+\pi)df/d\pi$. We see in particular that $\nabla(f(\pi)) \notin \mathbf{B}_K^\dagger$, and so it is necessary to extend the scalars to

$$\mathbf{B}_{\mathrm{rig},K}^{\dagger,r} = \{f(\pi_K) = \sum_{k=-\infty}^{+\infty} a_k \pi_K^k, \text{ where } a_k \in F' \text{ and } f(X) \text{ is convergent on } p^{-1/e_K r} \leq |X| < 1\}.$$

The definition is almost the same as that of $\mathbf{B}_K^{\dagger,r}$, but we have dropped the boundedness condition. A typical element of $\mathbf{B}_{\mathrm{rig},K}^{\dagger,r}$ is $t = \log(1+\pi)$. We see that $\mathbf{B}_{\mathrm{rig},K}^{\dagger,r}$ is a Fréchet space, with all the norms given by the sup norms on "closed" annuli, and that it contains $\mathbf{B}_K^{\dagger,r}$ as a dense subspace. The union $\mathbf{B}_{\mathrm{rig},K}^\dagger = \cup_{r\gg 0}\mathbf{B}_{\mathrm{rig},K}^{\dagger,r}$ is the *Robba ring* $\mathcal{R}_K$ of p-adic differential equations, and $\mathcal{E}_K^\dagger = \mathbf{B}_K^\dagger$ is the subring of $\mathcal{R}_K$ consisting of those functions which are bounded. The p-adic completion of $\mathcal{E}_K^\dagger = \mathbf{B}_K^\dagger$ is $\mathcal{E}_K = \mathbf{B}_K$.

This being done, we see that the formula $\nabla_V = \log(\gamma)/\log_p(\chi(\gamma))$ (this operator is defined in the same way as in paragraph II.1.2) gives the action of $\mathrm{Lie}(\Gamma_K)$ on $\mathbf{D}_{\mathrm{rig}}^\dagger(V) = \mathbf{B}_{\mathrm{rig},K}^\dagger \otimes_{\mathbf{B}_K^\dagger} \mathbf{D}^\dagger(V)$. Unfortunately, the action of $\mathrm{Lie}(\Gamma_K)$ on $\mathbf{B}_{\mathrm{rig},K}^\dagger$ is not very nice, because $\nabla(f(\pi)) = \log(1+\pi)(1+\pi)df/d\pi$ and this operator has zeroes at all the $\zeta-1$ with $\zeta \in \mu_{p^\infty}$. In particular, it is not a basis of $\Omega^1_{\mathbf{B}_{\mathrm{rig},K}^\dagger}$ and it is not the kind of differential operator that fits in the framework of p-adic differential equations. The "right" differential operator is $\partial_V = \frac{1}{\log(1+\pi)}\nabla_V$, but this operator acting on $\mathbf{D}_{\mathrm{rig}}^\dagger(V)$ has poles at all the $\zeta-1$. In the following paragraphs, we will see that one can "remove" these poles exactly when V is de Rham.

See [**Ber01, Ber02**] or for detailed constructions and the basic properties of those rings and operators.

IV.3. Crystalline and semi-stable representations

IV.3.1. Construction of $\mathbf{D}_{\mathrm{cris}}(V)$ *and of* $\mathbf{D}_{\mathrm{st}}(V)$. — We will start by studying the action of Γ_K on $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$, and our main result is that $\mathbf{D}_{\mathrm{cris}}(V) = (\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)[1/t])^{\Gamma_K}$, in a sense which will be made precise below. In addition, one can define $\mathbf{B}^{\dagger}_{\log} = \mathbf{B}^{\dagger}_{\mathrm{rig}}[\log(\pi)]$ with the obvious actions of φ and Γ_K, and we shall also see that $\mathbf{D}_{\mathrm{st}}(V) = (\mathbf{D}^{\dagger}_{\log}(V)[1/t])^{\Gamma_K}$. If the Hodge-Tate weights of V are negative (if V is *positive*), then $\mathbf{D}_{\mathrm{cris}}(V) = \mathbf{D}^{\dagger}_{\mathrm{rig}}(V)^{\Gamma_K}$ and $\mathbf{D}_{\mathrm{st}}(V) = \mathbf{D}^{\dagger}_{\log}(V)^{\Gamma_K}$.

Recall that $\mathbf{B}_{\mathrm{st}}$ is a subring of $\mathbf{B}_{\mathrm{dR}}$ equipped with a Frobenius. The periods of V are the elements of $\mathbf{B}_{\mathrm{st}}$ which "occur" in the coefficients of $\mathbf{D}_{\mathrm{st}}(V)$, they form a finite dimensional F-vector subspace of $\mathbf{B}_{\mathrm{st}}$, stable by Frobenius. Therefore, these periods live in $\cap_{n=0}^{+\infty}\varphi^n(\mathbf{B}^+_{\mathrm{st}})[1/t]$.

The main strategy for comparing the theory of (φ,Γ)-modules and p-adic Hodge theory is to construct a rather large ring $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$, which contains $\mathbf{B}^{\dagger}$, $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$ and $\cap_{n=0}^{+\infty}\varphi^n(\mathbf{B}^+_{\mathrm{cris}})$ so that $\mathbf{B}^{\dagger}_{\mathrm{rig},K} \otimes_{\mathbf{B}^{\dagger}_K} \mathbf{D}^{\dagger}(V) \subset \widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}} \otimes_{\mathbf{Q}_p} V$ and $\mathbf{D}_{\mathrm{cris}}(V) \subset (\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}} \otimes_{\mathbf{Q}_p} V)^{G_K}$. The result alluded to above, for positive crystalline representations, is that the two F-vector subspaces of $\widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}} \otimes_{\mathbf{Q}_p} V$, $\mathbf{D}_{\mathrm{cris}}(V)$ and $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)^{\Gamma_K}$, actually coincide. This means that if V is crystalline, then the Frobenius φ on $\mathbf{D}^{\dagger}(V)$ has a rather special form. We'll give an informal justification for the above result in the next paragraph.

IV.3.2. Rings of periods and limits of algebraic functions. — First of all, one should think of most rings of periods as rings of "limits of algebraic functions" on certain subsets of $\mathbf{C}$. For example, the formula $\mathbf{B} = \widehat{\mathbf{B}}^{\mathrm{unr}}_F$ tells us that $\mathbf{B}$ is the ring of limits of (separable) algebraic functions on the boundary of the open unit disk. The ring $\widetilde{\mathbf{B}}$ is then the ring of all limits of algebraic functions on the boundary of the open unit disk.

Heuristically, one should view other rings in the same fashion: the ring $\mathbf{B}^+_{\mathrm{cris}}$ "is" the ring of limits of algebraic functions on the disk $D(0, |\varepsilon^{(1)}-1|_p)$, and $\mathbf{B}^+_{\max}$ "is" the ring of limits of algebraic functions on a slightly smaller disk $D(0,r)$. One should therefore think of $\varphi^n(\mathbf{B}^+_{\mathrm{cris}})$ as the ring of limits of algebraic functions on the disk $D(0,|\varepsilon^{(n)}-1|_p)$, and finally $\widetilde{\mathbf{B}}^+_{\mathrm{rig}}$ "is" the ring of limits of algebraic functions on the open unit disk $D(0,1)$.

Similarly, $\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}$ "is" the ring of limits of algebraic functions on an annulus $C[s,1[$, where s depends on r, and $\varphi^{-n}(\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}})$ "is" the ring of limits of algebraic functions on an annulus $C[s_n,1[$, where $s_n \to 0$, so that $\cap_{n=0}^{+\infty}\varphi^{-n}(\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}})$ "is" the ring of limits of algebraic functions on the open unit disk $D(0,1)$ minus the origin; furthermore, if an element of that ring satisfies some simple growth properties near the origin, then it "extends" to the origin (remember that in complex analysis, a holomorphic function on $D(0,1^-)-\{0\}$ which is bounded near 0 extends to a holomorphic function on $D(0,1^-)$).

As for the ring $\mathbf{B}_{\mathrm{dR}}^+$, it behaves like a ring of local functions around a circle (in particular, there is no Frobenius map defined on it). Via the map $\varphi^{-n} : \mathbf{B}_{\mathrm{rig}}^{\dagger,r_n} \to \mathbf{B}_{\mathrm{dR}}^+$, we have for $n \geq 1$ a filtration on $\mathbf{B}_{\mathrm{rig}}^{\dagger,r_n}$, which corresponds to the order of vanishing at $\varepsilon^{(n)} - 1$. For instance, we can now give a short solution to the exercise in paragraph II.3.4: given a sequence r_n of integers, let $q = \varphi(\pi)/\pi$ and set $x_r = \pi^{r_0} \prod_{n=1}^{+\infty} \varphi^{n-1}(q/p)^{r_n}$. This infinite product converges to a "function" whose order of vanishing at $\varepsilon^{(n)} - 1$ is exactly r_n.

IV.3.3. Regularization and decompletion. — We shall now justify the above results on $\mathbf{D}_{\mathrm{cris}}(V)$. The analogous results on $\mathbf{D}_{\mathrm{st}}(V)$ follow by adding $\log(\pi)$ everywhere. We've already seen that the periods of positive crystalline representations live in $\widetilde{\mathbf{B}}_{\mathrm{rig}}^+$ (if we don't assume that V is positive, then they live in $\widetilde{\mathbf{B}}_{\mathrm{rig}}^+[1/t]$).

The elements of $(\widetilde{\mathbf{B}}_{\mathrm{rig}}^\dagger \otimes_{\mathbf{Q}_p} V)^{G_K}$ form a finite dimensional F-vector space, so that there is an r such that $(\widetilde{\mathbf{B}}_{\mathrm{rig}}^\dagger \otimes_{\mathbf{Q}_p} V)^{G_K} = (\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r} \otimes_{\mathbf{Q}_p} V)^{G_K}$, and furthermore this F-vector space is stable by Frobenius, so that the periods of V (in this setting) not only live in $\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r}$ but actually in $\cap_{n=0}^{+\infty} \varphi^{-n}(\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,r})$ and they also satisfy some simple growth conditions (depending, say, on the size of $\det(\varphi)$), which ensure that they too can be seen as limits of algebraic functions on the open unit disk $D(0,1^-)$, that is as elements of $\widetilde{\mathbf{B}}_{\mathrm{rig}}^+$. In particular, we have $(\widetilde{\mathbf{B}}_{\mathrm{rig}}^\dagger \otimes_{\mathbf{Q}_p} V)^{G_K} = (\widetilde{\mathbf{B}}_{\mathrm{rig}}^+ \otimes_{\mathbf{Q}_p} V)^{G_K}$. This is what we get by *regularization* (of the periods).

It's easy to show that $(\widetilde{\mathbf{B}}_{\mathrm{rig}}^\dagger \otimes_{\mathbf{Q}_p} V)^{H_K} = \widetilde{\mathbf{B}}_{\mathrm{rig},K}^\dagger \otimes_{\mathbf{B}_{\mathrm{rig},K}^\dagger} \mathbf{D}_{\mathrm{rig}}^\dagger(V)$, and the last step is to show that $(\widetilde{\mathbf{B}}_{\mathrm{rig},K}^\dagger \otimes_{\mathbf{B}_{\mathrm{rig},K}^\dagger} \mathbf{D}_{\mathrm{rig}}^\dagger(V))^{G_K} = \mathbf{D}_{\mathrm{rig}}^\dagger(V)^{G_K}$. This is akin to a *decompletion* process, going from $\widetilde{\mathbf{B}}_{\mathrm{rig},K}^\dagger$ to $\mathbf{B}_{\mathrm{rig},K}^\dagger$. The ring extension $\widetilde{\mathbf{B}}_{\mathrm{rig},K}^\dagger / \mathbf{B}_{\mathrm{rig},K}^\dagger$ looks very much like $\widehat{K}_\infty / K$, so that by using Colmez's decompletion maps, which are analogous to Tate's pr_{K_n} maps from paragraph II.1.1, one can finally show that in fact, $\mathbf{D}_{\mathrm{cris}}(V) = (\mathbf{B}_{\mathrm{rig},K}^\dagger \otimes_{\mathbf{B}_K^\dagger} \mathbf{D}^\dagger(V))^{G_K}$. In particular, V is crystalline if and only if $(\mathbf{B}_{\mathrm{rig},K}^\dagger \otimes_{\mathbf{B}_K^\dagger} \mathbf{D}^\dagger(V))^{G_K}$ is a d-dimensional F-vector space.

See [**Ber01, Ber02**]. For decompletion maps and the "Tate-Sen" conditions, see [**BC03**] and Colmez' Bourbaki talk [**Col01**].

IV.4. De Rham representations

In the previous paragraph, we have shown how to recognize crystalline and semi-stable representations in terms of their (φ, Γ)-modules. We shall now do the same for de Rham representations, and show that a representation V is positive de Rham if and only if there exists a free $\mathbf{B}_{\mathrm{rig},K}^\dagger$-submodule of rank d of $\mathbf{D}_{\mathrm{rig}}^\dagger(V)$, called $\mathbf{N}_{\mathrm{dR}}(V)$, which is stable by the operator ∂_V (when V is not positive, then $\mathbf{N}_{\mathrm{dR}}(V) \subset \mathbf{D}_{\mathrm{rig}}^\dagger(V)[1/t]$). Of course, when V is crystalline or semi-stable, one can simply take $\mathbf{N}_{\mathrm{dR}}(V) = \mathbf{B}_{\mathrm{rig},K}^\dagger \otimes_F \mathbf{D}_{\mathrm{cris}}(V)$ or $\mathbf{N}_{\mathrm{dR}}(V) = (\mathbf{B}_{\log,K}^\dagger \otimes_F \mathbf{D}_{\mathrm{st}}(V))^{N=0}$.

IV.4.1. Construction of $\mathbf{N}_{\mathrm{dR}}(V)$. — In general, let us give an idea of how one can construct $\mathbf{N}_{\mathrm{dR}}(V)$. In the paragraph II.2.4, we recalled Fontaine's construction of "Sen's theory for $\mathbf{B}_{\mathrm{dR}}^{+}$". The map φ^{-n} sends $\mathbf{D}_{\mathrm{rig}}^{\dagger,r_n}(V)$ into $(\mathbf{B}_{\mathrm{dR}}^{+} \otimes_{\mathbf{Q}_p} V)^{H_K}$, which should be thought of as "localizing at $\varepsilon^{(n)}-1$" in geometrical terms. The module $\mathbf{D}_{\mathrm{dif}}^{+}(V)$ of Fontaine is then equal to $K_\infty[[t]] \otimes_{\varphi^{-n}(\mathbf{B}_{\mathrm{rig},K}^{\dagger,r_n})} \varphi^{-n}(\mathbf{D}_{\mathrm{rig}}^{\dagger,r_n}(V))$. Recall that Fontaine has shown that a positive V is de Rham and if and only if the connection ∇_V has a full set of sections on $\mathbf{D}_{\mathrm{dif}}^{+}(V)$ (in which case the kernel of the connection is $K_\infty \otimes_K \mathbf{D}_{\mathrm{dR}}(V)$). In geometrical terms, this means that if V is positive and de Rham, then ∇_V has some "local" solutions around the $\varepsilon^{(n)} - 1$. In that case, one can glue all of those solutions together to obtain $\mathbf{N}_{\mathrm{dR}}(V)$. More precisely, there exists $n_0 \gg 0$ and $r \gg 0$ such that we have $\mathbf{N}_{\mathrm{dR}}(V) = \mathbf{B}_{\mathrm{rig},K}^{\dagger} \otimes_{\mathbf{B}_{\mathrm{rig},K}^{\dagger,r}} N_r(V)$ where $N_r(V)$ is the set of $x \in \mathbf{D}_{\mathrm{rig}}^{\dagger,r}(V)$ such that for every $n \geq n_0$, one has $\varphi^{-n}(x) \in K_n[[t]] \otimes_K \mathbf{D}_{\mathrm{dR}}(V)$. It's easy to see that $N_r(V)[1/t] = \mathbf{D}_{\mathrm{rig}}^{\dagger,r}(V)[1/t]$ and that $N_r(V)$ is a closed (for the Fréchet topology) $\mathbf{B}_{\mathrm{rig},K}^{\dagger,r}$-submodule of $\mathbf{D}_{\mathrm{rig}}^{\dagger,r}(V)$. The fact that $N_r(V)$ is free of rank d then follows form the following fact: if $M \subset (\mathbf{B}_{\mathrm{rig},K}^{\dagger,r})^d$ is a closed submodule, such that $\operatorname{Frac} \mathbf{B}_{\mathrm{rig},K}^{\dagger,r} \otimes_{\mathbf{B}_{\mathrm{rig},K}^{\dagger,r}} M = (\operatorname{Frac} \mathbf{B}_{\mathrm{rig},K}^{\dagger,r})^d$, then M is free of rank d.

One can then show that $\mathbf{N}_{\mathrm{dR}}(V)$ is uniquely determined by the requirement that it be free of rank d and stable by ∂_V, so that in particular $\varphi^*\mathbf{N}_{\mathrm{dR}}(V) = \mathbf{N}_{\mathrm{dR}}(V)$.

We therefore have the following theorem: if V is a de Rham representation, then there exists $\mathbf{N}_{\mathrm{dR}}(V) \subset \mathbf{D}_{\mathrm{rig}}^{\dagger}(V)[1/t]$, a $\mathbf{B}_{\mathrm{rig},K}^{\dagger}$-module free of rank d, stable by ∂_V and φ, such that $\varphi^*\mathbf{N}_{\mathrm{dR}}(V) = \mathbf{N}_{\mathrm{dR}}(V)$. Such an object is by definition a p-adic differential equation with Frobenius structure (see IV.5.2 below).

Using this theorem, one can construct a faithful and essentially surjective exact $\otimes$-functor from the category of de Rham representations to the category of p-adic differential equations with a Frobenius structure.

The above theorem is the main result of [**Ber02**]. For applications, see [**Ber02, Ber03c**]. The result on closed submodules of $(\mathbf{B}_{\mathrm{rig},K}^{\dagger,r})^d$ is proved in [**Ber02**, 4.2], see also [**For67**].

IV.4.2. Example: $\mathbf{C}$*-admissible representations.* — Let us give an example for which it is easy to characterize $\mathbf{N}_{\mathrm{dR}}(V)$. We've already seen that when V is crystalline or semi-stable, one can take $\mathbf{N}_{\mathrm{dR}}(V) = \mathbf{B}_{\mathrm{rig},K}^{\dagger} \otimes_F \mathbf{D}_{\mathrm{cris}}(V)$ or $\mathbf{N}_{\mathrm{dR}}(V) = (\mathbf{B}_{\log,K}^{\dagger} \otimes_F \mathbf{D}_{\mathrm{st}}(V))^{N=0}$. Another easy case is when V is $\mathbf{C}$-admissible. This was one of the examples in I.2.3 where we mentioned Sen's result: a representation V is $\mathbf{C}$-admissible if and only if it is potentially unramified. We'll give a proof of that result which relies on a theorem of Tsuzuki on differential equations.

Let V be a $\mathbf{C}$-admissible representation. This means that $\mathbf{C}\otimes_{\mathbf{Q}_p}V = \mathbf{C}\otimes_K(\mathbf{C}\otimes_{\mathbf{Q}_p}V)^{G_K}$, so that V is Hodge-Tate and all its weights are 0. In particular, Sen's map Θ_V is zero. Since we recovered Sen's map from ∇_V by localizing at $\varepsilon^{(n)}-1$, this implies that the coefficients of a matrix of ∇_V are holomorphic functions which are 0 at $\varepsilon^{(n)}-1$ for all $n \gg 0$. These functions are therefore multiples of $t = \log(1+\pi)$ in $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$ and so $\nabla_V(\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)) \subset \log(1+\pi)\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$ so that we have $\mathbf{N}_{\mathrm{dR}}(V) = \mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$.

The $\mathcal{R}_K$-module $\mathbf{N}_{\mathrm{dR}}(V)$ is then endowed with a differential operator ∂_V and a unit-root Frobenius map φ which is overconvergent. One can show that if φ is overconvergent, then so is ∂_V (because φ regularizes functions). The module $\mathbf{N}_{\mathrm{dR}}(V)$ is therefore an *overconvergent unit-root isocrystal*, and Tsuzuki proved that these are potentially trivial (that is, they become trivial after extending the scalars to $\mathcal{R}_L/\mathcal{R}_K$ for a finite extension L/K). This implies easily enough that the restriction of V to I_K is potentially trivial.

See [**Ber02**, 5.6]. For Tsuzuki's theorem, see his [**Tsk99**] and Christol's [**Chr01**]. Sen's theorem was first proved in Sen's [**Sen73**].

IV.5. The monodromy theorem

IV.5.1. ℓ-adic monodromy and p-adic monodromy. — As was pointed out in the introduction, ℓ-adic representations are forced to be well-behaved, while the group G_K has far too many p-adic representations. Over the years it became apparent that the only representations related to arithmetic geometry were the de Rham representations (see II.5.1).

In particular it was conjectured (and later proved) that all representations coming from geometry were de Rham. Among these, some are more pleasant, they are the semi-stable ones, which are the analogue of the ℓ-adic unipotent representations. Grothendieck has shown that all ℓ-adic representations are quasi-unipotent, and after looking at many examples, Fontaine was led to conjecture the following p-adic analogue of Grothendieck's ℓ-adic monodromy theorem: every de Rham representation is potentially semi-stable. We shall now explain the proof of that statement.

An excellent reference throughout this section is Colmez' Bourbaki talk [**Col01**].

IV.5.2. p-adic differential equations. — A *p-adic differential equation* is a module M, free of finite rank over the Robba ring $\mathcal{R}_K$, equipped with a connection $\partial_M : M \to M$. We say that M has a *Frobenius structure* if there is a semi-linear Frobenius $\varphi_M : M \to M$ which commutes with ∂_M.

A p-adic differential equation is said to be *quasi-unipotent* if there exists a finite extension L/K such that ∂_M has a full set of solutions on $\mathcal{R}_L[\log(\pi)] \otimes_{\mathcal{R}_K} M$. Christol and Mebkhout extensively studied p-adic differential equations. Crew and Tsuzuki conjectured that every p-adic differential equation with a Frobenius structure is quasi-unipotent. Three independent proofs were given in the summer of 2001. One by André, using Christol-Mebkhout's results and a Tannakian argument. One by Kedlaya, who proved a "Dieudonné-Manin" theorem for φ-modules over $\mathcal{R}_K$. And one by Mebkhout, relying on Christol-Mebkhout's results.

We refer the reader to Christol and Mebkhout's surveys [**CM00, CM02**] and Colmez's Bourbaki talk [**Col01**] for enlightening discussions of p-adic differential equations. The above theorem is proved independently in André's [**And02b**], Mebkhout's [**Meb02**] and Kedlaya's [**Ked00**]. See also André's [**And02a**] for a beautiful discussion of a special case.

IV.5.3. The monodromy theorem. — Using the previous results, one can give a proof of Fontaine's monodromy conjecture. Let V be a de Rham representation, then one can associate to V a p-adic differential equation $\mathbf{N}_{\mathrm{dR}}(V)$. By André, Kedlaya, and Mebkhout's theorem, this differential equation is quasi-unipotent. Therefore, there exists a finite extension L/K such that $(\mathcal{R}_L[\log(\pi)] \otimes_{\mathcal{R}_K} \mathbf{N}_{\mathrm{dR}}(V))^{G_L}$ is an F-vector space of dimension d and by the results of paragraph IV.3.3, V is potentially semi-stable.

See [**Ber02**, 5.5] for further discussion of the above result.

IV.5.4. Example: Tate's elliptic curve. — To finish this chapter, we will sketch this for Tate's elliptic curve (or indeed for all ordinary elliptic curves). For simplicity, assume that k is algebraically closed. If $q = q_0$ is the parameter associated to E_q, then there exists $q_n \in F_n = F(\varepsilon^{(n)})$ such that $N_{F_{n+1}/F_n}(q_{n+1}) = q_n$ (this is the only place where we use the fact that k is algebraically closed), and by a result of Coleman, there is a power series $\mathrm{Col}_q(\pi)$ such that $q_n = \mathrm{Col}_q^{\sigma_F^{-n}}(\varepsilon^{(n)} - 1)$. If $F_q(\pi) = (1+\pi)\,\mathrm{dlog}\,\mathrm{Col}_q(\pi)$, then $F_q(\pi) \in \pi^{-1}\mathcal{O}_F[[\pi]]$ and one can show that there is a basis (a,b) of the (φ,Γ)-module $\mathbf{D}(V)$ associated to V such that the action of $\Gamma_F = \langle\gamma\rangle$ is given by:

$$\mathrm{Mat}(\eta) = \begin{pmatrix} \chi(\eta) & \frac{1-\eta}{1-\gamma}F_q(\pi) \\ 0 & 1 \end{pmatrix}$$

Let ∇ be the differential operator giving the action of the Lie algebra of Γ_F on power series, so that we have $(\nabla f)(\pi) = (1+\pi)\log(1+\pi)f'(\pi)$ (recall that $t = \log(1+\pi)$). The Lie algebra of Γ_F then acts on $\mathbf{D}^{\dagger}_{\mathrm{rig}}(V)$ by an operator ∇_V given by

$$\mathrm{Mat}(\nabla_V) = \begin{pmatrix} 1 & \frac{\nabla}{1-\gamma}F_q(\pi) \\ 0 & 0 \end{pmatrix}$$

One then sees that $\partial_V(t^{-1}a) = 0$ and that $\partial_V(b)$ belongs to $\mathbf{B}^{\dagger}_{\mathrm{rig},F}(t^{-1}a)$, so that the p-adic differential equation $\langle t^{-1}a, b\rangle$ is unipotent. This shows that V is indeed semi-stable.

The extensions of $\mathbf{Q}_p$ by $\mathbf{Q}_p(1)$ are important and also a source of explicit examples. They are related to Kummer theory as in paragraph II.4.4, and Coleman series as above, among other topics. Some interesting computations can be found in Cherbonnier-Colmez's [**CC99**, V].

V. Appendix

V.1. Diagram of the rings of periods

The following diagram summarizes the relationships between the different rings of periods. The arrows ending with $\twoheadrightarrow$ are surjective, the dotted arrow $\dashrightarrow$ is an inductive limit of maps defined on subrings ($\iota_n : \widetilde{\mathbf{B}}_{\log}^{\dagger,r_n} \to \mathbf{B}_{\mathrm{dR}}^+$), and all the other ones are injective.

$$\begin{tikzcd}
 & & & & \mathbf{B}_{\mathrm{dR}}^+ \arrow[dddl, bend left, two heads, "\theta"] \\
 & \widetilde{\mathbf{B}}_{\log}^{\dagger} \arrow[urrr, dotted] & \widetilde{\mathbf{B}}_{\log}^{+} \arrow[l] \arrow[rr] & & \mathbf{B}_{\mathrm{st}}^+ \arrow[u] \\
 & \widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger} \arrow[u] & \widetilde{\mathbf{B}}_{\mathrm{rig}}^{+} \arrow[l] \arrow[u] \arrow[rr] & & \mathbf{B}_{\max}^+ \arrow[u] \\
\widetilde{\mathbf{B}} & \widetilde{\mathbf{B}}^{\dagger} \arrow[l] \arrow[u] & \widetilde{\mathbf{B}}^{+} \arrow[l] \arrow[u] \arrow[r, two heads, "\theta"] & \mathbf{C} & \\
\widetilde{\mathbf{A}} \arrow[u] \arrow[d] & \widetilde{\mathbf{A}}^{\dagger} \arrow[l] \arrow[u] & \widetilde{\mathbf{A}}^{+} \arrow[l] \arrow[u] \arrow[d] \arrow[r, two heads, "\theta"] & \mathscr{O}_{\mathbf{C}} \arrow[u] \arrow[d] & \\
\widetilde{\mathbf{E}} & & \widetilde{\mathbf{E}}^{+} \arrow[ll] \arrow[r, two heads, "\theta"] & \mathscr{O}_{\mathbf{C}}/p &
\end{tikzcd}$$

All the rings with tildes ($\widetilde{\ }$) also have versions without a tilde: one goes from the latter to the former by making Frobenius invertible and completing. For example, $\widetilde{\mathbf{E}}$ is the completion of the perfection of $\mathbf{E}$.

The three rings in the leftmost column (at least their tilde-free versions) are related to the theory of (φ, Γ_K)-modules. The three rings in the rightmost column are related to p-adic Hodge theory. To go from one theory to the other, one goes from one side to the other through all the intermediate rings. The best case is when one can work in the middle column. For example, from top to bottom: semi-stable, crystalline, or finite height representations. The ring that binds them all is $\widetilde{\mathbf{B}}_{\log}^{\dagger}$.

V.2. List of the rings of power series

Let us review the different rings of power series which occur in this article; let $C[r;1[$ be the annulus$\{z \in \mathbf{C},\ p^{-1/r} \leq |z|_p < 1\}$. We then have:

$\mathbf{E}_F^+$	$k[[T]]$
$\mathbf{A}_F^+$	$\mathcal{O}_F[[T]]$
$\mathbf{B}_F^+$	$F \otimes_{\mathcal{O}_F} \mathcal{O}_F[[T]]$
$\mathbf{E}_F$	$k((T))$
$\mathbf{A}_F$	$\mathcal{O}_F\widehat{[[T]][T^{-1}]}$
$\mathbf{B}_F$	$F \otimes_{\mathcal{O}_F} \mathcal{O}_F\widehat{[[T]][T^{-1}]}$
$\mathbf{A}_F^{\dagger,r}$	Laurent series $f(T)$, convergent on $C[r;1[$, and bounded by 1
$\mathbf{B}_F^{\dagger,r}$	Laurent series $f(T)$, convergent on $C[r;1[$, and bounded
$\mathbf{B}_{\mathrm{rig},F}^{\dagger,r}$	Laurent series $f(T)$, convergent on $C[r;1[$
$\mathbf{B}_{\log,F}^{\dagger,r}$	$\mathbf{B}_{\mathrm{rig},F}^{\dagger,r}[\log(T)]$
$\mathbf{B}_{\mathrm{rig},F}^+$	$f(T) \in F[[T]],\ f(T)$ converges on the open unit disk $D[0;1[$
$\mathbf{B}_{\log,F}^+$	$\mathbf{B}_{\mathrm{rig},F}^+[\log(T)]$

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Août 2004

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