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On the solutions of an equation involving the Smarandache function

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Abstract Let n be any positive integer, the Smarandache function $S(n)$ is defined as $S(n) = \min\{m : n|m!\}$. In this paper, we discussed the solutions of the following equation involving the Smarandache function: $S(m_1)+S(m_2)+\cdots+S(m_k) = S(m_1+m_2+\cdots+m_k)$, and proved that the equation has infinity positive integer solutions.

Keywords Smarandache function, equation, positive integer solutions.

§1. Introduction

For any positive integer n, the Smarandache function $S(n)$ is defined as follows:

$$S(n) = \min\{m : n|m!\}.$$

From this definition we know that $S(n) = \max_{1\le i\le r}\{S(p_i^{\alpha_i})\}$, if n has the prime powers factorization: $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}$. Of course, this function has many arithmetical properties, and they are studied by many people (see references [1], [4] and [5]).

In this paper, we shall use the elementary methods to study the solvability of the equation

$$S(m_1) + S(m_2) + \cdots + S(m_k) = S(m_1 + m_2 + \cdots + m_k),$$

and prove that it has infinity positive integer solutions for any positive integer k. That is, we shall prove the following main conclusion:

Theorem. For any integer $k \ge 1$, the equation

$$S(m_1) + S(m_2) + \cdots + S(m_k) = S(m_1 + m_2 + \cdots + m_k) \qquad (1)$$

has infinity positive integer solutions.

§2. Proof of the theorem

In this section, we shall give the proof of the theorem in two ways, the first proof of the theorem is based on the following:

Lemma 1. For any positive integer m, there exist positive integers $a_1^{(m)}, a_2^{(m)}, \cdots, a_m^{(m)}$ which are independent of x, satisfying

$$x^m = (x-1)(x-2)\cdots(x-m) + \sum_{l=1}^{m-1} a_l^{(m)}(x-1)(x-2)\cdots(x-m+l) + a_m^{(m)}, \qquad (2)$$

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where x is an arbitrary real number.

Proof. We use induction to prove this Lemma. It is clear that the lemma holds if $m = 1$. That is, $x = (x-1)+1$ holds for any real number x, so we get

$$a_1^{(1)} = 1.$$

Now we assume that the lemma holds for $m = k$ $(k \geq 1)$, then for $m = k+1$, we have

$$\begin{aligned}
x^{k+1} &= x(x-1)(x-2)\cdots(x-k) + \sum_{l=1}^{k-1} a_l^{(k)} x(x-1)(x-2)\cdots(x-k+l) + a_k^{(k)} x \\
&= (x-1)(x-2)\cdots(x-k-1) + (k+1)(x-1)(x-2)\cdots(x-k) + \\
&\quad + \sum_{l=1}^{k-1} a_l^{(k)} (x-1)(x-2)\cdots(x-k+l)(x-k+l-1) + \\
&\quad + \sum_{l=1}^{k-1} a_l^{(k)} (k-l+1)(x-1)(x-2)\cdots(x-k+l) + a_k^{(k)}(x-1) + a_k^{(k)} \\
&= (x-1)(x-2)\cdots(x-k-1) + (k+1+a_1^{(k)})(x-1)(x-2)\cdots(x-k) + \\
&\quad + \sum_{l=1}^{k-2} a_{l+1}^{(k)} (x-1)(x-2)\cdots(x-k+l) + \\
&\quad + \sum_{l=1}^{k-2} a_l^{(k)} (k-l+1)(x-1)(x-2)\cdots(x-k+l) + (2a_{k-1}^{(k)} + a_k^{(k)})(x-1) + a_k^{(k)} \\
&= (x-1)(x-2)\cdots(x-k-1) + (k+1+a_1^{(k)})(x-1)(x-2)\cdots(x-k) + \\
&\quad + \sum_{l=1}^{k-2} (a_{l+1}^{(k)} + a_l^{(k)}(k-l+1))(x-1)(x-2)\cdots(x-k+l) + \\
&\quad + (2a_{k-1}^{(k)} + a_k^{(k)})(x-1) + a_k^{(k)}
\end{aligned}$$

so we can take

$$a_1^{(k+1)} = k+1+a_1^{(k)}, \tag{3}$$

$$a_l^{(k+1)} = a_l^{(k)} + a_{l-1}^{(k)}(k-l+2),\ (2 \leq l \leq k), \tag{4}$$

$$a_{k+1}^{(k+1)} = a_k^{(k)}, \tag{5}$$

and it is obvious from the inductive assumption and (3), (4), (5) that $a_1^{(k+1)}, a_2^{(k+1)}, \cdots, a_{k+1}^{(k+1)}$ are positive integers which are independent of x, and so the lemma holds for $m = k+1$. This completes the proof of Lemma 1.

Now we complete the proof of the theorem. From Lemma 1 we know that for any positive integer k, there exist positive integers $a_1, a_2, \cdots, a_{k-1}$ such that

$$p^{k-1} = (p-1)(p-2)\cdots(p-k+1) + \sum_{l=1}^{k-2} a_l (p-1)(p-2)\cdots(p-k+l+1) + a_{k-1}.$$

Hence

$$p^k = p(p-1)(p-2)\cdots(p-k+1) + \sum_{l=1}^{k-2} a_l p(p-1)(p-2)\cdots(p-k+l+1) + a_{k-1}p. \tag{6}$$

Note that $a_1, a_2, \cdots, a_{k-1}$ are independent of p and p is a prime large enough, from the definition of $S(n)$ we have

$$S(p^k) = kp,$$

$$S(p(p-1)(p-2)\cdots(p-k+1)) = p,$$

$$S(a_l p(p-1)(p-2)\cdots(p-k+l+1)) = p,\ (1 \le l \le k-2)$$

$$S(a_{k-1}p) = p.$$

From these equations and (6) we know that $m_1 = p(p-1)(p-2)\cdots(p-k+1)$, $m_{l+1} = a_l p(p-1)(p-2)\cdots(p-k+l+1)$ $(1 \le l \le k-2)$, $m_k = a_{k-1}p$ is a solution of (1), and (1) has infinity positive integer solutions because p is arbitrary.

The second proof of the theorem is based on the Vinogradov's three-primes theorem which we describle as the following:

Lemma 2. Every odd integer bigger than c can be expressed as sum of three odd primes, where c is a constant large enough.

Proof. (see §20.2 and §20.3 of [2]).

Lemma 3. Let odd integer $k \ge 3$, then any sufficiently large odd integer n can be expressed as sum of k odd primes

$$n = p_1 + p_2 + \cdots + p_k. \tag{7}$$

Proof. We will prove this lemma by induction. From Lemma 2 we know that it is true for $k = 3$. If it is true for odd integer k, then we will prove that it is also true for $k+2$. In fact, from Lemma 2 we know that every sufficient large odd integer n can be expressed as

$$n = p^{(1)} + p^{(2)} + p^{(3)},$$

and we can assume that $p^{(1)}$ is also sufficiently large and then satisfying

$$p^{(1)} = p_1 + p_2 + \cdots + p_k,$$

so we have

$$n = p_1 + p_2 + \cdots + p_k + p^{(2)} + p^{(3)}.$$

This means that n can be expressed as sum of $k+2$ odd primes, and Lemma 3 follows from the induction.

Now we give the second proof of the theorem. From Lemma 3 we know that for any odd integer $k \ge 3$, every sufficient large prime p can be expressed as

$$p = p_1 + p_2 + \cdots + p_k.$$

So we have

$$S(p) = S(p_1) + S(p_2) + \cdots + S(p_k).$$

This means that the theorem is true for odd integer $k \ge 3$.

If $k \ge 4$ is even, then for every sufficiently large prime p, $p-2$ is odd, and by Lemma 3, we have

$$p - 2 = p_1 + p_2 + \cdots + p_{k-1},$$

so

$$p = 2 + p_1 + p_2 + \cdots + p_{k-1},$$

or

$$S(p) = S(2) + S(p_1) + S(p_2) + \cdots + S(p_{k-1}).$$

This means that the theorem is true for even integer $k \geq 4$.

At last, for any prime $p \geq 3$, we have

$$S(p^2) = S(p^2 - p) + S(p),$$

so the theorem is also true for $k = 2$.

This completes the second proof of Theorem.

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