An introduction to the trace formula

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1 Introduction

Let G be a unimodular locally compact topological group and Γ a discrete subgroup of G. We take a Haar measure dg on G and form the Hilbert space of square integrable complex valued functions,

$$L^2(\Gamma\backslash G) = \left\{ f : \Gamma\backslash G \to \mathbf{C} : \int_{\Gamma\backslash G} |f(g)|^2 \, dg < \infty \right\},$$

which affords a representation R of G via right translation, i.e. for $\varphi \in L^2(\Gamma\backslash G)$ and $x, g \in G$,

$$(R(g)\varphi)(x) = \varphi(xg).$$

Our goal is to understand this representation of G. We shall be interested in the case that G is a real Lie group and Γ is a lattice in G (e.g. $G = \mathbf{R}$ and $\Gamma = \mathbf{Z}$, or $G = \mathrm{SL}(2,\mathbf{R})$ and $\Gamma = \mathrm{SL}(2,\mathbf{Z})$) or G is a reductive algebraic group over $\mathbf{Q}$ and $G = G(\mathbf{A_Q})$ and $\Gamma = G(\mathbf{Q})$.

For example, if G is a compact topological group and $\Gamma = \{1\}$ then the Peter-Weyl theorem asserts that

$$L^2(G) \cong \bigoplus_{\pi\in\hat{G}} (\dim_{\mathbf{C}} \pi)\pi.$$

When studying representations it is natural to study the trace of $R(g)$, this works well for finite groups but of course doesn't make sense if the quotient $\Gamma\backslash G$ is infinite. Instead we can view $L^2(\Gamma\backslash G)$ as a representation for the algebra $C_c(G)$ of compactly supported functions on G. Let $f \in C_c(G)$ then we get a representation $R(f)$ by right convolution, i.e. if $\varphi \in L^2(\Gamma\backslash G)$ and $x \in G$, then,

$$(R(f)\varphi)(x) = \int_G f(y)\varphi(xy) \, dy.$$

The trace formula attempts to compute $\operatorname{tr} R(f)$. This need not make sense in general, $\operatorname{tr} R(f)$ may only make sense when restricted to a subspace of $L^2(\Gamma\backslash G)$; e.g. the space of cuspidal functions. However if the quotient is compact then it does.

Note that we can write

$$\begin{aligned}(R(f)\varphi)(x) &= \int_G f(y)\varphi(xy)\ dy \\ &= \int_G f(x^{-1}y)\varphi(y)\ dy \\ &= \int_{\Gamma\backslash G}\sum_{\gamma\in\Gamma} f(x^{-1}\gamma y)\varphi(\gamma y)\ dy \\ &= \int_{\Gamma\backslash G}\left(\sum_{\gamma\in\Gamma} f(x^{-1}\gamma y)\right)\varphi(y)\ dy.\end{aligned}$$

Thus, $R(f)$ is an integral operator with kernel

$$K_f(x,y) = \sum_{\gamma\in\Gamma} f(x^{-1}\gamma y).$$

Hence the trace can be computed as

$$\operatorname{tr} R(f) = \int_{\Gamma\backslash G} K_f(x,x)\ dx.$$

Let $\{\Gamma\}$ denote the set of representatives of conjugacy classes in Γ and for $\gamma\in\Gamma$ we set

$$\begin{aligned}\Gamma_\gamma &= \{\delta\in\Gamma : \delta^{-1}\gamma\delta = \gamma\} \\ G_\gamma &= \{g\in G : g^{-1}\gamma g = \gamma\}.\end{aligned}$$

We now compute,

$$\begin{aligned}\operatorname{tr} R(f) &= \int_{\Gamma\backslash G}\sum_{\gamma\in\Gamma} f(x^{-1}\gamma x)\ dx \\ &= \int_{\Gamma\backslash G}\sum_{\gamma\in\{\Gamma\}}\sum_{\delta\in\Gamma_\gamma\backslash\Gamma} f(x^{-1}\delta^{-1}\gamma\delta x)\ dx \\ &= \sum_{\gamma\in\{\Gamma\}}\int_{\Gamma\backslash G}\sum_{\delta\in\Gamma_\gamma\backslash\Gamma} f(x^{-1}\delta^{-1}\gamma\delta x)\ dx \\ &= \sum_{\gamma\in\{\Gamma\}}\int_{\Gamma_\gamma\backslash G} f(x^{-1}\gamma x)\ dx \\ &= \sum_{\gamma\in\{\Gamma\}}\operatorname{vol}(\Gamma_\gamma\backslash G_\gamma)\int_{G_\gamma\backslash G} f(x^{-1}\gamma x)\ dx.\end{aligned}$$

On the other hand if we write

$$L^2(\Gamma\backslash G) \cong \bigoplus_{\pi\in\hat{G}} m_\pi \pi$$

then,

$$\operatorname{tr} R(f) = \sum_{\pi\in\hat{G}} m_\pi \operatorname{tr} \pi(f).$$

The trace formula is then the equality of these two expressions,

$$\sum_{\gamma\in\{\Gamma\}} \operatorname{vol}(\Gamma_\gamma\backslash G_\gamma) \int_{G_\gamma\backslash G} f(x^{-1}\gamma x)\ dx = \sum_{\pi\in\hat{G}} m_\pi \operatorname{tr} \pi(f).$$

The left hand side is called the geometric expansion of the trace formula, the right hand side the spectral side. One should think of the left hand side as an explicit expression and the right hand side as the mysterious but interesting side of the trace formula.

What's this expression good for? Applications of the trace formula usually fall into one of two categories.

1. Using the trace formula in isolation. One can attempt to compute the geometric expansion of the trace formula for suitable test functions f. This leads to,
   (a) dimension formulas for spaces of automorphic forms,
   (b) closed formulas for traces of Hecke operators,
   (c) Existence of cusp forms, Weyl's law.
2. Comparing the trace formula as one varies the group G. One can perhaps imagine trying to match up the geometric sides of the trace formula for different groups. This leads to,
   (a) Langlands' functorialities, e.g. the Jacquet-Langlands correspondence,
   (b) decomposition of the L-function of a Shimura variety into products of automorphic L-functions when the trace formula is compared with the Lefschetz fixed point formula.

1.1 Plan for the course

February: Detailed proof of the trace formula in the case of a compact quotient. Applications to Weyl's law and some simple cases of functoriality.

March: Derivation of the trace formula for $\mathrm{GL}(2, \mathbf{A}_{\mathbf{Q}})$ and application to the Jacquet-Langlands correspondence.

April: Stabilization of the SL(2) trace formula. Work of Labesse-Langlands.

May: The trace formula in general.

References will be given each lecture. More details can be found on the course webpage with links to online references where available. As you'll see this turned out to be far too ambitious for a one semester course!

1.2 Acknowledgements

It's a pleasure to thank Kevin Buzzard for his numerous comments on an earlier draft of these notes.

2 Trace formula for finite groups

Much of this section is taken from [Joy].

Let G be a finite group and Γ a subgroup of G. We consider the space

$$V_\Gamma = \{\varphi : G \to \mathbf{C} : \varphi(\gamma g) = \varphi(g) \text{ for all } \gamma \in \Gamma, g \in G\}.$$

Clearly the group G acts on this space by right translation, i.e. for $g, x \in G$ and $\varphi \in V_\Gamma$,

$$(R(g)\varphi)(x) = \varphi(xg).$$

R is the representation of G obtained by induction from the trivial representation of Γ. Let $\hat{G}$ denote the finite set of isomorphism classes of irreducible representations of G. We can decompose

$$R = \bigoplus_{\pi \in \hat{G}} m_\pi^\Gamma \pi,$$

with $m_\pi \in \mathbf{N} \cup \{0\}$.

Let π be a representation of G on a vector space V. Let $f : G \to \mathbf{C}$, we define a linear map $\pi(f) : V \to V$ by

$$\pi(f)v = \sum_{g \in G} f(g)\pi(g)v,$$

for $v \in V$.

We shall now compute the trace of $R(f)$ for any function $f : G \to \mathbf{C}$. We note that $\operatorname{tr}\pi(f)$ depends only on the isomorphism class of π, hence we have,

$$\operatorname{tr} R(f) = \sum_{\pi \in \hat{G}} m_\pi^\Gamma \operatorname{tr}\pi(f).$$

We note that we have, by definition, for $f : G \to \mathbf{C}$, $\varphi \in V_\Gamma$ and $x \in G$,

$$(R(f)\varphi)(x) = \sum_{y \in G} f(y)\varphi(xy).$$

After making the change of variables $y \mapsto x^{-1}y$,

$$(R(f)\varphi)(x) = \sum_{y \in G} f(x^{-1}y)\varphi(y),$$

we can rewrite this as the double sum

$$(R(f)\varphi)(x) = \sum_{y \in \Gamma\backslash G} \sum_{\gamma \in \Gamma} f(x^{-1}\gamma y)\varphi(\gamma y).$$

We now use the fact that the function $\varphi \in V_\Gamma$ is left Γ invariant to deduce that

$$(R(f)\varphi)(x) = \sum_{y \in \Gamma\backslash G} \left(\sum_{\gamma \in \Gamma} f(x^{-1}\gamma y) \right) \varphi(y).$$

We define

$$K_f(x,y) = \sum_{\gamma \in \Gamma} f(x^{-1}\gamma y)$$

so that

$$(R(f)\varphi)(x) = \sum_{y \in \Gamma\backslash G} K_f(x,y)\varphi(y).$$

Let $\Gamma z \in \Gamma\backslash G$, we define $\delta_{\Gamma z} \in V_\Gamma$ by

$$\delta_{\Gamma z}(x) = \begin{cases} 1, & \text{if } x \in \Gamma z; \\ 0, & \text{otherwise.} \end{cases}$$

Then we can take a basis for V_Γ to be

$$\mathcal{B} = \{\delta_{\Gamma z} : \Gamma z \in \Gamma\backslash G\}.$$

From above we have, for $\Gamma z \in \Gamma\backslash G$ and $x \in G$,

$$(R(f)\delta_{\Gamma z})(x) = \sum_{y \in \Gamma\backslash G} K_f(x,y)\delta_{\Gamma z}(y) = K_f(x,z).$$

Hence,

$$R(f)\delta_{\Gamma z} = \sum_{x \in \Gamma\backslash G} K_f(x,z)\delta_{\Gamma x}.$$

Thus,

$$\operatorname{tr} R(f) = \sum_{z \in \Gamma\backslash G} K_f(z,z).$$

We now wish to compute $\operatorname{tr} R(f)$ using this expression.

$$\begin{aligned} \operatorname{tr} R(f) &= \sum_{x \in \Gamma\backslash G} K_f(x,x) \\ &= \sum_{x \in \Gamma\backslash G} \sum_{\gamma \in \Gamma} f(x^{-1}\gamma x). \end{aligned}$$

Before continuing we introduce the following notation.

$$\{\Gamma\} = \{ \text{ set of representatives for the conjugacy classes in } \Gamma \ \}$$

For $\gamma \in \Gamma$ we set

$$\Gamma_\gamma = \{\delta \in \Gamma : \delta^{-1}\gamma\delta = \gamma\}$$
$$G_\gamma = \{g \in G : g^{-1}\gamma g = \gamma\}.$$

We have,

$$\begin{aligned}
\operatorname{tr} R(f) &= \sum_{x \in \Gamma\backslash G} \sum_{\gamma \in \Gamma} f(x^{-1}\gamma x) \\
&= \sum_{x \in \Gamma\backslash G} \sum_{\gamma \in \{\Gamma\}} \sum_{\delta \in \Gamma_\gamma\backslash\Gamma} f(x^{-1}\delta^{-1}\gamma\delta x) \\
&= \sum_{\gamma \in \{\Gamma\}} \sum_{x \in \Gamma\backslash G} \sum_{\delta \in \Gamma_\gamma\backslash\Gamma} f(x^{-1}\delta^{-1}\gamma\delta x) \\
&= \sum_{\gamma \in \{\Gamma\}} \sum_{x \in \Gamma_\gamma\backslash G} f(x^{-1}\gamma x) \\
&= \sum_{\gamma \in \{\Gamma\}} \frac{\#G_\gamma}{\#\Gamma_\gamma} \sum_{x \in G_\gamma\backslash G} f(x^{-1}\gamma x).
\end{aligned}$$

Theorem 2.1. *(The trace formula for finite groups) With notation as above, for $f : G \to \mathbf{C}$ we have,*

$$\sum_{\pi \in \hat{G}} m_\pi^\Gamma \operatorname{tr} \pi(f) = \operatorname{tr} R(f) = \sum_{\gamma \in \{\Gamma\}} \frac{\#G_\gamma}{\#\Gamma_\gamma} \sum_{x \in G_\gamma\backslash G} f(x^{-1}\gamma x).$$

For comparison with the trace formula given above we can write this more suggestively using integration notation. We take all measures on finite groups to give points volume 1. Then,

$$\operatorname{tr} R(f) = \sum_{\gamma \in \{\Gamma\}} \operatorname{vol}(\Gamma_\gamma\backslash G_\gamma) \int_{G_\gamma\backslash G} f(x^{-1}\gamma x)\ dx.$$

As an application of the trace formula we have just developed we prove Frobenius reciprocity for the representation R.

Theorem 2.2. *(Frobenius reciprocity) For $\pi \in \hat{G}$, $m_\pi^\Gamma = \dim \pi^\Gamma$, where π^Γ denotes the subspace of (the space of) π fixed by Γ.*

Before giving the proof we recall that for a finite dimensional representation π of G we have,

$$\sum_{g \in G} \operatorname{tr} \pi(g) = \#G \dim \pi^G.$$

Note furthermore that

$$\sum_{g\in G} \operatorname{tr}\pi(g) = \sum_{g\in\{G\}} \frac{\#G}{\#G_\gamma} \operatorname{tr}\pi(g)$$

where $\{G\}$ denotes a set of representatives for the conjugacy classes in G. Hence,

$$\sum_{g\in\{G\}} \frac{1}{\#G_\gamma} \operatorname{tr}\pi(g) = \dim \pi^G.$$

Proof. We fix an irreducible representation $\sigma \in \hat{G}$. We set f_σ equal to the complex conjugate of the character of σ. Thus, $f_\sigma(g) = \overline{\operatorname{tr}\sigma(g)}$. Let $\pi \in \hat{G}$. Recall that,

$$\pi(f_\sigma) : \pi \to \pi : v \mapsto \sum_{g\in G} f_\sigma(g)\pi(g)v = \sum_{g\in G} \overline{\operatorname{tr}\sigma(g)}\pi(g)v.$$

Hence,

$$\operatorname{tr}\pi(f_\sigma) = \sum_{g\in G} \overline{\operatorname{tr}\sigma(g)} \operatorname{tr}\pi(g).$$

Thus, by character orthogonality,

$$\operatorname{tr}\pi(f_\sigma) = \begin{cases} \#G, & \text{if } \pi \cong \sigma; \\ 0, & \text{otherwise.} \end{cases}$$

Hence we deduce that

$$\operatorname{tr} R(f_\sigma) = \sum_{\pi\in\hat{G}} m_\pi^\Gamma \operatorname{tr}\pi(f_\sigma) = \#G.m_\sigma^\Gamma.$$

On the other hand we have, by the trace formula,

$$\begin{aligned}
\operatorname{tr} R(f_\sigma) &= \sum_{\gamma\in\{\Gamma\}} \frac{\#G_\gamma}{\#\Gamma_\gamma} \sum_{x\in G_\gamma\backslash G} f_\sigma(x^{-1}\gamma x) \\
&= \sum_{\gamma\in\{\Gamma\}} \frac{\#G_\gamma}{\#\Gamma_\gamma} \frac{\#G}{\#G_\gamma} \overline{\operatorname{tr}\sigma(\gamma)} \\
&= \#G \overline{\sum_{\gamma\in\{\Gamma\}} \frac{1}{\#\Gamma_\gamma} \operatorname{tr}\sigma(\gamma)} \\
&= \#G \dim \sigma^\Gamma.
\end{aligned}$$

The proof is now complete. $\square$

Exercise 2.3. Derive the trace formula for the representation of G induced from an arbitrary representation of Γ. Use it to deduce Frobenius reciprocity.

3 Trace formula for compact quotient

Much of this section is taken from notes produced by Jacquet for a course on the trace formula taught at Columbia University in 2004.

In this section we will derive the trace formula for a compact quotient $\Gamma\backslash G$.

For example one could take $G = \mathbf{R}$ and $\Gamma = \mathbf{Z}$. For a non-abelian example one could take D to be a quaternion division algebra over $\mathbf{Q}$ and consider $G = D^\times/Z(D^\times)$ as an algebraic group over $\mathbf{Q}$. Then, $G(\mathbf{Q})\backslash G(\mathbf{A_Q})$ is compact. More explicitly, consider D to be a quaternion division algebra over $\mathbf{Q}$ such that $D \otimes_{\mathbf{Q}} \mathbf{R} = M_2(\mathbf{R})$. Let $\mathcal{O}$ be an order in D and consider the subgroup Γ of $\mathrm{PGL}(2, \mathbf{R})$ give as the image of $\mathcal{O}^\times$ under the map,

$$\mathcal{O}^\times \hookrightarrow D^\times \hookrightarrow \mathrm{GL}(2, \mathbf{R}) \twoheadrightarrow \mathrm{PGL}(2, \mathbf{R}).$$

Then the quotient $\Gamma\backslash \mathrm{PGL}(2, \mathbf{R})$ is compact.

3.1 Some functional analysis

Let H be a complex Hilbert space. We let $\langle\ ,\ \rangle$ denote the inner product on H. Then

$$\|v\| := \langle v, v\rangle$$

is a norm on H. The space H is complete for $\|\ \|$. Equivalently there is a total orthonormal system $\{e_i : i \in I\}$. This means that

$$\langle e_i, e_j\rangle = \delta_{i,j}$$

and every vector $v \in H$ can be expanded in a sum

$$v = \sum_{i\in I} \langle v, e_i\rangle e_i$$

with

$$\sum_{i\in I} |\langle v, e_i\rangle|^2$$

finite. We shall assume throughout that H is separable so that the set I is countable. We have for $v, w \in H$,

$$\langle v, w\rangle = \sum_{i\in I} \langle v, e_i\rangle\langle e_i, w\rangle$$

and

$$\|v\|^2 = \sum_{i\in I} |\langle v, e_i\rangle|^2.$$

A linear operator $A : H \to H$ is called bounded if

$$\sup_{\|v\|=1} \|Av\|$$

is finite[1]. We recall that A is bounded if and only if it is continuous. One sets

$$\|A\| := \sup_{\|v\|=1} \|Av\|.$$

Then clearly for any $v \in H$,

$$\|Av\| \leq \|A\|\|v\|.$$

If A is a bounded operator then the adjoint of A is the bounded operator $A^* : H \to H$ such that

$$\langle A^* v_1, v_2 \rangle = \langle v_1, A v_2 \rangle$$

for all $v_1, v_2 \in H$.

An operator $A : H \to H$ is said to be compact if the set $\{Av : \|v\| \leq 1\}$ is relatively compact, i.e. its closure is compact. It amounts to the same to demand that every for every bounded sequence of vectors $\{v_n\}$ the sequence $\{Av_n\}$ has a convergent subsequence. For example, any operator of finite rank is compact.

Lemma 3.1. *Let $A_n : H \to H$ be a sequence of compact operators which converge to an operator A in the norm topology (i.e. $\|A - A_n\| \to 0$ as $n \to \infty$). Then A is compact.*

Proof. Proof left as an exercise. See, for example, [RS80, Theorem VI.12]. □

Lemma 3.2. *Let $A : H \to H$ be a bounded operator. Let $\mathcal{B}$ be an orthonormal basis for H. The quantity*

$$\sum_{b \in \mathcal{B}} \|Ab\|^2$$

is independent of the choice of $\mathcal{B}$. If this sum is finite we set

$$\|A\|_2 = \sqrt{\sum_{b \in \mathcal{B}} \|Ab\|^2}$$

and we say that A is Hilbert-Schmidt. Furthermore we note that $\|A\| \leq \|A\|_2$ and $\|A\|_2 = \|A^*\|_2$.

Proof. Suppose we chose another orthonormal basis $\mathcal{B}'$. Let $b \in \mathcal{B}$. Then we can write

$$Ab = \sum_{b' \in \mathcal{B}'} \langle Ab, b' \rangle b'.$$

and hence,

$$\|Ab\|^2 = \sum_{b' \in \mathcal{B}'} \|\langle Ab, b' \rangle\|^2.$$

[1]The supremum here should be taken in $\mathbf{R}_{\geq 0}$ so that the zero map between zero dimensional vector spaces is bounded.

Thus,

$$\begin{aligned}\sum_{b\in\mathcal{B}}\|Ab\|^2 &= \sum_{b\in\mathcal{B}}\sum_{b'\in\mathcal{B}'}\|\langle Ab,b'\rangle\|^2\\ &= \sum_{b\in\mathcal{B}}\sum_{b'\in\mathcal{B}'}\|\langle b,A^*b'\rangle\|^2\\ &= \sum_{b'\in\mathcal{B}'}\sum_{b\in\mathcal{B}}\|\langle b,A^*b'\rangle\|^2\\ &= \sum_{b'\in\mathcal{B}'}\|A^*b'\|^2.\end{aligned}$$

First by taking $\mathcal{B}'=\mathcal{B}$ we note that

$$\sum_{b\in\mathcal{B}}\|Ab\|^2=\sum_{b\in\mathcal{B}}\|A^*b\|^2,$$

for any orthonormal basis $\mathcal{B}$ of V. Hence from above,

$$\sum_{b\in\mathcal{B}}\|Ab\|^2=\sum_{b'\in\mathcal{B}'}\|A^*b'\|^2=\sum_{b'\in\mathcal{B}'}\|Ab'\|^2.$$

Finally we note that if $v\in H$ is a unit vector then we can find an orthonormal basis $\{e_i\}$ with $e_1=v$ and hence,

$$\|Av\|^2\le\sum_i\|Ae_i\|^2=\|A\|_2.$$

Thus since, by definition,

$$\|A\|=\sup_{\|v\|=1}\|Av\|$$

we see that $\|A\|\le\|A\|_2$. □

Lemma 3.3. *Let A be a Hilbert-Schmidt operator. Then A is compact.*

Proof. Let $\{e_i\}$ be an orthonormal basis of H. Then,

$$\|A\|_2^2=\sum_i\|Ae_i\|^2<\infty.$$

Given $n\ge1$ there exists N_n such that,

$$\sum_{i>N_n}\|Ae_i\|^2<\frac{1}{n^2}.$$

Let A_n be the operator defined by

$$A_nv=\sum_{i\le N_n}\langle Av,e_i\rangle e_i.$$

Since A_n is of finite rank it is compact. We have,

$$\|A - A_n\|_2 = \sum_{i > N_n} \|Ae_i\|^2 < \frac{1}{n^2}.$$

Hence,

$$\|A - A_n\| \leq \|A - A_n\|_2 < \frac{1}{n}.$$

Thus A is the limit of the A_n in the uniform norm. Hence A is compact by Lemma 3.1. □

We now come to the notion of the trace of a bounded operator. For a more complete discussion of trace class operators see [RS80, Section VI.6].

Definition 3.4. *Let $A : H \to H$ be a bounded operator. We say that A is of trace class if*

$$\sum_{b \in \mathcal{B}} |\langle Ab, b \rangle|$$

converges for every *orthonormal basis $\mathcal{B}$ of H.*

We remark that the condition that

$$\sum_{b \in \mathcal{B}} |\langle Ab, b \rangle|$$

converge for every orthonormal basis $\mathcal{B}$ of H is important. For example consider the following operator on $\ell^2(\mathbf{N})$. Let $\{e_i : i \in \mathbf{N}\}$ denote the standard basis of $\ell^2(\mathbf{N})$. We define $A : \ell^2(\mathbf{N}) \to \ell^2(\mathbf{N})$ by,

$$Ae_i = \begin{cases} e_{i+1}, & \text{if } i \text{ is odd;} \\ e_{i-1}, & \text{if } i \text{ is even.} \end{cases}$$

Clearly we have, $\langle Ae_i, e_i \rangle = 0$ for all i. On the other hand if we choose the basis $\mathcal{B} = \{b_i : i \in \mathbf{N}\}$ where

$$b_i = \begin{cases} (e_i + e_{i+1})/\sqrt{2}, & \text{if } i \text{ is odd;} \\ (e_i - e_{i-1})/\sqrt{2}, & \text{if } i \text{ is even.} \end{cases}$$

Then,

$$Ab_i = \begin{cases} b_i/\sqrt{2}, & \text{if } i \text{ is odd;} \\ -b_i/\sqrt{2}, & \text{if } i \text{ is even.} \end{cases}$$

Hence,

$$\sum_{i=1}^{\infty} |\langle Ab_i, b_i \rangle| = \sum_{i=1}^{\infty} \frac{1}{\sqrt{2}} = \infty.$$

Lemma 3.5. *Let $A : H \to H$ be of trace class. Then,*

$$\sum_{b \in \mathcal{B}} \langle Ab, b \rangle$$

is absolutely convergent and independent of the choice of orthonormal basis $\mathcal{B}$.

Proof. By definition the sum is absolutely convergent. A proof of the independence of the sum can be found in [RS80, Theorem VI.24]. In the proof of the next Proposition we will show that the trace is independent of the basis when A is the composition of two Hilbert-Schmidt operators. It will turn out that we'll only be interested in this case. $\square$

We can now define the trace of a trace class operator.

Definition 3.6. *Let $A : H \to H$ be of trace class. We define the trace of A to be*

$$\operatorname{Tr} A = \sum_{b \in \mathcal{B}} \langle Ab, b \rangle$$

where $\mathcal{B}$ is any orthonormal basis for H.

Proposition 3.7.

1. *Any trace class operator is Hilbert-Schmidt.*
2. *If A and B are Hilbert-Schmidt operators then AB is of trace class and* $\operatorname{Tr} AB = \operatorname{Tr} BA$.
3. $|\operatorname{Tr} AB| \leq \|A\|_2 \|B\|_2$.
4. *If A is of trace class then A^* is also of trace class and* $\operatorname{Tr} A^* = \overline{\operatorname{Tr} A}$.

Proof. The first part follows from [RS80, Theorem VI.21] and [RS80, Theorem VI.22(e)]. Suppose now A and B are Hilbert-Schmidt operators. Let $\mathcal{B}$ be any orthonormal basis for H. We have,

$$\begin{aligned}
\sum_{b \in \mathcal{B}} |\langle A(Bb), b \rangle| &= \sum_{b \in \mathcal{B}} |\langle Bb, A^* b \rangle| \\
&\leq \sum_{b \in \mathcal{B}} \|Bb\| \|A^* b\| \\
&\leq \|B\|_2 \|A^*\|_2 \\
&= \|B\|_2 \|A\|_2,
\end{aligned}$$

which is finite since A and B are assumed to be Hilbert Schmidt. We check here also that $\operatorname{Tr} AB$ is independent of the choice of basis. Let $\mathcal{B}'$ be another orthonormal basis for H. Then,

$$\langle A(Bb), b \rangle = \langle Bb, A^* b \rangle = \sum_{b' \in \mathcal{B}'} \langle Bb, b' \rangle \langle b', A^* b \rangle.$$

Hence,

$$\sum_{b \in \mathcal{B}} \langle A(Bb), b \rangle = \sum_{b \in \mathcal{B}} \sum_{b' \in \mathcal{B}'} \langle Bb, b' \rangle \overline{\langle A^* b, b' \rangle}.$$

We wish to apply Fubini's theorem to change the order of summation. We note that,

$$\begin{aligned}
\sum_{b\in\mathcal{B}}\sum_{b'\in\mathcal{B}'}|\langle Bb,b'\rangle||\langle A^*b,b'\rangle| &\leq \sum_{b\in\mathcal{B}}\left(\sum_{b'\in\mathcal{B}'}|\langle Bb,b'\rangle|^2\right)^{\frac{1}{2}}\left(\sum_{b'\in\mathcal{B}'}|\langle A^*b,b'\rangle|^2\right)^{\frac{1}{2}} \\
&= \sum_{b\in\mathcal{B}}\|Bb\|\|A^*b\| \\
&\leq \left(\sum_{b\in\mathcal{B}}\|Bb\|^2\right)^{\frac{1}{2}}\left(\sum_{b\in\mathcal{B}}\|A^*b\|^2\right)^{\frac{1}{2}} \\
&= \|B\|_2\|A\|_2.
\end{aligned}$$

Thus we can apply Fubini's theorem to yield,

$$\begin{aligned}
\sum_{b\in\mathcal{B}}\langle A(Bb),b\rangle &= \sum_{b'\in\mathcal{B}'}\sum_{b\in\mathcal{B}}\langle Bb,b'\rangle\langle b',A^*b\rangle \\
&= \sum_{b'\in\mathcal{B}'}\sum_{b\in\mathcal{B}}\langle Ab',b\rangle\langle b,B^*b'\rangle \\
&= \sum_{b'\in\mathcal{B}'}\langle Ab',B^*b'\rangle \\
&= \sum_{b'\in\mathcal{B}'}\langle B(Ab'),b'\rangle.
\end{aligned}$$

Thus, first by taking $\mathcal{B}=\mathcal{B}'$ we see that, for any orthonormal basis $\mathcal{B}$,

$$\sum_{b\in\mathcal{B}}\langle A(Bb),b\rangle = \sum_{b\in\mathcal{B}}\langle B(Ab),b\rangle.$$

And hence that for any bases $\mathcal{B}$ and $\mathcal{B}'$ we have,

$$\sum_{b\in\mathcal{B}}\langle A(Bb),b\rangle = \sum_{b'\in\mathcal{B}'}\langle B(Ab'),b'\rangle = \sum_{b'\in\mathcal{B}'}\langle A(Bb'),b'\rangle.$$

The third part of the Proposition now follows as well.

For the fourth part of the Proposition we note that if $A: H\to H$ is of trace class and $\mathcal{B}$ is an orthonormal basis of H then

$$\sum_{b\in\mathcal{B}}\langle Ab,b\rangle = \sum_{b\in\mathcal{B}}\langle b,A^*b\rangle = \sum_{b\in\mathcal{B}}\overline{\langle A^*b,b\rangle}$$

from which the fourth part of the Proposition follows. □

We remark that by [RS80, Theorem VI.22(h)] any trace class operator is the composition of two Hilbert-Schmidt operators.

3.1.1 Integral operators

We now take a locally compact measure space (X, μ) and set $H = L^2(X, \mu)$ with inner product

$$\langle f_1, f_2 \rangle = \int_X f_1(x)\overline{f_2(x)}\ dx.$$

We assume that H is separable. We set $\|f\|_2 = \langle f, f \rangle^{\frac{1}{2}}$. Let $K(x, y) \in L^2(X \times X, \mu \otimes \mu)$. The integral operator $A_K : H \to H$ with kernel K is defined by

$$(A_K f)(x) = \int_X K(x, y) f(y)\ dy,$$

for $f \in L^2(X)$ and $x \in X$. Note that if $f_1, f_2 \in L^2(X, \mu)$ then

$$\int_X (A_K f_1)(x)\overline{f_2(x)}\ dx = \int_X \int_X K(x, y) f_1(y)\overline{f_2(x)}\ dx\ dy$$

converges absolutely, with absolute value bounded by $\|K\|_2\|f_1\|_2\|f_2\|_2$. Hence the function

$$x \mapsto \int_X K(x, y) f(y)\ dy$$

is in $L^2(X, \mu)$. Furthermore since

$$|\langle A_K f_1, f_2 \rangle| \leq \|K\|_2\|f_1\|_2\|f_2\|_2$$

we see that A_K is bounded, and in fact $\|A_K\| \leq \|K\|_2$.

Proposition 3.8. *A_K is Hilbert-Schmidt, and in fact $\|A_K\|_2 = \|K\|_2$.*

Proof. Let $\mathcal{B}$ be an orthonormal basis of $L^2(X)$. We have,

$$\begin{aligned}
\|A_K\|_2^2 &= \sum_{f_1, f_2 \in \mathcal{B}} |\langle A_K f_1, f_2 \rangle|^2 \\
&= \sum_{f_1, f_2 \in \mathcal{B}} \left| \int_X \int_X K(x, y) f_1(x)\overline{f_2(y)}\ dx\ dy \right|^2 \\
&= \sum_{f_1, f_2 \in \mathcal{B}} |\langle K(x, y), \overline{f_1(x)} f_2(y) \rangle_{L^2(X \times X)}|^2.
\end{aligned}$$

Since the functions $(x, y) \mapsto \overline{f_1(x)} f_2(y)$ for $f_1, f_2 \in \mathcal{B}$ form an orthonormal basis of $L^2(X \times X)$ so,

$$\sum_{f_1, f_2 \in \mathcal{B}} |\langle K(x, y), \overline{f_1(x)} f_2(y) \rangle_{L^2(X \times X)}|^2 = \|K(x, y)\|_2^2.$$

Hence, $\|A_K\|_2 = \|K\|_2$. ◻

3.2 Aside on Haar measure

For more details see [KL06, Chapter 7]. Let G be a locally compact topological group. A left Haar measure on G is a measure μ such that

1. $\mu(A) = \mu(xA)$ for all $x \in G$ and all measureable sets A, and
2. $\mu(U) > 0$ for all non-empty open sets U of G.

It is a fundamental fact that any locally compact topological group has a Haar measure and such a measure is unique up to scaling. One can define a right Haar measure similarly. We say G is unimodular if left and right Haar measures agree. Abelian groups, discrete groups and compact groups are all unimodular; see [KL06, Proposition 7.7].

We now take H to be a closed unimodular subgroup of a unimodular group G with Haar measure dg. The quotient space $H\backslash G$ is locally compact and Hausdorff. Let dh be a Haar measure on H. Then there exists a unique measure $d\bar{g}$ on $H\backslash G$ such that for all $f \in C_c(G)$,

$$\int_G f(g)\, dg = \int_{H\backslash G} \left(\int_H f(h\bar{g})\, dh \right) d\bar{g}.$$

Furthermore the measure $d\bar{g}$ is right G-invariant, and positive on non-empty open sets. For the proof we refer to [KL06, Theorem 7.10]. Throughout these notes we will frequently write dg for $d\bar{g}$, with the dependence on a choice of Haar measure on H being understood.

We note that if H_1 is a closed unimodular subgroup of H then,

$$\int_{H_1\backslash G} f(g)\, dg = \int_{H\backslash H_1} \left(\int_{H_1\backslash G} f(hg)\, dh \right) dg.$$

See [KL06].

3.3 The geometric side of the trace formula

Let G be a unimodular locally compact topological group and Γ a discrete subgroup of G, e.g. $G = \mathbf{R}$ and $\Gamma = \mathbf{Z}$. We fix a Haar measure dg on G and take the counting measure on Γ. We form the space $L^2(\Gamma\backslash G)$ which affords a representation R of G via right translation, i.e.

$$(R(g)\varphi)(x) = \varphi(xg).$$

Let $C_c(G)$ denote the space of continuous compactly supported functions on G. We recall that $C_c(G)$ is an algebra under convolution, i.e. for $f_1, f_2 \in C_c(G)$,

$$(f_1 * f_2)(g) = \int_G f_1(gh^{-1}) f_2(h)\, dh.$$

Let $f \in C_c(G)$, a continuous function with compact support. Then f defines an operator $R(f)$ on $L^2(\Gamma\backslash G)$ by right convolution. That is if $\varphi \in L^2(\Gamma\backslash G)$ and $x \in G$ then

$$(R(f)\varphi)(x) := \int_G f(z)\varphi(xz)\ dz.$$

We wish to realize $R(f)$ as an integral operator. Making the change of variables $y = xz$ in the definition of $(R(f)\varphi)(x)$ yields,

$$\begin{aligned}
(R(f)\varphi)(x) &= \int_G f(x^{-1}y)\varphi(y)\ dy \\
&= \int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} f(x^{-1}\gamma y)\varphi(\gamma y)\ dy \\
&= \int_{\Gamma\backslash G} \left(\sum_{\gamma\in\Gamma} f(x^{-1}\gamma y) \right) \varphi(y)\ dy.
\end{aligned}$$

Thus $R(f)$ is an integral operator with kernel

$$K_f(x,y) = \sum_{\gamma\in\Gamma} f(x^{-1}\gamma y).$$

We note that since f is compactly supported, for x and y fixed there are only finitely many γ that contribute to the sum. Hence $K_f(x,y)$ is a continuous function on the compact space $\Gamma\backslash G$ and so is square integrable. Hence $R(f)$ is a Hilbert-Schmidt operator on $L^2(\Gamma\backslash G)$.

Lemma 3.9. *For $f = f_1 * f_2$ with $f_1, f_2 \in C_c(G)$,*

$$R(f) = R(f_1) \circ R(f_2).$$

Hence $R(f)$, being the composition of two Hilbert-Schmidt operators, is of trace class. Furthermore,

$$K_f(x,y) = \int_{\Gamma\backslash G} K_{f_1}(x,z)K_{f_2}(z,y)\ dz.$$

Proof. Let $\varphi \in L^2(\Gamma\backslash G)$ and $x \in G$, then

$$\begin{aligned}
(R(f)\varphi)(x) &= \int_G f(y)\varphi(xy)\ dy \\
&= \int_G \int_G f_1(yz^{-1})f_2(z)\varphi(xy)\ dz\ dy \\
&= \int_G f_1(y) \left(\int_G f_2(z)\varphi(xyz)\ dz \right)\ dy \\
&= \int_G f_1(y)(R(f_2)\varphi)(xy)\ dy \\
&= (R(f_1)R(f_2)\varphi)(x).
\end{aligned}$$

For the second part we note that,

$$\begin{aligned}
R(f_1)(R(f_2)\varphi)(x) &= \int_{\Gamma\backslash G} K_{f_1}(x,z)(R(f_2)\varphi)(z)\ dz \\
&= \int_{\Gamma\backslash G} K_{f_1}(x,z) \int_{\Gamma\backslash G} K_{f_2}(z,y)\varphi(y)\ dy\ dz \\
&= \int_{\Gamma\backslash G} \left(\int_{\Gamma\backslash G} K_{f_1}(x,z) K_{f_2}(z,y)\ dz \right) \varphi(y)\ dy.
\end{aligned}$$

□

Lemma 3.10. *Let $f_1, f_2 \in C_c(G)$ and set $f = f_1 * f_2$ then $R(f)$ is of trace class and,*

$$\operatorname{tr} R(f) = \int_{\Gamma\backslash G} K_f(x,x)\ dx.$$

Proof. We now take $f = f_1 * f_2 \in C_c(G)$. Let $\mathcal{B}$ be an orthonormal basis for $L^2(\Gamma\backslash G)$. Then,

$$\begin{aligned}
\operatorname{tr} R(f) = \operatorname{tr}(R(f_1) \circ R(f_2)) &= \sum_{b\in\mathcal{B}} \sum_{b'\in\mathcal{B}} \langle R(f_2)b, b'\rangle \langle b', R(f_1)^* b\rangle \\
&= \sum_{b\in\mathcal{B}} \sum_{b'\in\mathcal{B}} \langle R(f_2)b, b'\rangle \overline{\langle R(f_1)^* b, b'\rangle}.
\end{aligned}$$

Now $R(f_1)^*$ is the operator defined by the kernel

$$K^*_{f_1}(x,y) = \overline{K_{f_1}(y,x)}.$$

Note also that,

$$\begin{aligned}
\langle R(f_2)b, b'\rangle &= \int_{\Gamma\backslash G} (R(f_2)b)(x)\overline{b'(x)}\ dx \\
&= \int_{\Gamma\backslash G} \int_{\Gamma\backslash G} K_{f_2}(x,y) b(y) \overline{b'(x)}\ dy\ dx \\
&= \langle K_{f_2}, b \otimes \bar{b}'\rangle,
\end{aligned}$$

with the inner product taken in $L^2(\Gamma\backslash G \times \Gamma\backslash G)$.

Thus,

$$\begin{aligned}
\operatorname{tr} R(f) &= \sum_{b\in\mathcal{B}}\sum_{b'\in\mathcal{B}} \langle R(f_2)b, b'\rangle \overline{\langle R(f_1)^* b, b'\rangle} \\
&= \sum_{b\in\mathcal{B}}\sum_{b'\in\mathcal{B}} \langle K_{f_2}, b\otimes \bar{b}'\rangle \overline{\langle K_{f_1}^* b\otimes \bar{b}'\rangle} \\
&= \langle K_{f_2}, K_{f_1}^*\rangle \\
&= \int_{\Gamma\backslash G}\int_{\Gamma\backslash G} K_{f_2}(x,y)\overline{K_{f_1}^*(x,y)}\; dy\; dx \\
&= \int_{\Gamma\backslash G}\int_{\Gamma\backslash G} K_{f_2}(x,y)K_{f_1}(y,x)\; dy\; dx \\
&= \int_{\Gamma\backslash G} K_f(x,x)\; dx.
\end{aligned}$$

□

We will now make this expression for the trace of $R(f)$ more explicit and give the geometric side of the trace formula. Let $\{\Gamma\}$ denote a set of representatives for the conjugacy classes in Γ. For $\gamma \in \Gamma$ we define,

$$\Gamma_\gamma = \{\delta \in \Gamma : \delta^{-1}\gamma\delta = \gamma\}$$
$$G_\gamma = \{g \in G : g^{-1}\gamma g = \gamma\}.$$

Proposition 3.11. *(The geometric side of the trace formula) Let $f = f_1 * f_2$ with $f_1, f_2 \in C_c(G)$. Then, $R(f)$ is of trace class and*

$$\operatorname{tr} R(f) = \sum_{\gamma\in\{\Gamma\}} \int_{\Gamma_\gamma\backslash G} f(x^{-1}\gamma x)\; dx.$$

If, furthermore, G_γ is unimodular for each $\gamma \in \Gamma$, then,

$$\operatorname{tr} R(f) = \sum_{\gamma\in\{\Gamma\}} \operatorname{vol}(\Gamma_\gamma\backslash G_\gamma) \int_{G_\gamma\backslash G} f(x^{-1}\gamma x)\; dx.$$

Proof. We begin by grouping terms in $K_f(x,x)$ into conjugacy classes,

$$K_f(x,x) = \sum_{\gamma\in\Gamma} f(x^{-1}\gamma x) = \sum_{\gamma\in\{\Gamma\}}\sum_{\delta\in\Gamma_\gamma\backslash\Gamma} f(x^{-1}\delta^{-1}\gamma\delta x).$$

Now we have,

$$\begin{aligned}
\int_{\Gamma\backslash G} K_f(x,x)\; dx &= \int_{\Gamma\backslash G} \sum_{\gamma\in\{\Gamma\}}\sum_{\delta\in\Gamma_\gamma\backslash\Gamma} f(x^{-1}\delta^{-1}\gamma\delta x)\; dx \\
&= \sum_{\gamma\in\{\Gamma\}}\int_{\Gamma\backslash G} \sum_{\delta\in\Gamma_\gamma\backslash\Gamma} f(x^{-1}\delta^{-1}\gamma\delta x)\; dx \\
&= \sum_{\gamma\in\{\Gamma\}}\int_{\Gamma_\gamma\backslash G} f(x^{-1}\gamma x)\; dx.
\end{aligned}$$

If we assume that each G_γ is unimodular, then,

$$\operatorname{tr} R(f) = \sum_{\gamma \in \{\Gamma\}} \operatorname{vol}(\Gamma_\gamma \backslash G_\gamma) \int_{G_\gamma \backslash G} f(x^{-1}\gamma x)\, dx.$$

$\square$

We remark that if $G = G(\mathbf{A}_F)$ for a reductive algebraic group G over a number field F then $G_\gamma(\mathbf{A}_F)$ is unimodular for each $\gamma \in G(F)$. In fact, furthermore, if $G(F)\backslash G(\mathbf{A}_F)$ is compact then each $\gamma \in G(F)$ is semisimple, hence G_γ is reductive and $G_\gamma(\mathbf{A}_F)$ is unimodular.

Fact 3.12. If G is a Lie group or the adelic points of a reductive algebraic group then any C^∞ function of compact support is a finite sum of convolutions of continuous functions of compact support; see [DL71].

3.4 Spectral theory of compact operators

We return to the setting of a Hilbert space H and a bounded operator A of H. Recall that A is called Hermitian or self-adjoint if $A = A^*$. That is,

$$\langle Av, w\rangle = \langle v, Aw\rangle$$

for all $v, w \in H$.

Proposition 3.13. *Let $A \neq 0$ be a compact and self-adjoint operator. Then there exists a real non-zero number λ such that the eigenspace*

$$H(\lambda) = \{v \in H : Av = \lambda v\}$$

is non-zero. Moreover, $\dim H(\lambda)$ *is finite.*

We begin with a Lemma.

Lemma 3.14. *If A is self-adjoint, then*

$$\|A\| = \sup_{\|v\|=1} |\langle Av, v\rangle|.$$

Proof. Recall that, by definition,

$$\|A\| = \sup_{\|v\|=1} \|Av\| = \sup_{\|v\|=1} \langle Av, Av\rangle^{\frac{1}{2}}.$$

Set

$$k = \sup_{\|v\|=1} |\langle Av, v\rangle|.$$

Now,

$$|\langle Av, v\rangle| \leq \|A\|\|v\|^2.$$

Hence, $k \leq \|A\|$. To prove the reverse inequality it suffices to check that for all $v, w \in H$ with $\|v\|, \|w\| \leq 1$ that,

$$|\langle Av, w\rangle| \leq k.$$

It suffices to check this for v, w such that $\langle Av, w\rangle$ is real. Now,

$$4\langle Av, w\rangle = \langle A(v+w), v+w\rangle - \langle A(v-w), v-w\rangle + i\left(\langle A(v+iw), v+iw\rangle - \langle A(v-iw), v-iw\rangle\right).$$

Since the bracketed terms are imaginary, this reduces to,

$$\begin{aligned} 4\langle Av, w\rangle &= \langle A(v+w), v+w\rangle - \langle A(v-w), v-w\rangle \\ &\leq k(\|v+w\|^2 + \|v-w\|^2) \\ &= k(2\|v\|^2 + \|w\|^2) \\ &\leq 4k. \end{aligned}$$

The lemma now follows. □

Proof of Proposition 3.13. We now prove the Proposition. Let (v_n) be a sequence of unit vectors such that,

$$\lim_{n\to\infty} |\langle Av_n, v_n\rangle| = \|A\|.$$

After passing to a subsequence if necessary we may assume further that,

$$\lim_{n\to\infty} \langle Av_n, v_n\rangle = \alpha,$$

exists. Then, $|\alpha| = \|A\|$. By the compactness of A, passing to a subsequence, we may assume that,

$$Av_n \to v.$$

Then,

$$\|Av_n - \alpha v_n\|^2 = \langle Av_n, Av_n,\rangle - 2\alpha\langle Av_n, v_n\rangle + \alpha^2\langle v_n, v_n\rangle$$

because $\langle Av_n, v_n\rangle$ is real. But,

$$\langle Av_n, Av_n\rangle \leq \|A\|^2 = \alpha^2.$$

Hence,

$$\|Av_n - \alpha v_n\|^2 \leq \alpha^2 - 2\alpha\langle Av_n, v_n\rangle + \alpha^2 \to 0$$

as $n \to \infty$. Hence $Av_n - \alpha v_n \to 0$. Since $Av_n \to v$ we have $v_n \to v/\alpha$ and hence $v \in V(\alpha)$. Moreover $v \neq 0$ for otherwise $\langle Av_n, v_n\rangle \to 0$, hence $\alpha = 0$ and so $\|A\| = 0$. Finally, $\dim H(\alpha)$ is finite since the restriction of A to $H(\alpha)$ is a compact scalar operator. □

An immediate corollary of the proposition,

Corollary 3.15. *Let A be compact and self-adjoint. Then one has an orthogonal decomposition,*

$$H = H(0) \oplus_{\lambda \in S} H(\lambda),$$

where the $H(\lambda)$, for $\lambda \in S$, have finite dimension.

In particular if A is the product of two Hilbert-Schmidt operators then the trace of A is well defined and,

$$\operatorname{tr} A = \sum_{\lambda \in S} \dim H(\lambda).\lambda.$$

3.5 The spectral side of the trace formula

We again take G to be a locally compact unimodular group. Let H be a Hilbert space and let $U(H)$ denote the group of unitary operators on H; i.e. operators $A : H \to H$ such that,

$$\langle Av, Aw \rangle = \langle v, w \rangle$$

for all $v, w \in H$. A unitary representation π of G on H is a homomorphism $\pi : G \to U(H)$ such that for all $v \in H$ the function $g \mapsto \pi(g)v$ is continuous. The representation π is called irreducible if H contains no non-trivial closed invariant subspaces. Suppose (π, H) is a unitary representation of G, then we obtain a homomorphism $\pi : C_c(G) \to \operatorname{End}(H)$ by defining,

$$\pi(f)v = \int_G f(g)\pi(g)v \; dg.$$

We note that if, for $f \in C_c(G)$ we define $f^* \in C_c(G)$ by $f^*(g) = \overline{f(g^{-1})}$ then $\pi(f)^* = \pi(f^*)$. We let $\widehat{G}$ denote the set of equivalence classes of irreducible representations of G.

We again take Γ to be a discrete subgroup of G such that $\Gamma\backslash G$ is compact. We assume that $C_c(G)$ contains approximations of unity.[2] We take R to be the representation of G on $L^2(\Gamma\backslash G)$ via right translation. In this section we will prove the following.

Theorem 3.16. *As a representation of G,*

$$L^2(\Gamma\backslash G) \cong \bigoplus_{\pi \in \hat{G}} m_\pi \pi$$

with the multiplicities m_π being finite.

We begin with a lemma.

Lemma 3.17. *Let H be a non-zero closed G-invariant subspace of $L^2(\Gamma\backslash G)$. Then H contains a closed irreducible subspace.*

[2] In fact it seems this assumption is unnecessary as it follows from Urysohn's Lemma for locally compact Hausdorff spaces.

Proof. We follow [Bum97, Lemma 2.3.2]. We begin by finding a function $f \in C_c(G)$ such that $R(f)$ is self-adjoint and $R(f)$ is non-zero when restricted to H. We take $0 \neq v \in H$ to be a unit vector. Since the map $g \mapsto \pi(g)v$ is continuous we can find an open neighborhood of the identity in G such that,

$$\|R(g)v - v\| < \frac{1}{2}$$

for all $g \in U$. Let f be a real-valued non-negative function with compact support contained in U such that,

$$\int_G f(g)\ dg = 1.$$

Then we have,

$$\|R(f)v - v\| = \left\| \int_G f(g)(R(g)v - v)\ dg \right\| \leq \int_G f(g)\|R(g)v - v\|\ dg < \frac{1}{2}.$$

Clearly we could make the further assumption that $f(g) = f(g^{-1})$ so that $R(f)$ is self-adjoint.

Recall that $R(f)$ is a Hilbert-Schmidt operator on H. By Proposition 3.13 we can take λ to be a non-zero eigenvalue of $R(f)$ on H. We let $H(\lambda)$ denote the corresponding eigenspace which is finite dimensional by Proposition 3.13. Among all invariant subspaces M of H we choose one such that $\dim M(\lambda)$ is positive but minimal. Let $v \in M(\lambda)$ be a non-zero vector. Let E be the closed invariant subspace generated by v. Then we have,

$$M = E \oplus E^{\perp}.$$

The space $E^{\perp}$ is also closed and invariant and,

$$M(\lambda) = E(\lambda) \oplus E^{\perp}(\lambda).$$

By the minimality of the dimension of $M(\lambda)$ we have $E^{\perp}(\lambda) = \{0\}$. Thus, $M(\lambda) = E(\lambda)$. Thus we may as well replace M by E. Now suppose E_1 is a closed invariant subspace of E. Then we again have an orthogonal decomposition,

$$E = E_1 \oplus E_1^{\perp},$$

and

$$E(\lambda) = E_1(\lambda) \oplus E_1^{\perp}(\lambda).$$

By the minimality of the decomposition we have $E(\lambda) = E_1(\lambda)$ or $E(\lambda) = E_1^{\perp}(\lambda)$. In particular $v \in E_1$ or $v \in E_1^{\perp}$. This implies that $E_1 = E$ or $E_1^{\perp} = E$. $\square$

Proof of Theorem 3.16. By Zorn's lemma, let S be a maximal set of orthogonal closed irreducible subspaces. Let $H = \bigoplus_{V \in S} V$. If H is proper, applying Lemma 3.17 to its orthogonal complement contradicts the maximality of S.

The finiteness of the multiplicities m_π follows from the fact that $R(f)$ is Hilbert-Schmidt for any $f \in C_c(G)$. $\square$

Before writing down the trace formula we make the following observation.

Lemma 3.18. *Let π be a representation of G on a Hilbert space H. For $f \in C_c(G)$ we define,*

$$\pi(f) : H \to H : v \mapsto \int_G f(g)\pi(g)v \; dg.$$

If $\pi(f)$ is of trace class then $\operatorname{tr}\pi(f)$ depends only on the isomorphism class of π and the Haar measure dg.

Corollary 3.19. *(The spectral side of the trace formula) As a representation of G, we have,*

$$L^2(\Gamma\backslash G) = \bigoplus_{\pi \in \widehat{G}} m_\pi \pi$$

with each $m_\pi \geq 0$. Hence for $R(f)$ of trace class,

$$\operatorname{tr} R(f) = \sum_{\pi \in \widehat{G}} m_\pi \operatorname{tr}\pi(f).$$

Proof. By definition,

$$\operatorname{tr} R(f) = \sum_{b \in \mathcal{B}} \langle R(f)b, b\rangle$$

where $\mathcal{B}$ is any orthonormal basis for $L^2(\Gamma\backslash G)$. We form an orthonormal basis for $L^2(\Gamma\backslash G)$ by taking an orthonormal basis for each subrepresentation of $L^2(\Gamma\backslash G)$. The Corollary now follows. □

Combining Proposition 3.11 and Corollary 3.19 yields the following.

Theorem 3.20. *(The trace formula) Assume that $f = f_1 * f_2$ with $f_1, f_2 \in C_c(G)$. Then $R(f)$ is of trace class and,*

$$\sum_{\pi \in \hat{G}} m_\pi \operatorname{tr}\pi(f) = \operatorname{tr} R(f) = \sum_{\gamma \in \{\Gamma\}} \operatorname{vol}(\Gamma_\gamma\backslash G_\gamma) \int_{G_\gamma\backslash G} f(g^{-1}\gamma g) \; dg.$$

3.6 Example: Poisson Summation

We now give an explicit example of the trace formula. Suppose $G = \mathbf{R}$ and $\Gamma = \mathbf{Z}$. For each integer n let χ_n be the character of $\mathbf{R}$ given by

$$\chi_n : \mathbf{R} \to \mathbf{C}^\times : x \mapsto e^{2\pi i n x}.$$

Then the theory of Fourier series tells us that,

$$L^2(\mathbf{Z}\backslash\mathbf{R}) = \bigoplus_{n \in \mathbf{Z}} \mathbf{C}e^{2\pi i n x}.$$

Hence as a representation of $\mathbf{R}$ we have,

$$L^2(\mathbf{Z}\backslash\mathbf{R}) = \bigoplus_{n \in \mathbf{Z}} \chi_n.$$

Let $f \in C_c^\infty(\mathbf{R})$. Then we have,

$$\operatorname{tr} R(f) = \sum_{n \in \mathbf{Z}} \operatorname{tr} \chi_n(f).$$

To compute $\operatorname{tr} \chi_n(f)$ we note that if v is in the space of χ_n then, have,

$$\begin{aligned}\chi_n(f)v &= \int_{\mathbf{R}} f(y)\chi_n(y)v \; dy \\ &= \left(\int_{\mathbf{R}} f(y)e^{2\pi i n y} \; dy\right) v.\end{aligned}$$

Thus,

$$\operatorname{tr} \chi_n(f) = \int_{\mathbf{R}} f(y)e^{2\pi i n y} \; dy = \hat{f}(-n),$$

where $\hat{f}$ denotes the Fourier transform of f. Hence,

$$\operatorname{tr} R(f) = \sum_{n \in \mathbf{Z}} \hat{f}(n).$$

On the other hand from the geometric expansion for the trace formula we have,

$$\operatorname{tr} R(f) = \sum_{n \in \mathbf{Z}} f(n).$$

Thus we obtain,

Theorem 3.21. *(Poisson summation formula) Let $f \in C_c^\infty(\mathbf{R})$. Then,*

$$\sum_{n \in \mathbf{Z}} f(n) = \sum_{n \in \mathbf{Z}} \hat{f}(n).$$

4 Weyl's law

References for this section [Bum97, Section 2.3] and [Bum03]. One can also see the books of Iwaniec [Iwa97] and [Iwa02] for further details.

Let Ω be a bounded region in the plane $\mathbf{R}^2$ with smooth boundary $\partial\Omega$. Consider the Euclidean Laplacian,

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

One looks for solutions to the partial differential equation,

$$\Delta\varphi + \lambda\varphi = 0$$

with boundary condition $\varphi|_{\partial\Omega} \equiv 0$. We note that $\lambda \geq 0$ and we let $N_\Omega(T)$ denote the number of linearly independent solutions with $\lambda \leq T$. Weyl proved,

$$N_\Omega(T) \sim \frac{\operatorname{area}(\Omega)}{4\pi} T$$

as $T \to \infty$. More generally one may consider a compact Riemannian manifold (M, g) of dimension d and its Laplacian $\Delta = \text{div grad}$. Let $N_M(T)$ denote the number of eigenfunctions for Δ with $\lambda \leq T$. Then Weyl's law in this context is,

$$N_M(T) \sim \frac{\text{vol}(M)}{(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2}+1)} T^{\frac{d}{2}}$$

as $T \to \infty$.

Selberg originally developed the trace formula to extend Weyl's law to certain non-compact spaces of the form $\Gamma\backslash\mathcal{H}$ where Γ is a congruence subgroup of $\text{SL}(2, \mathbf{R})$, i.e. a subgroup of $\text{SL}(2, \mathbf{Z})$ containing

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbf{Z}) : a, d \equiv 1 \bmod N,\ b, c \equiv 0 \bmod N \right\}$$

for some N.

In this section we will explain how the trace formula can be used to prove Weyl's law for compact quotients of the upper half plane which includes the case of any compact Riemann surface of genus at least 2.

4.1 About $\text{SL}(2, \mathbf{R})$

We now fix a Haar measure on G. We set,

$$\begin{aligned} A &= \left\{ a(u) = \begin{pmatrix} e^{\frac{u}{2}} & 0 \\ 0 & e^{-\frac{u}{2}} \end{pmatrix} : u \in \mathbf{R} \right\} \\ N &= \left\{ n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbf{R} \right\} \\ K &= \left\{ k(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} : \theta \in [0, 2\pi] \right\}. \end{aligned}$$

On each of these groups we take Lebesgue measure on $\mathbf{R}$ and pull it back to the group. Using the Iwasawa decomposition, $G = ANK$, we define a measure on G by,

$$dg = \frac{1}{2\pi}\ du\ dx\ d\theta.$$

That is, if $f \in C_c(G)$, then,

$$\int_G f(g)\ dg = \frac{1}{2\pi} \int_{[0,2\pi]} \int_{\mathbf{R}} \int_{\mathbf{R}} f(a(u)n(x)k(\theta))\ du\ dx\ d\theta.$$

Lemma 4.1. *The measure dg defined above is a left and right Haar measure on G.*

We also recall the Cartan decomposition $G = KAK$.

We recall the well known classification of motions. Let $g \in G \setminus \{\pm I\}$. Then,

1. g is called parabolic if $|\operatorname{tr}(g)| = 2$,

2. g is called hyperbolic if $|\operatorname{tr}(g)| > 2$, and
3. g is called elliptic if $|\operatorname{tr}(g)| < 2$.

Equivalently,

1. g is parabolic if g is conjugate to an element in $\pm N$,
2. g is hyperbolic if g is conjugate to an element in $\pm A$, and
3. g is elliptic if g is conjugate to an element in K.

4.2 Maass forms

For more information see [Mil97, Chapter 2] or the extended online version of [Bum03], especially Sections 1 and 8.

We take

$$\mathcal{H} = \{z = x + iy : x \in \mathbf{R}, y \in \mathbf{R}_{>0}\}$$

to be the upper half plane. We take the metric on $\mathcal{H}$ to be,

$$ds^2 = y^{-2}(dx^2 + dy^2).$$

For $z, w \in \mathcal{H}$ we let $\rho(z, w)$ denote the distance between z and w with respect to this metric. The hyperbolic measure on $\mathcal{H}$ given by,

$$dz = \frac{dx\ dy}{y^2}.$$

The non-Euclidean Laplacian is given by,

$$\Delta = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

and acts on the space $C^\infty(\mathcal{H})$ of smooth functions on $\mathcal{H}$.

We recall that the group $\mathrm{SL}(2, \mathbf{R})$ acts on $\mathcal{H}$ on the left by Mobius transformation, i.e.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \mapsto \frac{az + b}{cz + d}.$$

It will be convenient to think of $\mathcal{H} \hookrightarrow \hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ and to extend the action of $\mathrm{SL}(2, \mathbf{R})$ to $\hat{\mathbf{C}}$. One can readily check that for $g \in \mathrm{SL}(2, \mathbf{R}) \setminus \{\pm I\}$,

1. g is parabolic if g has one fixed point on $\hat{\mathbf{R}}$,
2. g is hyperbolic if g has two distinct fixed points on $\hat{\mathbf{R}}$, and
3. g is elliptic if g has one fixed point in $\mathcal{H}$ (and one in $\bar{\mathcal{H}}$).

One can easily check,

Lemma 4.2. *The metric ds and measure dz are invariant under* $\mathrm{SL}(2, \mathbf{R})$*. The non-Euclidean Laplacian is invariant under* $\mathrm{SL}(2, \mathbf{R})$ *in the sense that,*

$$\Delta(f \circ g)(z) = (\Delta f)(gz)$$

for all $f \in C^\infty(\mathcal{H})$ and $g \in \mathrm{SL}(2, \mathbf{R})$.

We note that $\mathrm{SL}(2, \mathbf{R})$ acts transitively on $\mathcal{H}$ and the stabilizer of i is,

$$\mathrm{SO}(2) = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a^2 + b^2 = 1 \right\}.$$

This yields a homeomorphism,

$$\mathrm{SL}(2, \mathbf{R}) / \mathrm{SO}(2) \xrightarrow{\sim} \mathcal{H} : \gamma \mapsto \gamma(i).$$

Note that if we take $f \in C_c(\mathcal{H})$ then,

$$\int_{\mathrm{SL}(2,\mathbf{R})} f(g(i))\ dg = \int_{\mathcal{H}} f(z)\ dz.$$

Let Γ be a discrete subgroup of $\mathcal{H}$. The group Γ acts on $\mathcal{H}$ discontinuously, i.e. for each $z \in \mathcal{H}$ the orbit Γz has no limit point in $\mathcal{H}$. We restrict our interest to discrete groups Γ such that the quotient $\Gamma\backslash\mathcal{H}$ has finite volume (these are the Fuchsian groups of the first kind), equivalently such that $\Gamma\backslash \mathrm{SL}(2, \mathbf{R})$ has finite volume. For example, $\mathrm{SL}(2, \mathbf{Z})$ is a Fuchsian group of the first kind, but

$$\left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbf{Z} \right\}$$

isn't.

A connected open set $F \subset \mathcal{H}$ is called a fundamental domain for Γ if,

1. distinct points in F are inequivalent mod Γ, and
2. any orbit of Γ in $\mathcal{H}$ contains a point in $\overline{F}$.

We note that,

$$\operatorname{area}(\Gamma\backslash\mathcal{H}) = \int_F dz$$

is independent of the choice of fundamental domain F.

For Fuchsian groups of the first kind fundamental domains exist and can be constructed as follows: Pick a point $w \in \mathcal{H}$ such that $\gamma z \neq z$ for all $\gamma \in \Gamma$ and take,

$$D(w) = \{z \in \mathcal{H} : \rho(z, w) < \rho(z, \rho w) \text{ for all } \gamma \in \Gamma\}.$$

Lemma 4.3. *([Bum03, Proposition 1]) With w as above, $D(w)$ is a fundamental domain for Γ. Furthermore the boundary of $D(w)$ consists of pairs of geodesic arcs $\{\alpha_i, \gamma_i(\alpha_i)\}$ with $\gamma_i \in \Gamma$ such that if the boundary of $D(w)$ is traversed counterclockwise then the arcs α_i and $\gamma_i(\alpha_i)$ are traversed in opposite directions.*

Let F be a fundamental domain of Γ, a Fuchsian group of the first kind. We view $\mathcal{H} \subset \widehat{\mathbf{C}}$ so that $\partial\mathcal{H} = \widehat{\mathbf{R}} = \mathbf{R} \cup \{\infty\}$. A cusp for F is a point on $\partial H \cap \overline{F}$. Let S_F denote the set of cusps of F. We note that the fundamental domain is compact if and only if $S_F = \emptyset$. One can arrange F so that all cusps in S_F are inequivalent modulo Γ (see Section 8 of the online extended version of [Bum03]). We assume that F is chosen so that this is the case. Furthermore any cusp of F is fixed by a parabolic element of Γ and, furthermore, when all cusps of F are inequivalent modulo F we have a bijection,

$$S_F \longleftrightarrow \Gamma\backslash\{x \in \widehat{\mathbf{R}} : \gamma x = x \text{ for some parabolic } \gamma \in \Gamma\}.$$

Thus we have,

Lemma 4.4. *Let Γ be a Fuchsian group of the first kind. Then $\Gamma\backslash\mathcal{H}$ is compact if and only if Γ contains no parabolic elements.*

We set, $\mathcal{A}(\Gamma\backslash\mathcal{H})$ equal to the space of smooth functions on $\Gamma\backslash\mathcal{H}$ which can be identified with

$$\mathcal{A}(\Gamma\backslash\mathcal{H}) = \{\varphi \in C^\infty(\mathcal{H}) : \varphi(\gamma z) = \varphi(z) \text{ for all } \gamma \in \Gamma\}.$$

By Lemma 4.2 Δ acts on the space $\mathcal{A}(\Gamma\backslash\mathcal{H})$. We define,

$$\mathcal{C}(\Gamma\backslash\mathcal{H}) = \{\varphi \in \mathcal{A}(\Gamma\backslash\mathcal{H}) : \varphi \text{ vanishes at the cusps}\},$$

and

$$\mathcal{B}(\Gamma\backslash\mathcal{H}) = \{\varphi \in \mathcal{A}(\Gamma\backslash\mathcal{H}) : \varphi \text{ is bounded}\},$$

and

$$\mathcal{D}(\Gamma\backslash\mathcal{H}) = \{\varphi \in \mathcal{B}(\Gamma\backslash\mathcal{H}) : \Delta\varphi \in \mathcal{B}(\Gamma\backslash\mathcal{H})\}.$$

We note that $\mathcal{C}(\Gamma\backslash\mathcal{H}) \subset \mathcal{B}(\Gamma\backslash\mathcal{H})$.

We take the inner product $\langle\ ,\ \rangle$ on $\mathcal{B}(\Gamma\backslash\mathcal{H})$ defined by,

$$\langle\varphi_1, \varphi_2\rangle = \int_{\Gamma\backslash\mathcal{H}} \varphi_1(z)\overline{\varphi_2(z)}\, dz.$$

This can be computed as,

$$\langle\varphi_1, \varphi_2\rangle = \int_F \varphi_1(z)\overline{\varphi_2(z)}\, dz,$$

where F is any fundamental domain for Γ.

Lemma 4.5. *([Bum03, Proposition 2] when $\Gamma\backslash\mathcal{H}$ is compact or [Iwa02, Chapter 4] in general) The Laplacian Δ acts on $C(\Gamma\backslash\mathcal{H})$ and satisfies,*

1. *$\langle\Delta\varphi_1, \varphi_2\rangle = \langle\varphi_1, \Delta\varphi_2\rangle$ for all $\varphi_1, \varphi_2 \in \mathcal{D}(\Gamma\backslash\mathcal{H})$, and*
2. *$\langle\Delta\varphi, \varphi\rangle \leq 0$ for all $\varphi \in \mathcal{D}(\Gamma\backslash\mathcal{H})$*

Proof. For the first part we need to prove that,

$$\langle \Delta\varphi_1, \varphi_2 \rangle - \langle \varphi_1, \Delta\varphi_2 \rangle = 0.$$

Taking a fundamental domain F as in Lemma 4.3 this equals,

$$\int_F \left(\bar{\varphi}_2 \left(\frac{\partial^2 \varphi_1}{\partial x^2} + \frac{\partial^2 \varphi_1}{\partial y^2} \right) - \varphi_1 \left(\frac{\partial^2 \bar{\varphi}_2}{\partial x^2} + \frac{\partial^2 \bar{\varphi}_2}{\partial y^2} \right) \right) \, dx \, dy = \int_F d\omega$$

where

$$\omega = -\varphi_1 \frac{\partial \bar{\varphi}_2}{\partial x} \, dy + \varphi_1 \frac{\partial \bar{\varphi}_2}{\partial y} \, dx + \bar{\varphi}_2 \frac{\partial \varphi_1}{\partial x} \, dy - \bar{\varphi}_2 \frac{\partial \varphi_1}{\partial y} \, dx.$$

By Stokes' theorem,

$$\int_F d\omega = \int_{\partial F} \omega.$$

But by Lemma 4.3 the contributions of the boundary arcs cancel in pairs. The second part of the Lemma also follows from Stokes' theorem; see the cited references. □

Definition 4.6. *An function $\varphi \in \mathcal{A}(\Gamma \backslash \mathcal{H})$ is called an automorphic form if,*

$$\Delta\varphi + \lambda_\varphi \varphi = 0$$

for some $\lambda_\varphi \in \mathbf{R}$ and has polynomial growth at the cusps. We say that φ is a cusp form if φ vanishes at the cusps.

We note that if φ is a cusp form then $\varphi \in \mathcal{D}(\Gamma \backslash \mathcal{H})$ and by Lemma 4.5 $\lambda_\varphi \geq 0$. Furthermore if $\Gamma \backslash \mathcal{H}$ is compact then any automorphic form is automatically a cusp form.

A natural analogue of Weyl's law for the not necessarily compact surfaces $\Gamma \backslash \mathcal{H}$ would be the following: For $T \geq 0$ let,

$$N_\Gamma(T) = \#\{ \text{ linearly independent cusp forms } \varphi : \lambda_\varphi \leq T \}.$$

Then,

$$N_\Gamma(T) \sim \frac{\operatorname{area}(\Gamma \backslash \mathcal{H})}{4\pi} T$$

as $T \to \infty$.

In this generality Weyl's law is monstrously false. In fact for a generic Fuchsian group of the first kind it seems that the number of cusp forms is likely to be finite; see [PS92]. Selberg established Weyl's law in the case of a congruence subgroup of $\mathrm{SL}(2, \mathbf{Z})$. We will establish Weyl's law in the case of a cocompact lattice Γ over the next few sections using the trace formula for $L^2(\Gamma \backslash \mathrm{SL}(2, \mathbf{R}))$ as established above.

4.3 Connection between Maass forms and representation theory

We continue with Γ a discrete subgroup of $\mathrm{SL}(2, \mathbf{R})$ such that $\Gamma\backslash \mathrm{SL}(2, \mathbf{R})$ is compact. We will explain the connection between the representation of $\mathrm{SL}(2, \mathbf{R})$ on $L^2(\Gamma\backslash \mathrm{SL}(2, \mathbf{R}))$ and Maass forms. By Theorem 3.16 we have,

$$L^2(\Gamma\backslash \mathrm{SL}(2, \mathbf{R})) = \bigoplus_{\pi \in \widehat{G}} m_\pi \pi.$$

The goal of this section is to prove the following.

Theorem 4.7. *Let H be a closed $\mathrm{SL}(2, \mathbf{R})$-invariant irreducible subspace of $L^2(\Gamma\backslash \mathrm{SL}(2, \mathbf{R}))$. Then H^K is at most one-dimensional. Suppose $0 \neq \varphi \in H^K$. Then $\varphi \in C^\infty(\Gamma\backslash \mathrm{SL}(2, \mathbf{R}))$ and, as a function on $\Gamma\backslash\mathcal{H}$,*

$$\Delta\varphi + \lambda\varphi = 0$$

for some $\lambda \in \mathbf{R}$ which depends only on the isomorphism class of H.

For more complete proofs see [Bum97, Chapter 2]

We let $C_c^\infty(G//K)$ denote the algebra (under convolution) of bi-K-invariant functions on G.

Lemma 4.8. *Let $f \in C_c^\infty(G//K)$. Then,*

$$f(g) = f({}^t g)$$

for all $g \in G$.

Proof. We recall the Cartan decomposition,

$$G = KAK$$

where A is the diagonal torus in G. Thus we can write any $g \in G$ in the form,

$$g = k_1 \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} k_2.$$

Thus, taking transpose,

$${}^t g = {}^t k_2 \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} {}^t k_1.$$

Note that ${}^t k_1, {}^t k_2 \in K$. Thus, for $f \in C_c^\infty(G//K)$ and $g \in G$, $f({}^t g) = f(g)$. $\square$

Lemma 4.9. *The algebra $C_c^\infty(G//K)$ is commutative.*

Proof. We take $f_1, f_2 \in C_c^\infty(G)$ and $g \in G$, then,

$$\begin{aligned}(f_1 * f_2)(g) &= \int_G f_1(gh^{-1}) f_2(h)\ dh \\ &= \int_G f_1({}^t h^{-1\,t} g) f_2({}^t h)\ dh \\ &= \int_G f_1({}^t h^{-1}) f_2({}^t g\, {}^t h)\ dh.\end{aligned}$$

Finally we make the change of variables $h \mapsto {}^t h$ which preserves the Haar measure to obtain,

$$\begin{aligned}\int_G f_1({}^t h^{-1}) f_2({}^t g\, {}^t h)\ dh &= \int_G f_1(h^{-1}) f_2({}^t g h)\ dh \\ &= (f_2 * f_1)({}^t g) \\ &= (f_2 * f_1)(g).\end{aligned}$$

This completes the proof. $\square$

More generally, let χ be a character of K. We define,

$$C_c^\infty(G//K, \chi) = \{ f \in C_c^\infty(G) : f(k_1 g k_2) = \chi(k_1)\chi(k_2) f(g) \}.$$

Then one can show that $C_c^\infty(G//K, \chi)$ is also commutative; see [Bum97, Proposition 2.2.8].

Lemma 4.10. *([Bum97, Lemma 2.3.2]) Let π be a unitary representation of G. Let v be a non-zero vector in the space of π such that,*

$$\pi(k)v = \chi(k)v$$

for all $k \in K$. Then there exists $f \in C_c^\infty(G//K, \chi^{-1})$ such that $\pi(f)$ is self-adjoint and $\pi(f)v \neq 0$.

Proof. We will prove this Lemma in the case that χ is trivial, leaving the general case to the reader. We let $U \subset G$ be an open neighborhood of the identity such that,

$$\|\pi(g)v - v\| < \frac{1}{2}$$

for all $g \in U$. We next seek an open neighborhood V of the identity such that $kVk^{-1} \subset U$ for all $k \in K$. We note that the map,

$$\alpha : G \times K \to G : (g, k) \mapsto kgk^{-1}$$

is continuous. Hence $\alpha^{-1}(U)$ is open in $G \times K$. Furthermore for each $k \in K$, $(1, k) \in \alpha^{-1}(U)$. Hence for each $k \in K$ we can find open sets V_k and W_k such that,

$$(1, k) \in V_k \times W_k \subset \alpha^{-1}(U).$$

The open sets W_k cover K, hence by compactness we can find a finite set $\{k_1, \ldots, k_r\} \subset K$ such that,

$$W_{k_1} \cup \ldots \cup W_{k_r} = K.$$

Let $V = V_{k_1} \cap \ldots \cap V_{k_r}$, then V has the necessary property.

We now take f_1 to be a smooth function positive function supported in V such that $f(g) = f(g^{-1})$ for all $g \in G$. We define, $f_2 \in C_c^\infty(G)$ by,

$$f_2(g) = \int_K f(kgk^{-1})\ dk.$$

Then f_2 is supported in U. After scaling we can assume $\int_G f_2(g)\ dg = 1$ and then one can check, as in the proof of Lemma 3.17, that,

$$\pi(f_2)v \neq 0.$$

Finally if we define $f \in C_c^\infty(G)$ by

$$f(g) = \int_K f_2(gk)\ dk$$

then $f \in C_c^\infty(G//K)$, $f(g^{-1}) = f(g)$ and $\pi(f)v = \pi(f_2)v \neq 0$. $\square$

Definition 4.11. *Let π be a representation of G on a Hilbert space H. We call π admissible if,*

$$\pi|_K = \bigoplus_{\rho \in \widehat{K}} m_\rho \rho$$

with $m_\rho < \infty$.

Fact 4.12. Any irreducible unitary representation of G is admissible.

We are only interested in those irreducible unitary representations π which appear in $L^2(\Gamma\backslash G)$, and for those we can prove this fact directly.

Proposition 4.13. *Let π be an irreducible unitary representation of G appearing in $L^2(\Gamma\backslash G)$. Then π is admissible and, furthermore, π^K is at most one-dimensional.*

Let π be an irreducible representation of $\mathrm{SL}(2, \mathbf{R})$. Then, π becomes a representation of $C_c^\infty(G)$ by defining,

$$\pi(f)v = \int_G f(g)\pi(g)v\ dg.$$

We note that if $f(kg) = f(g)$ for all $g \in G$ and $k \in K$ then $\pi(f)v \in \pi^K$. Hence, π^K is a representation of $C_c^\infty(G//K)$.

Proof. We need to show that for each character χ of K,

$$\{v \in \pi : \pi(k)v = \chi(k)v\}$$

is finite dimensional.

We will verify this for the trivial character χ leaving the general case as an exercise.

We may as well assume $\pi^K \neq \{0\}$. We first show that $C_c^\infty(G//K)$ acts irreducibly on π^K. Suppose $L \subset \pi^K$ is a non-zero closed $C_c^\infty(G//K)$-invariant subspace of π^K. We take $v \in \pi^K$ and will show that $v \in L$. We fix $\varepsilon > 0$. Since π is irreducible the closure of $\pi(C_c^\infty(G))L$ is the whole of π. Hence there exists $f \in C_c^\infty(G)$ and $w \in L$ such that $\|\pi(f)w - v\| < \varepsilon$. We set $v_1 = \pi(f)w$ and let,

$$v_2 = \int_K \pi(k)v_1 \; dk.$$

Here dk is the Haar measure on K normalized to give K volume one. Then $v_2 \in \pi^K$ and,

$$\|v_2 - v\| = \left\| \int_K \pi(k)(v_1 - v) \; dk \right\| \leq \int_K \|\pi(k)(v_1 - v)\| \; dk = \int_K \|v_1 - v\| \; dk < \varepsilon.$$

Since w is K-fixed, so

$$\pi(k)\pi(f)\pi(k')w = \pi(k)\pi(f)w = \pi(k)v_1$$

for all $k, k' \in K$. Integrating over k and k' we get $\pi(f_0)w = v_2$ where $f_0 \in C_c^\infty(G//K)$ is defined by,

$$f_0(g) := \int_K \int_K f(kgk') \; dk \; dk'$$

Since L is invariant under $C_c^\infty(G//K)$ we see that $v_2 \in L$. Hence $v \in \pi^K$ can be arbitrarily closely approximated by elements of L. Since L is closed so $v \in L$. Thus $L = \pi^K$.

Let $0 \neq v \in \pi^K$. We can find by Lemma 4.10 a function $f \in C_c^\infty(G//K)$ such that $\pi(f) = \pi(f)^*$ and $\pi(f)v \neq 0$. Now we use the fact that π appears in $L^2(\Gamma\backslash G)$ to deduce that $\pi(f)$ is compact. (We note that $\pi(f)$ is the restriction of $R(f)$ to a subspace and since $R(f)$ is Hilbert-Schmidt so is $\pi(f)$, hence $\pi(f)$ is compact.) Now by Proposition 3.13 $\pi(f)$ has a non-zero eigenvalue on π^K, λ say, and the eigenspace,

$$\pi^K(\lambda) = \left\{v \in \pi^K : \pi(f)v = \lambda v\right\}$$

is finite dimensional. Now since $C_c^\infty(G//K)$ is commutative the space $\pi^K(\lambda)$ is preserved by $C_c^\infty(G//K)$. Hence $\pi^K(\lambda)$ is a $C_c^\infty(G//K)$-invariant subspace of π^K and since it's finite dimensional it's closed. Thus we see that $\pi^K = \pi^K(\lambda)$ is finite dimensional. Finally we note that π^K is an irreducible finite dimensional representation for the commutative algebra $C_c^\infty(G//K)$, hence it's one-dimensional. $\square$

We can now prove the first part of Theorem 4.7. Let H be an irreducible subrepresentation of $L^2(\Gamma\backslash G)$. So far we know that H^K is at most one-dimensional. Suppose $0 \neq \varphi \in H^K$. From Lemma 4.10 we can find $f \in C_c^\infty(G//K)$ such that $R(f)\varphi \neq 0$. We note that since $R(f)$ preserves H so $R(f)\varphi = \lambda\varphi$ for some non-zero λ. By definition, $R(f)\varphi$ is the convolution of φ with f which is a smooth function of compact support, hence $R(f)\varphi$ is smooth and so too is φ.

In order to finish the proof of the theorem we need to give a representation theoretic definition of the Laplacian.

Let $\mathfrak{g}$ denote the Lie algebra of G, i.e.

$$\mathfrak{g} = \{X \in M(2, \mathbf{R}) : \operatorname{tr} X = 0\}$$

with Lie bracket,

$$[X, Y] = XY - YX.$$

Let $U(\mathfrak{g})$ denote the universal enveloping algebra which is an associative ring constructed in the following way from $\mathfrak{g}$. Consider,

$$\bigoplus_{k=0}^{\infty} \otimes^k \mathfrak{g}$$

with multiplication coming from the obvious map,

$$\otimes^k \mathfrak{g} \times \otimes^l \mathfrak{g} \to \otimes^{k+l} \mathfrak{g}.$$

Let I denote the ideal generated by the elements,

$$X \otimes Y - Y \otimes X - [X, Y]$$

with $X, Y \in \mathfrak{g}$. We define,

$$U(\mathfrak{g}) = \left(\bigoplus_{k=0}^{\infty} \otimes^k \mathfrak{g} \right) / I.$$

We recall that G acts on $C^\infty(G)$ by right translation. The Lie algebra acts on $C^\infty(G)$ in the usual way,

$$(dX\varphi)(g) = \left. \frac{d}{dt} f(g \exp(tX)) \right|_{t=0}$$

giving a representation of $\mathfrak{g}$, i.e. $d : \mathfrak{g} \to \operatorname{End}(C^\infty(G))$ such that for all $X, Y \in \mathfrak{g}$,

$$dX \circ dY - dY \circ dX = d[X, Y].$$

This representation may therefore be extended to of $U(\mathfrak{g})$ by defining,

$$d(X_1 \otimes \ldots \otimes X_n)\varphi = d(X_1) \circ \ldots \circ d(X_n)\varphi,$$

noting that since π is a representation of $\mathfrak{g}$ this is well defined on $U(\mathfrak{g})$. This representation is faithful and in this way $U(\mathfrak{g})$ may be realized as the space of left-invariant differential operators on G. The center $Z(U(\mathfrak{g}))$ of the universal enveloping algebra may then be realized as the space of left and right invariant differential operators on G; see [Bum97, Proposition 2.2.4]

Consider the following basis of $\mathfrak{g}$,

$$R = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ L = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We define the element in $U(\mathfrak{g})$,

$$C = -\frac{1}{4}\left(H \otimes H + 2R \otimes L + 2L \otimes R\right).$$

Lemma 4.14. *([Bum97, Theorem 2.2.1]) The element C defined above lies in the center of $U(\mathfrak{g})$.*

Proof. Easy to check by straightforward calculation. □

Let π be a representation of G on a Hilbert space H. Let $v \in H$. We call v C^1 if for all $X \in \mathfrak{g}$,

$$\pi(X)v = \frac{d}{dt}\pi(\exp(tX))v\bigg|_{t=0},$$

exists. We say v is C^k if v is C^1 and for all $X \in \mathfrak{g}$, $\pi(X)v$ is C^{k-1}. Finally, v is called smooth if v is C^k for all k. We denote by H^∞ the space of smooth vectors in H. We note that G preserves H^∞.

Lemma 4.15. *([Bum97, Lemma 2.4.2]) For the representation of G on $L^2(\Gamma\backslash G)$ we have $\varphi \in L^2(\Gamma\backslash G)^\infty$ if and only if $\varphi \in C^\infty(\Gamma\backslash G)$.*

Lemma 4.16. *([Bum97, Proposition 2.4.1]) Let π be a representation of G on a Hilbert space H. The space H^∞ affords a Lie algebra representation of $\mathfrak{g}$, i.e. $\pi : \mathfrak{g} \to \mathrm{End}(H^\infty)$ such that for all $v \in H^\infty$ and $X, Y \in \mathfrak{g}$,*

$$\pi(X)\pi(Y)v - \pi(Y)\pi(X)v = \pi([X,Y])v.$$

Furthermore for $g \in G$, $X \in \mathfrak{g}$ and $v \in H^\infty$,

$$\pi(g)\pi(X)\pi(g)^{-1}v = \pi(Ad(g)X)v.$$

Hence for $g \in G$ and $D \in Z(U(\mathfrak{g}))$,

$$\pi(g) \circ \pi(D) = \pi(D) \circ \pi(g)$$

on H^∞.

Lemma 4.17. *([Bum97, Proposition 2.2.5]) Let G act on the space $C^\infty(G)$ by right translation. Let $\varphi \in C^\infty(G)$ be right invariant under K. Then we can consider φ as a function on $\mathcal{H}$ and with this identification we have,*

$$\Delta\varphi = -C\varphi,$$

where Δ denotes the non-Euclidean Laplacian on $C^\infty(\mathcal{H})$.

Proof. If one uses the coordinates,

$$g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{\frac{1}{2}} & 0 \\ 0 & y^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

with $y \in \mathbf{R}_{>0}$, $x \in \mathbf{R}$, $\theta \in \mathbf{R}$, coming from the Iwasawa decomposition for G, then,

$$\begin{aligned} dR &= y\cos(2\theta)\frac{\partial}{\partial x} + y\sin(2\theta)\frac{\partial}{\partial y} + \sin^2(\theta)\frac{\partial}{\partial\theta} \\ dL &= y\cos(2\theta)\frac{\partial}{\partial x} + y\sin(2\theta)\frac{\partial}{\partial y} - \cos^2(\theta)\frac{\partial}{\partial\theta} \\ dH &= -2y\sin(2\theta)\frac{\partial}{\partial x} + 2y\cos(2\theta)\frac{\partial}{\partial y} + \sin(2\theta)\frac{\partial}{\partial\theta}. \end{aligned}$$

Now one can compute the action of C on φ and check the statement of the lemma. □

We can now finally finish the proof of Theorem 4.7. We again take H to be a closed irreducible subspace of $L^2(\Gamma\backslash G)$. As we have already observed H^K is at most one-dimensional. If there exists $0 \neq \varphi \in H^K$ then we have already observed that φ is smooth, by Lemma 4.16 we have $C\varphi = \lambda\varphi$ for some $\lambda \in \mathbf{C}$ and by Lemma 4.17 if we view φ as an element of $\mathcal{A}(\Gamma\backslash\mathcal{H})$ then $\Delta\varphi + \lambda\varphi = 0$.

4.4 Spherical representations of $\mathrm{SL}(2, \mathbf{R})$

For more details see [Bum97, Sections 2.5 & 2.6].

With a view to returning to the trace formula we wish to enumerate the irreducible unitary representations π of $\mathrm{SL}(2, \mathbf{R})$ such that $\pi^K \neq \{0\}$. Furthermore we wish to determine the action of the Casimir operator on π^K and give an expression for $\operatorname{tr}\pi(f)$ for any $f \in C_c^\infty(G//K)$.

We recall the construction of the principal series representations. We set,

$$B = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \in \mathrm{SL}(2, \mathbf{R}) \right\} = \pm AN$$

equal to the upper triangular Borel subgroup of G. Let $s \in \mathbf{C}$ and consider the character

$$\chi_s : B \to \mathbf{C}^\times : \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mapsto |a|^s.$$

Let δ_B be the character,

$$\delta_B : B \to \mathbf{C}^\times : \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mapsto |a|^2.$$

We consider the representation of G (unitarily) induced from this character. That is we consider G acting by right translation on the space of functions,

$$V_s = \left\{ \varphi : G \to \mathbf{C} : \varphi(bg) = \chi_s(b)\delta_B(b)^{\frac{1}{2}}\varphi(g) \text{ for all } b \in B \right\}.$$

We note that a function φ in this space is determined completely, by the Iwasawa decomposition, by its restriction to K. We set,

$$H_s = \left\{\varphi \in V_s : \varphi|_K \in L^2(K)\right\}.$$

We note that given $\psi \in L^2(K)$ there exists a function $\varphi \in V_s$ such that $\varphi|_K = \psi$ if and only if $\psi(k(-\theta)) = \psi(k(\theta))$ for all $k(\theta) \in K$.

We let G act on H_s by right translation.

Theorem 4.18. *The action of G on H_s yields an admissible representation of G with $\dim H_s^K = 1$. This representation has a unique irreducible subquotient π_s with $\pi_s^K \neq \{0\}$. In fact H_s is irreducible unless s is an odd integer in which case $\pi_s \cong \mathrm{Sym}^{|s|-1} V$ where V denotes the standard 2-dimensional representation of G. Up to infinitesimal equivalence every irreducible admissible representation π of G with $\pi^K \neq \{0\}$ appears in this way. Furthermore π_s and $\pi_{s'}$ are infinitesimally equivalent if and only if $s = \pm s'$.*

We note that up to infinitesimal equivalence every irreducible admissible representation of G is a subquotient of a representation induced from the Borel subgroup. One can also consider the representation induced from characters of the form,

$$B \to \mathbf{C}^\times : \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mapsto \mathrm{sgn}(a)|a|^s.$$

However since these characters are non-trivial on the element $-I \in \mathrm{SL}(2, \mathbf{R})$ none of the induced representations will have a K-fixed vector.

We are of course only interested in those representations π_s which are unitary. Suppose $\varphi_1 \in H_{s_1}$ and $\varphi_2 \in H_{s_2}$ then we can consider

$$\langle \varphi_1, \varphi_2 \rangle = \int_K \varphi_1(k)\overline{\varphi_2(k)}\ dk.$$

This is clearly a K-invariant Hermitian pairing on $H_{s_1} \times H_{s_2}$ and is G-invariant precisely when $s_2 = -\bar{s}_1$. Thus when $s \in i\mathbf{R}$ this defines a G-invariant inner product on H_s. When s is real one can construct an intertwining map $M(s) : H_s \to H_{-s}$ (strictly speaking this is only defined on the smooth vectors) defined by analytic continuation of,

$$(M(s)\varphi)(g) = \int_{-\infty}^{\infty} \varphi\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right)\ dx.$$

for $\varphi \in H_s$. (The integral is absolutely convergent only if $\Re s > 0$.) One then obtains a G-invariant Hermitian product on H_s by,

$$\langle \varphi_1, \varphi_2 \rangle = \int_K \varphi_1(k)\overline{(M(s)\varphi_2)(k)}\ dk$$

for $\varphi_1, \varphi_2 \in H_s$. This pairing is positive definite if and only if $s \in (-1, 1)$. Thus π_s is unitary if $s \in i\mathbf{R} \cup [-1, 1]$ and in fact these are all the irreducible unitary representations of G which posses a K-fixed vector.

Finally we compute the action of the Casimir element. We take $\varphi \in \pi_s^K$ which we normalize by taking $\varphi(k) = 1$. Then we have,

$$\varphi\left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} k\right) = |a|^{s+1}.$$

If we think of the function φ as a function on the upper half plane then,

$$\varphi(x+iy) = y^{\frac{s+1}{2}}.$$

From Lemma 4.17 the Casimir operator acts as $-\Delta$ on functions on the upper half plane and we have,

$$\Delta\varphi = y^2 \frac{\partial^2 \varphi}{\partial y^2} = \frac{s+1}{2}\frac{s-1}{2} y^{\frac{s+1}{2}} = \frac{s^2-1}{4}\varphi.$$

Theorem 4.19. *Let Γ be a cocompact discrete subgroup of G. We write,*

$$L^2(\Gamma\backslash G) = \bigoplus_{\pi \in \widehat{G}} m_\pi \pi.$$

For $s \in \mathbf{C}$ we set $\lambda_s = \frac{1-s^2}{4}$. Then,

$$\dim\{\varphi \in \mathcal{A}(\Gamma\backslash\mathcal{H}) : \Delta\varphi + \lambda_s\varphi = 0\} = m_{\pi_s}.$$

(Of course $m_{\pi_s} = 0$ unless $s \in i\mathbf{R} \cup [-1,1]$.)

In order to use the trace formula we need to write down $\operatorname{tr}\pi_s(f)$ for $f \in C_c^\infty(G//K)$. We note the following,

Lemma 4.20. *Let $s \in \mathbf{C}$ and $f \in C_c^\infty(G//K)$. Then,*

$$\operatorname{tr}\pi_s(f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\begin{pmatrix} e^{\frac{u}{2}} & x \\ 0 & e^{-\frac{u}{2}} \end{pmatrix} e^{\frac{us}{2}}\, du\, dx.$$

Proof. Since π_s^K is one-dimensional we have $\operatorname{tr}\pi_s(f)$ equal to the scalar by which $\pi_s(f)$ acts on a non-zero vector in π_s^K. We again take $\varphi \in \pi_s^K$ normalized so that $\varphi(I) = 1$. Then,

$$\begin{aligned}
(\pi_s(f)\varphi)(I) &= \int_G f(g)\varphi(g)\, dg \\
&= \frac{1}{2\pi}\int_0^{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(a(u)n(x)k(\theta))\varphi(a(u)n(x)k(\theta))\, du\, dx\, d\theta \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(a(u)n(x))\varphi(a(u)n(x))\, du\, dx \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\begin{pmatrix} e^{\frac{u}{2}} & e^{\frac{u}{2}}x \\ 0 & e^{-\frac{u}{2}} \end{pmatrix} e^{\frac{u(1+s)}{2}}\, du\, dx \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\begin{pmatrix} e^{\frac{u}{2}} & x \\ 0 & e^{-\frac{u}{2}} \end{pmatrix} e^{\frac{us}{2}}\, du\, dx.
\end{aligned}$$

□

4.5 Bi-K-invariant functions

We will now endeavor to explicate the trace formula for $L^2(\Gamma\backslash \mathrm{SL}(2,\mathbf{R}))$ when the test function f lies in $C_c^\infty(G//K)$. In this section we give a concrete realization of $C_c^\infty(G//K)$ and recast the terms in the trace formula in terms of this realization.

We recall that we have set,

$$A = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbf{R}_{>0} \right\}.$$

Let $f \in C_c^\infty(G//K)$, we define its Harish-Chandra transform $Hf \in C_c^\infty(A)$ defined by,

$$\begin{aligned} Hf \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} &= a \int_{\mathbf{R}} f\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) \, dx \\ &= a \int_{\mathbf{R}} f\left(\begin{pmatrix} a & ax \\ 0 & a^{-1} \end{pmatrix} \right) \, dx \\ &= \int_{\mathbf{R}} f\left(\begin{pmatrix} a & x \\ 0 & a^{-1} \end{pmatrix} \right) \, dx. \end{aligned}$$

Let $W = N_G(A)/\pm A$ be the Weyl group of A which has representatives modulo the diagonal torus,

$$W = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}.$$

We let w denote the non-trivial element in W. We note that

$$w^{-1} \begin{pmatrix} a & x \\ 0 & a^{-1} \end{pmatrix} w = \begin{pmatrix} a^{-1} & 0 \\ -x & a \end{pmatrix}$$

Thus since $w \in K$,

$$f \begin{pmatrix} a & x \\ 0 & a^{-1} \end{pmatrix} = f \begin{pmatrix} a^{-1} & 0 \\ -x & a \end{pmatrix} = f \begin{pmatrix} a^{-1} & -x \\ 0 & a \end{pmatrix},$$

by Lemma 4.8. Hence we see that $Hf \in C_c^\infty(A)^W$.

Theorem 4.21. *The map $H : C_c^\infty(G//K) \to C_c^\infty(A)^W$ is an isomorphism of algebras.*

We leave as an exercise the verification that the map is a homomorphism (see [Lan85, Chapter V, Theorem 2]), we will check that the map is bijective. Before we begin we record the following lemma,

Lemma 4.22. *Let $f \in C_c^\infty(\mathbf{R}_{>0})$ such that $f(z) = f(z^{-1})$ for all $z \in \mathbf{R}_{>0}$. Define F on $\mathbf{R}_{\geq 1}$ by,*

$$F\left(\frac{a^2 + a^{-2}}{2} \right) = f(a).$$

Then $F \in C_c^\infty(\mathbf{R}_{\geq 1})$.

Proof. By definition, for $a \geq 1$,

$$F(a) = f(a + \sqrt{a^2 - 1}).$$

Hence we see that F is smooth at every point except possibly $a = 1$. We define $g \in C_c^\infty(\mathbf{R})$ by $g(x) = f(e^x)$. Hence $g(x) = g(-x)$. Then we have,

$$F(a) = g(\log(a + \sqrt{a^2 - 1})).$$

We write $a = b + 1$ so that,

$$F(b) = g(\log(b + 1 + \sqrt{b^2 + 2b})).$$

But now,

$$(\sqrt{b} + \sqrt{b+2})^2 = 2(b+1) + 2\sqrt{b^2 + 2b},$$

hence,

$$\begin{aligned} F(b) &= g\left(\log\left(\frac{1}{2}(\sqrt{b} + \sqrt{b+2})^2\right)\right) \\ &= g\left(2\log\left(\frac{1}{\sqrt{2}}(\sqrt{b} + \sqrt{b+2})\right)\right). \end{aligned}$$

One has,

$$\log\left(\frac{1}{\sqrt{2}}(\sqrt{b} + \sqrt{b+2})\right) = \sum_{k=0}^{\infty} a_k b^{\frac{2k+1}{2}}.$$

Hence when we substitute this expression into the smooth even function g we obtain a function of b which has continuous derivatives of all orders at $b = 0$. $\square$

Lemma 4.23. *Let $f \in C_c^\infty(G//K)$ then there exists $F_f \in C_c^\infty(\mathbf{R}_{\geq 1})$ such that,*

$$f(g) = F_f\left(\frac{1}{2}\operatorname{tr} g^t g\right).$$

Conversely given $F \in C_c^\infty(\mathbf{R}_{\geq 1})$ the function f defined by $f(g) = F\left(\frac{1}{2}\operatorname{tr} g^t g\right)$ lies in $C_c^\infty(G//K)$.

Proof. We recall that by the Cartan decomposition ($G = KAK$) we can write any $g \in G$ as $g = k_1 a k_2$ and we have,

$$g^t g = k_1 a k_2{}^t k_2{}^t a^t k_1 = k_1 a^2 k_1^{-1}.$$

If we write,

$$a = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$$

with $\alpha > 0$, then

$$f(g) = f\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} = f\begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{pmatrix},$$

and hence f depends only on $\alpha + \alpha^{-1}$. We have,

$$g^t g = k_1 \begin{pmatrix} \alpha^2 & 0 \\ 0 & \alpha^{-2} \end{pmatrix} k_1^{-1}$$

Hence, since f depends only on $\alpha+\alpha^{-1}$ it also depends only on $\alpha^2+\alpha^{-2} = \mathrm{tr}(g\text{-}{}^t g)$. Thus there exists a function F_f on $[1, \infty)$ such that

$$f(g) = F_f \left(\frac{1}{2} \operatorname{tr} g^t g \right)$$

for all $g \in G$. By Lemma 4.22 we have $F_f \in C_c^\infty(\mathbf{R}_{\geq 1})$.

Conversely given $F \in C_c^\infty(\mathbf{R})$ we note that if we define $f(g) = F(1/2 \operatorname{tr} g^t g)$ then for $k_1, k_2 \in K$ and $a \in A$,

$$f(k_1 a k_2) = F \left(\frac{1}{2} \operatorname{tr}(k_1 a^2 k_1^{-1}) \right) = F \left(\frac{1}{2} \operatorname{tr} a^2 \right) = f(a).$$

$\square$

If we write,

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then $\mathrm{tr}(g^t g) = a^2 + b^2 + c^2 + d^2$. Hence we have,

$$f \begin{pmatrix} a & b \\ c & d \end{pmatrix} = F_f \left(\frac{a^2 + b^2 + c^2 + d^2}{2} \right).$$

Thus,

$$\begin{aligned} Hf \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} &= \int_{\mathbf{R}} f \left(\begin{pmatrix} a & x \\ 0 & a^{-1} \end{pmatrix} \right) \, dx \\ &= \int_{\mathbf{R}} F_f \left(\frac{a^2 + a^{-2}}{2} + \frac{x^2}{2} \right) \, dx. \end{aligned}$$

Lemma 4.24. *Let $F \in C_c^\infty(\mathbf{R}_{\geq 1})$ and define, for $a \geq 1$,*

$$H(a) = \int_{\mathbf{R}} F(a + x^2/2) \, dx.$$

Then $H \in C_c^\infty(\mathbf{R}_{\geq 1})$ and

$$F(a) = -\frac{1}{2\pi} \int_{\mathbf{R}} H'(a + x^2/2) \, dx.$$

The converse also holds.

Proof. Differentiating under the integral sign gives,

$$H'(a) = \int_{\mathbf{R}} F'(a + x^2/2)\ dx.$$

Hence,

$$\begin{aligned}\int_{\mathbf{R}} H'(a + y^2/2)\ dy &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F'(a + (x^2 + y^2)/2)\ dx\ dy \\ &= \int_0^{2\pi}\int_0^{\infty} F'(a + r^2/2) r\ dr\ d\theta \\ &= 2\pi \int_0^{\infty} F'(a + x)\ dx \\ &= -2\pi F(a).\end{aligned}$$

The converse follows in the same way. □

Proof of Theorem 4.21. We can now complete the proof of Theorem 4.21. We note that given $H \in C_c^{\infty}(A)^W$ we can write

$$H\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix} = h\left(\frac{a^2 + a^{-2}}{2}\right)$$

for a unique $h \in C_c^{\infty}(\mathbf{R}_{\geq 1})$. Then by Lemma 4.24 there exists a unique $F \in C_c^{\infty}(\mathbf{R}_{\geq 1})$ such that,

$$h(a) = \int_{\mathbf{R}} F\left(a + \frac{x^2}{2}\right)\ dx$$

for all $a \geq 1$. Hence,

$$H\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix} = Hf\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix}$$

where,

$$f(g) = F\left(\frac{1}{2}\operatorname{tr} g^t g\right).$$

Which completes the proof. □

We now want to cast all the terms in the trace formula in terms of the Harish-Chandra transform Hf. From above if we write,

$$h_f\left(\frac{a^2 + a^{-2}}{2}\right) = Hf\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix}$$

then,

$$F_f(a) = -\frac{1}{2\pi}\int_{\mathbf{R}} h_f'(a + x^2/2)\ dx.$$

To save on notation we define, for $f \in C_c^\infty(G//K)$, $g_f \in C_c^\infty(\mathbf{R})^{even}$, by

$$g_f(u) = Hf\begin{pmatrix} e^{\frac{u}{2}} & 0 \\ 0 & e^{-\frac{u}{2}} \end{pmatrix} = h_f\left(\frac{e^u + e^{-u}}{2}\right) = h_f(\cosh(u)).$$

Hence,

$$g'_f(u) = h'_f(\cosh(u))\sinh(u).$$

Thus,

$$f\begin{pmatrix} e^{\frac{u}{2}} & 0 \\ 0 & e^{-\frac{u}{2}} \end{pmatrix} = F_f(\cosh u) = -\frac{1}{2\pi}\int_{\mathbf{R}} h'_f(\cosh u + x^2/2)\ dx.$$

We begin with the geometric terms.

Theorem 4.25. *(Plancherel Formula) Let $f \in C_c^\infty(G//K)$, then*

$$f\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2\pi}\int_0^\infty \widehat{g}_f(u) u \tanh(\pi u)\ du.$$

where,

$$\widehat{g}_f(u) = \int_{\mathbf{R}} g_f(v) e^{iuv}\ dv$$

is the Fourier transform of g_f.

Proof. We wish to compute,

$$f(I) = -\frac{1}{2\pi}\int_{\mathbf{R}} h'_f(1 + x^2/2)\ dx.$$

We write $x = e^{\frac{t}{2}} - e^{-\frac{t}{2}} = 2\sinh(t/2)$. Then $dx = \cosh(t/2)\ dt$ and,

$$\begin{aligned} f(I) &= -\frac{1}{2\pi}\int_{\mathbf{R}} h'_f(1 + 2\sinh^2(t/2))\cosh(t/2)\ dt \\ &= -\frac{1}{2\pi}\int_{\mathbf{R}} h'_f(\cosh(t))\cosh(t/2)\ dt \\ &= -\frac{1}{2\pi}\int_{\mathbf{R}} g'_f(t)\frac{\cosh(t/2)}{\sinh(t)}\ dt \\ &= -\frac{1}{2\pi}\int_{\mathbf{R}} \frac{g'_f(t)}{e^{\frac{t}{2}} - e^{-\frac{t}{2}}}\ dt \end{aligned}$$

Since g_f is even g'_f is odd and we have by the Fourier inversion theorem,

$$g_f(t) = \frac{1}{\pi}\int_0^\infty \widehat{g}_f(u) e^{-iut}\ du$$

and

$$g'_f(t) = \frac{1}{i\pi}\int_0^\infty u\widehat{g}_f(u) e^{-iut}\ du.$$

Hence,

$$
\begin{aligned}
f(I) &= -\frac{1}{2\pi^2 i}\int_{-\infty}^{\infty}\int_0^{\infty} u\widehat{g}_f(u)\frac{e^{-iut}}{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}\ du\ dt \\
&= -\frac{1}{2\pi^2 i}\int_0^{\infty} u\widehat{g}_f(u)\int_{-\infty}^{\infty}\frac{e^{-iut}}{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}\ dt\ du
\end{aligned}
$$

Finally we use that,

$$
\int_{-\infty}^{\infty}\frac{e^{-iut}}{e^{\frac{t}{2}}-e^{-\frac{t}{2}}}\ dt = -\pi i\tanh(\pi u)
$$

(which can be established by moving the line of integration into the lower half plane and counting residues) to deduce that,

$$
f(I) = \frac{1}{2\pi}\int_0^{\infty}\widehat{g}_f(u)u\tanh(\pi u)\ du.
$$

□

For $\gamma \in G$ we set,

$$
I(\gamma, f) = \int_{G_\gamma\backslash G} f(g^{-1}\gamma g)\ dg.
$$

Note that if,

$$
\gamma = \begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix}
$$

with $a \neq \pm 1$ then $G_\gamma = \pm A$. We take the Lebesgue measure du on,

$$
A = \left\{\begin{pmatrix} e^{\frac{u}{2}} & 0 \\ 0 & e^{-\frac{u}{2}}\end{pmatrix} : u \in \mathbf{R}\right\},
$$

and use this to define a measure on G_γ.

Lemma 4.26. *Let $f \in C_c^\infty(G//K)$. For $u \neq 0$,*

$$
g_f(2u) = \frac{|e^u - e^{-u}|}{2} I\left(\begin{pmatrix} e^u & 0 \\ 0 & e^{-u}\end{pmatrix}, f\right),
$$

with the measure on $G_\gamma = \pm A$ chosen as above.

Proof. For the first part we note that for $u \neq 0$ if we set

$$
\gamma = \begin{pmatrix} e^u & 0 \\ 0 & e^{-u}\end{pmatrix}
$$

then $G_\gamma = \pm A$. Hence,

$$
\begin{aligned}
I(\gamma, f) &= \int_{G_\gamma\backslash G} f(g^{-1}\gamma g)\ dg \\
&= \frac{1}{2}\int_{A\backslash G} f(g^{-1}\gamma g)\ dg
\end{aligned}
$$

We have written the measure dg on G as $dg = da\ dn\ dk$ hence,

$$\begin{aligned}\int_{A\backslash G} f(g^{-1}\gamma g)\ dg &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{0}^{2\pi} f(k(\theta)^{-1}n(x)^{-1}\gamma n(x)k(\theta))\ d\theta\ dx \\ &= \int_{-\infty}^{\infty} f(n(x)^{-1}\gamma n(x))\ dx,\end{aligned}$$

since $f \in C^{\infty}(G//K)$. We write this out explicitly as,

$$\int_{-\infty}^{\infty} f\begin{pmatrix} e^{u} & (e^{u}-e^{-u})x \\ 0 & e^{-u}\end{pmatrix}\ dx.$$

Hence making the change of variables $x \mapsto (e^{u}-e^{-u})^{-1}x$ gives,

$$\begin{aligned}I(\gamma, f) &= |e^{u}-e^{-u}|\int_{-\infty}^{\infty} f\begin{pmatrix} e^{u} & x \\ 0 & e^{-u}\end{pmatrix}\ dx \\ &= |e^{u}-e^{-u}|g_f(2u).\end{aligned}$$

□

We now look at the spectral terms,

Lemma 4.27. *For $f \in C_c^{\infty}(G//K)$ and $r \in \mathbf{C}$,*

$$\widehat{g}_f(r) = \operatorname{tr}\pi_{2ir}(f).$$

Proof. From Lemma 4.20,

$$\begin{aligned}\operatorname{tr}\pi_s(f) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\begin{pmatrix} e^{\frac{u}{2}} & x \\ 0 & e^{-\frac{u}{2}}\end{pmatrix} e^{\frac{us}{2}}\ du\ dx \\ &= \int_{-\infty}^{\infty} g_f(u)e^{\frac{us}{2}}\ du \\ &= \widehat{g}_f(s/2i).\end{aligned}$$

□

4.6 Explication of the trace formula

We again take a discrete cocompact subgroup Γ of $\mathrm{SL}(2, \mathbf{R})$. We know (Lemma 4.4) that Γ doesn't contain any parabolic elements and we will assume further that Γ is hyperbolic and contains $-I$, i.e. all elements $\gamma \in \Gamma \setminus \{\pm I\}$ are hyperbolic. For example if X is a compact Riemann surface of genus at least 2 then the universal cover of X is $\mathcal{H}$, $\Gamma = \pi_1(X)$ acts on $\mathcal{H}$ with quotient X and is a hyperbolic group.

Suppose $\gamma \neq \pm I$ is an element of Γ. Then, γ is conjugate in G to an element of the form,

$$\pm \begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{pmatrix},$$

with $t > 0$. Let's assume, for the sake of convenience, that in fact,

$$\gamma = \begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{pmatrix}.$$

Then γ acts on $\mathcal{H}$ by $z \mapsto e^t z$. Its fixed points on $\widehat{\mathbf{R}}$ are 0 and ∞. The geodesic in $\mathcal{H}$ between these fixed points is the imaginary axis in $\mathcal{H}$. The distance between i and γi is equal to t, hence also the distance between the points z and γz for any points on the geodesic. The image of this geodesic in $\Gamma\backslash\mathcal{H}$ will close on itself forming a closed geodesic in $\Gamma\backslash\mathcal{H}$ (which may wind around itself several times) of length $\ell(\gamma) = t$. We note that γ_1 and γ_2 give rise to the same geodesic if and only if they are conjugate in Γ. In this way the conjugacy classes in Γ correspond to the closed geodesics in $\Gamma\backslash\mathcal{H}$. We note that for each $\gamma \in \Gamma \setminus \{\pm I\}$ the centralizer of γ in Γ is of the form,

$$\Gamma_\gamma = \left\{\pm\gamma_0^k : k \in \mathbf{Z}\right\}$$

for some element $\gamma_0 \in \Gamma$ called primitive. (Note that Γ_γ is a discrete subgroup of $G_\gamma \cong \{\pm 1\} \times \mathbf{R}$ of infinite order.) We have $\mathrm{vol}(\Gamma_\gamma\backslash G_\gamma) = \log \ell(\gamma_0)$. We also have $\mathrm{vol}(\Gamma\backslash G) = \frac{1}{2}\,\mathrm{vol}(\Gamma\backslash\mathcal{H})$.

Theorem 4.28. *Let Γ be a cocompact hyperbolic subgroup of $\mathrm{SL}(2,\mathbf{R})$ containing $-I$. Let $\{\lambda_j\}$ denote the eigenvalues (appearing with multiplicity) of Δ acting on $\mathcal{A}(\Gamma\backslash\mathcal{H})$ ordered such that $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$. We write each $\lambda_j = \frac{1}{4} + r_j^2$ with $r_j \in \mathbf{R}_{\geq 0} \cup [0, \frac{1}{2}]i$. Then, for any $g \in C_c^\infty(\mathbf{R})^{even}$,*

$$\sum_{j=0}^{\infty} \widehat{g}(r_j) = \frac{\mathrm{area}(\Gamma\backslash\mathcal{H})}{2\pi} \int_0^\infty \widehat{g}(r) r \tanh r \; dr + \frac{1}{2} \sum_{\gamma \in \{\Gamma\}} \frac{\ell(\gamma_0)}{e^{\frac{\ell(\gamma)}{2}} - e^{-\frac{\ell(\gamma)}{2}}} g(\ell(\gamma)).$$

This is just the trace formula rewritten using the results above. The trace formula gave us,

$$\sum_{\pi \in \widehat{G}} m_\pi \operatorname{tr} \pi(f) = \mathrm{vol}(\Gamma\backslash G) f(I) + \mathrm{vol}(\Gamma\backslash G) f(-I) + \sum_{\gamma \in \{\Gamma\}, \gamma \neq \pm I} \mathrm{vol}(\Gamma_\gamma\backslash G_\gamma) \int_{G_\gamma\backslash G} f(g^{-1}\gamma g)\, dg.$$

The equality of the left hand sides follows from Theorem 4.19 and Lemma 4.27, the identity elements come from the Plancherel formula, Theorem 4.25 and the non-central elements come from Lemma 4.26.

4.7 Proof of Weyl's law

Theorem 4.29. *(Weyl's law) Let Γ be a discrete cocompact hyperbolic subgroup of $\mathrm{SL}(2,\mathbf{R})$. Let $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$ denote the eigenvalues of Δ (appearing* *with multiplicity) acting on $\mathcal{A}(\Gamma\backslash\mathcal{H})$. For $T \geq 0$ we set,*

$$N_\Gamma(T) = \#\{j : \lambda_j < T\}.$$

Then,

$$N_\Gamma(T) \sim \frac{\operatorname{area}(\Gamma\backslash\mathcal{H})}{4\pi} T$$

as $T \to \infty$.

Our proof of Theorem 4.29 follows [LV07, Section 5.1].

Lemma 4.30. *Let $0 < \varepsilon < 1$. Then there exists $g \in C_c^\infty(\mathbf{R})^{even}$ such that,*

1. *$\widehat{g}(s) = \widehat{g}(-s)$ for all $s \in \mathbf{C}$,*
2. *$\widehat{g}(s)$ is real and non-negative for $s \in \mathbf{R} \cup i\mathbf{R}$,*
3. *$\widehat{g}(s) \leq 1$ for all $s \in \mathbf{R}$,*
4. *$\widehat{g}(s) \geq 1 - \varepsilon$ whenever $s \in \mathbf{R}$ with $0 \leq |s|^2 \leq 1 - \varepsilon$,*
5. *for t sufficiently small,*
$$\left| t^2 \int_0^\infty \widehat{g}(tu) u \tanh(\pi u)\, du - \frac{1}{2} \right| \leq \varepsilon,$$
6. $$\sup_{r \geq 1} (1 + r)^3 |\widehat{g}(r)| < \varepsilon.$$

Proof. Let χ be the characteristic function of $[-1, 1]$ on $\mathbf{R}$. There exists a non-empty open set of Schwartz functions ψ on $\mathbf{R}$ such that,

1. $0 \leq \psi(x) < 1$ for $|x| \leq 1$,
2. $\psi(x) > \sqrt{1 - \varepsilon}$ if $|x| \leq \sqrt{1 - \varepsilon}$,
3. $\sup_{|x| \geq 1} (1 + |x|)^3 |\psi(x)| < \varepsilon/2$, and
4. $\int_{\mathbf{R}} |\psi(x) - \chi(x)|(1 + |x|)\, dx < \varepsilon/2$.

We recall that the Fourier transform is an isomorphism of the Schwartz space onto itself and the space $C_c^\infty(\mathbf{R})$ is dense in the Schwartz space. Hence we can find ψ satisfying these conditions whose Fourier transform is compactly supported. Furthermore we can assume (by replacing ψ by $\frac{1}{2}(\psi(x) + \psi(-x))$) that $\psi(x) = \psi(-x)$.

Since the Fourier transform of ψ is compactly supported it follows that ψ extends to a holomorphic function on $\mathbf{C}$. Hence as a function on $\mathbf{C}$ we see that $\psi(s)$ depends only on s^2. Furthermore we note that for ε sufficiently small conditions 1 and 3 ensure that $\sup_{r \in \mathbf{R}} |\psi(r)| \leq 1$.

There exists $g_1 \in C_c^\infty(\mathbf{R})^{even}$ such that $\widehat{g}_1 = \psi$. We set $g = g_1 * \overline{g}_1$. Then $\widehat{g}(s) = \psi(s)\overline{\psi(\overline{s})}$ hence we obtain claim 2. Since for $s \in \mathbf{R}$ we have $g(s) = |\psi(s)|^2$ we obtain 1, 3, 4 and 6. Finally we check claim 5.

We note that $|\tanh(u)| \leq 1$ and

$$\int_0^T u \tanh(\pi u)\ du \sim \frac{T^2}{2}$$

as $T \to \infty$.

Thus,

$$\begin{aligned}
\limsup_{t\to 0} \left| t^2 \int_0^\infty \widehat{g}(tu) u \tanh(\pi u)\ du - \frac{1}{2} \right| &= \limsup_{t\to 0} \left| t^2 \int_0^\infty \widehat{g}(tu) u \tanh(\pi u)\ du - t^2 \int_0^{t^{-1}} u \tanh(\pi u)\ du \right| \\
&= \limsup_{t\to 0} \left| \int_0^\infty (t^2 \widehat{g}(tu) - t^2 \chi(ut)) u \tanh(\pi u)\ du \right| \\
&\leq \limsup_{t\to 0} \int_0^\infty |t^2 \widehat{g}(tu) - t^2 \chi(tu)|(1+u) du \\
&= \limsup_{t\to 0} \int_0^\infty |\widehat{g}(u) - \chi(u)|(t+u) du \\
&\leq \limsup_{t\to 0} \int_0^\infty |\widehat{g}(u) - \chi(u)|(1+u) du.
\end{aligned}$$

We recall that,

$$\sup_{r\in\mathbf{R}} |\psi(r)| \leq 1$$

so that $|\widehat{g}(u) - \chi(u)| \leq 2|\psi(u) - \chi(u)|$ for $u \in \mathbf{R}$. Using the fact that

$$\int_{\mathbf{R}} |\psi(x) - \chi(x)|(1+|x|)\ dx < \varepsilon/2$$

we see that for t sufficiently small,

$$\left| t^2 \int_0^\infty \widehat{g}(tu) u \tanh(\pi u)\ du - \frac{1}{2} \right| \leq \varepsilon.$$

□

Proof of Theorem 4.29. Fix $\varepsilon > 0$ and let $g \in C_c^\infty(\mathbf{R})^{even}$ given by Lemma 4.30. For $0 < t \leq 1$ let $g_t \in C_c^\infty(\mathbf{R})^{even}$ such that $\widehat{g}_t(s) = \widehat{g}(ts)$. Thus, $g_t(u) = t^{-1} g(ut^{-1})$. Then as $t \to 0$ the support of g_t shrinks to 0. Hence for t sufficiently small, the trace formula gives us,

$$\sum_{j=0}^\infty \widehat{g}_t(r_j) = \frac{\text{area}(\Gamma\backslash\mathcal{H})}{2\pi} \int_0^\infty \widehat{g}_t(u) u \tanh(\pi u)\ du,$$

which is the same as,

$$\sum_{j=0}^\infty \widehat{g}(tr_j) = \frac{\text{area}(\Gamma\backslash\mathcal{H})}{2\pi} \int_0^\infty \widehat{g}(tu) u \tanh(\pi u)\ du,$$

By construction,

$$\left| t^2 \int_0^\infty \widehat{g}_t(u) u \tanh(\pi u)\ du - \frac{1}{2} \right| \leq \varepsilon,$$

hence for t sufficiently small,

$$\left| \sum_{j=0}^{\infty} \widehat{g}(tr_j) - \frac{\text{area}(\Gamma \backslash \mathcal{H})}{4\pi t^2} \right| \leq \varepsilon.$$

We note that for but finitely many j we have $r_j \in \mathbf{R}_{\geq 0}$ (i.e. those j such that $\lambda_j \leq \frac{1}{4}$) and hence $\widehat{g}(tr_j) \leq 1$. Furthermore for all j, $\widehat{g}(tr_j)$ is real, non-negative, and uniformly bounded for $t \in (0, 1]$. From (4) of Lemma 4.30 we have,

$$1 - \varepsilon \leq \widehat{g}(s)$$

for any real number s with $0 \leq s^2 \leq 1 - \varepsilon$. Hence,

$$(1 - \varepsilon)\#\{j : 0 \leq r_j < t^{-1}\sqrt{1 - \varepsilon}\} \leq \frac{\text{area}(\Gamma \backslash \mathcal{H})}{4\pi t^2} + \varepsilon.$$

Thus for t ($t \mapsto t\sqrt{1 - \varepsilon}$) sufficiently small,

$$\begin{aligned} \#\{j : 0 \leq r_j < t^{-1}\} &\leq \frac{\text{area}(\Gamma \backslash \mathcal{H})}{4\pi t^2 (1 - \varepsilon)^2} + \frac{\varepsilon}{(1 - \varepsilon)} \\ &\leq \frac{\text{area}(\Gamma \backslash \mathcal{H})}{4\pi t^2} (1 + \varepsilon). \end{aligned}$$

On the other hand applying this together with (6) of Lemma 4.30 gives,

$$\sum_{j : r_j > t^{-1}} \widehat{g}(tr_j) \leq \sum_{n=0}^{\infty} \sum_{j : tr_j \in [2^n, 2^{n+1}]} \widehat{g}(tr_j).$$

Now by (6) we have, for j such that $tr_j \in [2^n, 2^{n+1}]$,

$$\widehat{g}(tr_j) < \frac{\varepsilon}{(1 + tr_j)^3} \leq \frac{\varepsilon}{(1 + 2^n)^3}.$$

On the other hand, from above, we see that for t sufficiently small,

$$\#\{j : tr_j \in [2^n, 2^{n+1}]\} \leq \#\{j : r_j < (t2^{-n})^{-1}\} \leq \frac{\text{area}(\Gamma \backslash \mathcal{H}) 2^{2n}}{2\pi t^2}.$$

Thus,

$$\sum_{j : tr_j \in [2^n, 2^{n+1}]} \widehat{g}(tr_j) \leq \frac{\varepsilon}{(1 + 2^n)^3} \frac{\text{area}(\Gamma \backslash \mathcal{H}) 2^{2n}}{2\pi t^2}.$$

Hence,

$$\sum_{j : r_j > t^{-1}} \widehat{g}(tr_j) \leq \frac{\varepsilon\, \text{area}(\Gamma \backslash \mathcal{H})}{2\pi t^2} \sum_{n=0}^{\infty} \frac{2^{2n}}{(1 + 2^n)^3} = C\varepsilon \frac{\text{area}(\Gamma \backslash \mathcal{H})}{4\pi t^2}$$

for some constant C which doesn't depends on ε or t (provided t is sufficiently small).

Now we use that $\widehat{g}(u) \leq 1$ for $u \in \mathbf{R}$,

$$\begin{aligned}
\#\{j : r_j < t^{-1}\} &\geq \#\{j : \lambda_j = 1/4 + r_j^2 \leq 1/4\} + \sum_{j:0\leq r_j<t^{-1}} \widehat{g}(tr_j) \\
&\geq constant + \sum_{j=0}^{\infty} \widehat{g}(tr_j) - \sum_{j:r_j>t^{-1}} \widehat{g}(tr_j) \\
&= \frac{\operatorname{area}(\Gamma\backslash\mathcal{H})}{4\pi t^2}\,(1 - C\varepsilon)\,.
\end{aligned}$$

This completes the proof of Theorem 4.29. $\square$

Remarks:

1. One could ask for a remainder term in Theorem 4.29, the correct remainder is $O(\sqrt{T})$; see [Lap08, Section 6].

2. One can also find in [LV07, Section 5.1] a proof of the general case of compact quotient.

3. More generally one can consider the following setup. Let G be a semisimple group over $\mathbf{Q}$ and let K be a maximal compact subgroup of $G(\mathbf{R})$. The space $G(\mathbf{R})/K_\infty$ is a symmetric space and one may consider quotients of it of the form $M = \Gamma\backslash G(\mathbf{R})/K$ with $\Gamma \subset G(\mathbf{Q})$ a congruence subgroup. In this case let $N(T)$ denote the number of cuspidal eigenfunctions of the Laplacian with eigenvalue $\leq T$. Sarnak conjectured,

$$N(T) \sim c(M)T^{\frac{\dim(M)}{2}}$$

for some explicit constant $c(M)$ involving the volume of M. In the case of $\mathrm{SL}(2,\mathbf{R})$ this was proven by Selberg and was his original reason for developing the trace formula. For G a split adjoint group over $\mathbf{Q}$ Weyl's law was proven by Lindenstrauss and Venkatesh [LV07]. We set $X = G(\mathbf{R})/K$ when $\Gamma\backslash G(\mathbf{R})$ is not compact one has a decomposition,

$$L^2(\Gamma\backslash G) = L^2_{cusp}(\Gamma\backslash G) \oplus L^2_{Eisen}(\Gamma\backslash G).$$

The trace formula in the noncompact setting (which we will develop in the case of GL(2) later in the course), gives an expression,

$$\operatorname{tr} R_{cusp}(f) + \operatorname{tr} R_{Eis}(f) = \operatorname{tr} R(f) = \sum_{\gamma\in\{\Gamma\}} \text{orbital integrals of } f.$$

(In the non-compact case $\operatorname{tr} R(f)$ doesn't make sense, but the trace formula in this case gives an expression as suggested above, and on the cuspidal part of the spectrum it is the trace of $R(f)$.) One can play the same game as above, and pick test functions which only see the identity orbital integral on the geometric side. However, now to prove Weyl's law one needs to know that the Eisenstein terms don't dominate the spectral side of the trace formula. In general this is a very tricky issue. Instead Lindenstrauss and Venkatesh construct test functions f which kill the Eisenstein series without losing too much of the cuspidal spectrum (it has been known for a while that one can do this, but the usual method of killing the Eisenstein series kills far too much of the cuspidal spectrum to be of use in proving Weyl's law). The geometric side of the trace formula for these test functions will not just consist of the identity orbital integrals but the analysis of the geometric side is manageable. One caveat, the work of Lindenstrauss and Venkatesh doesn't give an error term to Weyl's law.

4. Weyl's law proved by localizing the geometric side of the trace formula, can do similar thing on the spectral side, localizing to the zero eigenvalue corresponding to the constant functions. In this way obtain asymptotics for the lengths of primitive closed geodesics on $\Gamma\backslash\mathcal{H}$ which are very similar to the distribution of prime numbers; see [Iwa02].

5. Other applications of the trace formula in isolation: use pseudo-coefficients of discrete series to get formulas for dimensions of the space of automorphic forms, throw in a Hecke operator to get formulas for the trace of Hecke operators acting on spaces of modular forms.

5 A case of functoriality

The previous section described an application of the trace formula used in isolation. We computed the geometric side of the trace formula for particular test functions in order to derive information about the spectrum of the Laplace operator Δ. In this section we will study a comparison of trace formulas. The aim of this section will be to prove the following.

Let F be a number field with ring of integers $\mathcal{O}_F$ and let $\mathbf{A}_F$ denote the ring of adeles of F. We recall that, $\mathbf{A}_F = \prod'_v F_v$ with the product taken over all places v of F and with the prime denoting that $\mathbf{A}_F$ consists of all tuples (x_v) with $x_v \in F_v$ for all v and $x_v \in \mathcal{O}_{F_v}$ for almost all (non archimedean) v. If G is an algebraic group over F then we can consider its adelic points $G(\mathbf{A}_F)$. We note that we can identify $G(\mathbf{A}_F)$ with $\prod'_v G(F_v)$ with the prime denoting that we consider tuples (g_v) with $g_v \in G(F_v)$ for all v and $g_v \in G(\mathcal{O}_{F_v})$ for almost all v. For a finite set of places S of F we let $\mathbf{A}_F^S$ denote the ring of adeles away from S and $\mathbf{A}_{F,S} = \prod_{v\in S} F_v$. The group $G(\mathbf{A}_F)$ has a natural topology coming from that of $\mathbf{A}_F$, a basis of open sets for $G(\mathbf{A}_F)$ is given by open sets of the form $\prod_v U_v$ with U_v open in $G(F_v)$ for all v and with $U_v = G(\mathcal{O}_{F_v})$ for almost all v.

Let D and D' be central division algebras of prime degree p (i.e. $\dim_F D = p^2$) over a number field D. We assume that D and D' are ramified at precisely the same set of places of F which we denote by S; i.e $v \in S$ if and only if $D_v = D \otimes_F F_v$ and $D'_v = D' \otimes_F F_v$ are division algebras. We let $G = D^\times/Z(D^\times)$ and $G' = D'^\times/Z(D'^\times)$ which we view as algebraic groups over F. If $v \notin S$ we have $D_v \cong D'_v \cong M(p, F_v)$ we fix an isomorphism $\alpha_v : D_v \xrightarrow{\sim} D'_v$ which is well defined up to conjugation and which we can assume is defined over

$\mathcal{O}_{F_v}$ for almost all $v \notin S$. Patching together the α_v gives an isomorphism $\alpha : G(\mathbf{A}_F^S) \xrightarrow{\sim} G'(\mathbf{A}_F^S)$.

The group $G(F)$ embeds discretely in $G(\mathbf{A}_F^S)$ and $G'(F)$ embeds discretely in $G'(\mathbf{A}_F^S)$. We let R denote the representation of $G(\mathbf{A}_F^S)$ on $L^2(G(F)\backslash G(\mathbf{A}_F))$ and R' denote the representation of $G'(\mathbf{A}_F^S)$ on $L^2(G(F)\backslash G(\mathbf{A}_F^S))$. The goal of this section is to prove the following Theorem.

Theorem 5.1. *With notation as above the representations R and $R' \circ \alpha$ of $G(\mathbf{A}_F^S)$ on $L^2(G(F)\backslash G(\mathbf{A}_F^S))$ and $L^2(G'(F)\backslash G'(\mathbf{A}_F^S))$ are isomorphic.*

We will often choose to identify $L^2(G(F)\backslash G(\mathbf{A}_F^S))$ with the subspace of $L^2(G(F)\backslash G(\mathbf{A}_F))$ which is invariant under $G(\mathbf{A}_{F,S})$ and similarly with G'. Since D and D' are division algebras the quotients $G(F)\backslash G(\mathbf{A}_F)$ and $G'(F)\backslash G'(\mathbf{A}_F)$ are compact.

Equivalently, suppose we decompose,

$$L^2(G(F)\backslash G(\mathbf{A}_F)) = \bigoplus_{\pi \in \widehat{G}} m_\pi \pi$$

and

$$L^2(G'(F)\backslash G'(\mathbf{A}_F)) = \bigoplus_{\pi' \in \widehat{G'}} m_{\pi'} \pi'.$$

Let π be an irreducible admissible representation of $G(\mathbf{A}_F)$. We can write $\pi = \otimes_v \pi_v$ with π_v an irreducible admissible representation of $G(F_v)$ which is unramified for almost all v. Suppose π_v is the trivial representation for all $v \in S$. We define $\pi' = \otimes_v \pi'_v$ in the following way. For a place $v \in S$ we set $\pi'_v = \mathbf{1}_v$, the trivial representation of $G'(F_v)$, and for $v \notin S$ we transport π_v to a representation π'_v of $G'(F_v)$ via the isomorphism $G(F_v) \cong G'(F_v)$. Then Theorem 5.1 tells us that $m_\pi = m_{\pi'}$.

We note that if $p = 2$ then if D and D' are ramified at the same set of places then $D \cong D'$, hence G and G' are isomorphic over F and the statement of the Theorem is trivial. If $p > 2$ then the set of ramification does not uniquely determine the division algebra.

Here's a concrete consequence of Theorem 5.1. Suppose $F = \mathbf{Q}$ and $p > 2$ so that the set S does not contain the real place of $\mathbf{Q}$. Let K be a compact open subgroup of $G(\mathbf{A}_{\mathbf{Q}}^\infty)$ of the form,

$$K = \prod_{p \notin S} K_p \times \prod_{p \in S} G(\mathbf{Q}_p),$$

with K_p a compact open subgroup of $G(\mathbf{Q}_p)$ for all $p \notin S$ and with $K_p = G(\mathbf{Z}_p)$ for almost all p. We let K' denote the corresponding compact open subgroup of $G'(\mathbf{A}_{\mathbf{Q}}^\infty)$.

By strong approximation $G(\mathbf{A}) = G(\mathbf{Q})G(\mathbf{R})K$ and hence,

$$G(\mathbf{Q})\backslash G(\mathbf{A})/K \cong \Gamma\backslash G(\mathbf{R}) \cong \Gamma\backslash \mathrm{PGL}(p, \mathbf{R})$$

where $\Gamma = K \cap G(\mathbf{Q})$ and similarly

$$G'(\mathbf{Q})\backslash G'(\mathbf{A})/K' \cong \Gamma'\backslash G'(\mathbf{R}) \cong \Gamma'\backslash \mathrm{PGL}(p, \mathbf{R})$$

where $\Gamma' = K' \cap G'(\mathbf{Q})$. The Theorem tells us that the representations of $\mathrm{PGL}(p, \mathbf{R})$ on

$$L^2(G(\mathbf{Q})\backslash G(\mathbf{A})/K) = L^2(G(\mathbf{Q})\backslash G(\mathbf{A}))^K \cong L^2(\Gamma\backslash \mathrm{PGL}(p, \mathbf{R}))$$

and

$$L^2(G'(\mathbf{Q})\backslash G'(\mathbf{A})/K') = L^2(G'(\mathbf{Q})\backslash G'(\mathbf{A}))^{K'} \cong L^2(\Gamma'\backslash \mathrm{PGL}(p, \mathbf{R}))$$

are isomorphic.

Let K_∞ be a maximal compact subgroup of $G(\mathbf{R})$ and let K'_∞ denote the corresponding subgroup of $G'(\mathbf{R})$. We let $X = G(\mathbf{R})/K_\infty$ and $X' = G(\mathbf{R})/K'_\infty$, these are symmetric spaces. The spaces $\Gamma\backslash X$ and $\Gamma\backslash X'$ are compact Riemannian manifolds. The eigenvalues of the Laplace operator acting on these spaces are related to the representations appearing in $L^2(\Gamma\backslash G(\mathbf{R}))$ and $L^2(\Gamma'\backslash G'(\mathbf{R}))$ as in the case of $p = 2$ discussed in Section 4. Hence as a consequence of the Theorem we see that these spaces are isospectral, i.e. they have the same eigenvalues for the Laplace operator, however they won't (in general) be isomorphic.

How might we prove Theorem 5.1?

The quotients $G(F)\backslash G(\mathbf{A})$ and $G'(F)\backslash G'(\mathbf{A})$ are compact. So if we decompose,

$$L^2(G(F)\backslash G(\mathbf{A})) = \bigoplus_{\pi \in \widehat{G(\mathbf{A})}} m_\pi \pi$$

then the trace formula tells us that, for $f \in C_c^\infty(G(\mathbf{A}))$ (at least if f is the convolution of two continuous functions of compact support),

$$\sum_{\pi \in \widehat{G(\mathbf{A})}} m_\pi \operatorname{tr} \pi(f) = \operatorname{tr} R(f) = \sum_{\gamma \in \Gamma(G(F))} \mathrm{vol}(G_\gamma(F)\backslash G_\gamma(\mathbf{A})) \int_{G_\gamma(\mathbf{A})\backslash G(\mathbf{A})} f(g^{-1}\gamma g)\, dg$$

and for $f' \in C_c^\infty(G(\mathbf{A}))$ (at least if f is the convolution of two continuous functions),

$$\sum_{\pi' \in \widehat{G'(\mathbf{A})}} m_{\pi'} \operatorname{tr} \pi'(f') = \operatorname{tr} R'(f') = \sum_{\gamma' \in \Gamma(G'(F))} \mathrm{vol}(G'_\gamma(F)\backslash G'_\gamma(\mathbf{A})) \int_{G'_\gamma(\mathbf{A})\backslash G'(\mathbf{A})} f'(g^{-1}\gamma' g)\, dg.$$

To ease notation we'll write,

$$I(\gamma, f) = \int_{G_\gamma(\mathbf{A})\backslash G(\mathbf{A})} f(g^{-1}\gamma g)\, dg$$

and

$$I(\gamma', f') = \int_{G'_\gamma(\mathbf{A})\backslash G'(\mathbf{A})} f'(g^{-1}\gamma' g)\, dg.$$

We want to be able to compare the two geometric sides of the trace formulas for G and G'. In order to do this we'll need to write down a bijection,

$$\alpha : \Gamma(G(F)) \xrightarrow{\sim} \Gamma(G'(F))$$

and a map on functions,

$$\beta : C_c^\infty(G(\mathbf{A})) \to C_c^\infty(G'(\mathbf{A}))$$

such that for all $\gamma \in G(F)$ and $f \in C_c^\infty(G(\mathbf{A}))$,

$$\mathrm{vol}(G_\gamma(F)\backslash G_\gamma(\mathbf{A})) = \mathrm{vol}(G'_{\alpha(\gamma)}(F)\backslash G'_{\alpha(\gamma)}(\mathbf{A}))$$

and

$$I(\gamma, f) = I(\alpha(\gamma), \beta(f)).$$

Then we'll get, via the trace formula that,

$$\sum_{\pi \in \widehat{G}} m_\pi \operatorname{tr} \pi(f) = \operatorname{tr} R(f) = \operatorname{tr} R'(\beta(f)) = \sum_{\pi' \in \widehat{G'}} m_{\pi'} \operatorname{tr} \pi'(\beta(f)),$$

out of which we'll hope to be able to establish Theorem 5.1.

Before we start the proof of Theorem 5.1 we'll prove the following Theorem which tells us that an identity of the form $\operatorname{tr} R(f) = \operatorname{tr} R'(\beta(f))$ is sufficient in order to be able to extract data about the representations R and R'.

Theorem 5.2. *Let G be a topological group and let π_1 and π_2 be unitary representations of G. Assume that $\pi_1(f)$ and $\pi_2(f)$ are Hilbert-Schmidt operators for all $f \in C_c(G)$. For $f \in C_c(G)$ let $f^* \in C_c(G)$ be defined by $f^*(g) = \overline{f(g^{-1})}$. Then π_2 is isomorphic to a subrepresentation of π_1 if and only if,*

$$\operatorname{tr} \pi_2(f * f^*) \leq \operatorname{tr} \pi_1(f * f^*)$$

for all $f \in C_c(G)$.

Clearly if π_1 is isomorphic to a subrepresentation of π_2 then we have such an inequality. We need to prove the converse. We begin with a Lemma.

Lemma 5.3. *Suppose $H = \bigoplus H_\alpha$ is a unitary representation of G with each H_α irreducible. Let $x_\alpha \in H_\alpha$ be a given vector such that*

$$\sum_\alpha \|\pi_\alpha(f) x_\alpha\|^2$$

is finite for all $f \in C_c(G)$. Let (τ, H_τ) be an irreducible unitary representation of G which is not isomorphic to any (π_α, H_α). Then for each $y \in H_\tau$ and any $\varepsilon > 0$ there exists $f \in C_c(G)$ such that,

$$\sum_\alpha \|\pi_i(f) x_\alpha\|^2 < \varepsilon \|\tau(f) y\|^2.$$

Proof. Suppose this is not the case. Then there exists $\varepsilon > 0$ such that for every $f \in C_c(G)$ we have,

$$\bigoplus_\alpha \|\pi_i(f)x_\alpha\|^2 \geq \varepsilon \|\tau(f)y\|^2.$$

Let H' be closure in H of,

$$\operatorname{span}\left\{ \bigoplus_\alpha \pi_\alpha(f)x_\alpha : f \in C_c(G) \right\}.$$

We define a map A' from H' to H_τ by,

$$\bigoplus_\alpha \pi_\alpha(f)x_\alpha \mapsto \tau(f)y.$$

We note that this map is well defined (if $\pi_\alpha(f)x_\alpha = 0$ for all α then $\tau(f)y = 0$ by the inequality above) and originally only defined on a dense subset of H' but since it is bounded it extends to a map from H' to H_τ. Furthermore it is a G-map and is non-zero. Let A be the map from H to H_τ which is A' on H' and zero on its orthogonal complement. Since each π_α is irreducible and not isomorphic to τ so $A|_{H_\alpha}$ must be zero for each α yielding a contradiction. $\square$

Proof of Theorem 5.2. We now prove the Theorem. Since $\pi_i(f)$ is Hilbert-Schmidt for all $f \in C_c(G)$ we can write,

$$\pi_1 = \bigoplus_\alpha \pi_{1,\alpha} \quad \pi_2 = \bigoplus_\beta \pi_{2,\beta}.$$

(One can apply the same proof as in Theorem 3.16 which proved that $L^2(\Gamma\backslash G)$ decomposes discretely when the quotient $\Gamma\backslash G$ is compact.) It suffices to prove the Theorem when no $\pi_{2,\beta}$ is isomorphic to a $\pi_{1,\alpha}$ in which case we need to show that π_2 is zero. Equivalently we may as well assume that π_2 is irreducible and derive a contradiction. We choose a function $h \in C_c(G)$ such that $\pi_2(h)$ is non-zero and set $f = h * h^*$. Then $\pi_2(f) = \pi_2(h)\pi_2(h)^*$ is a non-zero compact self adjoint map $\pi_2 \to \pi_2$ and so has only non-negative eigenvalues. After scaling we may as well assume that the largest eigenvalue is 1; cf Proposition 3.13. We choose a unit vector $x_0 \in \pi_2$ such that $\pi_2(f)x_0 = x_0$. On the other hand $\pi_1(f)$ is also a compact self-adjoint map and we let λ denote the largest eigenvalue of it. Since $\operatorname{tr}\pi_1(f) \geq \operatorname{tr}\pi_2(f)$ we have $\lambda > 0$. For each α let $\{x_{\alpha,\gamma}\}$ be a basis of eigenvectors for $\pi_{1,\alpha}(f)$. Then we have,

$$\sum_{\alpha,\gamma} \|\pi_1(f')x_{\alpha,\gamma}\|^2 < \infty.$$

for all $f' \in C_c(G)$. We apply the previous Lemma to the representation $\oplus_{\alpha,\gamma}\pi_\alpha$ to deduce the existence of $f_1 \in C_c(G)$ such that,

$$\sum_{\alpha,\gamma} \|\pi_{1,\alpha}(f_1)x_{\alpha,\gamma}\|^2 < \frac{1}{2\lambda^2}\|\pi_2(f_1)x_0\|^2.$$

Now we compute

$$\operatorname{tr}(\pi_1(f_1 * f)\pi_1(f_1 * f)^*) = \operatorname{tr}(\pi_1(f_1 * f) * (f_1 * f)^*).$$

This can be computed as,

$$\begin{aligned}
\sum_{\alpha,\gamma} \|\pi_{1,\alpha}(f_1)\pi_{1,\alpha}(f)x_{\alpha,\gamma}|\|^2 &\leq \lambda^2 \sum_{\alpha,\gamma} \|\pi_{1,\alpha}(f_1)x_{\alpha,\gamma}\|^2 \\
&< \frac{1}{2}\|\pi_2(f_1)x_0\|^2 \\
&= \frac{1}{2}\|\pi_2(f_1)\pi_2(f)x_0\|^2 \\
&= \frac{1}{2}\|\pi_2(f_1 * f)x_0\|^2 \\
&\leq \frac{1}{2}\operatorname{tr}(\pi_2(f_1 * f)\pi_2(f_1 * f)^*).
\end{aligned}$$

Hence we have,

$$\operatorname{tr}(\pi_1(f_1 * f)\pi_1(f_1 * f)^*) \leq \frac{1}{2}\operatorname{tr}(\pi_2(f_1 * f)\pi_2(f_1 * f)),$$

which yields a contradiction. □

Corollary 5.4. *Let G be a topological group and let π_1 and π_2 be unitary representations of G as in Theorem 5.2. Then π_1 and π_2 are isomorphic if and only if,*

$$\operatorname{tr}\pi_1(f * f^*) = \operatorname{tr}\pi_2(f * f^*)$$

for all $f \in C_c(G)$.

5.1 Central simple algebras

The goal of this section will be, in the setting of the previous section, to describe a bijection $\Gamma(G(F)) \leftrightarrow \Gamma(G'(F))$. For more details including proofs see [Mil08, Chapter IV].

Definition 5.5. *A simple algebra over a field F is a finite dimensional F-algebra A which contains no non-trivial two sided ideals. A is called central if $Z(D) = F$.*

For us F will either be a number field or a local field. The obvious example of a central simple algebra is $M_n(F)$ the algebra of $n \times n$ matrices over F. More generally one can take a division algebra D over F with $Z(D) = F$ and consider $M_n(D)$. In fact any central simple algebra over F is isomorphic to $M_n(D)$ for some integer n and a division algebra D. (Take a minimal left ideal S of A and consider the map $A \mapsto \operatorname{End}_F(S)$. Let D be the centralizer of A in $\operatorname{End}_F(S)$. The centralizer of D in $\operatorname{End}_F(S)$ is A, i.e. $A = \operatorname{End}_D(S)$. Schur's lemma implies that D is a division algebra and therefore S is a free D-module, say $S \cong D^n$, hence $A \cong \mathrm{End}_D(S) \cong M_n(D^{opp})$.)

Let D be a central simple algebra over F, the degree of D is defined as $\sqrt{\dim_F D}$. Note that $D \otimes_F \overline{F}$ is a central simple algebra over $\overline{F}$ and hence isomorphic to $M_n(\overline{F})$ for some n. Thus,

$$\dim_F D = \dim_{\overline{F}} D \otimes_F \overline{F} = \dim_{\overline{F}} M_n(\overline{F}) = n^2$$

hence the degree is an integer.

We recall,

Theorem 5.6. *(Skolem-Noether theorem) Let $f, g : A \to B$ be homomorphisms from an F-algebra A to an F-algebra B. If A is simple and B is central simple then there exists an invertible element $b \in B$ such that $f(a) = bg(a)b^{-1}$ for all $a \in A$.*

Proof. First consider the case that $B = M_n(F)$. Using f and g we may consider F^n as an A-module in two ways. Any two modules over A of the same dimension are isomorphic, but an isomorphism $F^n \to F^n$ is simply an invertible element of $M_n(F)$. In general consider the induced maps, $f \otimes 1$ and $g \otimes 1$ from $A \otimes_F B^{opp}$ to $B \otimes_F B^{opp}$. Then $B \otimes_F B^{opp}$ is a matrix algebra. Hence from the first part of the proof $f \otimes 1$ and $g \otimes 1$ are conjugate by an element of $b \in B \otimes_F B^{opp}$ from which it follows that f and g are conjugate in B.s □

We recall the definition of the Brauer group $\mathrm{Br}(F)$. Let A and B be cental simple algebras over F. We define an equivalence relation on central simple algebras by $A \sim B$ if and only if there exist integers m and n such that $A \otimes_F M_n(F) \cong B \otimes_F M_m(F)$ for some integers m and n. For equivalence classes $[A]$ and $[B]$ we define,

$$[A].[B] = [A \otimes_F B].$$

We note that $[A]^{-1} = [A^{opp}]$, where A^{opp} is the algebra with the same underlying set as A but with the multiplication reversed. We note that the class $[M_n(F)]$ (for any n) is the identity in $\mathrm{Br}(F)$.

We recall that the Brauer group has a natural interpretation in terms of Galois cohomology. We have $A \otimes_F \overline{F} \cong M(n, \overline{F})$. Having fixed such an isomorphism we have two actions of the absolute Galois group $\mathrm{Gal}(\overline{F}/F)$ on $M(F, \overline{F})$ coming from the two Galois actions. By the Skolem-Noether theorem these two actions differ by conjugation. Thus for each $\sigma \in \mathrm{Gal}(\overline{F}/F)$ there exists $g_\sigma \in \mathrm{GL}(n, \overline{F})$ such that,

$$\sigma_A(g) = g_\sigma \sigma(g) g_\sigma^{-1}$$

for all $g \in M(n, \overline{F})$, where σ_A denotes the action on $M(n, \overline{F})$ coming from the isomorphism. In this way we get a well defined element $(g_\sigma) \in H^1(F, \mathrm{PGL}(n))$. The short exact sequence,

$$1 \to \mathrm{GL}(1) \to \mathrm{GL}(n) \to \mathrm{PGL}(n) \to 1$$

gives a map $H^1(F, \mathrm{PGL}(n, \overline{F})) \to H^2(F, \overline{F}^\times)$. This gives an identification of the Brauer group $\mathrm{Br}(F)$ with $H^2(F, \overline{F}^\times)$.

For a local field F we have,

$$\mathrm{Br}(F) \cong \begin{cases} \mathbf{Q}/\mathbf{Z}, & \text{if } F \text{ is non-archimedean;} \\ \frac{1}{2}\mathbf{Z}/\mathbf{Z}, & \text{if } F = \mathbf{R}; \\ \mathbf{Z}/\mathbf{Z}, & \text{if } F = \mathbf{C}. \end{cases}$$

Suppose F is a non-archimedean local field and D is a central simple algebra of degree n. Then in $\mathrm{Br}(F)$, $[A] = \frac{c}{n}$ for some c with $0 \leq c \leq n-1$. We write $\frac{c}{n} = \frac{d}{e}$ in lowest terms with $n = ef$. Then $A \cong M_f(\Delta)$ with Δ a central division algebra of rank d. Hence, up to isomorphism, there are $\varphi(n)$ non-isomorphic central division algebras of rank n over F. If $F = \mathbf{R}$ the non-trivial element of $\mathrm{Br}(\mathbf{R})$ is the usual Hamilton quaternion algebra.

If F is a number field then the Brauer group of F fits into a short exact sequence,

$$0 \to \mathrm{Br}(F) \to \bigoplus_v \mathrm{Br}(F_v) \to \mathbf{Q}/\mathbf{Z} \to 0.$$

The first map comes from the maps,

$$\mathrm{Br}(F) \to \mathrm{Br}(F_v) : [A] \mapsto [A \otimes_F F_v]$$

for each place v of F and the second is given by summation inside $\mathbf{Q}/\mathbf{Z}$.

Suppose now D is a central division algebra over F. If we take $\gamma \in D$ then $F[\gamma]$ is a field extension of F (it's an integral domain of finite degree over F). In order to understand conjugacy classes in $D^\times$ we need to understand which fields embed in D. First we note that if D is a central simple algebra of degree n over F then if $L \hookrightarrow D$ then $[L : F]$ divides n. This is a general fact, however we're only interested in the case that $n = p$ is prime. In this case it's obvious, since D is a vector space over L that $[L : F]$ must divide p^2. But if $[L : F] = p^2$ then $D = L$ which is commutative and hence $Z(D) = L$ yielding a contradiction.

Lemma 5.7. *Let L/F be an extension of degree n which embeds in a central simple algebra D of degree n over F. Then the centralizer of L in D is equal to L.*

Proof. If we let $C(L)$ denote the centralizer of L then we have,

$$n^2 = [D : F] = [L : F][C(L) : F] = n[C(L) : F]$$

by [Mil08, Chapter IV, Theorem 3.1]. Hence $[C(L) : F] = n$ and since $C(L) \supset L$ so we must have $C(L) = L$. □

Theorem 5.8. *Let D be a central division algebra over F of degree n. Then an extension L of degree n over F embeds in D if and only if D splits over L (i.e. $D \otimes_F L \cong M_n(L)$).*

Proof. See [Mil08, Chapter IV, Corollary 3.7]. □

Theorem 5.9. *Let A be a central simple algebra over a number field F of degree n. Then an extension L/F of degree n embeds in A if and only if $L_v = L \otimes_F F_v \hookrightarrow A_v$ for all places v of F.*

Proof. Use the injectivity of $\mathrm{Br}(F) \hookrightarrow \bigoplus_v \mathrm{Br}(F_v)$ together with Theorem 5.8 applied to F and F_v. □

Theorem 5.10. *Let A be a central simple algebra over a local field F of degree n. Then every field extension L/F of degree n embeds in D.*

Proof. We may as well assume that F is p-adic. Under the identification of $\mathrm{Br}(F)$ and $\mathrm{Br}(L)$ with $\mathbf{Q}/\mathbf{Z}$ the map $\mathrm{Br}(F) \mapsto \mathrm{Br}(L) : [A] \mapsto [A \otimes_F L]$ is multiplication by n. Since the class of a central simple algebra of degree n over F in $\mathrm{Br}(F)$ is of the form $\frac{a}{n}$ it follows that $[A]$ lies in the kernel of this map. □

Suppose now D is a division algebra of prime degree p over a number field F. Let S denote the set of places where D is ramified, i.e. the places v of F such that D_v is a division algebra. Let L be an extension of degree p over F. Then $L \hookrightarrow D$ if and only if $L_v \hookrightarrow D_v$ for all v by the previous Theorem. Clearly if $v \notin S$ then $D_v \cong M(p, F_v)$ and $L_v \hookrightarrow D_v$. On the other hand if $v \in S$ then D_v is a division algebra and $L_v = \prod_{w|v} L_w$ embeds in D_v if and only if L_v is a field. Thus, an extension L of degree p over F embeds in D if and only if the extension is inert at all places of F in S. Let $X(S,p)$ denote a set of representatives for the isomorphism classes of field extensions of F which are of degree p and inert at the places inside S.

Let γ be a non-zero element of D. Then γ is invertible and generates a subfield $F[\gamma]$ of D. If γ is non-central then the extension $F[\gamma]$ will be of degree p. Suppose now that $L \in X(S,p)$ so L is an extension of F of degree p which embeds in D. Then any two embeddings of L into D differ by conjugation by an element of $D^\times$ by the Noether-Skolem Theorem (Theorem 5.6). Hence each non-zero $\gamma \in L$ gives a well defined conjugacy class in $D^\times$. Furthermore two elements $\gamma_1, \gamma_2 \in L$ yield the same conjugacy class in $D^\times$ if and only if there is an automorphism of L over F mapping γ_1 to γ_2. Thus we get a well defined bijection,

$$\coprod_{L \in X(S,p)} \{L \backslash F \text{ modulo } F\text{-automorphisms}\} \longleftrightarrow \{ \text{ conjugacy classes of non-central elements in } D^\times \ \}.$$

Of course,

$$F^\times \longleftrightarrow \{ \text{ conjugacy classes of central elements in } D^\times \ \}.$$

Suppose now that D and D' are division algebras of prime degree p over F which ramify at the same set of places S of F. Then we have a well defined bijection,

$$\beta : \Gamma(D^\times) \longleftrightarrow \Gamma(D'^\times).$$

Given an element $\gamma \in D^\times$ we consider the field extension $F[\gamma]$ of F. We pick an embedding $\iota : F[\gamma] \hookrightarrow D'$. Then $\beta(\gamma)$ is defined to be the conjugacy class of $\iota(\gamma)$. This is well defined.

For each place $v \notin S$ we have $D_v \cong D'_v \cong M(p, F_v)$. We fix an isomorphism $\alpha_v : D_v \xrightarrow{\sim} D'_v$ which is unique up to conjugation by the Skolem-Noether theorem. Patching the α_v together gives us an isomorphism $\alpha : D(\mathbf{A}_F^S) \xrightarrow{\sim} D'(\mathbf{A}_F^S)$. Suppose that $\gamma \in D^\times$ then we obtain a conjugacy class $\beta(\gamma)$ in $D'^\times$.

Lemma 5.11. *Let $\gamma \in D^\times$. Then $\alpha(\gamma) \in D'^\times(\mathbf{A}_F^S)$ is conjugate in $D'^\times(\mathbf{A}_F^S)$ to $\beta(\gamma)$.*

Proof. By the Skolem-Noether theorem we see that $\alpha(\gamma)$ and $\beta(\gamma)$ when viewed as elements of $D'^\times(\mathbf{A}_F^S)$ are conjugate under $\prod_{v \notin S} D'^\times(F_v)$. In order to deduce that they are actually conjugate in $D' \times (\mathbf{A}_F^S)$ one can use [Kot86, Proposition 7.2] which applies for almost all $v \notin S$. □

5.2 Facts about reductive groups

Let F be a field and G/F a reductive algebraic group over F. (G is reductive if the unipotent radical $R_u(G)$, the maximal connected unipotent normal subgroup of G, is trivial.) For an algebraic group G we let $X(G)$ denote the group of characters of G, i.e. homomorphisms of algebraic groups $\chi : G \to \mathrm{GL}(1)$. Let $X(G)_F$ denote the group of characters defined over F.

Definition 5.12. *Let T be a torus defined over F. We say that T is F-anisotropic if $X(T)_F = \{0\}$, i.e. the only character of T defined over F is the trivial character. Let G be a reductive algebraic group over F. We say G is F-anisotropic over if every (maximal) torus in G defined over F is F-anisotropic.*

We note that if G is F-anisotropic then $G(F)$ contains only semisimple elements.

Lemma 5.13. *Let D be a central division algebra over F. The group $G = D^\times / Z(D^\times)$ is F-anisotropic.*

Proof. Let n denote the degree of D. The maximal tori in $D^\times$ correspond to the degree n field extensions L of F which embed in D. Let L be a field extension of F of degree n and let T be the corresponding torus. Then,

$$T(\overline{F}) = (L \otimes_F \overline{F})^\times \cong \prod_{\sigma \in X} \overline{F}^\times$$

with the product taken over the embeddings $\sigma : L \hookrightarrow \overline{F}$ over F. The action of $\mathrm{Gal}(\overline{F}/F)$ is by permuting the factors, i.e. if $\tau \in \mathrm{Gal}(\overline{F}/F)$ and $(x_\sigma) \in \prod_\sigma \overline{F}^\times$ then $\tau(x_\sigma) = (x_{\tau\sigma})$. The characters of $T(\overline{F})$ are of the form,

$$(x_\sigma) \mapsto \prod_\sigma x_\sigma^{n_\sigma}$$

with $n_\sigma \in \mathbf{Z}$. The characters of T defined over F are precisely those which are left invariant under $\mathrm{Gal}(\overline{F}/F)$. Since the absolute Galois group permutes the

$\overline{F}^{\times}$ factors we see that such a character is defined over F if and only if $n_\sigma = n$ for all σ.

We now consider the group $G = D^\times/Z(D^\times)$ the maximal tori are of the form $L^\times/F^\times$ with L/F a field extension of degree n. Then we have,

$$T(\overline{F}) \cong \prod_{\sigma \in X} \overline{F}^{\times}/\Delta\overline{F}^{\times}$$

where $\Delta\overline{F}^{\times}$ denotes the image of $\overline{F}^{\times}$ diagonally embedded in the product. We note that the characters of $T(\overline{F})$ are as before with the condition that $\sum_{\sigma\in X} n_\sigma = 0$. Hence we see that the only character defined over F is the trivial character. $\square$

Suppose now F is a local field. Let G be an algebraic group over F. The group of F-points $G(F)$ inherits a topology from that of F. One can realize G as a closed subgroup of $\mathrm{GL}(n)$ and then $G(F)$ inherits the topology from that of $\mathrm{GL}(n, F)$.

Theorem 5.14. *([PR94, Theorem 3.1]) Let F be a local field and G a reductive algebraic group defined over F. Then $G(F)$ is compact if and only if G is F-anisotropic.*

We leave the proof to the cited reference. We give the following example of an anisotropic group. Let D be a division algebra of degree 2 over a local field F. We can realize D in the following way. Let E/F be a quadratic extension and let $\varepsilon \in F^\times \setminus N_{E/F}E^\times$. Then,

$$D \cong \left\{ \begin{pmatrix} \alpha & \varepsilon\beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} : \alpha, \beta \in E \right\}.$$

One can then readily check that the group $G = D^\times/Z(D^\times)$ is compact.

Suppose now F is a number field. Let G be an algebraic group over F. The adelic points of G, $G(\mathbf{A}_F)$ inherits a topology from that of $\mathbf{A}_F$. The group $G(\mathbf{A}_F)$ is a locally compact topological group and hence has a Haar measure, if G is reductive then $G(\mathbf{A}_F)$ is unimodular. The group of F-rational points $G(F)$ embeds discretely in $G(\mathbf{A}_F)$ and so the quotient $G(F)\backslash G(\mathbf{A}_F)$ possesses a right $G(\mathbf{A}_F)$ invariant measure, unique up to scaling.

Theorem 5.15. *([PR94, Theorem 5.5]) Let F be a number field and G a reductive algebraic group defined over F. Then $G(F)\backslash G(\mathbf{A}_F)$ is compact if and only if G is F-anisotropic and $G(F)\backslash G(\mathbf{A}_F)$ has finite volume if and only if the center of G is F-anisotropic.*

We leave the proof to the cited reference which relies on reduction theory to construct a fundamental domain for $G(F)\backslash G(\mathbf{A}_F)$. When $G = \mathrm{GL}(2)$ we will touch on this later in the term.

In general if G is reductive and K is a compact open subgroup of $G(\mathbf{A}_{F,fin})$ then

$$G(\mathbf{A}_F) = \coprod_{i=1}^{k} G(F)x_i K G(\mathbf{A}_{F,\infty})$$

with $x_i \in G(\mathbf{A}_{F,fin})$ and $k < \infty$; see [PR94, Theorem 5.1]. (When $G = \mathrm{GL}(1)$ this is equivalent to the statement that the class group of F is finite.) Hence, projection onto $G(\mathbf{A}_{F,\infty})$ yields a homeomorphism,

$$G(F)\backslash G(\mathbf{A}_F) = \coprod_{i=1} \Gamma_i \backslash G(\mathbf{A}_{F,\infty})$$

where $\Gamma_i = G(F) \cap x_i K x_i^{-1}$. Hence $G(F)\backslash G(\mathbf{A}_F)$ is compact if and only if $\Gamma_i\backslash G(\mathbf{A}_{F,\infty})$ is compact for all i. If G is F-anisotropic then $G(F)$ contains only semisimple elements. In the case of $G = \mathrm{SL}(2)$ we showed (Lemma 4.4) that a finite volume quotient $\Gamma\backslash \mathrm{SL}(2, \mathbf{R})$ is compact if and only if Γ doesn't contain parabolic (i.e. unipotent) elements.

5.3 Comparison of trace formulas

We now fix a number field F and denote by $\mathbf{A}$ the ring of adeles of F. Again we take D and D' to be division algebras of prime degree p over a number field F such that $S = \mathrm{Ram}(D) = \mathrm{Ram}(D')$. We let $G = D^\times/Z(D^\times)$ and $G' = D'^\times/Z(D'^\times)$ which we view as algebraic groups over F. For each $v \notin S$ we fix an isomorphism $\alpha_v : D_v \xrightarrow{\sim} D'_v$ which is unique up to conjugation by the Skolem-Noether theorem. This clearly yields an isomorphism $\alpha_v : G(F_v) \xrightarrow{\sim} G'(F_v)$ which we denote with the same notation. Furthermore we can ensure, after conjugating the map α_v, that for almost all v the map α_v maps $G(\mathcal{O}_{F_v})$ into $G'(\mathcal{O}_{F_v})$. Thus taking together all the α_v we get an isomorphism,

$$\alpha : G(\mathbf{A}^S) \xrightarrow{\sim} G'(\mathbf{A}^S)$$

which is well defined up to conjugation.

The trace formula involves choices of measures. For each place $v \notin S$ we fix a Haar measure dg_v on $G(F_v)$ and transport it to a measure on $G'(F_v)$ via the isomorphism $\alpha_v : G(F_v) \to G'(F_v)$. We assume that the measures are chosen so that $G(\mathcal{O}_{F_v})$ is given volume one for almost all v. At the places inside S we take the Haar measures on $G(F_v)$ and $G'(F_v)$ to give these groups volumes one (which is okay since by Theorem 5.14 they are compact). We take the product of these measures to give the Haar measures on $G(\mathbf{A})$ and $G'(\mathbf{A})$. Let $\gamma \in G(F)$. In writing down the trace formula we also need to choose measures on $G_\gamma(\mathbf{A}_F)$. If $\gamma = e$ then $G_\gamma = G$ and we have already chosen our measures. If $\gamma \in G(F)$ is non-trivial then $F[\gamma]$ is an extension of F of degree p inside D and $G_\gamma = F[\gamma]^\times/F^\times$ by Lemma 5.7. We fix a Haar measure on $G_\gamma(\mathbf{A})$. We recall that $\beta(\gamma)$ is defined up to conjugacy by picking an embedding $F[\gamma] \hookrightarrow D'$ which gives an isomorphism $G_\gamma \cong G_{\beta(\gamma)}$ defined over F. We use this to transport the measure on G_γ to one on $G_{\beta(\gamma)}$. With this choice we have,

$$\mathrm{vol}(G_\gamma(F)\backslash G_\gamma(\mathbf{A})) = \mathrm{vol}(G'_{\beta(\gamma)}(F)\backslash G'_{\beta(\gamma)}(\mathbf{A})).$$

We denote by R the representation of $G(\mathbf{A})$ on $L^2(G(F)\backslash G(\mathbf{A}))$ and by R' the representation of $G'(\mathbf{A})$ on $L^2(G'(F)\backslash G'(\mathbf{A}))$. Let $f \in C_c^\infty(G(\mathbf{A}^S))$. We view f as a function on $G(\mathbf{A})$ by composing f with the projection $G(\mathbf{A}) \to G(\mathbf{A}^S)$. We note that since $G(\mathbf{A}_S)$ is compact so f, as a function on $G(\mathbf{A})$, is still compactly supported.

Theorem 5.16. *For all $f \in C_c^\infty(G(\mathbf{A}^S))$,*

$$\operatorname{tr} R(f) = \operatorname{tr}(R' \circ \alpha)(f).$$

Proof. We note that $\operatorname{tr}(R' \circ \alpha)(f) = \operatorname{tr} R'(f \circ \alpha^{-1})$.

By the trace formula,

$$\operatorname{tr} R(f) = \operatorname{vol}(G(F)\backslash G(\mathbf{A}))f(e) + \sum_{\gamma \in \Gamma(G(F)), \gamma \neq e} \operatorname{vol}(G_\gamma(F)\backslash G_\gamma(\mathbf{A}))I(\gamma, f)$$

and

$$\operatorname{tr} R'(f') = \operatorname{vol}(G'(F)\backslash G'(\mathbf{A}))f'(e') + \sum_{\gamma' \in \Gamma(G'(F)), \gamma' \neq e'} \operatorname{vol}(G'_\gamma(F)\backslash G'_\gamma(\mathbf{A}))I(\gamma', f')$$

We note that for $f \in C_c^\infty(G(\mathbf{A}^S))$,

$$I(\gamma, f) = \prod_{v \in S} \operatorname{vol}(G_\gamma(F_v)\backslash G(F_v)) \int_{G_\gamma(\mathbf{A}^S)\backslash G(\mathbf{A}^S)} f(g^{-1}\gamma g)\ dg.$$

We recall the bijection,

$$\beta : \Gamma(G(F)) \xrightarrow{\sim} \Gamma(G'(F)).$$

Let $\gamma \in G(F)$ be a non-identity element. Then $\beta(\gamma)$ is conjugate in $G'(\mathbf{A}_F^S)$ to $\alpha(\gamma)$, hence with our choice of measures,

$$\begin{aligned}\int_{G_\gamma(\mathbf{A}^S)\backslash G(\mathbf{A}^S)} f(g^{-1}\gamma g)\ dg &= \int_{G'_{\alpha(\gamma)}(\mathbf{A}^S)\backslash G'(\mathbf{A}^S)} f \circ \alpha^{-1}(g^{-1}\alpha(\gamma)g)\ dg \\ &= \int_{G'_{\beta(\gamma)}(\mathbf{A}^S)\backslash G'(\mathbf{A}^S)} f \circ \alpha^{-1}(g^{-1}\beta(\gamma)g)\ dg,\end{aligned}$$

the second equality coming from Lemma 5.11. We also have,

$$\prod_{v \in S} \operatorname{vol}(G_\gamma(F_v)\backslash G(F_v)) = \prod_{v \in S} \operatorname{vol}(G'_\gamma(F_v)\backslash G'(F_v)).$$

Thus for all $\gamma \in \Gamma(G(F))$, $I(\gamma, f) = I(\beta(\gamma), f \circ \alpha^{-1})$.

As discussed above if γ is a non-trivial element of $G(F)$ then,

$$\operatorname{vol}(G_\gamma(F)\backslash G_\gamma(\mathbf{A})) = \operatorname{vol}(G'_{\beta(\gamma)}(F)\backslash G'_{\beta(\gamma)}(\mathbf{A})).$$

Hence we see that,

$$\operatorname{tr} R(f) - \operatorname{tr} R'(f \circ \alpha^{-1}) = (\operatorname{vol}(G(F)\backslash G(\mathbf{A})) - \operatorname{vol}(G'(F)\backslash G'(\mathbf{A})))\, f(e).$$

It remains to show that,

$$\operatorname{vol}(G(F)\backslash G(\mathbf{A})) = \operatorname{vol}(G'(F)\backslash G'(\mathbf{A})).$$

We note that this equality is not clear since the groups G and G' are not isomorphic over F.

The identity above can be written in the form,

$$\sum_{\pi \in \widehat{G(\mathbf{A})}} n_\pi \operatorname{tr} \pi(f) = (\operatorname{vol}(G(F)\backslash G(\mathbf{A})) - \operatorname{vol}(G'(F)\backslash G'(\mathbf{A})))\, f(e)$$

with $n_\pi \in \mathbf{Z}$. Fix a place $u \notin S$ (either p-adic or archimedean) and functions $f_v \in C_c^\infty(G(F_v))$ for $v \neq S \cup \{u\}$ with $f_v(e) \neq 0$ and $f_v = 1_{G(\mathcal{O}_{F_v})}$ for almost all v. Taking $f = f_u \prod_{v \neq u} f_v$ we can then treat this identity as an identity of distributions on $G(F_u)$. The identity above is of the form,

$$\sum_{\pi_u \in \widehat{G(F_u)}} c_{\pi_u} \operatorname{tr} \pi_u(f_u) = C f_u(e)$$

for some constant C. The Plancherel formula gives an explicit expression for $f_u(e)$ as a sum of integrals of $\operatorname{tr} \pi_u(f_u)$ for π_u tempered against a continuous measure. For example in the case of $\mathrm{SL}(2,\mathbf{R})$ and $f \in C_c^\infty(\mathrm{SL}(2,\mathbf{R})//\,\mathrm{SO}(2))$ we have from Theorem 4.25 and Lemma 4.27

$$f\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{4\pi} \int_{-\infty}^{\infty} \operatorname{tr} \pi_{2ui}(f) u \tanh(\pi u)\, du.$$

An argument of Langlands then concludes that $C = 0$. Hence $\operatorname{vol}(G(F)\backslash G(\mathbf{A})) = \operatorname{vol}(G'(F)\backslash G'(\mathbf{A}))$ and $\operatorname{tr} R(f) = \operatorname{tr}(R' \circ \alpha)(f)$ for all $f \in C_c^\infty(G(\mathbf{A}^S))$.. $\square$

Applying Corollary 5.4 yields,

Corollary 5.17. *The representations of $G(\mathbf{A})$ on $L^2(G(F)G(\mathbf{A}_S)\backslash G(\mathbf{A}))$ and $L^2(G'(F)G'(\mathbf{A}_S)\backslash G'(\mathbf{A}))$ are isomorphic.*

As a consequence of the proof of Theorem 5.16,

Corollary 5.18. $\operatorname{vol}(G(F)\backslash G(\mathbf{A})) = \operatorname{vol}(G'(F)\backslash G'(\mathbf{A}))$.

Remarks:

1. One could relax the condition that the division algebras be of prime degree. Instead one could work with division algebras D and D' of composite degree n. One would then need to demand that at any place where the division algebras are not isomorphic that they both be division algebras. One could also allow a central character.

2. In fact Corollary 5.18 is true in great generality. Kottwitz [Kot88] proved, via the trace formula, that if G and G' are inner forms then with respect to the Tamagawa measure $\mathrm{vol}(G(F)\backslash G(\mathbf{A}_F)) = \mathrm{vol}(G'(F)\backslash G'(\mathbf{A}_F))$.

3. Extend Corollary 5.17 to a correspondence between full automorphic spectra. The result above applies to automorphic representations $\pi = \otimes\pi_v$ of $G(\mathbf{A})$ such that π_v is the trivial representation of $G(F_v)$ for all $v \in S$. We established a correspondence between conjugacy classes,

$$\beta : \Gamma(G(F)) \xrightarrow{\sim} \Gamma(G'(F)).$$

In order to match up the trace formulas one needed to establish a map,

$$\delta : C_c^\infty(G(\mathbf{A})) \to C_c^\infty(G'(\mathbf{A}))$$

such that,

$$I(\gamma, f) = I(\beta(\gamma), \delta(f)).$$

We could handle this for functions $f \in C_c^\infty(\mathbf{A}^S))$ by using the isomorphism $\alpha : G(\mathbf{A}^S) \to G'(\mathbf{A}^S)$. In order to obtain a correspondence between all automorphic representations one would need to extend this map to all of $C_c^\infty(G(\mathbf{A}))$. However this is now a more delicate issue at the places outside of S the groups $G(F_v)$ and $G'(F_v)$ are not isomorphic.

4. Extend to a correspondence between all inner forms of $\mathrm{GL}(n)$. We took D and D' to be division algebras in order to ensure that the quotients $G(F)\backslash G(\mathbf{A})$ and $G'(F)\backslash G'(\mathbf{A})$ were compact so we could apply the trace formula we have developed above. However it would be natural to consider any central simple algebra of degree p. Note that if we have D and D' two central simple algebras of degree p then we have an injection,

$$\Gamma(D^\times) \hookrightarrow \Gamma(D'^\times)$$

if and only if $\mathrm{Ram}(D') \subset \mathrm{Ram}(D)$. In particular if we take $D' = M(p, F)$ so that $D^\times = \mathrm{GL}(p)$ then

$$\Gamma(D^\times) \hookrightarrow \Gamma(\mathrm{GL}(p))$$

The quotient $\mathrm{PGL}(p, F)\backslash \mathrm{PGL}(p, \mathbf{A}_F)$ is no longer compact although it does have finite volume by Theorem 5.15.

6 Problems when $\Gamma\backslash G$ is not compact

Suppose we take G to be a unimodular topological group and Γ a discrete subgroup of G. As before the quotient $\Gamma\backslash G$ possesses a (unique up to scaling) right G invariant measure and we can consider the space $L^2(\Gamma\backslash G)$ of square integrable functions on the quotient. This space admits a representation R of G by right translation and one of $C_c(G)$ by,

$$(R(f)\varphi)(x) = \int_G f(y)\varphi(xy)\, dy.$$

This is an integral operator with kernel,

$$K_f(x,y) = \sum_{\gamma\in\Gamma} f(x^{-1}\gamma y) \in C(\Gamma\backslash G \times \Gamma\backslash G).$$

In the case that the quotient $\Gamma\backslash G$ is compact we showed that $R(f)$ is Hilbert-Schmidt and hence if $f = f_1 * f_2$ then $R(f) = R(f_1)R(f_2)$, being the composition of two Hilbert-Schmidt operators, is of trace class and

$$\operatorname{tr} R(f) = \int_{\Gamma\backslash G} K_f(x,x)\ dx.$$

Interchanging summation and integration gave us the geometric side of the trace formula,

$$\operatorname{tr} R(f) = \sum_{\gamma\in\{\Gamma\}} \operatorname{vol}(\Gamma_\gamma\backslash G_\gamma) \int_{G_\gamma\backslash G} f(g^{-1}\gamma g)\ dg.$$

We then used the fact that the operators $R(f)$ are Hilbert-Schmidt, and hence compact, to prove that as a representation of G the space $L^2(\Gamma\backslash G)$ decomposed into a (Hilbert space) direct sum of irreducible representations each appearing with finite multiplicities (Theorem 3.16). This gave us the spectral side of the trace formula,

$$\operatorname{tr} R(f) = \sum_{\pi\in\widehat{G}} m_\pi \operatorname{tr} \pi(f).$$

Now, $R(f)$ is Hilbert-Schmidt if and only if,

$$K_f(x,y) \in L^2(\Gamma\backslash G \times \Gamma\backslash G).$$

Of course if $\Gamma\backslash G$ is compact then this is automatic since we know that $K_f(x,y)$ is continuous. However if $\Gamma\backslash G$ is not compact then this need not be the case.

How do this lack of compactness manifest itself in the trace formula?

On the spectral side of the trace formula the fact that the operators $R(f)$ are Hilbert-Schmidt allowed us to prove that $L^2(\Gamma\backslash G)$ decomposed as a (Hilbert space) direct sum of irreducible representations. This will no longer be the case if the quotient is not compact. For example consider the case that $G = \mathbf{R}$. If we take $\Gamma = \mathbf{Z}$ then the theory of Fourier series tells us that any function $f \in L^2(\mathbf{Z}\backslash\mathbf{R})$ can be written uniquely as,

$$f(x) = \sum_{n\in\mathbf{Z}} \hat{f}(n)e^{2\pi inx}.$$

Hence as a representation of $\mathbf{R}$ we have,

$$L^2(\mathbf{Z}\backslash\mathbf{R}) = \bigoplus_{n\in\mathbf{Z}} \mathbf{C}e^{2\pi inx}.$$

On the other hand if $\Gamma = \{0\}$ then the theory of the Fourier transform tells us that each $f \in L^2(\mathbf{R})$ can be written as,

$$f(x) = \frac{1}{2\pi}\int_{\mathbf{R}} \hat{f}(y)e^{iyx}\ dy.$$

Hence as a representation of $\mathbf{R}$ the space $L^2(\mathbf{R})$ is a direct integral of the irreducible unitary characters $x \mapsto e^{iyx}$ of $\mathbf{R}$.

What goes wrong on the geometric side of the trace formula? Suppose now we work with $G = \mathrm{PGL}(2)/\mathbf{Q}$. In this case the quotient $\mathrm{PGL}(2,\mathbf{Q})\backslash \mathrm{PGL}(2,\mathbf{A})$ is not compact although does have finite volume by Theorem 5.15. The geometric side of the trace formula in the compact case would give us,

$$\sum_{\gamma \in \Gamma(G(\mathbf{Q}))} \mathrm{vol}(G_\gamma(\mathbf{Q})\backslash G_\gamma(\mathbf{A})) \int_{G_\gamma(\mathbf{A})\backslash G(\mathbf{A})} f(g^{-1}\gamma g)\, dg.$$

Which of this terms no longer converge? If G were anisotropic over $\mathbf{Q}$ then every element in $G(\mathbf{Q})$ would lie in a $\mathbf{Q}$-anisotropic torus. In $\mathrm{PGL}(2,\mathbf{Q})$ we have two types of element which are not present in the anisotropic case. Namely, diagonal elements,

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

which lie in a split torus, and unipotent elements,

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

In the first case if we take,

$$\gamma = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

with $a \neq b$, then

$$G_\gamma = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \right\}.$$

Hence,

$$\mathrm{vol}(G_\gamma(\mathbf{Q})\backslash G_\gamma(\mathbf{A})) = \mathrm{vol}(\mathbf{Q}^\times \backslash \mathbf{A}^\times) = \infty.$$

On the other hand suppose we take a unipotent element,

$$\gamma = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

with $x \neq 0$. Then,

$$G_\gamma = \left\{ \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \right\}.$$

Since the quotient $\mathbf{Q}\backslash\mathbf{A}$ is compact so $\mathrm{vol}(G_\gamma(\mathbf{Q})\backslash G_\gamma(\mathbf{A})) < \infty$. On the other hand suppose we take $f = \prod_v f_v \in C_c^\infty(G(\mathbf{A}))$ then,

$$\int_{G_\gamma(\mathbf{A})\backslash G(\mathbf{A})} f(g^{-1}\gamma g)\, dg = \prod_v \int_{G_\gamma(\mathbf{Q}_v)\backslash G(\mathbf{Q}_v)} f_v(g_v^{-1}\gamma g_v)\, dg_v.$$

For almost all finite p we have $f_p = \mathbf{1}_{\mathrm{PGL}(2,\mathbf{Z}_p)}$. The Iwasawa decomposition gives us

$$\mathrm{PGL}(2,\mathbf{Q}_p) = N(\mathbf{Q}_p)M(\mathbf{Q}_p)\,\mathrm{PGL}(2,\mathbf{Z}_p)$$

where,

$$N = \left\{ \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \right\}$$

and

$$M = \left\{ \begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

Using this decomposition we can write the Haar measure on $\mathrm{PGL}(2, \mathbf{Q}_p)$ as

$$dg = |\beta|^{-1} d\alpha \; d^\times \beta \; dk$$

where $d\alpha$ denotes an additive Haar measure on $\mathbf{Q}_p$, $d^\times\beta$ a multiplicative Haar measure on $\mathbf{Q}_p^\times$ and dk a Haar measure on the compact group $\mathrm{PGL}(2, \mathbf{Z}_p)$. Now we have,

$$\int_{G_\gamma(\mathbf{Q}_p)\backslash G(\mathbf{Q}_p)} f_p(g_p^{-1}\gamma g_p)\, dg_p = \int_{\mathrm{PGL}(2,\mathbf{Z}_p)} \int_{\mathbf{Q}_p^\times} f_p\left(k^{-1} \begin{pmatrix} \beta^{-1} & 0 \\ 0 & 1 \end{pmatrix} \gamma \begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix} k \right) |\beta|^{-1} \, d^\times\beta \, dk.$$

If we take $f_p = \mathbf{1}_{\mathrm{PGL}(2,\mathbf{Z}_p)}$ then this integral is equal to,

$$\int_{\mathbf{Q}_p^\times} \mathbf{1}_{\mathrm{PGL}(2,\mathbf{Z}_p)} \begin{pmatrix} 1 & \beta^{-1}x \\ 0 & 1 \end{pmatrix} |\beta|^{-1} \; d^\times \beta.$$

For almost all p we have x will lie in $\mathbf{Z}_p^\times$ and hence this integral becomes

$$\int_{\beta^{-1}\in\mathbf{Z}_p} |\beta^{-1}| \; d^\times\beta = 1 + \frac{1}{p} + \frac{1}{p^2} + \ldots = \frac{1}{1-p^{-1}}.$$

Thus we have,

$$\int_{G_\gamma(\mathbf{A})\backslash G(\mathbf{A})} f(g^{-1}\gamma g) \; dg = C(f) \prod_p \frac{1}{1-p^{-1}} = \infty.$$

Thus elements of $\mathrm{PGL}(2, \mathbf{Q})$ which lie in the Borel subgroup give us divergent terms in the trace formula, in the case of semisimple elements we get infinite volumes and in the case of unipotent elements we get infinite orbital integrals.

How to rectify this? We'll work with $G = \mathrm{PGL}(2)$ over a number field F and consider the quotient $G(F)\backslash G(\mathbf{A})$ which has finite volume. We consider again $L^2(G(F)\backslash G(\mathbf{A}))$. We define the space of cuspidal functions $L^2_{cusp}(G(F)\backslash G(\mathbf{A}))$ to be those functions $\varphi \in L^2(G(F)\backslash G(\mathbf{A}))$ such that,

$$\int_{F\backslash\mathbf{A}} \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) \; dx = 0$$

for (almost) all $g \in G(\mathbf{A})$. We note that $L^2_{\mathrm{cusp}}(G(F)\backslash G(\mathbf{A}))$ is a closed $G(\mathbf{A})$-invariant subspace of $L^2(G(F)\backslash G(\mathbf{A}))$. Thus we will have an orthogonal decomposition,

$$L^2(G(F)\backslash G(\mathbf{A})) = L^2_{\mathrm{cusp}}(G(F)\backslash G(\mathbf{A})) \oplus L^2_{\mathrm{Eis}}(G(F)\backslash G(\mathbf{A})).$$

We will see that if $f \in C^\infty(G(\mathbf{A}))$ then the restriction of $R(f)$ to $L^2_{\mathrm{cusp}}(G(F)\backslash G(\mathbf{A}))$ is Hilbert-Schmidt and hence that,

$$L^2_{\mathrm{cusp}}(G(F)\backslash G(\mathbf{A})) = \bigoplus_{\pi \in \widehat{G(\mathbf{A})}} m_\pi \pi$$

with $m_\pi < \infty$ (in fact $m_\pi \in \{0,1\}$). On the other hand the theory of Eisenstein series gives an explicit decomposition for $L^2_{\mathrm{Eis}}(G(F)\backslash G(\mathbf{A}))$ in terms of continuous integrals involving Eisenstein series. We shall see that for $f \in C_c^\infty(G(\mathbf{A}))$ the kernel has a spectral expansion,

$$K_f(x,y) = K_{f,\mathrm{cusp}}(x,y) + K_{f,\mathrm{Eis}}(x,y)$$

which describes the action of $R(f)$ on $L^2_{\mathrm{cusp}}(G(F)\backslash G(\mathbf{A}))$ and $L^2_{\mathrm{Eis}}(G(F)\backslash G(\mathbf{A}))$ respectively.

One way is to write down a trace formula is to restrict the choice of test functions f. One can consider functions f which avoid the problems on the geometric side, by being supported on the regular elliptic elements, and on the spectral side by killing the Eisenstein series. Suppose we take $f = \prod_v f_v \in C_c^\infty(G(\mathbf{A}))$ such that f_{v_1} is the matrix coefficient of some supercuspdial representation of $G(F_{v_1})$ and f_{v_2} is supported on the elliptic regular elements in $G(F_{v_2})$. In this case the image of $R(f)$ will lie in $L^2_{\mathrm{cusp}}(G(F)\backslash G(\mathbf{A}))$ and hence will be of trace class. One can then compute the trace of $R(f)$ and one will obtain the following trace formula,

$$\operatorname{tr} R(f) = \sum_{\pi \text{ cuspidal}} m_\pi \operatorname{tr} \pi(f).$$

and

$$\operatorname{tr} R(f) = \sum_{\gamma \in \Gamma(G(F)),\text{ell. reg.}} \mathrm{vol}(G_\gamma(F)\backslash G_\gamma(\mathbf{A})) \int_{G_\gamma(\mathbf{A})\backslash G(\mathbf{A})} f(g^{-1}\gamma g)\, dg$$

This simple trace formula has many applications, for example to obtaining cases of functoriality, however it's clear that one will be losing information. For example one will always kill representations which are everywhere unramified. In particular this simple trace formula will be of no use in trying to establish Weyl's law, say for $\mathrm{SL}(2,\mathbf{Z})\backslash\mathcal{H}$.

Following Arthur we will consider a truncation of the kernel $K_f^T(x,y)$ depending on a truncation parameter $T \gg 0$ such that,

$$J^T(f) = \int_{G(F)\backslash G(\mathbf{A})} K_f^T(x,x)\, dx$$

is absolutely integrable. Of course this integral no longer represents the trace of anything. However, the truncation we define will not affect the cuspidal part of the kernel and so,

$$J^T(f) = \operatorname{tr} R_{\mathrm{cusp}}(f) + \int_{G(F)\backslash G(\mathbf{A})} K_{f,\mathrm{Eis}}^T(x,x)\, dx,$$

where, $R_{\text{cusp}}(f)$ denotes the restriction of $R(f)$ to $L^2_{\text{cusp}}(G(F)\backslash G(\mathbf{A}))$.

Similarly using the geometric expansion for $K_f(x, y)$ we will obtain a geometric expansion for

$$\int_{G(F)\backslash G(\mathbf{A})} K_f^T(x,x)\ dx = \sum_{\mathfrak{o}\in\mathfrak{O}} J_{\mathfrak{o}}^T(f).$$

Here the sum is over equivalence classes $\mathfrak{O}$ of elements $\gamma \in G(F)$ defined in the following way. Given $\gamma_1, \gamma_2 \in G(F)$ we write $\gamma_1 = \gamma_{1,s}\gamma_{1,u}$ and $\gamma_2 = \gamma_{2,s}\gamma_{2,u}$ in their Jordan decompositions. We define an equivalence relation by $\gamma_1 \sim \gamma_2$ if and only if $\gamma_{1,s}$ and $\gamma_{2,s}$ are conjugate. The terms $J_{\mathfrak{o}}^T(f)$ will involve *weighted* orbital integrals. By analogy with the spectral side of the trace formula some terms will be untouched by the truncation. Suppose $\gamma \in G(F)$ is a regular elliptic element then the equivalence class of γ is the same as its conjugacy class and,

$$J_{\mathfrak{o}}^T(f) = \mathrm{vol}(G_\gamma(F)\backslash G_\gamma(\mathbf{A})) \int_{G_\gamma(\mathbf{A})\backslash G(\mathbf{A})} f(g^{-1}\gamma g)\ dg.$$

On the geometric side the truncation will affect only those elements which are conjugate to elements in the Borel subgroup.

Thus the trace formula we will develop will have an expansion of the form,

$$J_{ell}(f) + J_{par}^T(f) = J^T(f) = J_{cusp}(f) + J_{Eis}^T(f).$$

Where $J_{ell}(f)$ looks like the geometric side of the trace formula for compact quotient, and $J_{cusp}(f)$ looks like the spectral side of the trace formula for compact quotient. If one wishes this can be written as,

$$J_{cusp}(f) = J_{ell}(f) + J_{par}^T(f) - J_{Eis}^T(f),$$

which computes the trace of $R(f)$ on $L^2_{\text{cusp}}(G(F)\backslash G(\mathbf{A}))$, although now the right hand side contains both geometric and spectral terms.

7 Geometric side of the trace formula for $\mathrm{GL}(2)$

The main references for this section are [GJ79] and [Gel96].

We fix the following notation throughout this section. We take F to be a number field with ring of adeles $\mathbf{A}$. We set $G = \mathrm{GL}(2)$ which we view as an algebraic group over F. We define the following F-subgroups of G,

$$Z = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right\},\quad B = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \right\},\quad M = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \right\},\quad N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right\}.$$

For each place v of F we fix a maximal compact subgroup K_v of $\mathrm{GL}(2, F_v)$. For v p-adic we take,

$$K_v = \mathrm{GL}(2, \mathcal{O}_{F_v})$$

where $\mathcal{O}_{F_v}$ denotes the ring of integers in F_v. For v real we set,

$$K_v = \mathrm{O}(2)$$

and for v complex we take,

$$K_v = \mathrm{U}(2).$$

We set,

$$K = \prod_v K_v \subset G(\mathbf{A}).$$

The Iwasawa decomposition gives us,

$$G(\mathbf{A}) = B(\mathbf{A})K.$$

The Bruhat decomposition gives us,

$$G = B \amalg BwB$$

where,

$$w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

For $x = (x_v) \in \mathbf{A}^\times$ we define,

$$|x| = \prod_v |x_v|_v.$$

The local $|\ \ |_v$ are normalized in the usual way so that,

$$|x| = \prod_v |x|_v = 1$$

for any $x \in F^\times$. We let,

$$\mathbf{A}^1 = \left\{x \in \mathbf{A}^\times : |x| = 1\right\}.$$

7.1 Geometry of $G(F)\backslash G(\mathbf{A})$

We define, $H_M : M(\mathbf{A}) \to \mathbf{R}$ by,

$$H_M \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} = \log |ad^{-1}|.$$

We extend H_M to a map $H : G(\mathbf{A}) \to \mathbf{R}$ by using the Iwasawa decomposition $G(\mathbf{A}) = N(\mathbf{A})M(\mathbf{A})K$ and defining,

$$H(nmk) = H_M(m).$$

We note that

$$H(g) = \sum_v H_v(g_v)$$

where H_v denotes the corresponding height function on $G(F_v)$.

It's perhaps worth illustrating what this map does for elements in $G(\mathbf{R})$. Suppose $g \in G(\mathbf{R})$ and we write

$$g = \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} k$$

with $b \in B(\mathbf{R})$ with $\det b > 0$. Recalling that $G(\mathbf{R})$ acts on $\mathcal{H}$ then $g(i) = b + |a| i$ and $H(g) = \log |a|$ is the logarithm of the height of $g(i)$ above the x-axis. (More correctly on $\mathcal{H} \cup \mathcal{H}^-$, the union of the upper and lower half planes, but after modding out by complex conjugation we get an action of $G(\mathbf{R})$ on $\mathcal{H}$.)

We now introduce the notion of a Siegel domain which will play the role of a fundamental domain for $G(F)\backslash G(\mathbf{A})$. We write elements $g \in G(\mathbf{A})$ in the following way using the Iwasawa decomposition,

$$g = znh_t m_a k$$

with $z \in Z(A)$, $n \in N(\mathbf{A})$, $h_t = (h_{t,v}) \in M(\mathbf{A})$ such that,

$$h_{t,v} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

for each finite v, and

$$h_{t,v} = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$

at each infinite place v,

$$m_a = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$$

with $a \in \mathbf{A}^1$ and $k \in K$.

Let C_1 be a compact subset of $N(\mathbf{A})$ and let C_2 be a compact subset of $\mathbf{A}^1$. Given $c > 0$ we define a Siegel set $\mathfrak{S}_c$ to be the set of all $g \in G(\mathbf{A})$ of the form

$$g = znh_t m_a k$$

as above, with $n \in C_1$, $a \in C_2$, and $t > \frac{c}{2}$. Note that,

$$\mathfrak{S}_c \subset \{g \in \mathrm{GL}(2, \mathbf{A}) : H(g) > c\}.$$

Theorem 7.1. *([Gel96, Lecture II, Facts 1 & 2]) For any $c > 0$, $\mathfrak{S}_c \cap \gamma \mathfrak{S}_c \neq \emptyset$ for only finitely many $\gamma \in \mathrm{GL}(2, F)$ modulo $Z(F)$. If $\mathfrak{S}_c$ is sufficiently large (i.e. if C_1 and C_2 are sufficiently large, and c is sufficiently small), then $G(\mathbf{A}) = G(F)\mathfrak{S}_c$.*

This theorem is due to Godement. We'll explain how one can see this theorem in the case of $\mathrm{GL}(2)/\mathbf{Q}$. First we claim that,

$$G(\mathbf{A}) = G(\mathbf{Q})G(\mathbf{R})K_{\mathrm{fin}}.$$

Note that by the Iwasawa decomposition it suffices to verify this for $\gamma \in B(\mathbf{A}_{\mathrm{fin}})$. It suffices to show that

$$B(\mathbf{A}_{fin}) = B(\mathbf{Q})(B(\mathbf{A}_{\mathrm{fin}}) \cap K_{\mathrm{fin}}).$$

Which follows from the fact that,

$$\mathbf{A}_{fin}^{\times} = \mathbf{Q}^{\times} \prod_p \mathbf{Z}_p^{\times}$$

which is equivalent to the assertion that $\mathbf{Q}$ has class number one, and

$$\mathbf{A}_{fin} = \mathbf{Q} + \prod_p \mathbf{Z}_p.$$

Using $G(\mathbf{A}) = G(\mathbf{Q})G(\mathbf{R})K_{\mathrm{fin}}$ we see that projection onto $G(\mathbf{R})$ yields a homeomorphism,

$$G(\mathbf{Q})\backslash G(\mathbf{A})/K_{\mathrm{fin}} \xrightarrow{\sim} K_{\mathrm{fin}} \cap G(\mathbf{Q})\backslash G(\mathbf{R}) = \mathrm{GL}(2, \mathbf{Z})\backslash \mathrm{GL}(2, \mathbf{R}).$$

We will translate the assertions of the Theorem into questions about $\mathrm{GL}(2, \mathbf{R})$.

We now choose compact sets C_1 and C_2. We take,

$$C_1 = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x_p \in \mathbf{Z}_p, x_\infty \in [-\alpha, \alpha] \right\}$$

for some fixed $\alpha > 0$, and we take,

$$C_2 = \prod_p \mathbf{Z}_p^{\times}.$$

Suppose we have $g = znh_tmk \in \mathfrak{S}_c$. We will separate out these elements of $G(\mathbf{A})$ into their finite and infinite parts. We have,

$$g = (z_f, z_\infty) \left(\begin{pmatrix} 1 & x_f \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & x_\infty \\ 0 & 1 \end{pmatrix} \right) \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \right) \left(\begin{pmatrix} a_f & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) (k_f, k_\infty),$$

which equals

$$g = (z_f, z_\infty) \left(\begin{pmatrix} 1 & x_f \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_f & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & x_\infty \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \right) (k_f, k_\infty).$$

By our choice of C_1 and C_2 we see that,

$$\begin{pmatrix} 1 & x_f \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \in \prod_p K_p.$$

Hence after changing k_f we can rewrite g as,

$$g = \left(z_f k_f, z_\infty \begin{pmatrix} 1 & x_\infty \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} k_\infty \right),$$

with $x_\infty \in [-\alpha, \alpha]$ and $t > \frac{c}{2}$. We write $\mathfrak{S}_c = \mathfrak{S}_{c,\mathrm{fin}} \times \mathfrak{S}_{c,\infty}$, so $\mathfrak{S}_{c,\mathrm{fin}} = Z(\mathbf{A}_{\mathrm{fin}})K_{\mathrm{fin}}$. Suppose $\gamma \in G(\mathbf{Q})$ is such that $\mathfrak{S}_c \cap \gamma\mathfrak{S}_c \neq \emptyset$. Since $\gamma\mathfrak{S}_{c,\mathrm{fin}} \cap \mathfrak{S}_{c,\mathrm{fin}} \neq \emptyset$ so $\gamma \in G(\mathbf{Q}) \cap Z(\mathbf{A}_{\mathrm{fin}})K_{\mathrm{fin}} = Z(\mathbf{Q})G(\mathbf{Z})$. If we now project $\mathfrak{S}_{c,\infty}$ onto $\mathcal{H} = Z(\mathbf{R})\backslash G(\mathbf{R})/K_\infty$ then we see that the image of $\mathfrak{S}_{c,\infty}$ is,

$$\{x + iy \in \mathcal{H} : x \in [-\alpha, \alpha], y > e^c\}.$$

And hence there are only finitely many $\gamma \in G(\mathbf{Z})$ such that $\gamma\mathfrak{S}_{c,\infty} \cap \mathfrak{S}_{c,\infty} \neq \emptyset$. As is well known if $\alpha \geq \frac{1}{2}$ and $c < \log(\sqrt{3}2)$ then $G(\mathbf{R}) = G(\mathbf{Z})\mathfrak{S}_{c,\infty}$. We claim that then also $G(\mathbf{A}) = G(\mathbf{Q})\mathfrak{S}_c$. Suppose $g = (g_f, g_\infty) \in G(\mathbf{A})$. Then we know we can write $g_f = \gamma k_f$ with $\gamma \in G(\mathbf{Q})$ and $k_f \in K_{\mathrm{fin}}$. Hence,

$$g = (\gamma k_f, \gamma\gamma^{-1}g_\infty).$$

Now $\gamma^{-1}g_\infty = \delta x_\infty$ with $x_\infty \in \mathfrak{S}_{c,\infty}$ and $\delta \in G(\mathbf{Z}) = G(\mathbf{Q}) \cap K_{\mathrm{fin}}$. Hence,

$$g = (\gamma k_f, \gamma\delta x_\infty) = (\gamma\delta, \gamma\delta)(\delta^{-1}k_f, x_\infty) \in G(\mathbf{Q})\mathfrak{S}_c.$$

Theorem 7.2. *The space $Z(\mathbf{A})G(F)\backslash G(\mathbf{A})$ has finite volume.*

Using the decomposition,

$$g = znh_t m_a k$$

we can write the Haar measure of $G(\mathbf{A})$ in the following way,

$$\int_{Z(\mathbf{A})\backslash G(\mathbf{A})} f(g)\, dg = \int_K \int_{N(\mathbf{A})} \int_{\mathbf{A}^1} \int_{-\infty}^{\infty} f(nh_t mk) e^{-2t}\, dt\, d^\times a\, dn\, dk.$$

Thus we see that for a suitable Siegel domain $\mathfrak{S}_c$,

$$\mathrm{vol}(Z(\mathbf{A})G(F)\backslash G(\mathbf{A})) \leq \int_{Z(\mathbf{A})\backslash \mathfrak{S}_c} dg = \int_K \int_{C_1} \int_{C_2} \int_c^{\infty} e^{-2t}\, dt\, d^\times a\, dn\, dk,$$

which is finite since K, C_1 and C_2 are all compact.

7.2 Truncation operator

Let $\varphi \in C(G(F)\backslash G(\mathbf{A}))$. We define its constant term along N to be the function $\varphi_N \in C(M(F)N(\mathbf{A})\backslash G(F))$ defined by,

$$\varphi_N(g) = \int_{N(F)\backslash N(\mathbf{A})} \varphi(ng)\, dg.$$

We take the measure on $N(\mathbf{A})$ to be such that $\mathrm{vol}(N(F)\backslash N(\mathbf{A})) = 1$. By definition φ is cuspidal if and only if $\varphi_N \equiv 0$. Let τ be the characteristic function on $\mathbf{R}$ of the interval $(0, \infty)$. Let $T > 0$. We obtain a function on $B(F)N(\mathbf{A})\backslash G(\mathbf{A})$,

$$\varphi_N(g)\tau(H(g) - T).$$

In order to obtain a function on $G(F)\backslash G(\mathbf{A})$ we need to average over $B(F)\backslash G(F)$. We define $\Lambda^T\varphi$ on $G(F)\backslash G(\mathbf{A})$ by,

$$\Lambda^T\varphi(g) = \varphi(g) - \sum_{\delta \in B(F)\backslash G(F)} \varphi_N(\delta g)\tau(H(\delta g) - T).$$

The sum on the right hand side is always finite. In fact more is true.

Lemma 7.3. *Let $\mathfrak{S}_c$ be a Siegel domain for G and let $T \in \mathbf{R}$. Then the set of $\gamma \in G(F)$ such that $H(\gamma g) > T$ for some $g \in \mathfrak{S}_c$ is finite modulo $B(F)$.*

We'll prove this lemma for $F = \mathbf{Q}$ and functions f on $\mathrm{SL}(2,\mathbf{Z})\backslash\mathcal{H}$ provided T is sufficiently large. We use the homeomorphism,

$$Z(\mathbf{A})G(\mathbf{Q})\backslash G(\mathbf{A})/K \xrightarrow{\sim} \mathrm{SL}(2,\mathbf{Z})\backslash\mathcal{H}$$

to pull f back to a function φ on $G(\mathbf{Q})\backslash G(\mathbf{A})$ which is right K invariant. Since $G(\mathbf{A}) = G(\mathbf{Q})B(\mathbf{R})K$ so φ is completely determined by its values on $B(\mathbf{R})$ and we have,

$$\varphi\begin{pmatrix} a_\infty & b_\infty \\ 0 & 1 \end{pmatrix} = f(b_\infty + a_\infty i)$$

for $a_\infty \in \mathbf{R}_{>0}$, $b_\infty \in \mathbf{R}$. Furthermore,

$$\varphi_N\begin{pmatrix} a_\infty & b_\infty \\ 0 & 1 \end{pmatrix} = \int_0^1 f(x + a_\infty i)\ dx.$$

Suppose now we look at,

$$\Lambda^T\varphi\begin{pmatrix} a_\infty & b_\infty \\ 0 & 1 \end{pmatrix} = \varphi\begin{pmatrix} a_\infty & b_\infty \\ 0 & 1 \end{pmatrix} - \sum_{\delta \in B(F)\backslash G(F)} \varphi_N\left(\delta\begin{pmatrix} a_\infty & b_\infty \\ 0 & 1 \end{pmatrix}\right)\tau\left(H\left(\delta\begin{pmatrix} a_\infty & b_\infty \\ 0 & 1 \end{pmatrix}\right) - T\right).$$

When $\delta \in B(F)$ the term in the sum above is 0 if $|a_\infty| \leq e^T$ and if $|a_\infty| > e^T$ then it is

$$\int_0^1 f(x + a_\infty i)\ dx.$$

On the other hand suppose we take $b_\infty + ia_\infty$ to lie in the usual fundamental domain,

$$F = \{z = x + iy \in \mathcal{H} : x \in [-1/2, 1/2], |z| > 1\},$$

for the action of $\mathrm{SL}(2,\mathbf{Z})$ on $\mathcal{H}$. Suppose $\delta \in G(F) \setminus B(F)$ is such that,

$$\tau\left(H\left(\delta\begin{pmatrix} a_\infty & b_\infty \\ 0 & 1 \end{pmatrix}\right) - T\right) = 1,$$

i.e. assume that

$$H\left(\delta\begin{pmatrix} a_\infty & b_\infty \\ 0 & 1 \end{pmatrix}\right) > T.$$

Note that,

$$H\left(\delta\begin{pmatrix} a_\infty & b_\infty \\ 0 & 1 \end{pmatrix}\right) = \sum_p H_p(\delta) + H_\infty\left(\delta\begin{pmatrix} a_\infty & b_\infty \\ 0 & 1 \end{pmatrix}\right).$$

Using the Iwasawa decomposition we can assume

$$\delta = wn = \begin{pmatrix} 0 & 1 \\ -1 & -t \end{pmatrix}.$$

Then, (see [KL06, Lemma 7.19]),

$$H_p(\delta) = -\log\max\{1, |t|_p^2\},$$

and

$$H_\infty\left(\delta\begin{pmatrix} a_\infty & b_\infty \\ 0 & 1 \end{pmatrix}\right) = \log\left(\frac{a_\infty}{a_\infty^2 + (b_\infty + t)^2}\right).$$

Thus we need,

$$\frac{a_\infty^2 + (b_\infty + t)^2}{|a_\infty|}\prod_p \max\{1, |t|_p^2\} \le e^{-T}.$$

and hence,

$$\prod_p \max\{1, |t|_p^2\} \le \frac{a_\infty}{a_\infty^2 + (b_\infty + t)^2}e^{-T} \le a_\infty^{-1}e^{-T} \le \frac{2}{\sqrt{3}}e^{-T}$$

since $b_\infty + a_\infty i \in F$ and hence $a_\infty \ge \frac{\sqrt{3}}{2}$. Thus if T is sufficiently large then we see that we must have $t \in \mathbf{Z}$. But then we have,

$$H\left(\delta\begin{pmatrix} a_\infty & b_\infty \\ 0 & 1 \end{pmatrix}\right) = H_\infty\left(\delta\begin{pmatrix} a_\infty & b_\infty \\ 0 & 1 \end{pmatrix}\right),$$

which is the logarithm of the height of the point,

$$\left(\delta\begin{pmatrix} a_\infty & b_\infty \\ 0 & 1 \end{pmatrix}\right)(i) = \delta(b_\infty + ia_\infty)$$

above the x-axis. But now since $b_\infty + ia_\infty$ is constrained to lie in F we see that for $T > 0$,

$$H_\infty\left(\delta\begin{pmatrix} a_\infty & b_\infty \\ 0 & 1 \end{pmatrix}\right) > e^T$$

only if $\delta \in B(\mathbf{Z})$ which gives a contradiction. Thus only $\delta \in B(\mathbf{Q})$ contribute to the sum and we have, for $b_\infty + ia_\infty \in F$,

$$\Lambda^T\varphi\begin{pmatrix} a_\infty & b_\infty \\ 0 & 1 \end{pmatrix} = f(b_\infty + a_\infty i),$$

if $b_\infty \le e^T$, and

$$\Lambda^T\varphi\begin{pmatrix} a_\infty & b_\infty \\ 0 & 1 \end{pmatrix} = f(b_\infty + a_\infty i) - \int_0^1 f(x + a_\infty i)\, dx$$

if $b_\infty > e^T$.

7.3 Geometric side of the trace formula

We now take $f \in C_c^\infty(Z(\mathbf{A})\backslash G(\mathbf{A}))$ and let,

$$K_f(x,y) = \sum_{\gamma \in Z(F)\backslash G(F)} f(x^{-1}\gamma y)$$

denote the corresponding kernel function. We define $K_f^T(x,y) = \Lambda_2^T K_f(x,y)$ where Λ_2^T denotes that the truncation is taken in the second variable. Thus,

$$\begin{aligned} K_f^T(x,y) &= K_f(x,y) - \sum_{\delta \in B(F)\backslash G(F)} K_{f,N}(x,\delta y)\tau(H(\delta y) - T) \\ &= \sum_{\gamma \in Z(F)\backslash G(F)} f(x^{-1}\gamma y) - \sum_{\delta \in B(F)\backslash G(F)} \left(\int_{N(F)\backslash N(\mathbf{A})} \sum_{\gamma \in Z(F)\backslash G(F)} f(x^{-1}\gamma n \delta y)\ dn \right) \tau(H(\delta y) - T). \end{aligned}$$

Our goal in this section is to prove that,

$$\int_{Z(\mathbf{A})G(F)\backslash G(\mathbf{A})} K_f^T(x,x)\ dx$$

is absolutely convergent and to provide a geometric expansion for this integral.

If the quotient $Z(\mathbf{A})G(F)\backslash G(\mathbf{A})$ were compact then in order to write down the geometric side of the trace formula we would write,

$$K_f(x,x) = \sum_{\gamma \in \Gamma(Z(F)\backslash G(F))} \sum_{\delta \in G_\gamma(F)\backslash G(F)} f(x^{-1}\delta x)$$

and then interchange integration over $Z(\mathbf{A})G(F)\backslash G(\mathbf{A})$ with the summation over $\gamma \in \Gamma(Z(F)\backslash G(F))$ which is legitimate since,

$$x \mapsto \sum_{\delta \in G_\gamma(F)\backslash G(F)} f(x^{-1}\delta x)$$

is a function on $Z(\mathbf{A})G(F)\backslash G(\mathbf{A})$.

In the same way we would like to write out a geometric expansion for $K_f^T(x,x)$. Note that,

$$K_f^T(x,x) = \sum_{\gamma \in G(F)} f(x^{-1}\gamma x) - \sum_{\delta \in B(F)\backslash G(F)} \left(\int_{N(F)\backslash N(\mathbf{A})} \sum_{\gamma \in Z(F)\backslash G(F)} f(x^{-1}\gamma n \delta x)\ dn \right) \tau(H(\delta x) - T).$$

We note that we need to be more careful here because of the integral over $N(F)\backslash N(\mathbf{A})$. The naive geometric expansion for $K_f^T(x,x)$ could yield terms which are not well defined on $N(F)\backslash N(\mathbf{A})$.

We define the following equivalence on elements of $G(F)$. Let $\gamma_1, \gamma_2 \in G(F)$. We write $\gamma_1 = \gamma_{1,s}\gamma_{1,u}$ and $\gamma_2 = \gamma_{2,s}\gamma_{2,u}$ in their Jordan decompositions. We define $\gamma_1 \sim \gamma_2$ if and only if $\gamma_{1,s}$ and $\gamma_{2,s}$ are conjugate. We let $\mathfrak{O}$ denote the set of equivalence classes with respect to this equivalence relation.

Let $\gamma \in G(F)$. If γ is regular elliptic (i.e. its eigenvalues don't lie in F) then γ is automatically semisimple and the equivalence class of γ is just the conjugacy class of γ. If γ is not regular elliptic then it is conjugate to an element of $B(F)$. We may assume,

$$\gamma = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}.$$

If $a \neq d$ then the equivalence class of γ is the same as the conjugacy class of γ, which is the same as the conjugacy class of,

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}.$$

On the other hand if $a = d$ then the equivalence class of γ is equal to the union of the conjugacy classes of,

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}.$$

We write,

$$K_f(x,x) = \sum_{\mathfrak{o} \in \mathfrak{O}} K_{f,\mathfrak{o}}(x,x)$$

where,

$$K_{f,\mathfrak{o}}(x,x) = \sum_{\gamma \in \mathfrak{o}} f(x^{-1}\gamma x).$$

We want to write,

$$K_f^T(x,x) = \sum_{\mathfrak{o} \in \mathfrak{O}} K_{f,\mathfrak{o}}^T(x,x).$$

The following lemma tells us that if T is sufficiently large then many of the terms in

$$\sum_{\delta \in B(F)\backslash G(F)} \left(\int_{N(F)\backslash N(\mathbf{A})} \sum_{\gamma \in Z(F)\backslash G(F)} f(x^{-1}\delta^{-1}\gamma n \delta x)\ dn \right) \tau(H(\delta x) - T)$$

vanish.

Lemma 7.4. *([Gel96, Lecture II, Lemma 2.2]) Suppose $\Omega \subset G(\mathbf{A})$ is compact modulo $Z(\mathbf{A})$. Then there exists $d_\Omega > 0$ with the following property: if $\gamma \in G(F)$ is such that,*

$$g^{-1}\gamma n g \in \Omega$$

for some $n \in N(\mathbf{A})$ and $g \in G(\mathbf{A})$ with $H(g) > d_\Omega$, then $\gamma \in B(F)$.

Before giving a proof we'll motivate the lemma by considering the real case. We take $\Omega \subset G(\mathbf{R})$ to be a compact set modulo $Z(\mathbf{R})$. Clearly we are free to replace Ω by $K_\infty \Omega K_\infty$. We may as well then assume that $g \in B(\mathbf{R})$. We write,

$$g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

so $H_\infty(g) = \log |ad^{-1}|$. Recall that g acts on $\mathcal{H}$ by,

$$g(z) = bd^{-1} + |ad^{-1}|z.$$

Suppose $g^{-1}\gamma ng \in \Omega$ with $\gamma \in G(\mathbf{Z})$, $n \in G(\mathbf{R})$. We project Ω onto a compact subset of $\mathcal{H}$ which we again denote by Ω. Then we have,

$$(g^{-1}\gamma ng)(i) \in \Omega.$$

Or equivalently, $(\gamma ng)(i) \in g(\Omega)$. If $H_\infty(g) = \log |ad^{-1}|$ is sufficiently large all elements in $g(\Omega)$ will have imaginary part larger than 1. But also $g(i)$ has large imaginary part, equal to $|ad^{-1}|$ which we can assume if larger than 1. We recall $N(\mathbf{R})$ acts by translation and hence doesn't effect the imaginary part of $g(i)$. Then since $\gamma \in G(\mathbf{Z})$ we know that the imaginary part of $(\gamma ng)(i)$ will be less than 1 unless $\gamma \in B(\mathbf{Z})$.

Proof. We follow the proof from [Gel96, Lecture II, Lemma 2.2]. Let V denote the space of 2×2 trace zero matrices and consider the adjoint representation,

$$\rho : G \to \mathrm{GL}(V)$$

defined by,

$$\rho(g)X = gXg^{-1}.$$

We fix the basis,

$$e_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$

of V. Let,

$$\alpha : \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mapsto ab^{-1}.$$

Then we have,

$$\rho(a)e_j = \alpha(a)^{1-j}e_j,$$

for all $a \in M$,

$$\rho(n)e_0 = e_0$$

for all $n \in N$, and

$$\rho(w)e_0 = e_2.$$

For $\xi \in V(\mathbf{A})$ we define,

$$\|\xi\| = \prod_v \|\xi_v\|_v$$

where, for v finite,

$$\|a_0e_0 + a_1e_1 + a_2e_2\|_v = \max\{|a_0|_v, |a_1|_v, |a_2|_v\},$$

and for v infinite we define $\| \ \|_v$ via the Hilbert space structure that makes $\{e_0, e_1, e_2\}$ into an orthonormal basis.

Suppose, with the notation of the lemma, we have,

$$g^{-1}\gamma ng \in \Omega.$$

We write $g = n_1ak$ using the Iwasawa decomposition. So that $H(g) = H(a)$ and,

$$a^{-1}n_1^{-1}\gamma nn_1a \in K\Omega K.$$

Since the map $g \mapsto \rho(g)e_0$ is continuous with respect to the norm $\| \ \|$ and ρ is trivial on $Z(\mathbf{A})$ so there exists d_0 such that,

$$\|\rho(a^{-1}n_1^{-1}\gamma nn_1a)e_0\| \leq e^{2d_0}.$$

Suppose now that $\gamma \notin B(F)$. Then by Bruhat's decomposition we can write

$$g = b_0wn_0$$

with $b_0 = n_0'm_0 \in B(F)$ and $n_0 \in N(F)$. For any $b \in B(\mathbf{A})$ and $g \in G(\mathbf{A})$,

$$\|\rho(gb)e_0\| = e^{H(b)}\|\rho(g)e_0\|,$$

and

$$\|\rho(gb)e_2\| = e^{-H(b)}\|\rho(g)e_2\|,$$

We now compute,

$$\begin{aligned}
\|\rho(a^{-1}n_1^{-1}\gamma nn_1a)e_0\| &= \|\rho(a^{-1}n_1^{-1}b_0wn_0nn_1a)e_0\| \\
&= \|\rho((a^{-1}n_1^{-1}b_0w)n_0nn_1a)e_0\| \\
&= e^{H(a)}\|\rho(a^{-1}n_1^{-1}b_0w)e_0\| \\
&= e^{H(a)}\|\rho(a^{-1}n_1^{-1}n_0'aa^{-1}bm0)e_2\| \\
&= e^{2H(a)}\|\rho(a^{-1}n_1^{-1}n_0'a)e_2\|.
\end{aligned}$$

We note that,

$$\rho\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} e_2 = -x^2e_0 + xe_1 + e_2$$

and hence that $\|\rho(n')e_2\| \geq \|e_2\| = 1$ for all $n' \in N(\mathbf{A})$. Hence,

$$\|\rho(a^{-1}n_1^{-1}\gamma nn_1a)e_0\| \geq e^{2H(a)}.$$

Hence the lemma follows if we take $d_\Omega > d_0$. $\square$

Applying this lemma to the support of f we see that if T is sufficiently large, then

$$K_f^T(x,x) = \sum_{\gamma \in Z(F)\backslash G(F)} f(x^{-1}\gamma x) - \sum_{\delta \in B(F)\backslash G(F)} \left(\int_{N(F)\backslash N(\mathbf{A})} \sum_{\gamma \in Z(F)\backslash B(F)} f(x^{-1}\delta^{-1}\gamma n \delta x)\ dn \right) \tau(H(\delta x) - T).$$

Since we have $B(F) = N(F)M(F)$ we can exchange the sum and integral to give,

$$K_f^T(x,x) = \sum_{\gamma \in Z(F)\backslash G(F)} f(x^{-1}\gamma x) - \sum_{\delta \in B(F)\backslash G(F)} \left(\sum_{\gamma \in Z(F)\backslash M(F)} \int_{N(\mathbf{A})} f(x^{-1}\delta^{-1}\gamma n \delta x)\ dn \right) \tau(H(\delta x) - T).$$

For each equivalence class $\mathfrak{o} \in \mathfrak{O}$ we define,

$$K_{f,\mathfrak{o}}^T(x,x) = \sum_{\gamma \in \mathfrak{o}} f(x^{-1}\gamma x) - \sum_{\delta \in B(F)\backslash G(F)} \left(\sum_{\gamma \in Z(F)\backslash M(F) \cap \mathfrak{o}} \int_{N(\mathbf{A})} f(x^{-1}\delta^{-1}\gamma n \delta x)\ dn \right) \tau(H(\delta x) - T)$$

so that for T sufficiently large with respect to the support of f,

$$K_f^T(x,x) = \sum_{\mathfrak{o} \in \mathfrak{O}} K_{f,\mathfrak{o}}^T(x,x).$$

We want to prove the following,

Theorem 7.5. *Let $f \in C_c^\infty(Z(\mathbf{A})\backslash G(\mathbf{A}))$. Then for T sufficiently large (with respect to the support of f),*

$$\sum_{\mathfrak{o} \in \mathfrak{O}} \int_{Z(\mathbf{A})G(F)\backslash G(\mathbf{A})} |K_{f,\mathfrak{o}}^T(x,x)|\ dx < \infty.$$

We begin with the elliptic conjugacy classes. By Lemma 7.4 if T is sufficiently large,

$$K_{f,ell}^T(x,x) = \sum_{\substack{\mathfrak{o} \in \mathfrak{O} \\ \text{elliptic}}} K_{f,\mathfrak{o}}^T(x,x) = \sum_{\substack{\mathfrak{o} \in \mathfrak{O} \\ \text{elliptic}}} K_{f,\mathfrak{o}}(x,x) = K_{f,ell}(x,x).$$

Proposition 7.6. *The function $K_{f,ell}(x,x)$ is absolutely integrable over $Z(\mathbf{A})G(F)\backslash G(\mathbf{A})$. Furthermore,*

$$\int_{Z(\mathbf{A})G(F)\backslash G(\mathbf{A})} K_{f,ell}(x,x)\ dx = \sum_{\substack{\gamma \in \Gamma(Z(F)\backslash G(F)) \\ elliptic}} \mathrm{vol}(Z(\mathbf{A})G_\gamma(F)\backslash G_\gamma(\mathbf{A})) \int_{G_\gamma(\mathbf{A})\backslash G(\mathbf{A})} f(g^{-1}\gamma g)\ dg.$$

Proof. It suffices to prove the first assertion. We note that $K_{f,ell}(x,x)$ is a smooth function on $Z(\mathbf{A})G(F)\backslash G(\mathbf{A})$ and we will show that it is compactly supported on $Z(\mathbf{A})G(F)\backslash G(\mathbf{A})$. Recall,

$$K_{f,ell}(x,x) = \sum_{\substack{\gamma \in \Gamma(Z(F)\backslash G(F)) \\ \text{elliptic}}} f(x^{-1}\gamma x).$$

By Lemma 7.4 there exists d such that,

$$K_{f,ell}(x,x) = 0$$

for all $x \in G(\mathbf{A})$ with $H(x) > d$. We take a Siegel domain $\mathfrak{S}_c$ for G such that $G(F)\mathfrak{S}_c = G(\mathbf{A})$. Recall, that in the notation above,

$$\mathfrak{S}_c = \{znh_t m_a k : z \in Z(\mathbf{A}), n \in C_1, t > c, a \in C_2, k \in K\}.$$

Hence on $\mathfrak{S}_c$, $K_{f,ell}(x,x)$ is supported on,

$$\mathfrak{S}_c = \{znh_t m_a k : z \in Z(\mathbf{A}), n \in C_1, c \leq t \leq d, a \in C_2, k \in K\}.$$

Thus $K_{f,ell}(x,x)$ is compactly supported on $Z(\mathbf{A})\backslash\mathfrak{S}_c$ and hence also on $Z(\mathbf{A})G(F)\backslash G(\mathbf{A})$. $\square$

We now consider the non-elliptic terms. Let $\mathfrak{o}$ be a non-elliptic equivalence class. Then, by definition,

$$K^T_{f,\mathfrak{o}}(x,x) = \sum_{\gamma \in \mathfrak{o}} f(x^{-1}\gamma x) - \sum_{\delta \in B(F)\backslash G(F)} \left(\sum_{\gamma \in Z(F)\backslash M(F)\cap\mathfrak{o}} \int_{N(\mathbf{A})} f(x^{-1}\delta^{-1}\gamma n\delta x)\ dn \right) \tau(H(\delta x) - T).$$

We define, $F_f \in C_c^\infty(\mathbf{A})$ by,

$$F_f(a) = \int_K f\left(k^{-1}\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} k\right)\ dk.$$

Before continuing we recall some details from Tate's Thesis; see [Bum97, Section3.1]. Let $f \in C_c^\infty(\mathbf{A})$. One defines a zeta integral,

$$Z(f,s) = \int_{\mathbf{A}^\times} f(a)|a|^s\ d^\times a.$$

This defines an analytic function of s provided $\Re s > 1$ and has a meromorphic continuation to $\mathbf{C}$. This is obtained by writing,

$$Z(f,s) = \int_{F^\times\backslash\mathbf{A}^\times} \sum_{\alpha \in F^\times} f(\alpha a)|a|^s\ d^\times a$$

which we then break up as, the sum of

$$\int_{|a|\geq 1} \sum_{\alpha \in F^\times} f(\alpha a)|a|^s\ d^\times a$$

and

$$\int_{|a|\leq 1} \sum_{\alpha\in F^\times} f(\alpha a)|a|^s \; d^\times a.$$

The first integral defines an analytic function on $\mathbf{C}$. For the second we obtain the meromorphic continuation by using Poisson summation. We fix a non-trivial additive character $\psi : F\backslash\mathbf{A} \to \mathbf{C}^\times$ and define,

$$\widehat{f}(x) = \int_{\mathbf{A}} f(y)\psi(xy) \; dy.$$

Then Poisson summation gives us, for an appropriate choice of measure,

$$\sum_{\alpha\in F} f(\alpha) = \sum_{\alpha\in F} \widehat{f}(\alpha)$$

or more generally, for $a \in \mathbf{A}^\times$,

$$\sum_{\alpha\in F^\times} f(a\alpha) + f(0) = |a|^{-1} \sum_{\alpha\in F^\times} \widehat{f}(a^{-1}\alpha) + |a|^{-1}\widehat{F}_f(0).$$

Hence the second integral above is equal to,

$$\int_{|a|\leq 1} \left(|a|^{-1} \sum_{\alpha\in F^\times} \widehat{f}(a^{-1}\alpha) + |a|^{-1}\widehat{f}(0) - f(0) \right) |a|^s \; d^\times a,$$

which is equal to,

$$\int_{|a|\leq 1} \sum_{\alpha\in F^\times} \widehat{f}(a^{-1}\alpha)|a|^{s-1} + \widehat{f}(0)|a|^{s-1} - f(0)|a|^s \; d^\times a.$$

The first part of the integral defines an analytic function of $\mathbf{C}$. The second and third are equal to,

$$\widehat{f}(0)\frac{\mathrm{vol}(F^\times\backslash\mathbf{A}^1)}{s} - f(0)\frac{\mathrm{vol}(F^\times\backslash\mathbf{A}^1)}{1-s}.$$

Thus we see that,

$$Z(f,s) + f(0)\frac{\mathrm{vol}(F^\times\backslash\mathbf{A}^1)}{1-s}$$

is analytic at $s = 1$. We denote the value of this function at $s = 1$ as,

$$\mathrm{f.p.}_{s=1}(Z(f,s))$$

Proposition 7.7. *Let $\mathfrak{o}$ be the equivalence class of the identity. Then $K^T_{f,\mathfrak{o}}(x,x)$ is absolutely integrable over $Z(\mathbf{A})G(F)\backslash G(\mathbf{A})$ and,*

$$\int_{Z(\mathbf{A})G(F)\backslash G(\mathbf{A})} K^T_{f,\mathfrak{o}}(x,x) \; dx$$

is equal to,

$$\mathrm{vol}(Z(\mathbf{A})G(F)\backslash G(\mathbf{A}))f(I) + f.p._{s=1}(Z(F_f,s)) + T\,\mathrm{vol}(F^\times\backslash\mathbf{A}^1)\int_{\mathbf{A}} F_f(y)\ dy.$$

We remark that $Z(F_f,1)$ is equal to the orbital integral of f at a unipotent element.

Proof. We have,

$$\mathfrak{o} = \left\{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right\}\bigcup\left\{\delta^{-1}\begin{pmatrix}1 & \alpha\\ 0 & 1\end{pmatrix}\delta : \alpha\in F, \delta\in B(F)\backslash G(F)\right\}.$$

Thus we have,

$$K^T_{f,\mathfrak{o}}(x,x) = f\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \sum_{\delta\in B(F)\backslash G(F)}\sum_{\alpha\in F} f\left(x^{-1}\delta^{-1}\begin{pmatrix}1 & \alpha\\ 0 & 1\end{pmatrix}\delta x\right)$$
$$- \sum_{\delta\in B(F)\backslash G(F)}\left(\int_{N(\mathbf{A})} f(x^{-1}\delta^{-1}n\delta x)\ dn\right)\tau(H(\delta x)-T).$$

The first term gives the first contribution to our integral. For the second part we can replace our integral over $Z(\mathbf{A})G(F)\backslash G(\mathbf{A})$ by an integral over $Z(\mathbf{A})B(F)\backslash G(\mathbf{A})$ of,

$$\sum_{\alpha\in F} f\left(x^{-1}\begin{pmatrix}1 & \alpha\\ 0 & 1\end{pmatrix}x\right) - \left(\int_{N(\mathbf{A})} f(x^{-1}nx)\ dn\right)\tau(H(x)-T).$$

We again use the Iwasawa decomposition $G(\mathbf{A}) = N(\mathbf{A})M(\mathbf{A})K$ to write,

$$x = \begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}\begin{pmatrix}b & 0\\ 0 & 1\end{pmatrix}k$$

and we can decompose the Haar measure dg on $G(\mathbf{A})$ as $dg = |b|^{-1}da\ d^\times b\ dk$. The measure da is chosen so that $\mathrm{vol}(F\backslash\mathbf{A}) = 1$. Note that in this notation, we have $H(x) = \log|b|$ and

$$x^{-1}\begin{pmatrix}1 & \alpha\\ 0 & 1\end{pmatrix}x = k^{-1}\begin{pmatrix}1 & b^{-1}\alpha\\ 0 & 1\end{pmatrix}k.$$

We see that the integral of

$$\sum_{\alpha\in F} f\left(x^{-1}\begin{pmatrix}1 & \alpha\\ 0 & 1\end{pmatrix}x\right) - \left(\int_{N(\mathbf{A})} f(x^{-1}nx)\ dn\right)\tau(H(x)-T).$$

over $Z(\mathbf{A})B(F)\backslash G(\mathbf{A})$ is equal to,

$$\int_{F^\times\backslash\mathbf{A}^\times}\left(\sum_{\alpha\in F^\times} F_f(b^{-1}\alpha) - \int_{\mathbf{A}} F_f(b^{-1}a)\ da\ \tau(\log|b|-T)\right)\ |b|^{-1}\ d^\times b$$

which equals,

$$\int_{F^\times\backslash \mathbf{A}^\times} \left(\sum_{\alpha \in F^\times} F_f(b^{-1}\alpha) - \int_{\mathbf{A}} F_f(a)\ da\ |b|\tau(\log|b| - T) \right)\ |b|^{-1}\ d^\times b$$

after making a change of variables in the integral over $\mathbf{A}$.

We can rewrite our integral as,

$$\int_{F^\times\backslash \mathbf{A}^\times} \left(\sum_{\alpha \in F^\times} F_f(b^{-1}\alpha) - \widehat{F}_f(0)|b|\tau(\log|b| - T) \right)\ |b|^{-1}\ d^\times b.$$

which we break up as, the sum of,

$$\int_{|b|\leq 1} \left(\sum_{\alpha \in F^\times} F_f(b^{-1}\alpha) \right)\ |b|^{-1}\ d^\times b$$

and

$$\int_{|b|\geq 1} \left(\sum_{\alpha \in F^\times} F_f(b^{-1}\alpha) - \widehat{F}_f(0)|b|\tau(\log|b| - T) \right)\ |b|^{-1}\ d^\times b.$$

We apply Poisson summation,

$$\sum_{\alpha \in F^\times} F_f(b^{-1}\alpha) + F_f(0) = |b| \sum_{\alpha \in F^\times} \widehat{F}_f(b\alpha) + |b|\widehat{F}_f(0)$$

to the second integral to yield,

$$\int_{|b|\geq 1} \left(\sum_{\alpha \in F^\times} \widehat{F}_f(b\alpha) - F_f(0)|b|^{-1} + \widehat{F}_f(0)(1 - \tau(\log|b| - T)) \right)\ d^\times b.$$

Summing up gives the integral as,

$$\int_{|b|\geq 1} \sum_{\alpha \in F^\times} F_f(b\alpha)|b| + \sum_{\alpha \in F^\times} \widehat{F}_f(b\alpha) - F_f(0)|b|^{-1} + \widehat{F}_f(0)(1 - \tau(\log|b| - T))\ d^\times b.$$

This proves the absolute convergence of the integral and we see that it is equal to,

$$\text{f.p.}_{s=1}(Z(f,s)) + T\,\mathrm{vol}(F^\times\backslash \mathbf{A}^1)\widehat{F}_f(0).$$

□

We now look at the contributions from non-identity elements lying in a split torus.

Proposition 7.8. *Let $\mathfrak{o}$ be the equivalence class of a non-identity element $\gamma \in M(F)$. Then $K^T_{f,\mathfrak{o}}(x,x)$ is absolutely integrable over $Z(\mathbf{A})G(F)\backslash G(\mathbf{A})$ and,*

$$\int_{Z(\mathbf{A})G(F)\backslash G(\mathbf{A})} K^T_{f,\mathfrak{o}}(x,x)\ dx$$

is equal to,

$$2T\,\mathrm{vol}(F^\times\backslash\mathbf{A}_F^1)\int_{G_\gamma(\mathbf{A})\backslash G(\mathbf{A})} f\left(g^{-1}\gamma g\right)\,dg - \mathrm{vol}(F^\times\backslash\mathbf{A}_F^1)\int_{G_\gamma(\mathbf{A})\backslash G(\mathbf{A})} f(g^{-1}\gamma g)H(wg)\,dg$$

if $w\notin\mathfrak{o}$ and is equal to $\frac{1}{2}$ of this is $w\in\mathfrak{o}$.

Proof. We take the absolute convergence of the integral for granted, see [Gel96, Lecture II, Section 4] for a proof along the same lines as the one above. We will instead derive the formula given in the Proposition.

We now take a semisimple element of the form,

$$\gamma=\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}$$

with $\alpha\neq 1$. Then,

$$\mathfrak{o}=\left\{\delta^{-1}\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\delta : \delta\in M(F)\backslash G(F)\right\}.$$

We note that since we are working mod the center if $\alpha=-1$ then as δ runs through $M(F)\backslash G(F)$ each element of $\mathfrak{o}$ is counted twice. We will assume throughout that $\alpha\neq -1$, otherwise one should insert $\frac{1}{2}$ in appropriate places. We recall,

$$K_{f,\mathfrak{o}}^T(x,x)=\sum_{\delta\in M(F)\backslash G(F)} f\left(x^{-1}\delta^{-1}\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\delta x\right)$$
$$-\sum_{\delta\in B(F)\backslash G(F)}\left(\int_{N(\mathbf{A})} f\left(x^{-1}\delta^{-1}\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}n\delta x\right)\,dn\right)\tau(H(\delta x)-T)$$
$$-\sum_{\delta\in B(F)\backslash G(F)}\left(\int_{N(\mathbf{A})} f\left(x^{-1}\delta^{-1}\begin{pmatrix}\alpha^{-1} & 0\\ 0 & 1\end{pmatrix}n\delta x\right)\,dn\right)\tau(H(\delta x)-T).$$

We rewrite the first sum as,

$$\sum_{\delta\in B(F)\backslash G(F)}\sum_{\nu\in N(F)} f\left(x^{-1}\delta^{-1}\nu^{-1}\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\nu\delta x\right).$$

We make the change of variables,

$$n\mapsto\begin{pmatrix}\alpha^{-1} & 0\\ 0 & 1\end{pmatrix}n^{-1}\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}n$$

in the first integral to yield,

$$\sum_{\delta\in B(F)\backslash G(F)}\left(\int_{N(\mathbf{A})} f\left(x^{-1}\delta^{-1}n^{-1}\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}n\delta x\right)\,dn\right)\tau(H(\delta x)-T)$$

and we make a similar change in the second to yield,

$$\sum_{\delta \in B(F)\backslash G(F)} \left(\int_{N(\mathbf{A})} f\left(x^{-1}\delta^{-1}n^{-1} \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & 1 \end{pmatrix} n\delta x \right) \, dn \right) \tau(H(\delta x) - T).$$

We again replace our integral over $Z(\mathbf{A})G(F)\backslash G(\mathbf{A})$ with one over $Z(\mathbf{A})B(F)\backslash G(\mathbf{A})$ of

$$\begin{aligned} &\sum_{\nu \in N(F)} f\left(x^{-1}\nu^{-1} \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \nu x \right) \\ &- \left(\int_{N(\mathbf{A})} f\left(x^{-1}n^{-1} \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} nx \right) \, dn \right) \tau(H(x) - T) \\ &- \left(\int_{N(\mathbf{A})} f\left(x^{-1}n^{-1} \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & 1 \end{pmatrix} nx \right) \, dn \right) \tau(H(x) - T). \end{aligned}$$

We can compute this by first integrating over $N(F)\backslash N(\mathbf{A})$ and then over $Z(\mathbf{A})N(\mathbf{A})M(F)\backslash G(\mathbf{A})$. The first integral yields,

$$\begin{aligned} &\int_{N(\mathbf{A})} f\left(x^{-1}n^{-1} \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} nx \right) \, dn \\ &- \left(\int_{N(\mathbf{A})} f\left(x^{-1}n^{-1} \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} nx \right) \, dn \right) \tau(H(x) - T) \\ &- \left(\int_{N(\mathbf{A})} f\left(x^{-1}n^{-1} \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & 1 \end{pmatrix} nx \right) \, dn \right) \tau(H(x) - T). \end{aligned}$$

The integral over $Z(\mathbf{A})N(\mathbf{A})M(F)\backslash G(\mathbf{A})$ of this function is the same as the integral over $Z(\mathbf{A})M(F)\backslash G(\mathbf{A})$ of,

$$f\left(x^{-1} \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} x \right) - f\left(x^{-1} \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} x \right) \tau(H(x) - T) - f\left(x^{-1} \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & 1 \end{pmatrix} x \right) \tau(H(x) - T),$$

which equals the integral over $Z(\mathbf{A})M(F)\backslash G(\mathbf{A})$ of,

$$f\left(x^{-1} \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} x \right) \left(1 - \tau(H(x) - T) - \tau(H(wx) - T)\right).$$

We again use the Iwasawa decomposition $G(\mathbf{A}) = M(\mathbf{A})N(\mathbf{A})K$ to write our elements in the form,

$$g = \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} k$$

so that $H(g) = \log |b|$ and we decompose the measure on $G(\mathbf{A})$ as,

$$dg = d^\times b \; da \; dk.$$

The integral becomes,

$$\int_K \int_{N(\mathbf{A})} f\left(k^{-1}n^{-1}\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix} nk\right)\, dn\, dk \int_{F^\times\backslash\mathbf{A}^\times} (1-\tau(\log|b|-T)-\tau(H(wn)-\log|b|-T))d^\times b.$$

We note that the integrand in the integral over $F^\times\backslash\mathbf{A}^\times$ is zero unless $H(wn) - T < \log|b| < T$ in which case it is one. Hence our integral is equal to,

$$\begin{aligned} 2T\,\mathrm{vol}(F^\times\backslash\mathbf{A}_F^1)\int_K \int_{N(\mathbf{A})} f\left(k^{-1}n^{-1}\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix} nk\right)\, dn\, dk \\ -\mathrm{vol}(F^\times\backslash\mathbf{A}_F^1)\int_K \int_{N(\mathbf{A})} f\left(k^{-1}n^{-1}\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix} nk\right) H(wn)\, dn\, dk. \end{aligned}$$

□

Finally we finish the proof of Theorem 7.5. We have,

$$K_f^T(x,x) = \sum_{\mathfrak{o}\in\mathfrak{O}} K_{f,\mathfrak{o}}^T(x,x).$$

Propositions 7.6, 7.7 and 7.8 tells us that each $K_{f,\mathfrak{o}}(x,x)$ is absolutely integrable over $Z(\mathbf{A})G(F)\backslash G(\mathbf{A})$. It remains to observe that for non elliptic $\mathfrak{o}$ the functions $K_{f,\mathfrak{o}}^T(x,x)$ are identically zero except for finitely many $\mathfrak{o}$. For this we note that if $\Omega \subset G(\mathbf{A})$ is compact modulo $Z(\mathbf{A})$ then,

$$x^{-1}\delta^{-1}\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\delta x,\, x^{-1}\delta^{-1}\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix} n\delta x,$$

belong to Ω for only finitely many α.

Putting everything together we have,

Theorem 7.9. *For T sufficiently large with respect to the support of f,*

$$\int_{Z(\mathbf{A})G(F)\backslash G(\mathbf{A})} K_f^T(x,x)\, dx$$

is absolutely integrable, and is equal to the sum of the following terms,

$$\mathrm{vol}(Z(\mathbf{A})G(F)\backslash G(\mathbf{A}))f(I) + \sum_{\substack{\gamma\in\Gamma(Z(F)\backslash G(F))\\ ell.\ reg.}} \mathrm{vol}(G_\gamma(F)\backslash G_\gamma(\mathbf{A}))\int_{G_\gamma(\mathbf{A})\backslash G(\mathbf{A})} f(g^{-1}\gamma g)\, dg$$

$$f.p._{s=1}(Z(F_f,s)) - \frac{1}{2}\mathrm{vol}(F^\times\backslash\mathbf{A}^1)\sum_{\gamma\in M(F)}\int_{M(\mathbf{A})\backslash G(\mathbf{A})} f(g^{-1}\gamma g)H(wg)\, dg$$

$$T\,\mathrm{vol}(F^\times\backslash\mathbf{A}^1)\sum_{\gamma\in M(F)}\int_{M(\mathbf{A})\backslash G(\mathbf{A})} f(g^{-1}\gamma g)\, dg + T\,\mathrm{vol}(F^\times\backslash\mathbf{A}^1)\int_{\mathbf{A}} F_f(y)\, dy$$

8 Spectral side of the trace formula for GL(2)

We will now derive a spectral expansion for,

$$\int_{Z(\mathbf{A})G(F)\backslash G(\mathbf{A})} K_f^T(x,x)\ dx.$$

8.1 The space of cusp forms

We recall that in the case of a compact quotient we used the fact that the operators $R(f)$ were Hilbert-Schmidt to deduce that,

$$L^2(Z(\mathbf{A})G(F)\backslash G(\mathbf{A})) = \bigoplus_{\pi} m_\pi \pi$$

with the multiplicities m_π being finite. We then had,

$$\operatorname{tr} R(f) = \sum_{\pi} m_\pi \operatorname{tr} \pi(f).$$

Recall we have defined,

$$L^2_{\text{cusp}}(\overline{G}(F)\backslash\overline{G}(\mathbf{A}))$$

to be the closed $\overline{G}(\mathbf{A})$-invariant subspace of $L^2(\overline{G}(F)\backslash\overline{G}(\mathbf{A}))$ of functions $\varphi \in L^2(\overline{G}(F)\backslash\overline{G}(\mathbf{A}))$ such that,

$$\int_{N(F)\backslash N(\mathbf{A})} \varphi(ng)\ dn \equiv 0.$$

Theorem 8.1. *As a representation of $G(\mathbf{A})$,*

$$L^2_{\text{cusp}}(Z(\mathbf{A})G(F)\backslash G(\mathbf{A})) = \bigoplus_{\pi\in\widehat{G(\mathbf{A})}} m_\pi \pi$$

with finite multiplicities m_π.

This will follow as in Theorem 3.16 if we can prove that the operators $R(f)$ when restricted to $L^2_{\text{cusp}}(G)$ are Hilbert-Schmidt.

We again take $f \in C_c^\infty(\overline{G}(\mathbf{A}))$. Let $\varphi \in L^2(\overline{G}(F)\backslash\overline{G}(\mathbf{A}))$ then we have,

$$(R(f)\varphi)(x) = \int_{\overline{G}(\mathbf{A})} f(y)\varphi(xy)\ dy = \int_{N(F)\backslash\overline{G}(\mathbf{A})} H_f(x,y)\varphi(y)\ dy$$

where we define

$$H_f(x,y) = \sum_{\gamma\in N(F)} f(x^{-1}\gamma y).$$

We set,

$$H'_f(x,y) = H_f(x,y) - \int_{N(\mathbf{A})} f(x^{-1}ny)\ dn.$$

(Again taking the measure on $N(\mathbf{A})$ such that $\mathrm{vol}(N(F)\backslash N(\mathbf{A})) = 1$.) Then for $\varphi \in L^2_{\mathrm{cusp}}(\overline{G}(F)\backslash\overline{G}(\mathbf{A}))$,

$$(R(f)\varphi)(x) = \int_{N(F)\backslash\overline{G}(\mathbf{A})} H'_f(x,y)\varphi(y)\ dy.$$

For this it suffices to note that for $\varphi \in L^2_{\mathrm{cusp}}(\overline{G}(F)\backslash\overline{G}(\mathbf{A}))$,

$$\int_{N(F)\backslash\overline{G}(\mathbf{A})} \varphi(y) \int_{N(\mathbf{A})} f(x^{-1}ny)\ dn\ dy = \int_{N(\mathbf{A})\backslash\overline{G}(\mathbf{A})} f(x^{-1}y) \int_{N(F)\backslash N(\mathbf{A})} \varphi(n^{-1}y)\ dn\ dy = 0.$$

We now need to estimate the size of $H'_f(x,y)$. We again take a Siegel domain,

$$\mathfrak{S} = \left\{ znh_t m_a k : z \in Z(\mathbf{A}), n \in C_1, a \in C_2, t > \frac{c}{2}, k \in K \right\}$$

such that $G(F)\mathfrak{S} = G(\mathbf{A})$. Here C_1 and C_2 are suitable compact subsets of $N(\mathbf{A})$ and $\mathbf{A}^1$ respectively. We recall that,

$$m_a = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}.$$

For $t \in \mathbf{R}$ we set $x_t = (x_{t,v}) \in \mathbf{A}$ such that $x_{t,v} = 1$ if v is finite and $x_{t,v} = e^t$ if v is infinite. Then,

$$h_t = \begin{pmatrix} x_t & 0 \\ 0 & x_t^{-1} \end{pmatrix}.$$

We identify $\mathfrak{S}$ with its image in $N(F)\backslash G(\mathbf{A})$ and we denote by $L^2(\mathfrak{S})$ the Hilbert space of functions $f : Z(\mathbf{A})N(F)\backslash\mathfrak{S} \to \mathbf{C}$ such that,

$$\int_{N(F)Z(\mathbf{A})\backslash\mathfrak{S}} |f(g)|^2\ dg < \infty.$$

We note that restricting a function

$$\varphi \in L^2(\overline{G}(F)\backslash\overline{G}(\mathbf{A}))$$

to $Z(\mathbf{A})N(F)\backslash\mathfrak{S}$ allows us to identify, $L^2(\overline{G}(F)\backslash\overline{G}(\mathbf{A})$ with a closed subspace of $L^2(\mathfrak{S})$. Furthermore since $\mathfrak{S}\cap\gamma\mathfrak{S} \neq \emptyset$ for only finitely many $\gamma \in Z(F)\backslash G(F)$ so the inclusion map,

$$L^2(\overline{G}(F)\backslash\overline{G}(\mathbf{A})) \hookrightarrow L^2(\mathfrak{S})$$

and the orthogonal projection

$$L^2(\mathfrak{S}) \to L^2(\overline{G}(F)\backslash\overline{G}(\mathbf{A}))$$

are bounded operators.

Let Ω_f denote a subset of $G(\mathbf{A})$ which is compact modulo $Z(\mathbf{A})$ and such that f is supported on Ω_f. Suppose $x \in \mathfrak{S}$ and $y \in G(\mathbf{A})$ are such that $N'(x,y) \neq 0$. Then there exists $n' \in N(\mathbf{A})$ such that,

$$x^{-1}n'y \in \Omega_f.$$

We write $x = znm_ah_tk$ so that we have,

$$y \in (n')^{-1}x\Omega_f = z(n')^{-1}nm_ah_tk\Omega_f.$$

Enlarging Ω_f if needs be we can assume that,

$$y \in (n')^{-1}x\Omega_f = z(n')^{-1}nm_ah_t\Omega_f.$$

Using the Iwasawa decomposition we have $\Omega_f \subset CK$ with $C \subset B(\mathbf{A})$ which is compact modulo $Z(\mathbf{A})$. Using this we see that there exists a Siegel set $\mathfrak{S}'$ such that once we identify $\mathfrak{S}'$ with its image in $Z(\mathbf{A})N(F)\backslash G(\mathbf{A})$ so $H'(x,y) \neq 0$ implies that y lies in $\mathfrak{S}'$. Again after enlarging $\mathfrak{S}'$ if needs be we may assume that $G(F)\mathfrak{S}' = G(\mathbf{A})$.

We wish to show that,

$$\int_{N(F)Z(\mathbf{A})\backslash\mathfrak{S}} \int_{N(F)Z(\mathbf{A})\backslash\mathfrak{S}'} |H'_f(x,y)|^2 \, dx \, dy < \infty.$$

Having fixed $f \in C_c^\infty(\overline{G}(\mathbf{A}))$ we let Ω_f be a compact subset of $G(\mathbf{A})$ such that the support of f is contained in $Z(\mathbf{A})\Omega_f$.

For $x, y \in G(\mathbf{A})$ and $a \in \mathbf{A}$ we define,

$$\Phi_{x,y}(a) = f\left(x^{-1}\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}y\right).$$

Then $\Phi_{x,y} \in C_c^\infty(\mathbf{A})$. The Fourier transform of $\Phi_{x,y}$ is defined by,

$$\widehat{\varphi}_{x,y}(\alpha) = \int_{F\backslash\mathbf{A}} \Phi_{x,y}(a)\psi(a\alpha) \, da.$$

Here $\psi : F\backslash\mathbf{A} \to \mathbf{C}^\times$ is a character such that the measure da on $\mathbf{A}$ which gives $F\backslash\mathbf{A}$ volume one is self-dual with respect to ψ. We have,

$$H'_f(x,y) = \sum_{\alpha\in F} \varphi_{x,y}(\alpha) - \widehat{\Phi}_{x,y}(0).$$

Hence after applying Poisson summation we have,

$$H'_f(x,y) = \sum_{\alpha\in F^\times} \widehat{\Phi}_{x,y}(\alpha).$$

We write,

$$x = z\begin{pmatrix}1 & n_x\\ 0 & 1\end{pmatrix}m_ah_tk \in \mathfrak{S}$$

and

$$y = z'\begin{pmatrix}1 & n_y\\ 0 & 1\end{pmatrix}m_{a'}h_{t'}k' \in \mathfrak{S}'.$$

Then we have,

$$\Phi_{x,y}(b) = f\left(k^{-1}h_t^{-1}m_a^{-1}\begin{pmatrix}1 & b-n_x+n_y\\ 0 & 1\end{pmatrix}m_{a'}h_{t'}k'\right).$$

Thus,

$$\widehat{\Phi}_{x,y}(\alpha) = \int_{\mathbf{A}} f\left(k^{-1}h_t^{-1}m_a^{-1}\begin{pmatrix}1 & b-n_x+n_y\\ 0 & 1\end{pmatrix}m_{a'}h_{t'}k'\right)\psi(b\alpha)\ db.$$

We make a change of variables in b to give,

$$\widehat{\Phi}_{x,y}(\alpha) = \psi(\alpha(n_x-n_y))\int_{\mathbf{A}} f\left(k^{-1}h_t^{-1}m_a^{-1}\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}m_{a'}h_{t'}k'\right)\psi(b\alpha)\ db$$

which equals,

$$\widehat{\Phi}_{x,y}(\alpha) = \psi(\alpha(n_x-n_y))\int_{\mathbf{A}} f\left(k^{-1}\begin{pmatrix}1 & a^{-1}x_{2t}^{-1}b\\ 0 & 1\end{pmatrix}m_{a'a^{-1}}h_{t'-t}k'\right)\psi(b\alpha)\ db.$$

We make another change of variables to yield,

$$\widehat{\Phi}_{x,y}(\alpha) = \psi(\alpha(n_x-n_y))|x_{2t}|\int_{\mathbf{A}} f\left(k^{-1}\begin{pmatrix}1 & c\\ 0 & 1\end{pmatrix}m_{a'a^{-1}}h_{t'-t}k'\right)\psi(\alpha a x_{2t}c)\ dc.$$

For $a \in \mathbf{A}^1$, $t \in \mathbf{R}$, $k \in K$ and $k' \in K$ we define,

$$F_{k,k',a,t}(b) = f\left(k^{-1}\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}m_a h_t k\right) \in C_c^\infty(\mathbf{A}).$$

Thus we have,

$$\widehat{\Phi}_{x,y}(\alpha) = \psi(\alpha(n_x-n_y))|x_{2t}|\widehat{F}_{k,k',a^{-1}a',-t+t'}(\alpha a x_{2t}).$$

We note that since f is compactly supported modulo $Z(\mathbf{A})$ so there exist compact subsets $\Omega_1 \subset \mathbf{A}^1$ and $\Omega_2 \subset \mathbf{R}$ such that, if

$$F_{k,k',a,t}(b) \neq 0$$

for some b and any k, k' then $a \in \Omega_1$ and $b \in \Omega_2$.

We have,

$$|H_f'(x,y)| \leq |x_{2t}| \sum_{\alpha \in F^\times} |\widehat{F}_{k,k',a^{-1}a',-t+t'}(\alpha a x_{2t})|,$$

where $\widehat{F}_{k,k',a,t}(y)$ is a Schwartz function of y, depending continuously on k, k', a and t and vanishing identically unless (k,k',a,t) lie in the compact set $K \times K \times \Omega_1 \times \Omega_2$. Thus given $N > 0$ there exists a constant C_N such that,

$$|H_f'(x,y)| \leq C_N e^{-Nt}.$$

We recall that using the Iwasawa decomposition, $g = znm_a h_t k$ we can write the measure on $\overline{G}(\mathbf{A})$ as,

$$dg = |x_{-2t}|\ dt\ d^{\times}a\ dn\ dk.$$

Using this, together with the bound on $|H'_f(x,y)|$ provided above we see that,

$$\int_{Z(\mathbf{A})N(F)\backslash\mathfrak{S}} \int_{Z(\mathbf{A})N(F)\backslash\mathfrak{S}'} |H'_f(x,y)|^2\ dx\ dy < \infty.$$

Hence $H'_f(x,y)$ is the kernel of a Hilbert-Schmidt operator,

$$B : L^2(\mathfrak{S}) \to L^2(\mathfrak{S}').$$

Thus we can realize $R(f)$, on $L^2_{\text{cusp}}(\overline{G}(F)\backslash\overline{G}(\mathbf{A}))$ as the composition of the maps,

$$L^2_{\text{cusp}}(\overline{G}(F)\backslash\overline{G}(\mathbf{A})) \hookrightarrow L^2(\mathfrak{S}) \to L^2(\mathfrak{S}') \to L^2_{\text{cusp}}(\overline{G}(F)\backslash\overline{G}(\mathbf{A})).$$

Hence we see that $R(f)$ is a Hilbert-Schmidt operator on $L^2_{\text{cusp}}(\overline{G}(F)\backslash\overline{G}(\mathbf{A}))$.

Corollary 8.2. *Let $f \in C_c^\infty(\overline{G}(\mathbf{A}))$ then (at least if $f = f_1 * f_2$) if we decompose,*

$$L^2_{\text{cusp}}(Z(\mathbf{A})G(F)\backslash G(\mathbf{A})) = \bigoplus_{\pi\in\widehat{G(\mathbf{A})}} m_\pi \pi$$

we have

$$\operatorname{tr} R(f)|_{L^2_{\text{cusp}}(\overline{G}(F)\backslash\overline{G}(\mathbf{A}))} = \sum_{\pi\in\widehat{\overline{G}(\mathbf{A})}} m_\pi \operatorname{tr}\pi(f).$$

8.2 The orthogonal complement of the cusp forms

The main reference for this section is [GJ79, Sections 3, 4 & 5].

We want to give an explicit construction of the space orthogonal to $L^2_{\text{cusp}}(\overline{G}(F)\backslash\overline{G}(\mathbf{A}))$.

Let f be a function on $G(\mathbf{A})$ such that,

$$f(n\gamma zg) = f(g)$$

for all $n \in N(\mathbf{A})$, $\gamma \in B(F)$, $z \in Z(\mathbf{A})$ and $g \in G(\mathbf{A})$. We form the series,

$$F(g) = \sum_{\gamma\in B(F)\backslash G(F)} f(\gamma g).$$

Recall from Lemma 7.3 that if $\mathfrak{S}$ is a Siegel set and $T \in \mathbf{R}$ then the set of $\gamma \in G(F)$ such that $H(\gamma g) > T$ for some $g \in \mathfrak{S}$ is finite modulo $B(F)$. Thus if f is supported in,

$$\{g \in G(\mathbf{A}) : H(g) > T\}$$

for some $T \in \mathbf{R}$ then the series defining $F(g)$ is finite. In particular if f is compactly supported modulo $N(\mathbf{A})Z(\mathbf{A})B(F)$ then the series defining F is finite and moreover F will be compactly supported modulo $Z(\mathbf{A})G(F)$, this follows from the fact that if $H(g)$ is large then $H(\gamma g)$ is small for all $\gamma \in G(F) \setminus B(F)$. Suppose $\varphi \in L^2(\overline{G}(F)\backslash\overline{G}(\mathbf{A}))$. Then,

$$\begin{aligned}
(\varphi, F) &= \int_{Z(\mathbf{A})G(F)\backslash G(\mathbf{A})} \varphi(g)\overline{F(g)}\, dg \\
&= \int_{Z(\mathbf{A})G(F)\backslash G(\mathbf{A})} \varphi(g) \sum_{\gamma\in B(F)\backslash G(F)} \overline{f(\gamma g)}\, dg \\
&= \int_{Z(\mathbf{A})G(F)\backslash G(\mathbf{A})} \sum_{\gamma\in B(F)\backslash G(F)} \varphi(\gamma g)\overline{f(\gamma g)}\, dg \\
&= \int_{Z(\mathbf{A})B(F)\backslash G(\mathbf{A})} \varphi(g)\overline{f(g)}\, dg \\
&= \int_{Z(\mathbf{A})N(\mathbf{A})B(F)\backslash G(\mathbf{A})} \int_{N(F)\backslash N(\mathbf{A})} \varphi(ng)\overline{f(ng)}\, dn\, dg \\
&= \int_{Z(\mathbf{A})N(\mathbf{A})B(F)\backslash G(\mathbf{A})} \overline{f(g)} \int_{N(F)\backslash N(\mathbf{A})} \varphi(ng)\, dn\, dg \\
&= \int_{Z(\mathbf{A})N(\mathbf{A})B(F)\backslash G(\mathbf{A})} \varphi_N(g)\overline{f(g)}\, dg.
\end{aligned}$$

Thus we see that if φ is cuspidal then φ is orthogonal to F. Conversely if φ is orthogonal to all series F coming from f which are compactly supported modulo $Z(\mathbf{A})N(\mathbf{A})B(F)$ then φ is cuspidal. Thus the series associated to compactly supported f span a dense subspace of the complement of $L^2_{\text{cusp}}(\overline{G}(F)\backslash\overline{G}(\mathbf{A}))$.

Suppose now we have two series F_1 and F_2 associated to f_1 and f_2. We now compute,

$$(F_1, F_2) = \int_{Z(\mathbf{A})N(\mathbf{A})B(F)\backslash G(\mathbf{A})} F_{1,N}(g)\overline{F_2(g)}\, dg.$$

For a series F we have,

$$\begin{aligned}
F_N(g) &= \int_{N(F)\backslash N(\mathbf{A})} F(ng)\, dn \\
&= \int_{N(F)\backslash N(\mathbf{A})} \sum_{\gamma\in B(F)\backslash G(F)} f(\gamma ng)\, dn.
\end{aligned}$$

From the Bruhat decomposition we have,

$$G(F) = B(F) \amalg B(F)wN(F).$$

Hence,

$$F(g) = f(g) + \sum_{\gamma\in N(F)} f(w\gamma g).$$

Thus,

$$F_N(g) = \int_{N(F)\backslash N(\mathbf{A})} f(ng)\ dn + \int_{N(F)\backslash N(\mathbf{A})} \sum_{\gamma \in N(F)} f(w\gamma ng)\ dn.$$

As usual we normalize the measure on $N(\mathbf{A})$ so that $\mathrm{vol}(N(F)\backslash N(\mathbf{A})) = 1$. Thus,

$$F_N(g) = f(g) + f'(g)$$

where,

$$f'(g) = \int_{N(\mathbf{A})} f(wng)\ dn.$$

Thus,

$$(F_1, F_2) = \int_{Z(\mathbf{A})N(\mathbf{A})B(F)\backslash G(\mathbf{A})} f_1(g)\overline{f_2(g)}\ dg + \int_{Z(\mathbf{A})N(\mathbf{A})B(F)\backslash G(\mathbf{A})} f_1'(g)\overline{f_2(g)}\ dg.$$

These manipulations are valid if f_1 and f_2 are compactly supported modulo $Z(\mathbf{A})N(\mathbf{A})M(F)$, although note that f_1' need not be compactly supported.

We fix measures such that,

$$\int_{\overline{G}(\mathbf{A})} f(g)\ dg = \int_K \int_{\mathbf{A}}^{\times} \int_{N(\mathbf{A})} f\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} k\right) |a|^{-1}\ dx\ d^{\times}a\ dk.$$

We define,

$$F_\infty^+ = \left\{ x_t \in \mathbf{A}^\times : t \in \mathbf{R}_{>0} \right\}$$

where $x_t = (x_{t,v})$ is such that, $x_{t,v} = 1$ if v is finite and $x_{t,v} = t$ if v is infinite.

For $s \in \mathbf{C}$ we consider functions $\varphi : G(\mathbf{A}) \to \mathbf{C}$ such that,

$$\varphi\left(\begin{pmatrix} \alpha a u & x \\ 0 & \beta a v \end{pmatrix} g\right) = \left|\frac{u}{v}\right|^{s+\frac{1}{2}} \varphi(g)$$

for all $\alpha, \beta \in F^\times$, $x \in \mathbf{A}$, $a \in \mathbf{A}^\times$ and $u, v \in F_\infty^+$. By the Iwasawa decomposition these functions are determined by their restriction to elements of the form,

$$g = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} k$$

with $a \in \mathbf{A}^1$ and $k \in K$. We set $H(s)$ equal to the space of such φ for which,

$$\int_K \int_{F^\times\backslash \mathbf{A}^1} \left|\varphi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} k\right)\right|^2\ da\ dk < \infty.$$

We have a representation π_s of $G(\mathbf{A})$ on $H(s)$ via right translation.

We note that we can identify each space $H(s)$ with $H(0)$. For $\varphi \in H(0)$ we obtain an element φ_s of $H(s)$ by defining,

$$\varphi_s(g) = e^{sH(g)}\varphi(g).$$

We think of $H(s)$ as a fibre bundle over $\mathbf{C}$. We note that this bundle is trivial since every $\varphi \in H(0)$ defines a section of the bundle via,

$$(g, s) \mapsto \varphi_s(g).$$

We define a pairing on $H(s) \times H(-\bar{s})$ by,

$$(\varphi_1, \varphi_2) = \int_{F^\times \backslash \mathbf{A}^1} \int_K \varphi_1 \left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} k \right) \overline{\varphi_2 \left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} k \right)} \, da \, dk.$$

We note that, for all $g \in G(\mathbf{A})$,

$$(\pi_s(g)\varphi_1, \pi_{-\bar{s}}(g)\varphi_2) = (\varphi_1, \varphi_2).$$

(See [Bum97, Lemma 2.6.1].)

We recall the Mellin transform. Suppose $f \in C_c(\mathbf{R}_{>0})$, one defines for $s \in \mathbf{C}$,

$$\varphi(s) = \int_0^\infty f(x) x^s \frac{dx}{x}.$$

This defines a holomorphic function of s and one can recover f from φ via,

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} \varphi(s) \, ds$$

for any $c \in \mathbf{R}$. One also has the Plancherel formula,

$$\int_0^\infty f_1(t) \overline{f_2(t)} \ fracdtt = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \varphi_1(s) \overline{\varphi_2(-\bar{s})} \, ds.$$

More generally consider functions f on $\mathbf{R}_{>0}$ which vanish for x sufficiently large and are $O(x^u)$ as $u \to 0$. In this case the Mellin transform,

$$\varphi(s) = \int_0^\infty f(x) x^s \frac{dx}{x}$$

is well defined for $\Re s > -u$. The inversion formula

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} \varphi(s) \, ds$$

will also hold provided $c > -u$.

Let f be a function on $G(\mathbf{A})$ such that,

$$f(zn\gamma g) = f(g)$$

for all $n \in \mathbf{N}(\mathbf{A})$, $\gamma \in B(F)$, $z \in Z(\mathbf{A})$ and $g \in G(\mathbf{A})$. We can assume further that f is smooth and compactly supported modulo $Z(\mathbf{A})N(\mathbf{A})B(F)$. We define the Mellin transform of f by,

$$\widehat{f}(g, s) = \int_{F_\infty^+} f\left(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} g \right) |t|^{-s-\frac{1}{2}} \, d^\times t,$$

which gives a section of the bundle $H(s)$. The Haar measure is chosen so that,

$$f(g) = \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} \widehat{f}(g,s)\ ds.$$

We again take compactly supported functions f_1 and f_2 on $Z(\mathbf{A})N(\mathbf{A})M(F)\backslash G(\mathbf{A})$ which give rise to the series F_1 and F_2. From above we have,

$$(F_1,F_2) = \int_{Z(\mathbf{A})N(\mathbf{A})B(F)\backslash G(\mathbf{A})} f_1(g)\overline{f_2(g)}\ dg + \int_{Z(\mathbf{A})N(\mathbf{A})B(F)\backslash G(\mathbf{A})} f_1'(g)\overline{f_2(g)}\ dg.$$

Since f_1 and f_2 are compactly supported so their Mellin transforms $\hat{f}_1(s)$ and $\hat{f}_2(s)$ are defined for all $s \in \mathbf{C}$. We have,

$$f_1'(g) = \int_{N(\mathbf{A})} f_1(wng)\ dn.$$

Now f_1' is left invariant under $Z(\mathbf{A})N(\mathbf{A})M(F)$ however it need not be compactly supported modulo this group. In particular the Mellin transform $\hat{f}_1'(s)$ is only defined if $\Re s \ll 0$.

For the first part of the expression for (F_1,F_2) we have,

$$\begin{aligned}\int_{Z(\mathbf{A})N(\mathbf{A})B(F)\backslash G(\mathbf{A})} f_1(g)\overline{f_2(g)}\ dg &= \int_K \int_{F^\times\backslash\mathbf{A}^1}\int_{F_\infty^+} f_1\left(\begin{pmatrix} at & 0 \\ 0 & 1\end{pmatrix}k\right)\overline{f_2\left(\begin{pmatrix} at & 0 \\ 0 & 1\end{pmatrix}k\right)}|t|^{-1}\ d^\times t\ da\ dk \\ &= \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty}\int_K\int_{F^\times\backslash\mathbf{A}^1} \hat{f}_1\left(\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix}k, s\right).\overline{\hat{f}}_2\left(\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix}k, -\bar{s}\right)\ da\ dk\ ds \\ &= \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} (\hat{f}_1(s), \hat{f}_2(-\bar{s}))\ ds.\end{aligned}$$

This expression is valid for any $x \in \mathbf{R}$. In the same way the second part of the expression for (F_1,F_2) is equal to,

$$\frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} (\widehat{f_1'}(-s), \hat{f}_2(\bar{s}))\ ds.$$

Here we need to integrate over a line where $\hat{f}_1'(-s)$ is defined by a convergent integral, i.e. we must have $x > \frac{1}{2}$.

Furthermore we compute, for $\Re s > \frac{1}{2}$,

$$\begin{aligned}
\hat{f}'(-s) &= \int_{F_\infty^+} f'\left(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} g\right) |t|^{s-\frac{1}{2}}\, d^\times t \\
&= \int_{F_\infty^+} |t|^{s-\frac{1}{2}}\, d^\times t \int_{\mathbf{A}} f\left(w \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & t^{-1}x \\ 0 & 1 \end{pmatrix} g\right)\, dx \\
&= \int_{F_\infty^+} |t|^{s+\frac{1}{2}}\, d^\times t \int_{\mathbf{A}} f\left(\begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} w \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right)\, dx \\
&= \int_{F_\infty^+} |t|^{-s-\frac{1}{2}}\, d^\times t \int_{\mathbf{A}} f\left(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} w \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right)\, dx \\
&= \int_{\mathbf{A}} \left(\int_{F_\infty^+} f\left(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} w \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) |t|^{-s-\frac{1}{2}}\, d^\times t\right)\, dx \\
&= \int_{\mathbf{A}} \hat{f}\left(w \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g, s\right)\, dx.
\end{aligned}$$

We define an operator $M(s) : H(s) \to H(-s)$ (at least on the smooth vectors) by,

$$(M(s)\varphi)(g) = \int_{N(\mathbf{A})} \varphi(wng)\, dn.$$

This integral converges when $\Re s > \frac{1}{2}$ and in this range we have,

$$\hat{f}'(-s) = M(s)\hat{f}(s).$$

Proposition 8.3. *([GJ79, (3.19)]) Let F_1 and F_2 be the series associated to f_1 and f_2 which are assumed to be compactly supported modulo $Z(\mathbf{A})N(\mathbf{A})B(F)$. Then,*

$$(F_1, F_2) = \frac{1}{2\pi i} \int (\hat{f}_1(s), \hat{f}_2(-\bar{s}))\, ds + \frac{1}{2\pi i} \int (M(s)\hat{f}_1(s), \hat{f}_2(\bar{s}))\, ds.$$

The integrals are taken over any vertical lines with $\Re s > \frac{1}{2}$.

We wish to move these integrals to the imaginary axis. To begin with we analytically continue $M(s)$ to the whole complex plane.

Theorem 8.4. *([GJ79, Theorem 4.19]) The operator $M(s)$ has a meromorphic continuation to $\mathbf{C}$ and satisfies the functional equation,*

$$M(-s)M(s) = Id.$$

Its only pole in the half-plane $\Re s \geq 0$ is at $s = \frac{1}{2}$ and the residue there is such that,

$$(\mathrm{Res}_{s=\frac{1}{2}} M(s)\hat{f}_1(1/2), \hat{f}_2(1/2)) = c \sum_{\chi:\chi^2=1} (\hat{f}_1(1/2), \chi \circ \det)\overline{(\hat{f}_2(1/2), \chi \circ \det)}$$

for some constant c.

We refer to the reference for the proof and just make some vague comments here. Recall that the group $F^\times\backslash\mathbf{A}^1$ is compact and so the space $H(s)$ decomposes as a Hilbert space direct sum,

$$H(s) = \bigoplus_\chi H(s)_\chi$$

taken over the characters $\chi : F^\times\backslash\mathbf{A}^1 \to \mathbf{C}^\times$ and where $H(s)_\chi$ consists of functions $\varphi : G(\mathbf{A}) \to \mathbf{C}$ such that,

$$\varphi\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g\right) = \left|\frac{a}{d}\right|^{s+\frac{1}{2}} \chi\left(\frac{a}{d}\right)\varphi(g).$$

Here we view χ as a character of $\mathbf{A}^\times$ by using the decomposition $\mathbf{A}^\times = \mathbf{A}^1 \times F_\infty^+$ and making χ trivial on F_∞^+. The operator $M(s)$ then maps $H(s)_\chi$ to $H(-s)_{\chi^{-1}}$. We now note that everything factors. The space $H(s)_\chi$ is spanned by functions of the form,

$$\varphi = \prod_v \varphi_v$$

with $\varphi_v \in H_v(s)_{\chi_v}$ (defined in the obvious fashion as functions on $G(F_v)$) and such that φ_v is invariant under K_v and $\varphi_v(I) = 1$ for almost all v. In the same way the intertwining operator $M(s)$ factors as a product of local intertwining operators defined by,

$$(M_v(s)_{\chi_v}\varphi_v)(g) = \int_{N(F_v)} \varphi_v(wng)\ dn = \int_{F_v} \varphi_v\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right)\ dx$$

for $\varphi_v \in H_v(s)_{\chi_v}$. We take a section,

$$\varphi_s = \prod_v \varphi_{s,v} \in H(s)_\chi.$$

Consider a partition of unity of the form,

$$\varphi_1(x) + \varphi_2(x^{-1}) = 1$$

with φ_1, φ_2 smooth and compactly supported. So we have,

$$(M_v(s)_{\chi_v}\varphi_{s,v})(g) = \int_{F_v} \varphi_{s,v}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right)\varphi_1(x)\ dx + \int_{F_v} \varphi_{s,v}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right)\varphi_2(x^{-1})\ dx.$$

The first integral converges for $\varphi_{s,v} \in H(s)_{\chi_v}$ for any s. For the second we note that

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -x \end{pmatrix} = \begin{pmatrix} -x^{-1} & 1 \\ 0 & -x \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x^{-1} & 1 \end{pmatrix}.$$

Hence,

$$\varphi_{s,v}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) = \varphi_{s,v}\left(\begin{pmatrix} -x^{-1} & 1 \\ 0 & -x \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x^{-1} & 1 \end{pmatrix} g\right) = |x|_v^{-2s-1}\chi_v(x)^{-2}\varphi_{s,v}\left(\begin{pmatrix} 1 & 0 \\ x^{-1} & 1 \end{pmatrix} g\right).$$

Thus the second integral is equal to,

$$\int_{F_v} \varphi_{s,v}\left(\begin{pmatrix} 1 & 0 \\ x^{-1} & 1 \end{pmatrix} g\right) \varphi_2(x^{-1})|x|_v^{-2s-1}\chi_v(x)^{-2}\ dx$$

which equals, after the change of variables $x \mapsto x^{-1}$,

$$\int_{F_v} \varphi_{s,v}\left(\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} g\right) \varphi_2(x)|x|_v^{2s-1}\chi_v(x)^{2}\ dx.$$

This is equal to a zeta integral of the type considered by Tate and is equal to a holomorphic multiple of $L(2s, \chi_v^2)$. Thus we see that the local intertwining operators $M(s)_v$ have meromorphic continuation to $\mathbf{C}$ with a pole at $s = 0$ if $\chi_v^2 = 1$. Furthermore if v is a non-archimedean place such that χ_v is unramified, dx is unramified and $\varphi_{s,v} = \varphi_{s,v}^0$ is the spherical vector in $H_v(s)_{\chi_v}$ such that $\varphi_{s,v}(I) = 1$ then,

$$M_v(s)_{\chi_v}\varphi_{s,v}^0 = \frac{L(2s,\chi^2)}{L(2s+1,\chi^2)}\varphi'^{0}_{-\bar{s},v}$$

where $\varphi'^{0}_{-\bar{s},v}$ is the spherical vector in $H_v(s)_{\chi_v^{-1}}$ such that $\varphi'^{0}_{-\bar{s},v}(I) = 1$. One defines normalized intertwining operators $R_v(s)_{\chi_v}$ by,

$$M_v(s)_{\chi_v} = \frac{L(2s,\chi_v^2)}{L(2s+1,\chi_v^2)\varepsilon(2s,\chi_v^2,\psi_v)}R(s)_v.$$

Then $R_v(s)_{\chi_v}$ are holomorphic on $\mathbf{C}$ and at an unramified place,

$$R_v(s)_{\chi_v}\varphi_{s,v}^0 = \varphi_{-\bar{s},v}^0.$$

Globally we take $R(s)_\chi = \prod_v R_v(s)_{\chi_v}$ then $R(s) : H(s)_\chi \to H(-s)_{\chi^{-1}}$ is well defined and holomorphic for all s. Since,

$$M(s)_\chi = \frac{L(2s,\chi^2)}{L(2s+1,\chi^2)\varepsilon(2s,\chi^2)}R(s)$$

we see that $M(s)$ is holomorphic if $\chi^2 \neq 1$ and if $\chi^2 = 1$ then it has a pole at $s = \frac{1}{2}$. Furthermore one obtains the functional equation for $M(s)_\chi$ from the functional equation,

$$L(1-s,\chi^2)\varepsilon(s,\chi^2) = L(s,\chi^2)$$

and the local functional equations $R_v(s)_{\chi_v}R_v(-s)_{\chi_v^{-1}} = 1$.

Finally it remains to analyze the poles of the intertwining operators $M(s)_\chi$. It's clear that if $\Re s \geq 0$ then the only pole of $M(s)_\chi$ will occur when $s = \frac{1}{2}$ and $\chi^2 = 1$. In this case we see that,

$$\operatorname{Res}_{s=\frac{1}{2}} M(s)_\chi = cR(1/2)_\chi.$$

Now we need to examine,

$$R\left(1/2\right)_\chi = \otimes_v R\left(1/2\right)_{\chi_v}.$$

Recall,

$$R_v\,(1/2)_{\chi_v} : H_v(1/2)_{\chi_v} \to H_v(-1/2)_{\chi_v^{-1}}.$$

Now the representations $H_v(1/2)_{\chi_v}$ and $H_v(-1/2)_{\chi_v^{-1}}$ are reducible. The space $H_v(1/2)_{\chi_v}$ contains an irreducible subspace of codimension one,

$$\left\{\varphi \in H_v(1/2)_{\chi_v} : \int_{F_v} \varphi\left(w\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right)\, dx = 0\right\}.$$

Thus $R_v(1/2)_{\chi_v}$ is trivial on this subspace. Similarly the space $H_v(-1/2)_{\chi_v^{-1}}$ contains an irreducible 1-dimensional subspace generated by the function $\chi_v^{-1} \circ \det$.

Thus given f_1 and f_2 with compact support giving rise to the series F_1 and F_2 we have,

$$(F_1, F_2) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big((\hat{f}_1(iy), \hat{f}_2(iy)) + (M(iy)\hat{f}_1(iy), \hat{f}_2(-iy))\Big)\, dy + c\sum_{\chi^2=1}(\hat{f}_1(1/2), \chi\circ\det)\overline{(\hat{f}_2(1/2), \chi\circ\det)}.$$

Given f we define, the section a_f of $H(s)$ over $i\mathbf{R}$ by,

$$a_f(iy) = \frac{1}{2}\left(\hat{f}(iy) + M(-iy)\hat{f}(-iy)\right).$$

Then,

$$M(-iy)a_f(-iy) = a_f(iy).$$

Hence,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\Big((\hat{f}_1(iy), \hat{f}_2(iy)) + (M(iy)\hat{f}_1(iy), \hat{f}_2(-iy))\Big)\, dy = \frac{1}{\pi}\int_{-\infty}^{\infty}(a_{f_1}(iy), a_{f_2}(iy))\, dy.$$

Furthermore we note that,

$$\begin{aligned}
(\hat{f}_1(1/2), \chi\circ\det) &= \int_{F^\times\backslash\mathbf{A}^1}\int_K \hat{f}_1\left(\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}k, \frac{1}{2}\right)\chi(a\det k)\, d^\times a\, dk\\
&= \int_{F^\times\backslash\mathbf{A}^1}\int_K\int_{F_\infty^+} f_1\left(\begin{pmatrix}at & 0\\ 0 & 1\end{pmatrix}k\right)\chi(a\det k)\, d^\times t\, d^\times a\, dk\\
&= \int_{Z(\mathbf{A})N(\mathbf{A})M(F)\backslash G(\mathbf{A})} f_1(g)\overline{\chi(\det g)}\, dg\\
&= \int_{Z(\mathbf{A})G(F)\backslash G(\mathbf{A})}\left(\sum_{\gamma\in B(F)\backslash G(F)} f_1(\gamma g)\right)\overline{\chi(\det g)}\, dg\\
&= \int_{Z(\mathbf{A})G(F)\backslash G(\mathbf{A})} F_1(g)\overline{\chi(\det g)}\, dg\\
&= (F_1, \chi\circ\det).
\end{aligned}$$

Hence we have,

$$(F_1, F_2) = \frac{1}{\pi}\int_{-\infty}^{\infty} (a_{f_1}(iy), a_{f_2}(iy))\ dy + c\sum_{\chi^2=1}(F_1, \chi\circ\det)(\chi\circ\det, F_2).$$

Note that for $g \in G(\mathbf{A})$, $R(g)F$ is the series obtained from $R(g)f$ given by translating f on the right by g and we have,

$$a_{R(g)f}(iy) = \pi_{iy}(g)a_f(iy).$$

We define $\mathcal{L}$ to be the Hilbert space of square integrable sections $a : i\mathbf{R} \to H$ such that $M(-iy)a(-iy) = a(iy)$. We define the inner product,

$$(a_1, a_2) = \frac{1}{\pi}\int_{-\infty}^{\infty} (a_1(iy), a_2(iy))\ dy.$$

on $\mathcal{L}$. We obtain a unitary representation π of $G(\mathbf{A})$ on $\mathcal{L}$ by defining,

$$(\pi(g)a)(iy) = \pi_{iy}(g)(a(iy)).$$

Theorem 8.5. *The map,*

$$F \mapsto (a_f, \hat{f}(1/2))$$

yields an isomorphism,

$$L^2_{\text{cusp}}(\overline{G}(F)\backslash\overline{G}(\mathbf{A}))^{\perp} \xrightarrow{\sim} \mathcal{L} \oplus \bigoplus_{\chi^2=1}\chi$$

as representations of $G(\mathbf{A})$.

Thus we have a further decomposition,

$$L^2(\overline{G}(F)\backslash\overline{G}(\mathbf{A})) = L^2_{\text{cusp}}(\overline{G}(F)\backslash\overline{G}(\mathbf{A}))\oplus L^2_{\text{res}}(\overline{G}(F)\backslash\overline{G}(\mathbf{A}))\oplus L^2_{\text{cont}}(\overline{G}(F)\backslash\overline{G}(\mathbf{A}))$$

where

$$L^2_{\text{res}}(\overline{G}(F)\backslash\overline{G}(\mathbf{A})) \cong \bigoplus_{\chi^2=1}\chi$$

and

$$L^2_{\text{cont}}(\overline{G}(F)\backslash\overline{G}(\mathbf{A})) \cong \mathcal{L}.$$

We now want to write down the kernel of $R(f)$ restricted to

$$L^2_{\text{res}}(\overline{G}(F)\backslash\overline{G}(\mathbf{A})) \cong \bigoplus_{\chi^2=1}\chi.$$

We see that

$$c\sum_{\chi^2=1}(F_1, \chi\circ\det)(\chi\circ\det F_2)$$

is the scalar product of the orthogonal projections of F_1 and F_2 onto $L^2_{\mathrm{res}}(\overline{G}(F)\backslash\overline{G}(\mathbf{A}))$. Therefore,

$$c = \frac{1}{\mathrm{vol}(\overline{G}(F)\backslash\overline{G}(\mathbf{A}))}.$$

Furthermore the kernel for $R(f)$ restricted to $L^2_{\mathrm{res}}(\overline{G}(F)\backslash\overline{G}(\mathbf{A}))$ is given by,

$$K_{f,\mathrm{res}}(x,y) = \frac{1}{\mathrm{vol}(\overline{G}(F)\backslash\overline{G}(\mathbf{A}))}\sum_{\chi^2=1}\chi(\det x)\overline{\chi(\det y)}\int_{\overline{G}(\mathbf{A})} f(g)\overline{\chi(\det g)}\ dg.$$

We now wish to write down the kernel for $R(f)$ restricted to $L^2_{\mathrm{cont}}(\overline{G}(F)\backslash\overline{G}(\mathbf{A}))$ using the isomorphism of this space with $\mathcal{L}$. We denote by,

$$S : L^2(\overline{G}(F)\backslash\overline{G}(\mathbf{A})) \to \mathcal{L}$$

given by extending the map defined above to be zero on $L^2_{\mathrm{cusp}}(\overline{G}(F)\backslash\overline{G}(\mathbf{A}))$. Then S^*S is equal to the projection of $L^2(\overline{G}(F)\backslash\overline{G}(\mathbf{A}))$ onto $L^2_{\mathrm{cont}}(\overline{G}(F)\backslash\overline{G}(\mathbf{A}))$. For $f \in C_c^\infty(\overline{G}(\mathbf{A}))$ we have,

$$S^*SR(f)S^*S = S^*\pi(f)S.$$

If $K_{f,\mathrm{cont}}(x,y)$ denotes the kernel of $R(f)$ restricted to $L^2_{\mathrm{cont}}(\overline{G}(F)\backslash\overline{G}(\mathbf{A}))$ then,

$$(S^*SR(f)S^*SF_1, F_2) = \int_{\overline{G}(F)\backslash\overline{G}(\mathbf{A})}\int_{\overline{G}(F)\backslash\overline{G}(\mathbf{A})} K_{f,\mathrm{cont}}(x,y)F_1(x)\overline{F_2(y)}\ dx\ dy.$$

We want to find an expression for $K_{f,\mathrm{cont}}(x,y)$.

For series F_1 and F_2 coming from f_1 and f_2 we wish to compute,

$$(S^*SR(f)S^*SF_1, F_2) = (S^*\pi(f)SF_1, F_2) = (\pi(f)SF_1, SF_2).$$

By definition of the inner product on $\mathcal{L}$,

$$(\pi(f)SF_1, SF_2) = \frac{1}{\pi}\int_{-\infty}^{\infty}(\pi_{iy}(f)SF_1(iy), SF_2(iy))\ dy.$$

We take an orthonormal basis $\{\varphi_\alpha\}$ for $H(0)$ which we extend to a constant section of our fiber bundle by defining,

$$\varphi_\alpha(s,g) = e^{sH(g)}\varphi_\alpha(g).$$

We note that for $y \in \mathbf{R}$, $\{\varphi_\alpha(iy)\}$ is an orthonormal basis of $H(iy)$. Then we have,

$$\begin{aligned}(\pi(f)SF_1, SF_2) &= \frac{1}{\pi}\int_{-\infty}^{\infty}\sum_\alpha(\pi_{iy}(f)SF_1(iy), \varphi_\alpha(iy))(\varphi_\alpha(iy), SF_2(iy))\ dy\\ &= \frac{1}{\pi}\int_{-\infty}^{\infty}\sum_\alpha(SF_1(iy), \pi^*_{iy}(f)\varphi_\alpha(iy))(\varphi_\alpha(iy), SF_2(iy))\ dy\end{aligned}$$

Thus we need an expression for

$$(h, SF(iy))$$

with $h \in H(iy)$.

We now need to introduce Eisenstein series. Let φ be a holomorphic section of the bundle $H(s)$. Then we define an Eisenstein series,

$$E(\varphi(s), g) = \sum_{\gamma \in B(F)\backslash G(F)} \varphi(\gamma g, s).$$

This series converges for $\Re s > \frac{1}{2}$ (see [Bum97, Proposition 3.7.2]) and defines a holomorphic function there.

Suppose we take f as usual to be a function on

$$Z(\mathbf{A})N(\mathbf{A})M(F)\backslash G(\mathbf{A})$$

of compact support. We let $\hat{f}(g, s)$ be the section of $H(s)$ defined as usual. Then we have,

$$f(g) = \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} \hat{f}(g, s)\ ds.$$

Hence,

$$\begin{aligned}
F(g) &= \sum_{\gamma \in B(F)\backslash G(F)} f(\gamma g) \\
&= \frac{1}{2\pi i} \sum_{\gamma \in B(F)\backslash G(F)} \int_{x-i\infty}^{x+i\infty} \hat{f}(\gamma g, s)\ ds \\
&= \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} \sum_{\gamma \in B(F)\backslash G(F)} \hat{f}(\gamma g, s)\ ds \\
&= \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} E(\hat{f}(s), g)\ ds,
\end{aligned}$$

provided $x > \frac{1}{2}$.

As before we want to analytically continue this integral to one over the imaginary axis. In order to do this we need to analytically continue the Eisenstein series.

Theorem 8.6. *Let $\varphi(s)$ be a holomorphic section of $H(s)$. Then the Eisenstein series $E(\varphi(s), g)$ has a meromorphic continuation with the only poles in $\Re s \geq 0$ appearing at $s = \frac{1}{2}$ and satisfies the functional equation,*

$$E(\varphi(s), g) = E(M(s)\varphi(s), g).$$

For the case when $\varphi(s)$ is a flat section see [Bum97, Theorem 3.7.1]. We say a few words here relating the analytic continuation of the Eisenstein series to the analytic continuation of the intertwining operator $M(s)$.

We will use the truncation operator to obtain the analytic continuation of $E(\varphi(s), g)$. We write

$$E(\varphi(s), g) = \Lambda^T E(\varphi(s), g) + \sum_{\gamma \in B(F)\backslash G(F)} E_N(\varphi(s), \gamma g)\tau(H(\gamma g) - T).$$

We can compute the constant term $E_N(\varphi(s), g)$ as before by the formula,

$$E_N(\varphi(s), g) = \varphi(g, s) + \int_{N(\mathbf{A})} \varphi(wng, s)\ dn = \varphi(s)(g) + (M(s)\varphi(s))(g).$$

Thus we see that,

$$\sum_{\gamma \in B(F)\backslash G(F)} E_N(\varphi(s), \gamma g)\tau(H(\gamma g) - T)$$

has meromorphic continuation to $\mathbf{C}$ with singularities at most those of $M(s)$. On the other hand,

$$\Lambda^T E(\varphi(s), g)$$

is square integrable and one can analytically continue it as a square integrable function to all of $\mathbf{C}$ (see [GJ79, Section 5]). For the functional equation we note that,

$$E_N(M(s)\varphi(s), g) = E_N(\varphi(s), g)$$

which follows from the functional equation $M(s)M(-s) = Id$. Hence,

$$E(\varphi(s), g) - E(M(s)\varphi(s), g)$$

has zero constant term, i.e. it is a cuspidal function. On the other hand it we know that it is orthogonal to all cuspidal functions. Hence we deduce that,

$$E(\varphi(s), g) - E(M(s)\varphi(s), g) = 0.$$

See [GJ79, Section 5.C] for more details.

We return to the formula,

$$F(g) = \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} E(\hat{f}(s), g)\ ds.$$

We have,

$$\mathrm{res}_{s=\frac{1}{2}} E(\hat{f}(s), g) = \mathrm{res}_{s=\frac{1}{2}} M(s)\hat{f}(1/2).$$

Proposition 8.7. *Let $h \in H(0)$ and F a series associated to a compactly supported function f as usual. Then,*

$$(h(iy), SF(iy)) = \frac{1}{2} \int_{\overline{G}(F)\backslash \overline{G}(\mathbf{A})} E(h(iy), g)\overline{F(g)}\ dg.$$

Proof. Suppose we take f_1 and f_2 compactly supported giving rise to the series F_1 and F_2. Then,

$$\begin{aligned}(F_1, F_2) &= \int_{\overline{G}(F)\backslash\overline{G}(\mathbf{A})} \left(\frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} E(\hat{f}_1(s), g)\ ds \right) \overline{F_2(g)}\ dg \\ &= \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} \int_{\overline{G}(F)\backslash\overline{G}(\mathbf{A})} E(\hat{f}_1(s), g)\overline{F_2(g)}\ dg\ ds\end{aligned}$$

provided $x > \frac{1}{2}$. We now shift the integral to the imaginary axis to obtain,

$$(F_1, F_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{\overline{G}(F)\backslash\overline{G}(\mathbf{A})} E(\hat{f}_1(iy), g)\overline{F_2(g)}\ dg + c \sum_{\chi^2=1} (F_1, \chi\circ\det)(\chi\circ\det, F_2).$$

Now we use that,

$$E(M(s)\varphi(s), g) = E(\varphi(s), g)$$

and the definition,

$$a_{f_1}(iy) = \frac{1}{2}\left(\hat{f}_1(iy) + M(-iy)\hat{f}_1(-iy)\right).$$

To deduce that,

$$(F_1, F_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{\overline{G}(F)\backslash\overline{G}(\mathbf{A})} E(a_{f_1}(iy), g)\overline{F_2(g)}\ dg\ dy + c \sum_{\chi^2=1} (F_1, \chi\circ\det)(\chi\circ\det, F_2).$$

On the other hand we have,

$$(F_1, F_2) = \frac{1}{\pi} \int_{-\infty}^{\infty} (a_{f_1}(iy), a_{f_2}(iy))\ dy + c \sum_{\chi^2=1} (F_1, \chi \circ \det)(\chi \circ \det, F_2).$$

Since the space of a_{f_1} is dense in $\mathcal{L}$ we conclude that,

$$\int_{-\infty}^{\infty} (a(iy), SF(iy))\ dy = \frac{1}{2} \int_{-\infty}^{\infty} \int_{\overline{G}(F)\backslash\overline{G}(\mathbf{A})} E(a(iy), g)\overline{F(g)}\ dg\ dy$$

for any $a \in \mathcal{L}$. Taking $a(iy) = c(y)h(iy)$ with $c(y)$ a constant function in $L^2(\mathbf{R})$ gives

$$\int_{-\infty}^{\infty} c(y)(h(iy), SF(iy))\ dy = \frac{1}{2} \int_{-\infty}^{\infty} \int_{\overline{G}(F)\backslash\overline{G}(\mathbf{A})} c(y)E(h(iy), g)\overline{F(g)}\ dg\ dy$$

from which we can conclude, by varying $c(y)$, that

$$(h(iy), SF(iy)) = \frac{1}{2} \int_{\overline{G}(F)\backslash\overline{G}(\mathbf{A})} E(h(iy), g)\overline{F(g)}\ dg.$$

$\square$

We now make the further assumption that $f \in C_c^\infty(Z(\mathbf{A})\backslash G(\mathbf{A}))$ is right and left K-finite and we pick our orthonormal basis $\{\varphi_\alpha\}$ of $H(0)$ to be K-finite. Applying this Proposition we see that,

$$(\pi(f)SF_1, SF_2) = \frac{1}{4\pi}\int_{-\infty}^{\infty}\left(\sum_\alpha \int_{\overline{G}(F)\backslash\overline{G}(\mathbf{A})} F_1(g)\bar{E}(\pi_{iy}^*(f)\varphi_\alpha(iy), g)\, dg \int_{\overline{G}(F)\backslash\overline{G}(\mathbf{A})} \bar{F}_2(h)E(\varphi_\alpha(iy), h)\, dh\right)\, dy.$$

Interchanging summation and integration yields,

$$(\pi(f)SF_1, SF_2) = \int_{\overline{G}(F)\backslash\overline{G}(\mathbf{A})}\int_{\overline{G}(F)\backslash\overline{G}(\mathbf{A})} F_1(g)\bar{F}_2(h)\, dg\, dh\left(\sum_\alpha \frac{1}{4\pi}\int_{-\infty}^{\infty} E(\varphi_\alpha(iy), h)\bar{E}(\pi_{iy}^*(f)\varphi_\alpha(iy), g)dy\right).$$

Thus the kernel for the restriction of $R(f)$ to $L^2_{\text{cont}}(\overline{G}(F)\backslash\overline{G}(\mathbf{A}))$ is,

$$K_{f,\text{cont}}(h,g) = \frac{1}{4\pi}\sum_\alpha \int_{-\infty}^{\infty} E(\varphi_\alpha(iy), h)\bar{E}(\pi_{iy}^*(f)\varphi_\alpha(iy), g)dy$$

Note that we have,

$$\pi_{iy}^*(f)\varphi_\alpha = \sum_\beta (\pi_{iy}^*(f)\varphi_\alpha(iy), \varphi_\beta(iy))\varphi_\beta(iy)$$

and so we can rewrite

$$K_{f,\text{cont}}(h,g) = \frac{1}{4\pi}\sum_{\alpha,\beta}\int_{-\infty}^{\infty}(\pi_{iy}(f)\varphi_\beta(iy), \varphi_\alpha(iy))E(\varphi_\alpha(iy), h)\overline{E(\varphi_\beta(iy), g)}\, dg.$$

8.3 Spectral side of the trace formula

In the previous section we have written down the kernel $K_{f,\text{cont}}(x,y)$ for the map,

$$P_{\text{cont}}R(f)P_{\text{cont}} : L^2(\overline{G}(F)\backslash\overline{G}(\mathbf{A})) \to L^2(\overline{G}(F)\backslash\overline{G}(\mathbf{A}))$$

where P_{cont} denotes the orthogonal projection,

$$P_{\text{cont}} : L^2(\overline{G}(F)\backslash\overline{G}(\mathbf{A})) \to L^2_{\text{cont}}(\overline{G}(F)\backslash\overline{G}(\mathbf{A})).$$

We can also write down the kernels for $P_{\text{cusp}}R(f)P_{\text{cusp}}$ and $P_{\text{res}}R(f)P_{\text{res}}$ where P_{cusp} and P_{res} denote the orthogonal projections of $L^2(\overline{G}(F)\backslash\overline{G}(\mathbf{A}))$ on to the cuspidal and residual spaces respectively. We recall the following lemma,

Lemma 8.8. *Let $A : L^2(X) \to L^2(X)$ be a Hilbert-Schmidt operator and let $\{\varphi_i\}$ be an orthonormal basis of $L^2(X)$, then the function,*

$$K_A(x,y) = \sum_i (A\varphi_i)(x)\overline{\varphi_i(y)} \in L^2(X\times X)$$

and is a kernel for A,

Thus we have,

$$K_{f,\mathrm{res}}(x,y) = \frac{1}{\mathrm{vol}(\overline{G}(F)\backslash\overline{G}(\mathbf{A}))} \sum_{\substack{\chi: F^\times\backslash\mathbf{A}^\times\to\mathbf{C}^\times\\ \chi^2=1}} \chi(\det x)\overline{\chi(\det y)} \int_{\overline{G}(\mathbf{A})} f(g)\chi(\det g)\, dg.$$

We note that for a fixed function f the sum over χ is finite, since such a function f will be invariant under translation by a compact open subgroup of $\overline{G}(\mathbf{A}_{\mathrm{fin}})$.

If we decompose,

$$L^2_{\mathrm{cusp}}(\overline{G}(F)\backslash\overline{G}(\mathbf{A})) = \bigoplus_{\pi\in\widehat{\overline{G}(\mathbf{A})}} m_\pi \pi.$$

We set

$$\mathcal{A}_{\mathrm{cusp}}(G) = \left\{\pi \in \widehat{\overline{G}(\mathbf{A})} : m_\pi > 0\right\}$$

and for each $\pi \in \mathcal{A}_{\mathrm{cusp}}(G)$ let $\mathcal{B}(\pi)$ denote an orthonormal basis of the π-isotypic subspace of $L^2_{\mathrm{cusp}}(\overline{G}(F)\backslash\overline{G}(\mathbf{A}))$. Then since $R(f)$ is a Hilbert-Schmidt operator when restricted to $L^2_{\mathrm{cusp}}(\overline{G}(F)\backslash\overline{G}(\mathbf{A}))$ so,

$$K_{f,\mathrm{cusp}}(x,y) = \sum_{\pi\in\mathcal{A}_{\mathrm{cusp}}(G)} \sum_{\varphi\in\mathcal{B}(\pi)} (R(f)\varphi)(x)\overline{\varphi(y)}.$$

We now have obtained the spectral decomposition of the kernel,

$$K_f(x,y) = K_{f,\mathrm{cusp}}(x,y) + K_{f,\mathrm{res}}(x,y) + K_{f,\mathrm{cont}}(x,y).$$

We recall that we defined,

$$\begin{aligned} K_f^T(x,y) &= \Lambda_2^T K_f(x,y) \\ &= K_f(x,y) - \sum_{\delta\in B(F)\backslash G(F)} K_{f,N}(x,\delta y)\tau(H(\delta y) - T) \end{aligned}$$

where,

$$K_{f,N}(x,y) = \int_{N(F)\backslash N(\mathbf{A})} K_f(x,ny)\, dn.$$

Thus we have,

$$K_f^T(x,y) = K^T_{f,\mathrm{cusp}}(x,y) + K^T_{f,\mathrm{res}}(x,y) + K^T_{f,\mathrm{cont}}(x,y).$$

One now wants to prove that each term is absolutely integrable over the diagonal and provide an expression for each integral. We will largely ignore the problem of absolute convergence for which we refer to [Gel96, Lecture III] instead we'll focus on computing the integrals.

Before doing so we recall that we know from our geometric calculations that,

$$\int_{\overline{G}(F)\backslash\overline{G}(\mathbf{A})} K_f^T(x,x)\, dx$$

is a polynomial in T. In particular we need only compute each term in the spectral expansion up to terms which vanish as $T \to \infty$.

To begin with we see that,

$$K^T_{f,\mathrm{cusp}}(x,y) = K_{f,\mathrm{cusp}}(x,y).$$

We denote by $R_{\mathrm{cusp}}(f)$ the restriction of $R(f)$ to

$$L^2_{\mathrm{cusp}}(\overline{G}(F)\backslash\overline{G}(\mathbf{A})).$$

Then we have, from the results of Section 8.1,

Proposition 8.9. *Let $f \in C_c^\infty(\overline{G}(\mathbf{A}))$ then*

$$\int_{\overline{G}(F)\backslash\overline{G}(\mathbf{A})} K_{f,\mathrm{cusp}}(x,x)\ dx$$

is absolutely integrable and is equal to,

$$\operatorname{tr} R_{\mathrm{cusp}}(f).$$

We now deal with the residual terms. We recall,

$$K_{f,\mathrm{res}}(x,y) = \frac{1}{\mathrm{vol}(\overline{G}(F)\backslash\overline{G}(\mathbf{A}))} \sum_{\substack{\chi:F^\times\backslash\mathbf{A}^\times\to\mathbf{C}^\times \\ \chi^2=1}} \chi(\det x)\overline{\chi(\det y)} \int_{\overline{G}(\mathbf{A})} f(g)\chi(\det g)\ dg$$

and hence $K^T_{f,\mathrm{res}}(x,y)$ is equal to,

$$\frac{1}{\mathrm{vol}(\overline{G}(F)\backslash\overline{G}(\mathbf{A}))} \sum_{\substack{\chi:F^\times\backslash\mathbf{A}^\times\to\mathbf{C}^\times \\ \chi^2=1}} \chi(\det x)\overline{\chi(\det y)} \int_{\overline{G}(\mathbf{A})} f(g)\chi(\det g)\ dg \left(1 - \sum_{\delta\in B(F)\backslash G(F)} \tau(H(\delta y) - T)\right).$$

Proposition 8.10. *As $T \to \infty$,*

$$\int_{\overline{G}(F)\backslash\overline{G}(\mathbf{A})} K^T_{f,\mathrm{res}}(x,x)\ dx \to \int_{\overline{G}(F)\backslash\overline{G}(\mathbf{A})} K_{f,\mathrm{res}}(x,x)\ dx = \operatorname{tr} R_{\mathrm{res}}(f).$$

Proof. We note that

$$K^T_{f,\mathrm{res}}(x,x) = \frac{1}{\mathrm{vol}(\overline{G}(F)\backslash\overline{G}(\mathbf{A}))} \sum_{\substack{\chi:F^\times\backslash\mathbf{A}^\times\to\mathbf{C}^\times \\ \chi^2=1}} \int_{\overline{G}(\mathbf{A})} f(g)\chi(\det g)\ dg \left(1 - \sum_{\delta\in B(F)\backslash G(F)} \tau(H(\delta x) - T)\right).$$

Thus we need to show that,

$$\int_{\overline{G}(F)\backslash\overline{G}(\mathbf{A})} \sum_{\delta\in B(F)\backslash G(F)} \tau(H(\delta x) - T)\ dx \to 0$$

as $T \to \infty$. We note that this integral is equal to,

$$\int_{Z(F)B(F)\backslash G(\mathbf{A})} \tau(H(x)-T)\ dx.$$

We again use the Iwasawa decomposition to write,

$$x = n \begin{pmatrix} at & 0 \\ 0 & 1 \end{pmatrix} k$$

with $n \in N(\mathbf{A})$, $a \in \mathbf{A}^1$, $t \in F_\infty^+$ and $k \in K$. Then the integral is equal to,

$$\mathrm{vol}(F^\times \backslash \mathbf{A}^1)\,\mathrm{vol}(K) \int_{F_\infty^+} \tau(\log|t| - T)|t|^{-1} d^\times t.$$

This integral is equal to,

$$C \int_{e^T}^\infty \frac{dt}{t^2} = Ce^{-T}$$

which tends to zero as $T \to \infty$. □

Finally we deal with the continuous spectrum. We recall,

$$K_{f,\mathrm{cont}}(h,g) = \frac{1}{4\pi} \sum_{\alpha,\beta} \int_{-\infty}^\infty (\pi_{iy}(f)\varphi_\beta(iy), \varphi_\alpha(iy)) E(\varphi_\alpha(iy), h)\overline{E(\varphi_\beta(iy), g)}\ dg.$$

Hence,

$$K^T_{f,\mathrm{cont}}(h,g) = \frac{1}{4\pi} \sum_{\alpha,\beta} \int_{-\infty}^\infty (\pi_{iy}(f)\varphi_\beta(iy), \varphi_\alpha(iy)) E(\varphi_\alpha(iy), h)\overline{E^T(\varphi_\beta(iy), g)}\ dy.$$

Taking absolute convergence for granted we have,

$$\int_{\overline{G}(F)\backslash\overline{G}(\mathbf{A})} K^T_{f,\mathrm{cont}}(g,g)\ dg = \frac{1}{4\pi} \sum_{\alpha,\beta} \int_{-\infty}^\infty (\pi_{iy}(f)\varphi_\beta(iy), \varphi_\alpha(iy)) \left(\int_{\overline{G}(F)\backslash\overline{G}(\mathbf{A})} E(\varphi_\alpha(iy), g)\overline{E^T(\varphi_\beta(iy), g)}\ dg \right) d$$

Lemma 8.11. *We have, for $y \neq 0$,*

$$\int_{\overline{G}(F)\backslash\overline{G}(\mathbf{A})} E(\varphi_\alpha(iy), g)\overline{E^T(\varphi_\beta(iy), g)}\ dg$$

equal to

$$2(\varphi_\alpha, \varphi_\beta)T - (M(-iy)M'(iy)\varphi_\alpha, \varphi_\beta) + \left((\varphi_\alpha, M(iy)\varphi_\beta)e^{2iyT} - (M(iy)\varphi_\alpha, \varphi_\beta)e^{-2iyT} \right) \frac{1}{2iy}.$$

Proof. Recall that $\varphi_\alpha \in H(0)$ and we define $\varphi_\alpha(s) \in H(s)$ by,

$$\varphi_\alpha(s,g) = e^{sH(g)}\varphi_\alpha(g).$$

By definition,

$$E(\varphi_\alpha(s), g) = \sum_{\gamma \in B(F)\backslash G(F)} \varphi_\alpha(s, \gamma g).$$

By definition,

$$E^T(\varphi_\alpha(s), g) = E(\varphi_\alpha(s), g) - \sum_{\delta \in B(F)\backslash G(F)} E_N(\varphi_\alpha(s), \delta g)\tau(H(\delta g) - T)$$

and

$$E_N(\varphi_\alpha(s), g) = \varphi_\alpha(s)(g) + (M(s)\varphi_\alpha(s))(g).$$

Hence,

$$\begin{aligned} E^T(\varphi_\alpha(s), g) &= \sum_{\gamma \in B(F)\backslash G(F)} \varphi_\alpha(s, \gamma g) - \sum_{\delta \in B(F)\backslash G(F)} \left(\varphi_\alpha(s, \delta g) + (M(s)\varphi_\alpha(s))(\delta g)\right)\tau(H(\delta g) - T) \\ &= \sum_{\gamma \in B(F)\backslash G(F)} \varphi_\alpha(s, \gamma g)(1 - \tau(H(\gamma g) - T)) + \sum_{\gamma \in B(F)\backslash G(F)} (M(s)\varphi_\alpha(s))(\gamma g)\tau(H(\gamma g) - T). \end{aligned}$$

Thus, since $E^T(\varphi_\alpha(s), g)$ is a P-series so we have $(E(\varphi_\alpha(s_1)), E^T(\varphi_\beta(\bar{s}_2))$ equal to,

$$\int_{Z(\mathbf{A})N(\mathbf{A})M(F)\backslash G(\mathbf{A})} E_N(\varphi_\alpha(s_1), g)\overline{(\varphi_\beta(\bar{s}_2, g)(1 - \tau(H(g) - T)) - (M(\bar{s}_2)\varphi_\beta(\bar{s}_2))(g)\tau(H(g) - T))}\, dg$$

which equals

$$\int_{Z(\mathbf{A})N(\mathbf{A})M(F)\backslash G(\mathbf{A})} (\varphi_\alpha(s_1, g) + (M(s_1)\varphi_\alpha(s_1))(g))\overline{(\varphi_\beta(\bar{s}_2, g)(1 - \tau(H(g) - T)) - (M(\bar{s}_2)\varphi_\beta(\bar{s}_2))(g)\tau(H(g) - T}$$

We use the Iwasawa decomposition to write $g \in Z(\mathbf{A})N(\mathbf{A})M(F)\backslash G(\mathbf{A})$ as

$$g = \begin{pmatrix} at & 0 \\ 0 & 1 \end{pmatrix} k$$

with $a \in F^\times\backslash\mathbf{A}^1$, $t \in F_\infty^+$ and $k \in K$ so that we can decompose the measure as,

$$dg = |t|^{-1} d^\times a\ d^\times t\ dk.$$

Recall that for g written in this form we have,

$$\varphi_\alpha(s, g) = |t|^{s+\frac{1}{2}}\varphi_\alpha\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} k\right)$$

and

$$H(g) = \log|t|.$$

We also recall the pairing on $H(0) \times H(0)$ defined by,

$$(h_1, h_2) = \int_{F^\times\backslash\mathbf{A}^1}\int_K h_1\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} k\right)\overline{h_2\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} k\right)}\, d^\times a\ dk.$$

Thus we see that $(E(\varphi_\alpha(s_1)), E^T(\varphi_\beta(\bar{s}_2))$ is equal to the integral of

$$(\varphi_\alpha, \varphi_\beta)|t|^{s_1+s_2}(1-\tau(\log|t|-T))-(M(s_1)\varphi_\alpha(s_1), M(\bar{s}_2)\varphi_\beta(\bar{s}_2))|t|^{-(s_1+s_2)}\tau(\log|t|-T)$$

plus

$$-(\varphi_\alpha, M(\bar{s}_2)\varphi_\beta)|t|^{s_1-s_2}\tau(\log|t|-T)+(M(s_1)\varphi_\alpha, \varphi_\beta)|t|^{-s_1+s_2}(1-\tau(\log|t|-T))$$

with respect to $d^\times t$ which equals,

$$\Big((\varphi_\alpha, \varphi_\beta)e^{(s_1+s_2)T} - (M(s_1)\varphi_\alpha(s_1), M(\bar{s}_2)\varphi_\beta(\bar{s}_2))e^{-(s_1+s_2)T}\Big)\frac{1}{s_1+s_2}$$

plus

$$\Big((\varphi_\alpha, M(\bar{s}_2)\varphi_\beta)e^{(s_1-s_2)T} - (M(s_1)\varphi_\alpha, \varphi_\beta)e^{-(s_1-s_2)T}\Big)\frac{1}{s_1-s_2}.$$

From the functional equation for $M(s)$ and the fact that,

$$(M(s)\varphi_1, \varphi_2) = (\varphi_1, M(\bar{s})\varphi_2)$$

one sees that this expression is meromorphic in s with singularities at most those of $M(s)$. Furthermore we can rewrite this as,

$$\Big((\varphi_\alpha, \varphi_\beta)e^{(s_1+s_2)T} - (M(s_2)M(s_1)\varphi_\alpha, \varphi_\beta)e^{-(s_1+s_2)T}\Big)\frac{1}{s_1+s_2}$$

plus

$$\Big((\varphi_\alpha, M(\bar{s}_2)\varphi_\beta)e^{(s_1-s_2)T} - (M(s_1)\varphi_\alpha, \varphi_\beta)e^{-(s_1-s_2)T}\Big)\frac{1}{s_1-s_2}.$$

Specializing to the case that $s_1 = -s_2 = s \neq 0$ we see that the second term is equal to,

$$\big((\varphi_\alpha, M(-\bar{s})\varphi_\beta)e^{2sT} - (M(s)\varphi_\alpha, \varphi_\beta)e^{-2sT}\big)\frac{1}{2s}.$$

while applying L'Hopital's rule the first term is equal to,

$$2(\varphi_\alpha, \varphi_\beta)T - (M(-s)M'(s)\varphi_\alpha, \varphi_\beta).$$

$\square$

From this lemma we see that,

$$\int_{\overline{G}(F)\backslash\overline{G}(\mathbf{A})} K^T_{f,\mathrm{cont}}(g,g)\ dg$$

is equal to the sum of

$$\frac{T}{2\pi}\sum_{\alpha,\beta}\int_{-\infty}^{\infty}(\pi_{iy}(f)\varphi_\beta, \varphi_\alpha)(\varphi_\alpha, \varphi_\beta)\ dy,$$

$$-\frac{1}{4\pi}\sum_{\alpha,\beta}\int_{-\infty}^{\infty}(\pi_{iy}(f)\varphi_\beta,\varphi_\alpha)(M(-iy)M'(iy)\varphi_\alpha,\varphi_\beta)\ dy$$

and

$$\frac{1}{4\pi}\sum_{\alpha,\beta}\int_{-\infty}^{\infty}(\pi_{iy}(f)\varphi_\beta,\varphi_\alpha)\left((\varphi_\alpha,M(iy)\varphi_\beta)e^{2iyT}-(M(iy)\varphi_\alpha,\varphi_\beta)e^{-2iyT}\right)\frac{1}{2iy}\ dy.$$

After interchanging summation and integration we see that the first term is equal to,

$$\begin{aligned}\frac{T}{2\pi}\int_{-\infty}^{\infty}\sum_{\alpha,\beta}(\pi_{iy}(f)\varphi_\beta,\varphi_\alpha)(\varphi_\alpha,\varphi_\beta)\ dy &= \frac{T}{2\pi}\int_{-\infty}^{\infty}\sum_{\alpha}(\pi_{iy}(f)\varphi_\alpha,\varphi_\alpha)\ dy\\ &= \frac{T}{2\pi}\int_{-\infty}^{\infty}\operatorname{tr}\pi_{iy}(f)\ dy.\end{aligned}$$

Lemma 8.12. *We have,*

$$\operatorname{tr}\pi_{iy}(f)=\operatorname{vol}(F^\times\backslash\mathbf{A}^1)\int_K\int_{N(\mathbf{A})}\int_{F_\infty^+}\sum_{\alpha\in F^\times} f\left(k^{-1}\begin{pmatrix}t\alpha & 0\\ 0 & 1\end{pmatrix}nk\right)|t|^{\frac{1}{2}+iy}\ d^\times t\ dn\ dk.$$

Proof. Let $\varphi\in H(iy)$. Thus $\varphi:\overline{G}(\mathbf{A})\to\mathbf{C}$ such that,

$$\varphi\left(\begin{pmatrix}a & x\\ 0 & 1\end{pmatrix}g\right)=|a|^{\frac{1}{2}+iy}\varphi(g)$$

for $a\in F^\times F_\infty^+$, $x\in\mathbf{A}$ and $g\in\overline{G}(\mathbf{A})$. We recall that by the Iwasawa decomposition we can identify the space $H(iy)$ with square integrable functions,

$$\varphi\left(\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}k\right)$$

with $a\in F^\times\backslash\mathbf{A}^1$ and $k\in K$. In order to compute the trace of $\pi_{iy}(f)$ we will realize this operator, via this identification of $H(iy)$, as an integral operator and then compute the trace by integrating the kernel over the diagonal.

By definition,

$$(\pi_{iy}(f)\varphi)(h)=\int_{\overline{G}(\mathbf{A})}f(g)\varphi(hg)\ dg=\int_{\overline{G}(\mathbf{A})}f(h^{-1}g)\varphi(g)\ dg.$$

As usual we write,

$$g=\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}at & 0\\ 0 & 1\end{pmatrix}k$$

with $x\in\mathbf{A}$, $a\in\mathbf{A}^1$, $t\in F_\infty^+$ and $k\in K$ and we decompose the measure as,

$$dg=|t|^{-1}\ dx\ d^\times a\ d^\times t\ dk.$$

Then we have,

$$
\begin{aligned}
(\pi_{iy}(f)\varphi)(h) &= \int_K \int_{F_\infty^+} \int_{\mathbf{A}^1} \int_{\mathbf{A}} f\left(h^{-1}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}at & 0\\ 0 & 1\end{pmatrix}k\right)\varphi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}at & 0\\ 0 & 1\end{pmatrix}k\right)|t|^{-1}\,dx\,d^\times a\,d^\times t\,dk\\
&= \int_K \int_{F_\infty^+} \int_{\mathbf{A}^1} \int_{\mathbf{A}} f\left(h^{-1}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}at & 0\\ 0 & 1\end{pmatrix}k\right)\varphi\left(\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}k\right)|t|^{-\frac{1}{2}+iy}\,dx\,d^\times a\,d^\times t\,dk\\
&= \int_K \int_{F^\times \mathbf{A}^1}\left(\int_{\mathbf{A}}\int_{F_\infty^+}\sum_{\alpha\in F^\times} f\left(h^{-1}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}\alpha at & 0\\ 0 & 1\end{pmatrix}k\right)|t|^{-\frac{1}{2}+iy}\,d^\times t\,dx\right)\varphi\left(\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}k\right)\,d^\times a
\end{aligned}
$$

Thus, $\pi_{iy}(f)$ is an integral operator on $F^\times\backslash\mathbf{A}^1\times K$ with kernel,

$$
H_f(h,g) = \int_{\mathbf{A}}\int_{F_\infty^+}\sum_{\alpha\in F^\times} f\left(h^{-1}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}\alpha t & 0\\ 0 & 1\end{pmatrix}g\right)|t|^{-\frac{1}{2}+iy}\,d^\times t\,dx.
$$

Hence we have,

$$
\begin{aligned}
\operatorname{tr}\pi_{iy}(f) &= \int_K\int_{F^\times\backslash\mathbf{A}^1} H_f\left(\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}k, \begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}k\right)\,d^\times a\,dk\\
&= \int_K\int_{F^\times\backslash\mathbf{A}^1}\int_{\mathbf{A}}\int_{F_\infty^+}\sum_{\alpha\in F^\times} f\left(k^{-1}\begin{pmatrix}a^{-1} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}\alpha t & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}k\right)|t|^{-\frac{1}{2}+iy}\,d^\times t\,dx\,d^\times a\,dk\\
&= \int_K\int_{F^\times\backslash\mathbf{A}^1}\int_{\mathbf{A}}\int_{F_\infty^+}\sum_{\alpha\in F^\times} f\left(k^{-1}\begin{pmatrix}\alpha t & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & a^{-1}\alpha^{-1}t^{-1}x\\ 0 & 1\end{pmatrix}k\right)|t|^{-\frac{1}{2}+iy}\,d^\times t\,dx\,d^\times a\,dk\\
&= \int_K\int_{F^\times\backslash\mathbf{A}^1}\int_{\mathbf{A}}\int_{F_\infty^+}\sum_{\alpha\in F^\times} f\left(k^{-1}\begin{pmatrix}\alpha t & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}k\right)|t|^{\frac{1}{2}+iy}\,d^\times t\,dx\,d^\times a\,dk\\
&= \operatorname{vol}(F^\times\backslash\mathbf{A}^1)\int_K\int_{\mathbf{A}}\int_{F_\infty^+}\sum_{\alpha\in F^\times} f\left(k^{-1}\begin{pmatrix}\alpha t & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}k\right)|t|^{\frac{1}{2}+iy}\,d^\times t\,dx\,dk.
\end{aligned}
$$

□

Thus we have,

$$
\frac{T}{2\pi}\int_{-\infty}^{\infty}\operatorname{tr}\pi_{iy}(f)\,dy = \frac{\operatorname{vol}(F^\times\backslash\mathbf{A}^1)T}{2\pi}\int_K\int_{N(\mathbf{A})}\sum_{\alpha\in F^\times}\int_{-\infty}^{\infty}\int_{F_\infty^+} f\left(k^{-1}\begin{pmatrix}t\alpha & 0\\ 0 & 1\end{pmatrix}nk\right)|t|^{\frac{1}{2}+iy}\,d^\times t\,dy\,dn\,dk.
$$

We apply the Mellin inversion formula to give,

$$
\frac{T}{2\pi}\int_{-\infty}^{\infty}\operatorname{tr}\pi_{iy}(f)\,dy = T\operatorname{vol}(F^\times\backslash\mathbf{A}^1)\int_K\int_{N(\mathbf{A})}\sum_{\alpha\in F^\times} f\left(k^{-1}\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}nk\right)\,dn\,dk.
$$

For the second term after interchanging summation and integration we have,

$$
-\frac{1}{4\pi}\int_{-\infty}^{\infty}\sum_{\alpha,\beta}(\pi_{iy}(f)\varphi_\beta,\varphi_\alpha)(M(-iy)M'(iy)\varphi_\alpha,\varphi_\beta)\,dy
$$

and we note that,

$$\begin{aligned}\sum_{\alpha,\beta}(\pi_{iy}(f)\varphi_\beta,\varphi_\alpha)(M(-iy)M'(iy)\varphi_\alpha,\varphi_\beta) &= \sum_{\alpha,\beta}(\pi_{iy}(f)\varphi_\beta,\varphi_\alpha)(\varphi_\alpha,(M(-iy)M'(iy))^*\varphi_\beta)\\ &= \sum_{\beta}(\pi_{iy}(f)\varphi_\beta,(M(-iy)M'(iy))^*\varphi_\beta)\\ &= \sum_{\beta}(M(-iy)M'(iy)\pi_{iy}(f)\varphi_\beta,\varphi_\beta)\\ &= \operatorname{tr}(M(-iy)M'(iy)\pi_{iy}(f)).\end{aligned}$$

Hence the second term is equal to

$$-\frac{1}{4\pi}\int_{-\infty}^{\infty}\operatorname{tr}(M(-iy)M'(-y)\pi_{iy}(f))\ dy.$$

Finally the third term is,

$$\frac{1}{4\pi}\sum_{\alpha,\beta}\int_{-\infty}^{\infty}(\pi_{iy}(f)\varphi_\beta,\varphi_\alpha)\left((\varphi_\alpha,M(iy)\varphi_\beta)e^{2iyT}-(M(iy)\varphi_\alpha,\varphi_\beta)e^{-2iyT}\right)\frac{1}{2iy}\ dy$$

which equals,

$$\frac{1}{4\pi}\int_{-\infty}^{\infty}\sum_{\beta}\left((\pi_{iy}(f)\varphi_\beta,M(iy)\varphi_\beta)e^{2iyT}-(\pi_{iy}(f)\varphi_\beta,M(-iy)\varphi_\beta)e^{-2iyT}\right)\frac{1}{2iy}\ dy$$

which equals,

$$\frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{\operatorname{tr}(M(-iy)\pi_{iy}(f))e^{2iyT}-\operatorname{tr}(M(iy)\pi_{iy}(f))e^{-2iyT}}{2iy}\ dy.$$

In order to compute this we note the following,

Lemma 8.13. *Let Φ_1 and Φ_2 be continuous, integrable functions on* $\mathbf{R}$ *which are differentiable at zero. Suppose further that,*

$$\Phi_1(0)=\Phi_2(0).$$

Then,

$$\lim_{T\to\infty}\int_{-\infty}^{\infty}\frac{\Phi_1(y)e^{2iyT}-\Phi_2(y)e^{-2iyT}}{y}\ dy = 2\pi i\Phi_1(0).$$

Proof. We break up the integral as the sum of,

$$\int_{-\infty}^{\infty}\frac{(\Phi_1(y)-\Phi_2(y))e^{2iyT}}{y}\ dy+\int_{-\infty}^{\infty}\frac{\Phi_2(y)(e^{2iyT}-e^{-2iyT})}{y}\ dy = 2\pi i\Phi_1(0).$$

For the first term we note that,

$$\frac{\Phi_1(y)-\Phi_2(y)}{y}$$

is continuous and integrable on $\mathbf{R}$, hence the term

$$\int_{-\infty}^{\infty} \frac{(\Phi_1(y) - \Phi_2(y))e^{2iyT}}{y} \, dy$$

being the Fourier transform of an integrable function goes to zero as $T \to \infty$. For the second term we define,

$$G(T) = \int_{-\infty}^{\infty} \frac{\Phi_2(y)(e^{2iyT} - e^{-2iyT})}{y} \, dy.$$

Then $G(0) = 0$, and

$$G'(T) = 2i \int_{-\infty}^{\infty} \Phi_2(y)(e^{2iyT} + e^{-2iyT}) \, dy = 2i(\hat{\Phi}_2(T/\pi) + \hat{\Phi}_2(-T/\pi)).$$

Hence,

$$\begin{aligned} G(T) &= 2i \int_0^T (\hat{\Phi}_2(T/\pi) + \hat{\Phi}_2(-T/\pi)) \, dT \\ &= 2i \int_{-T}^{T} \hat{\Phi}_2(T/\pi) \, dT \end{aligned}$$

Hence as $T \to \infty$, $G(T)$ tends to $2\pi i \Phi_2(0)$. $\square$

Lemma 8.14. *The functions,*

$$y \mapsto \operatorname{tr}(M(-iy)\pi_{iy}(f))$$

and

$$y \mapsto \operatorname{tr}(M(iy)\pi_{iy}(f))$$

are integrable over $\mathbf{R}$.

Proof. See [Gel75]. $\square$

Thus we see that as $T \to \infty$,

$$\frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{\operatorname{tr}(M(-iy)\pi_{iy}(f))e^{2iyT} - \operatorname{tr}(M(iy)\pi_{iy}(f))e^{-2iyT}}{2iy} \, dy$$

tends to,

$$\frac{1}{4} \operatorname{tr} M(0)\pi_0(f).$$

Proposition 8.15. *We have,*

$$\int_{\overline{G}(F)\backslash\overline{G}(\mathbf{A})} K_{f,\mathrm{cont}}^T(x,x) \, dx$$

equal to the sum of

$$-\frac{1}{4\pi}\int_{-\infty}^{\infty} \operatorname{tr}(M(-iy)M'(iy)\pi_{iy}(f))\ dy + \frac{1}{4}\operatorname{tr}(M(0)\pi_0(f)).$$

and

$$\frac{T}{2\pi}\operatorname{vol}(F^\times\backslash\mathbf{A}^1)\int_K\int_{N(\mathbf{A})}\sum_{\alpha\in F^\times} f\left(k^{-1}\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}nk\right)\ dn\ dk.$$

and a term which tends to 0 as $T\to\infty$.

We can now write down the spectral side of the trace formula. We note that from our calculation of,

$$\int_{\overline{G}(F)\backslash\overline{G}(\mathbf{A})} K_f^T(x,x)\ dx$$

using the geometric expansion for the truncated kernel that this is a linear polynomial in T. Hence from the calculations above we conclude,

Theorem 8.16. *(Spectral side of the trace formula) We have,*

$$\int_{\overline{G}(F)\backslash\overline{G}(\mathbf{A})} K_f^T(x,x)\ dx$$

equal to the sum of

$$\operatorname{tr} R_{\text{cusp}}(f)+\operatorname{tr} R_{\text{res}}(f)-\frac{1}{4\pi}\int_{-\infty}^{\infty}\operatorname{tr}(M(-iy)M'(iy)\pi_{iy}(f))\ dy+\frac{1}{4}\operatorname{tr}(M(0)\pi_0(f))$$

and

$$\frac{T}{2\pi}\operatorname{vol}(F^\times\backslash\mathbf{A}^1)\int_K\int_{N(\mathbf{A})}\sum_{\alpha\in F^\times} f\left(k^{-1}\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}nk\right)\ dn\ dk.$$

9 The trace formula for GL(2)

We can now finally give the statement of the trace formula for GL(2) which is given by equating the constant terms of

$$\int_{\overline{G}(F)\backslash\overline{G}(\mathbf{A})} K_f^T(x,x)\ dx.$$

Theorem 9.1. *(The trace formula for* GL(2)*) Let $f\in C_c^\infty(\overline{G}(\mathbf{A}))$. Then, the sum of*

$$\operatorname{vol}(Z(\mathbf{A})G(F)\backslash G(\mathbf{A}))f(I)+\sum_{\substack{\gamma\in\Gamma(Z(F)\backslash G(F))\\ \text{ell. reg.}}}\operatorname{vol}(G_\gamma(F)\backslash G_\gamma(\mathbf{A}))\int_{G_\gamma(\mathbf{A})\backslash G(\mathbf{A})} f(g^{-1}\gamma g)\ dg$$

$$f.p._{s=1}(Z(F_f,s)) - \frac{1}{2}\,\mathrm{vol}(F^\times\backslash\mathbf{A}^1)\sum_{\gamma\in M(F)}\int_{M(\mathbf{A})\backslash G(\mathbf{A})} f(g^{-1}\gamma g)H(wg)\;dg$$

is equal to

$$\mathrm{tr}\,R_{\mathrm{cusp}}(f)+\mathrm{tr}\,R_{\mathrm{res}}(f)-\frac{1}{4\pi}\int_{-\infty}^{\infty}\mathrm{tr}(M(-iy)M'(iy)\pi_{iy}(f))\;dy+\frac{1}{4}\,\mathrm{tr}(M(0)\pi_0(f)).$$

9.1 Simpler form of the trace formula

For our application to the Jacquet-Langlands correspondence we record the following simpler version of the trace formula. Suppose we now take a function $f=\prod_v f_v \in C_c^\infty(\overline{G}(\mathbf{A}))$ such that for two distinct places v_1 and v_2 and any $\gamma \in M(F_{v_i})\setminus Z(F_{v_i})$, we have,

$$\int_{M(F_{v_i})\backslash G(F_{v_i})} f_{v_i}(g^{-1}\gamma g)\;dg.$$

We begin with the geometric side of the trace formula. We recall for $g=(g_v)\in G(\mathbf{A})$,

$$H(g)=\sum_v H_v(g_v).$$

Thus we have for $f=\prod_v f_v$ and $\gamma\in M(F)$,

$$\begin{aligned}\int_{M(\mathbf{A})\backslash G(\mathbf{A})} f(g^{-1}\gamma g)H(wg)\;dg &= \sum_v \int_{M(\mathbf{A})\backslash G(\mathbf{A})} f(g^{-1}\gamma g)H_v(wg)\;dg\\ &= \sum_v \int_{M(F_u)\backslash G(F_u))} f_v(g_v^{-1}\gamma g_v)H_v(wg_v)\;dg_v \prod_{u\neq v}\int_{M(F_u)\backslash G(F_u))} f_u(g_u^{-1}\gamma g_u)\;dg_u\\ &= 0.\end{aligned}$$

For the unipotent term we recall that we defined, $F_f\in C_c^\infty(\mathbf{A})$ by

$$F_f(a)=\int_K f\left(k^{-1}\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}k\right)\;dk.$$

Then,

$$\mathrm{f.p.}_{s=1}(Z(F_f,s))$$

is defined to be the finite part of the zeta integral

$$Z(F_f,s)=\int_{\mathbf{A}^\times} F_f(a)|a|^s\;d^\times a$$

which converges provided $\Re s>1$. We note that for a factorizable function f we have,

$$F_f=\prod_v F_{f_v}$$

where

$$F_{f_v}(a) = \int_{K_v} f_v \left(k^{-1} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} k \right) \, dk.$$

And we have, for $\Re s > 1$,

$$Z(F_f, s) = \prod_v \int_{F_v^\times} F_{f_v}(a) |a|_v^s \, d^\times a.$$

We define,

$$\theta(F_f, s) = \frac{Z(F_f, s)}{L(s, 1_F)}$$

which is holomorphic at $s = 1$. We define,

$$\theta(F_{f_v}, s) = \frac{Z(F_{f_v}, s)}{L(s, 1_{F_v})}$$

so that

$$\theta(F_f, s) = \prod_v \theta(F_{f_v}, s).$$

Almost all factors in this product are identically one and so,

$$\theta(F_f, s) = \prod_v \theta(F_{f_v}, s).$$

We expand $L(s, 1_F)$ in its Laurent expansion around $s = 1$ as,

$$L(s, 1_F) = \frac{a_{-1}}{s-1} + a_0 + a_1(s-1) + \ldots.$$

Then we have

$$\text{f.p.}_{s=1}(Z(F_f, s)) = a_{-1}\theta'(F_f, 1) + a_0\theta(F_f, 1).$$

We have,

$$\theta'(F_f, s) = \sum_v \prod_{u \neq v} \theta(F_{f_u}, s) \times \theta'(F_{f_v}, s).$$

Now we note that,

$$\begin{aligned}
\theta(F_{f_v}, 1) &= \frac{1}{L(1, 1_{F_v})} \int_{F_v^\times} \int_{K_v} f_v \left(k^{-1} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} k \right) |a| \, dk \, d^\times a \\
&= C \int_{F_v^\times} \int_{K_v} f_v \left(k^{-1} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix} k \right) |a| \, dk \, d^\times a \\
&= C \int_{Z(F_v)N(F_v)\backslash G(F_v)} f_v \left(g^{-1} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} g \right) \, dg.
\end{aligned}$$

But now we note that,

$$\int_{Z(F_v)N(F_v)\backslash G(F_v)} f_v \left(g^{-1} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} g \right) \, dg = \lim_{a \to 1} |1 - a^{-1}| \int_{M(F_v)\backslash G(F_v)} f_v \left(g^{-1} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g \right) \, dg.$$

Thus at the places $v \in \{v_1, v_2\}$ we have

$$\theta(F_{f_v}, 1) = 0$$

and hence,

$$\text{f.p.}_{s=1}(Z(F_f, s)) = 0.$$

Thus under the assumptions above the geometric side of the trace formula reduces to,

$$\mathrm{vol}(Z(\mathbf{A})G(F)\backslash G(\mathbf{A}))f(I) + \sum_{\substack{\gamma \in \Gamma(Z(F)\backslash G(F)) \\ \text{ell. reg.}}} \mathrm{vol}(G_\gamma(F)\backslash G_\gamma(\mathbf{A})) \int_{G_\gamma(\mathbf{A})\backslash G(\mathbf{A})} f(g^{-1}\gamma g)\, dg.$$

We now consider the spectral side of the trace formula for functions f satisfying the condition above at the places v_1 and v_2. We begin with a lemma,

Lemma 9.2. *Suppose $f \in C_c^\infty(\overline{G}(F_v))$ is such that,*

$$\int_{M(F_v)\backslash G(F_v)} f(g^{-1}\gamma g)\ dg = 0$$

for all $\gamma \in M(F_v) \setminus Z(F_v)$. Then for all characters $\chi : Z(F_v)\backslash M(F_v) \to C^\times$ we have,

$$\operatorname{tr} \pi_\chi(f) = 0$$

where π_χ is the representation of $G(F_v)$ induced from χ.

Proof. Via the Iwasawa decomposition we identify the space of π_χ with a closed subspace of $L^2(K)$. Suppose $\varphi \in \pi_\chi$. Then by definition,

$$(\pi_\chi(f)\varphi)(x) = \int_{\overline{G}(F_v)} f(y)\varphi(xy)\ dy = \int_{\overline{G}(F_v)} f(x^{-1}y)\varphi(y)\ dy.$$

We use the Iwasawa decomposition and write,

$$y = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} k$$

and decompose the Haar measure on $\overline{G}(F_v)$ as,

$$dy = d^\times a\ db\ dk.$$

Then we have,

$$\begin{aligned}(\pi_\chi(f)\varphi)(x) &= \int_K \int_{F_v} \int_{F_v^\times} f\left(x^{-1}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} k\right) \varphi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} k\right)\ d^\times a\ db\ dk \\ &= \int_K \left(\int_{F_v} \int_{F_v^\times} f\left(x^{-1}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} k\right) |a|^{\frac{1}{2}}\chi(a)\ d^\times a\ db\right) \varphi(k)\ dk\end{aligned}$$

Thus map $\pi_\chi(f)$ is given by the kernel,

$$K_f(x,y) = \int_{F_v^\times} \int_{F_v} f\left(y^{-1}\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & 1\end{pmatrix} x\right)\chi(a)|a|^{\frac{1}{2}}\ d^\times a\ db.$$

After a change of variables this is equal to

$$K_f(x,y) = \int_{F_v^\times} \int_{F_v} f\left(y^{-1}\begin{pmatrix} 1 & -b \\ 0 & 1\end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & 1\end{pmatrix} x\right)\chi(a)\frac{|a-1|}{|a|^{\frac{1}{2}}}\ d^\times a\ db.$$

We can compute the trace by integrating $K_f(x,y)$ over the diagonal to yield,

$$\operatorname{tr}\pi_\chi(f) = \int_{F_v^\times}\left(\int_{M(F_v)\backslash G(F_v)} f\left(g^{-1}\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix} g\right)\ dg\right)\chi(a)\frac{|a-1|}{|a|^{\frac{1}{2}}}\ d^\times a$$

which is zero by hypothesis. $\square$

To begin with we claim that,

$$\operatorname{tr}(M(0)\pi_0(f)) = 0.$$

We recall that we can decompose,

$$\pi_0 = \bigoplus_{\chi: F^\times\backslash \mathbf{A}^1 \to \mathbf{C}^\times} \pi_{0,\chi}$$

by considering the left action of $M(\mathbf{A})$ on π_0. We recall that the intertwining operator $M(0)$ maps $\pi_{0,\chi}$ into $\pi_{0,\chi^{-1}}$. Hence,

$$\operatorname{tr}(M(0)\pi_0(f)) = \sum_{\chi:\chi^2=1} \operatorname{tr}(M_\chi(0)\pi_{0,\chi}(f)).$$

For each such χ the representation $\pi_{0,\chi}$ is irreducible and hence $M(0)$ must act on $\pi_{0,\chi}$ by a scalar c_χ. Thus we have,

$$\operatorname{tr}(M(0)\pi_0(f)) = \sum_{\chi:\chi^2=1} c_\chi \operatorname{tr}\pi_{0,\chi}(f).$$

But now $\operatorname{tr}\pi_{0,\chi}(f) = \prod_v \operatorname{tr}\pi_{0,\chi_v}(f_v) = 0$ by Lemma 8.12.

We now consider the term

$$\int_{-\infty}^{\infty} \operatorname{tr}(M(-iy)M'(iy)\pi_{iy}(f))\ dy.$$

Again we can decompose,

$$\pi_{iy} = \bigoplus_{\chi: F^\times\backslash \mathbf{A}^1 \to \mathbf{C}^\times} \pi_{iy,\chi}$$

so that our term becomes

$$\sum_{\chi:F^\times\backslash\mathbf{A}^1\to\mathbf{C}^\times}\int_{-\infty}^{\infty}\operatorname{tr}(M_{\chi^{-1}}(-iy)M'_\chi(iy)\pi_{iy,\chi}(f))\ dy.$$

We recall that when we established the meromorphic continuation of the intertwining operators $M_\chi(s)$ we wrote them in the form,

$$M(s)_\chi = \frac{L(2s,\chi^2)}{L(2s+1,\chi^2)\varepsilon(2s,\chi^2)}R(s)_\chi$$

with $R(s)_\chi$ a normalized intertwining operator which was defined as the product of local normalized intertwining operators $R_v(s)_{\chi_v}$. For ease of notation we write this in the form

$$M(s)_\chi = m(s)_\chi R(s)_\chi.$$

We apply the product rule to obtain,

$$\begin{aligned} M'(s)_\chi &= m'(s)_\chi R(s)_\chi + m(s)_\chi R'(s)_\chi \\ &= m'(s)_\chi R(s)_\chi + m(s)_\chi \sum_v R'_v(s)_{\chi_v}\otimes_{u\neq v} R_v(s)_{\chi_v}. \end{aligned}$$

Using the functional equation $M(-s)_{\chi^{-1}}M(s)_\chi = Id$ and $R(-s)_{\chi^{-1}}R(s)_\chi$ we see that,

$$M_{\chi^{-1}}(-iy)M'_\chi(iy) = \frac{m'(iy)_\chi}{m(iy)_\chi}Id + \sum_v R_v(iy)_{\chi_v{}^{-1}}R'_v(iy)_{\chi_v}\otimes_{u\neq v} I_{v,\chi_v},$$

where

$$I_{v,\chi} : H_v(s)_{\chi_v} \to H_v(s)_{\chi_v}$$

denotes the identity. Thus, for a factorizable function $f=\prod_v f_v$,

$$\operatorname{tr}(M_{\chi^{-1}}(-iy)M'_\chi(iy)\pi_{iy,\chi}(f)) = \frac{m'(iy)_\chi}{m(iy)_\chi}\operatorname{tr}\pi_{iy,\chi}(f)+\sum_v \operatorname{tr}(R_v(iy)_{\chi_v{}^{-1}}R'_v(iy)_{\chi_v}\pi_{iy,\chi_v})\prod_{u\neq v}\operatorname{tr}\pi_{iy,\chi_v}(f_v).$$

Now at the places $v\in\{v_1,v_2\}$ we have,

$$\operatorname{tr}\pi_{iy,\chi_v}(f_v)=0$$

by Lemma 9.2. Hence we conclude that for such functions f we have

$$\int_{-\infty}^{\infty}\operatorname{tr}(M(-iy)M'(iy)\pi_{iy}(f))\ dy = 0.$$

Theorem 9.3. *Let $f\in C_c^\infty(\overline{G}(\mathbf{A}))$ satisfy the condition above. Then,*

$$\operatorname{vol}(Z(\mathbf{A})G(F)\backslash G(\mathbf{A}))f(I)+\sum_{\substack{\gamma\in\Gamma(Z(F)\backslash G(F))\\ ell.\ reg.}}\operatorname{vol}(G_\gamma(F)\backslash G_\gamma(\mathbf{A}))\int_{G_\gamma(\mathbf{A})\backslash G(\mathbf{A})} f(g^{-1}\gamma g)\ dg$$

is equal to

$$\operatorname{tr}R_{\text{cusp}}(f)+\operatorname{tr}R_{\text{res}}(f).$$

10 The Jacquet-Langlands correspondence

We now take D to be a quaternion division algebra over F. We denote by S the set (finite and of even cardinality) of places v such that $D_v = D \otimes_F F_v$ is a division algebra. We set $G_D = D^\times$ which we view as an algebraic group over F. We let Z_D denote the center of G_D and we set $\overline{G}_D = Z_D \backslash G_D$. Since the quotient $\overline{G}_D(F)\backslash\overline{G}_D(\mathbf{A})$ is compact so

$$L^2(\overline{G}_D(F)\backslash\overline{G}_D(\mathbf{A})) \cong \bigoplus_{\pi^D} m_{\pi^D}\pi^D$$

as a Hilbert direct sum of irreducible representations of $G_D(\mathbf{A})$. We define,

$$\mathcal{A}(G_D) = \left\{\pi^D : m_{\pi^D} > 0\}\right\}.$$

We write,

$$L^2_{\mathrm{disc}}(\overline{G}(F)\backslash\overline{G}(\mathbf{A})) = L^2_{\mathrm{cusp}}(\overline{G}(F)\backslash\overline{G}(\mathbf{A})) \oplus L^2_{\mathrm{res}}(\overline{G}(F)\backslash\overline{G}(\mathbf{A})) \cong \bigoplus_{\pi} m_\pi \pi$$

and define

$$\mathcal{A}_{\mathrm{disc}}(G) = \{\pi : m_\pi > 0\}\}.$$

For each place $v \notin S$ we fix an isomorphism,

$$\alpha_v : D_v \xrightarrow{\sim} M(2, F_v)$$

which is well defined, by the Skolem-Noether theorem, up to conjugacy. This induces,

$$\alpha_v : G_D(F_v) \xrightarrow{\sim} G(F_v)$$

and we assume that for almost all v,

$$\alpha_v : G_D(\mathcal{O}_{F_v}) \xrightarrow{\sim} G(\mathcal{O}_{F_v}).$$

A weak form of the Jacquet-Langlands correspondence is given by the following,

Theorem 10.1. *Let $\pi^D \in \mathcal{A}(G_D)$. We write $\pi^D = \otimes_v \pi_v^D$. Then there exists $\pi = \otimes_v \pi_v \in \mathcal{A}_{\mathrm{disc}}(G)$ such that for all $v \notin S$,*

$$\pi_v = \pi_v^D \circ \alpha_v^{-1}.$$

The stronger version of the Jacquet-Langlands correspondence determines the representation π_v at the places $v \in S$, asserts that the representations appear with the same multiplicity and determines the image of $\mathcal{A}(G_D)$ in $\mathcal{A}(G)$. This stronger version can also be obtained via the trace formula but requires more local inputs.

The trace formula in the compact quotient case tells us that for $f \in C_c^\infty(\overline{G}_D(\mathbf{A}))$,

$$\sum_{\pi^D} m_{\pi^D} \operatorname{tr} \pi^D(f) = \sum_{\gamma \in \Gamma(\overline{G}_D)} \operatorname{vol}(\overline{G}_{D,\gamma}(F)\backslash \overline{G}_{D,\gamma}(\mathbf{A})) I(\gamma, f),$$

where we have set,

$$I(\gamma, f) = \int_{\overline{G}_{D,\gamma}(\mathbf{A})\backslash \overline{G}_D(\mathbf{A})} f(g^{-1}\gamma g)\ dg.$$

Furthermore if $f = \prod_v f_v$ then,

$$I(\gamma, f) = \prod_v I(\gamma, f_v).$$

Our goal is to compare the geometric side of the trace formula for G_D with the geometric side of the trace formula for G. We begin by establishing a map,

$$\iota : \Gamma(G_D(F)) \hookrightarrow \Gamma(G(F)).$$

Since all quadratic extensions of F embed in $M(2, F)$ we can pick an embedding,

$$\beta : F[\gamma] \hookrightarrow M(2, F)$$

of F-algebras. By the Skolem-Noether theorem such a map is well defined up to conjugacy and hence the conjugacy class of $\beta(\gamma)$ in $G(F)$ is well defined. Thus we get a well defined map,

$$\iota : \Gamma(G_D(F)) \hookrightarrow \Gamma(G(F)).$$

Similarly if $v \in S$ then we have maps

$$\iota_v : \Gamma(G_D(F_v)) \hookrightarrow \Gamma(G(F_v))$$

and for places $v \notin S$ we have maps,

$$\iota_v : \Gamma(G_D(F_v)) \hookrightarrow \Gamma(G(F_v))$$

given by the isomorphism $\alpha_v : D_v \xrightarrow{\sim} M(2, F_v)$.

Lemma 10.2. *Let $\gamma \in G(F)$. Then $\gamma \in \iota(\Gamma(G_D(F)))$ if and only if $\gamma \in \iota_v(\Gamma(G_D(F_v)))$ for all places $v \in S$.*

Proof. It suffices to recall that a quadratic extension E of F embeds in D if and only if E_v embeds in D_v for all places v of F. □

Suppose $\gamma \in D \setminus F$. Then $\iota(\gamma)$ is defined by picking an embedding

$$\beta : F[\gamma] \hookrightarrow M(2, F).$$

Thus β induces an isomorphism

$$\beta : G_{D,\gamma} \xrightarrow{\sim} G_{\beta(\gamma)}$$

of algebraic groups over F. This allows us to relate the measure on $G_{D,\gamma}$ and $G_{\beta(\gamma)}$ which are implicit in the statement of the trace formula.

Next time we will establish the following,

Theorem 10.3. *Let $v \in S$ and $f_v \in C_c^\infty(\overline{G}_D(F_v))$. Then there exists $f'_v \in C_c^\infty(\overline{G}(F_v))$ such that for all $\gamma \in G_D(F_v)$*

$$I(\gamma, f_v) = I(\iota_v(\gamma), f'_v).$$

Furthermore if δ is a semisimple element of $G(F_v)$ which doesn't lie in the image of the map ι_v then,

$$I(\delta, f'_v) = 0.$$

We will take this result for granted for the moment and use it to prove Theorem 10.1. Implicit in the statement of the theorem is a choice of Haar measure on $G(F_v)$ and $G_D(F_v)$. At the places $v \notin S$ we fix a Haar measure on $G(F_v)$ and use the isomorphism α_v to transport it to a Haar measure on $G_D(F_v)$.

Let $f = \prod_v f_v \in C_c^\infty(G_D(\mathbf{A}))$. For each place v we define $f'_v \in C_c^\infty(G(\mathbf{A}))$ in the following way. If $v \notin S$ we define,

$$f'_v = f_v \circ \alpha_v^{-1},$$

for places $v \in S$ we take f_v as in Theorem 10.3. Thus if we take $\gamma \in \Gamma(G_D(F))$ then

$$I(\gamma, f) = \prod_v I(\gamma, f_v) = \prod_v I(\iota(\gamma), f'_v) = I(\iota(\gamma), f').$$

Furthermore if $\delta \in G(F)$ does not lie in the image of ι then, by Lemma 10.2 there exists a place $v \in S$ such that γ doesn't lie in the image of ι_v. Thus,

$$I(\delta, f_v) = 0$$

and hence $I(\delta, f) = 0$. We can apply Theorem 9.3 to f' to deduce that,

$$\begin{aligned}
\operatorname{tr} R_{\text{disc}}(f') &= \operatorname{vol}(\overline{G}(F)\backslash\overline{G}(\mathbf{A}))f'(I) + \sum_{\substack{\delta\in\Gamma(\overline{G}(F))\\ \text{ell. reg.}}} \operatorname{vol}(\overline{G}_\delta(F)\backslash\overline{G}_\delta(\mathbf{A}))I(\delta, f') \\
&= \operatorname{vol}(\overline{G}(F)\backslash\overline{G}(\mathbf{A}))f(I) + \sum_{\substack{\gamma\in\Gamma(\overline{G}_D(F))\\ \text{ell. reg.}}} \operatorname{vol}(\overline{G}_{D,\gamma}(F)\backslash\overline{G}_{D,\gamma}(\mathbf{A}))I(\gamma, f).
\end{aligned}$$

Thus we see that,

$$\operatorname{tr} R_{\text{disc}}(f') - \operatorname{tr} R(f) = \operatorname{vol}(\overline{G}(F)\backslash\overline{G}(\mathbf{A})) - \operatorname{vol}(\overline{G}_D(F)\backslash\overline{G}_D(\mathbf{A}))f(I).$$

Now one fixes a place $u \notin S$ and functions f_v for $v \neq u$ and considers this an identity of distributions on $C_c^\infty(\overline{G}_D(F_v)) \cong C_c^\infty(\overline{G}(F_v))$. Applying the Plancherel formula allows on to deduce that,

$$\operatorname{vol}(\overline{G}(F)\backslash\overline{G}(\mathbf{A})) = \operatorname{vol}(\overline{G}_D(F)\backslash\overline{G}_D(\mathbf{A}))$$

and hence that,

$$\operatorname{tr} R_{\mathrm{disc}}(f') = \operatorname{tr} R(f).$$

Suppose now that we are given $\sigma = \otimes\sigma_v \in \mathcal{A}(G_D)$. For each place $v \in S$ the group $\overline{G}_D(F_v)$ is compact and hence σ_v is finite dimensional. For each place $v \in S$ we let f_v be a matrix coefficient of σ_v. Then if τ is an irreducible representation of $\overline{G}_D(F_v)$ we have,

$$\operatorname{tr} \tau(f_v) = 0$$

unless $\tau \cong \sigma_v$. Thus,

$$\operatorname{tr} R(f) = \sum_{\pi^D \in \mathcal{A}(G_D)} m_{\pi^D} \operatorname{tr} \pi^D(f) = \operatorname{tr} \sigma_S(f_S) \sum_{\substack{\pi^D \in \mathcal{A}(G_D) \\ \pi^D_S \cong \sigma_S}} m_{\pi^D} \operatorname{tr} \pi^{D,S}(f^S).$$

On the other hand we have,

$$\begin{aligned}\operatorname{tr} R_{\mathrm{disc}}(f') = \sum_{\pi \in \mathcal{A}_{\mathrm{disc}}(G)} m_\pi \operatorname{tr} \pi(f') &= \sum_{\pi \in \mathcal{A}_{\mathrm{disc}}(G)} m_\pi \operatorname{tr} \pi_S(f'_S) \operatorname{tr} \pi^S(f'^S) \\ &= \sum_{\pi \in \mathcal{A}_{\mathrm{disc}}(G)} m_\pi \operatorname{tr} \pi_S(f'_S) \operatorname{tr}(\pi^S \circ \alpha^S)(f^S).\end{aligned}$$

Thus the distribution on $C_C^\infty(G_D(\mathbf{A}^S))$,

$$f^S \mapsto \operatorname{tr} \sigma_S(f_S) \sum_{\substack{\pi^D \in \mathcal{A}(G_D) \\ \pi^D_S \cong \sigma_S}} m_{\pi^D} \operatorname{tr} \pi^{D,S}(f^S) - \sum_{\pi \in \mathcal{A}_{\mathrm{disc}}(G)} m_\pi \operatorname{tr} \pi_S(f'_S) \operatorname{tr}(\pi^S \circ \alpha^S)(f^S)$$

is identically zero. We can rewrite this as

$$\sum_{\pi^{D,S}} \left(\sigma_S(f_S) m_{\sigma_S \otimes \pi^{D,S}} - \sum_{\pi_S} m_{\pi_S \otimes \pi^{D,S} \circ \alpha^{-1}} \operatorname{tr} \pi_S(f'_S) \right) \operatorname{tr} \pi^{D,S}(f^S).$$

Applying linear independence of characters [JL70, Lemma 16.1.1] we deduce that for every representation $\pi^{D,S}$ of $\overline{G}_D(\mathbf{A}^S)$ we have,

$$\sigma_S(f_S) m_{\sigma_S \otimes \pi^{D,S}} - \sum_{\pi_S} m_{\pi_S \otimes \pi^{D,S} \circ \alpha^{-1}} \operatorname{tr} \pi_S(f'_S).$$

Applying this identity to $\pi^{D,S} = \sigma^S$ we see that there exists $\pi \in \mathcal{A}_{\mathrm{disc}}(G)$ such that $\pi^S = \sigma^S \circ \alpha^{-1}$ and $m_\pi > 0$. This concludes the proof of Theorem 10.1

10.1 On the proof of Theorem 10.3

For complete results see [Lan80, Chapter 6] and [Lan73, Section 6].

We now switch to the local setting. We take F to be a local field (for us we'll restrict to the case that F is nonarchimedean) and D to be the quaternion division algebra over F. We take,

$$\iota : \Gamma(G_D(F)) \hookrightarrow \Gamma(G(F)).$$

In order to establish the existence of a function f' on $G(F)$ matching f we need to a classification theorem for functions on $G(F)$ which arise as orbital integrals of smooth compactly supported functions on $G(F)$.

Suppose we fix a set of representatives $\{T\}$ for the tori in $G(F)$. Let f be a smooth compactly supported function on $G(F)$. For a regular element $\gamma \in T(F)$, i.e. for $\gamma \in T(F) \setminus Z(F)$ we define normalized orbital integrals by,

$$F^T(\gamma, f) = \Delta(\gamma) \int_{T(F)\setminus G(F)} f(g^{-1}\gamma g)\ dg.$$

where if γ_1 and γ_2 are the eigenvalues of γ then,

$$\Delta(\gamma) = \left| \frac{(\gamma_1 - \gamma_2)^2}{\gamma_1 \gamma_2} \right|^{\frac{1}{2}}.$$

Of course these integrals depends on a choice of measure on $T(F)$ which we suppress from our notations.

Suppose we have a collection of functions $\{a^T\}$ for each T defined on the regular elements $T(F)$. We need to determine when such a family comes from the orbital integrals of a smooth function of compact support.

In order for such a collection of functions to come from orbital integrals we need to have some compatibility condition as the regular elements $\gamma \in T(F)$ approach other tori, i.e. as $\gamma \to Z(F)$.

We begin with T being the diagonal torus M in G. Then for a regular element

$$\gamma = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \in M(F)$$

we have,

$$\begin{aligned}
\int_{M(F)\setminus G(F)} f(g^{-1}\gamma g)\ dg &= \int_{N(F)} \int_K f(k^{-1}n^{-1}\gamma n k)\ dk\ dn \\
&= \int_F \int_K f\left(k^{-1} \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} k\right)\ dk\ dx \\
&= \int_F \int_K f\left(k^{-1} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} 1 & x(1-\alpha^{-1}\beta) \\ 0 & 1 \end{pmatrix} k\right)\ dk\ dx \\
&= \frac{1}{|1-\alpha^{-1}\beta|} \int_F \int_K f\left(k^{-1} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} k\right)\ dk\ dx.
\end{aligned}$$

Thus we see that

$$F^M(\gamma) = \left|\frac{\alpha}{\beta}\right|^{\frac{1}{2}} \int_F \int_K f\left(k^{-1} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} k\right)\ dk\ dx,$$

as a function on $M^{reg}(F)$, is locally constant and compactly supported. Furthermore if γ is sufficiently close to

$$\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} \in Z(F)$$

then,

$$\begin{aligned}
F^M(\gamma) &= \int_F \int_K f\left(k^{-1}\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} k\right)\ dk\ dx \\
&= \int_{F^\times} \int_K f\left(k^{-1}\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} z & 1 \\ 0 & z \end{pmatrix}\begin{pmatrix} x^{-1} & 0 \\ 0 & 1 \end{pmatrix} k\right)\ dk\ dx \\
&= \int_{F^\times} \int_K f\left(k^{-1}\begin{pmatrix} x^{-1} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} z & 1 \\ 0 & z \end{pmatrix}\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} k\right) \frac{1}{|x|^2}\ dk\ dx \\
&= \int_{F^\times} \int_K f\left(k^{-1}\begin{pmatrix} x^{-1} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} z & 1 \\ 0 & z \end{pmatrix}\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} k\right) \frac{1}{|x|}\ dk\ d^\times x \\
&= \int_{G_{\gamma'}(F)\backslash G(F)} f\left(g^{-1}\begin{pmatrix} z & 1 \\ 0 & z \end{pmatrix} g\right)\ dg,
\end{aligned}$$

where,

$$\gamma' = \begin{pmatrix} z & 1 \\ 0 & z \end{pmatrix}.$$

Suppose now that T is a non-split torus in G. In this case again $F^T(\gamma)$ is locally constant and compactly supported on $T(F)$. In this case for $z \in Z(F)$ there is a neighborhood U of z such that if $t \in T^{reg}(F)$ then,

$$F^T(t) = \int_{G_{\gamma'}(F)\backslash G(F)} f\left(g^{-1}\begin{pmatrix} z & 1 \\ 0 & z \end{pmatrix} g\right)\ dg - c_T f(z)$$

where c_T is a constant depending on T; see [JL70, Section 7].

This is a special case of a general phenomenon, the Shalika germ expansion. Let G be a connected reductive algebraic group over a nonarchimedean field F and let $\mathcal{O}_1, \ldots, \mathcal{O}_r$ be the unipotent conjugacy classes in $G(F)$. Let T be a maximal torus defined over F. Then there exist functions $\Gamma_1, \ldots, \Gamma_r$ on $T^{reg}(F)$ such that: for every $f \in C_c^\infty(G(F))$ there exists an open neighborhood of 1 such that for all $\gamma \in T^{reg}(F) \cap U_f$,

$$O_\gamma(f) = \sum_{i=1}^r \mu_i(f)\Gamma_i(\gamma)$$

where $\mu_i(f)$ denotes the orbital integral of f over $\mathcal{O}_i$.

However, the Shalika germ expansion doesn't determine the functions Γ_i. In [JL70, Lemma 7.3.1] the orbital integrals of the characteristic function of $G(\mathcal{O}_F)$ are computed which is then sufficient to determine the functions Γ_i.

Conversely given a family of functions $\{a^T\}$ such that each a^T is a smooth function on $T^{reg}(F)$ and is compactly supported function on $T(F)$. Then there exists a smooth compactly supported function f on $G(F)$ such that $\{a^T\} = \{F^T\}$ if and only if there exists locally constant functions of compact support ζ and ξ on $Z(F)$ such that

1. $a^M(\gamma) = \xi(z)$ if γ is a regular element in $M(F)$ close to $z \in Z(F)$, and
2. $a^T(\gamma) = \xi(z) - c_T\zeta(z)$, for T a non-split torus and γ a regular element in $T(F)$ close to $Z(F)$.

Given functions ζ and ξ Langlands explicitly constructs a smooth compactly supported function $f = f_{\zeta,\xi} \in C_c^\infty(G(F))$ such that the associated family $\{F^T\}$ satisfies the above conditions for that particular choice of ζ and ξ. Thus it suffices to establish the claim for families $\{a^T\}$ which vanish near $Z(F)$.

Thus it suffices to prove that a family $\{a^T\}$ which vanishes near elements of $Z(F)$ arises as the orbital integrals of a smooth compactly supported function on $G(F)$.

One has a similar result for orbital integrals on $D^\times$, although now there is no non-split torus to worry about. The point is that the constants c_T that appear are the same for $D^\times$. Thus given a function $f \in C_c^\infty(G_D(F))$ we form a family $\mathfrak{a}^T\}$ on G by setting a^T equal to the normalized orbital integral of f and set $a^M \equiv 0$. Then this family of functions arises from the orbital integrals of a function $f' \in C_c^\infty(G(F))$.

References

[Bum97] Daniel Bump, *Automorphic forms and representations*, Cambridge Studies in Advanced Mathematics, vol. 55, Cambridge University Press, Cambridge, 1997. MR MR1431508 (97k:11080)

[Bum03] ______, *Spectral theory and the trace formula*, An introduction to the Langlands program (Jerusalem, 2001), Birkhäuser Boston, Boston, MA, 2003, pp. 153–196. MR MR1990379

[DL71] Michel Duflo and Jean-Pierre Labesse, *Sur la formule des traces de Selberg*, Ann. Sci. École Norm. Sup. (4) **4** (1971), 193–284. MR MR0437462 (55 #10392)

[Gel75] Stephen S. Gelbart, *Automorphic forms on adèle groups*, Princeton University Press, Princeton, N.J., 1975, Annals of Mathematics Studies, No. 83. MR MR0379375 (52 #280)

[Gel96] Stephen Gelbart, *Lectures on the Arthur-Selberg trace formula*, University Lecture Series, vol. 9, American Mathematical Society, Providence, RI, 1996. MR MR1410260 (98d:22017)

[GJ79] Stephen Gelbart and Hervé Jacquet, *Forms of* GL(2) *from the analytic point of view*, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979, pp. 213–251. MR MR546600 (81e:10024)

[Iwa97] Henryk Iwaniec, *Topics in classical automorphic forms*, Graduate Studies in Mathematics, vol. 17, American Mathematical Society, Providence, RI, 1997. MR MR1474964 (98e:11051)

[Iwa02] ______, *Spectral methods of automorphic forms*, second ed., Graduate Studies in Mathematics, vol. 53, American Mathematical Society, Providence, RI, 2002. MR MR1942691 (2003k:11085)

[JL70] Hervé Jacquet and Robert P. Langlands, *Automorphic forms on* GL(2), Springer-Verlag, Berlin, 1970, Lecture Notes in Mathematics, Vol. 114. MR MR0401654 (53 #5481)

[Joy] David Joyner, *Notes on trace formulas for finite groups*, Unpublished.

[KL06] Andrew Knightly and Charles Li, *Traces of Hecke operators*, Mathematical Surveys and Monographs, vol. 133, American Mathematical Society, Providence, RI, 2006. MR MR2273356 (2008g:11090)

[Kot86] Robert E. Kottwitz, *Stable trace formula: elliptic singular terms*, Math. Ann. **275** (1986), no. 3, 365–399. MR MR858284 (88d:22027)

[Kot88] ______, *Tamagawa numbers*, Ann. of Math. (2) **127** (1988), no. 3, 629–646. MR MR942522 (90e:11075)

[Lan73] R. P. Langlands, *Modular forms and ℓ-adic representations*, Modular functions of one variable, II (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), Springer, Berlin, 1973, pp. 361–500. Lecture Notes in Math., Vol. 349. MR MR0354617 (50 #7095)

[Lan80] Robert P. Langlands, *Base change for* GL(2), Annals of Mathematics Studies, vol. 96, Princeton University Press, Princeton, N.J., 1980. MR MR574808 (82a:10032)

[Lan85] Serge Lang, $\mathrm{SL}_2(\mathbf{R})$, Graduate Texts in Mathematics, vol. 105, Springer-Verlag, New York, 1985, Reprint of the 1975 edition. MR MR803508 (86j:22018)

[Lap08] Erez Lapid, *Introductory notes on the trace formula*, Preprint, 2008.

[LV07] Elon Lindenstrauss and Akshay Venkatesh, *Existence and Weyl's law for spherical cusp forms*, Geom. Funct. Anal. **17** (2007), no. 1, 220–251. MR MR2306657 (2008c:22016)

[Mil97] James Milne, *Modular forms and modular functions*, 1997.

[Mil08] ______, *Class field theory*, 2008.

[PR94] Vladimir Platonov and Andrei Rapinchuk, *Algebraic groups and number theory*, Pure and Applied Mathematics, vol. 139, Academic Press Inc., Boston, MA, 1994, Translated from the 1991 Russian original by Rachel Rowen. MR MR1278263 (95b:11039)

[PS92] R. Phillips and P. Sarnak, *Automorphic spectrum and Fermi's golden rule*, J. Anal. Math. **59** (1992), 179–187, Festschrift on the occasion of the 70th birthday of Shmuel Agmon. MR MR1226958 (94i:58203)

[RS80] Michael Reed and Barry Simon, *Methods of modern mathematical physics. I*, second ed., Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1980, Functional analysis. MR MR751959 (85e:46002)