Lecture Notes for Complex Analysis

Frank Neubrander

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> Analysis does not owe its really significant successes of the last century to any mysterious use of $\sqrt{-1}$, but to the quite natural circumstance that one has infinitely more freedom of mathematical movement if he lets quantities vary in a plane instead of only on a line.
>
> Leopold Kronecker

Recommended Readings:

1. Walter Rudin, *Real and Complex Analysis* (paperback), McGraw-Hill Publishing Co., 1987

2. John B. Conway, *Functions of One Complex Variable*, Springer Verlag, 1986

3. Jerold E. Marsden, Michael J. Hoffman, *Basic Complex Analysis*, Freeman, 1987

4. Reinhold Remmert, *Theory of Complex Functions*, Springer Verlag, 1991

5. E.C. Titchmarsh, *The Theory of Functions*, Oxford University Press, 1975

6. Joseph Bak, Donald J. Newman, *Complex Analysis*, Second Edition, Springer-Verlag New York, 1996

Tentative Table of Contents

Chapter 1

The Basics

1.1 The Field of Complex Numbers

The two dimensional $\mathbb{R}$-vector space $\mathbb{R}^2$ of ordered pairs $z = (x, y)$ of real numbers with multiplication $(x_1, y_1)(x_2, y_2) := (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1)$ is a commutative field denoted by $\mathbb{C}$. We identify a real number x with the complex number $(x, 0)$. Via this identification $\mathbb{C}$ becomes a field extension of $\mathbb{R}$ with the unit element $1 := (1, 0) \in \mathbb{C}$. We further define $i := (0, 1) \in \mathbb{C}$. Evidently, we have that $i^2 = (-1, 0) = -1$. The number i is often called the *imaginary unit* of $\mathbb{C}$ although nowadays it is hard to see anything imaginary in the plane point $(0, 1)$.[1] Every $z \in \mathbb{C}$ admits a unique representation

$$z = (x, y) = x(1, 0) + (0, 1)y = x + iy,$$

where x is called the real part of z, $x = Re(z)$, and y the imaginary part of z, $y = Im(z)$. The number $\overline{z} = x - iy$ is the conjugate of z and

$$|z| := \sqrt{x^2 + y^2} = \sqrt{z\overline{z}}$$

is called the absolute value or norm. The multiplicative inverse of $0 \neq z \in \mathbb{C}$ is given by

$$z^{-1} = \frac{1}{z} = \frac{\overline{z}}{|z|^2}.$$

[1] The situation was different in 1545 when Girolamo Cardano introduced complex numbers in his *Ars Magna* only to dismiss them immediately as *subtle as they are useless.* In 1702 Leibnitz described the square root of -1 as *that amphibian between existence and nonexistence* and in 1770 Euler was still sufficiently confused to make mistakes like $\sqrt{-2}\sqrt{-3} = \sqrt{6}$. The first answer to the question *"What is a complex number"* that satisfied human senses was given in the late eighteenth century by Gauss. Since then we have the rock-solid geometric interpretation of a complex number as a point in the plane. With Gauss, the algebraically mysterious imaginary unit $i = \sqrt{-1}$ became the geometrically obvious, boring point $(0, 1)$. Many teachers introduce complex numbers with the convenient half-truth that they are useful since they allow to solve all quadratic equations. If this were their main purpose of existence, they would truly be subtle as they were useless. The first one to see the true usefullness of the complex numbers was Rafael Bombelli in his *L'Algebra* from 1572. Investigating Cardano's formula, which gives a solution of the cubic equation $x^3 - 3px - 2q = 0$ by $x_0 = \sqrt[3]{q + \sqrt{q^2 - p^3}} + \sqrt[3]{q - \sqrt{q^2 - p^3}}$, he noticed that the solution of $x^3 - 15x - 4 = 0$ is given by $x_0 = \sqrt[3]{2 + 11\sqrt{-1}} + \sqrt[3]{2 - 11\sqrt{-1}}$. It was Bombelli's famous *wild thought* that led him to recognize that $x_0 = 4$. This was the first manifestation of one of the truly powerful properties of complex numbers: real solutions of real problems can be determined by computations in the complex domain. See also: T. Needham, *Visual Complex Analysis* [1997] and J. Stillwell, *Mathematics and Its History* [1989].

The set of 2×2-matrices $\left\{ \begin{pmatrix} x & -y \\ y & x \end{pmatrix}; x, y \in \mathbb{R} \right\}$ with the usual matrix addition and multiplication is a field isomorphic to $\mathbb{C}$. A mapping $T : \mathbb{C} \to \mathbb{C}$ is $\mathbb{C}$-linear iff

$$Tz = \mu z \text{ for some } \mu \in \mathbb{C} \text{ and all } z \in \mathbb{C}.$$

A mapping $T : \mathbb{R}^2 \to \mathbb{R}^2$ is $\mathbb{R}$-linear iff

$$T \simeq \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ for some } a, b, c, d \in \mathbb{R}.$$

A $\mathbb{R}$-linear mapping $T : \mathbb{R}^2 \to \mathbb{R}^2$ is $\mathbb{C}$-linear iff

$$T \simeq \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \text{ for some } a, b \in \mathbb{R}.$$

The function $d : \mathbb{C} \times \mathbb{C} \to \mathbb{R}_+$, $d(z, w) := |z - w|$ defines a metric on $\mathbb{C}$. Since $\mathbb{C}$ is $\mathbb{R}^2$ with the extra algebraic structure of multiplication, many geometric and topological concepts can be translated from $\mathbb{R}^2$ into complex notation. In particular, $\mathbb{C}$ is a complete metric space in which the Heine-Borel theorem holds (compact $\iff$ closed and bounded). Let $M \subset \mathbb{C}$ and $I = [a, b] \subset \mathbb{R}$. Every *continuous* function $\gamma : I \to M$ is called a path in M. The set M is called path-connected if every two points in M are in the image of a path in M and M is called connected if for any two *disjoint open* sets $U, V \subset \mathbb{C}$ with $M \subset U \cup V$ one has either $M \subset U$ or $M \subset V$. Any open and connected subset D of the complex plane is called a region.

Proposition 1.1.1. *Any two points of a region D can be connected by a smooth path.*

Proof. Fix $z_0 \in D$ and consider the sets $A = \{z \in D : z \text{ can be connected to } z_0 \text{ by a smooth path}\}$ and $B = \{z \in D : z \text{ can not be connected to } z_0 \text{ by a smooth path}\}$. Then A, B are open.[2] Clearly, $D \subset A \cup B$ and $A \cap B = \emptyset$. Since D is connected it follows that either $D \subset A$ or $D \subset B$. Since $z_0 \in D \cap A$ we obtain that $D \subset A$. $\square$

Problems. (0) Prepare a 20 minute lecture on "Complex Numbers" suitable for a College Algebra (Math 1021) course.

(1) Show that every path-connected set is connected. Find a connected set which is not path-connected.

(2) *Towards a Functional Calculus.* Let $A = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$. Compute A^n ($n \in \mathbb{N}_0$) using a) the Jordan normal form of A, b) the fact that A is in a field isomorphic to $\mathbb{C}$, and try to do it without any use of complex numbers. For which functions f other than $f(z) = z^n$ can one define $f(A)$? Compute e^{tA} and find the solution of $x'(t) = Ax(t)$ with initial value $x(0) = (1, -1)$. Let A be an arbitrary $2x2$-matrix and let $\sigma(A) \subset \mathbb{C}$ denote the set of eigenvalues of A. Show that the asymptotic behavior of the solutions of $x'(t) = Ax(t)$ can be characterized in terms of the so-called spectrum $\sigma(A)$.

(3) *Towards the Mandelbrot set.* Let $f(z) = z^2 + c$ for some $c \in \mathbb{C}$ and $a_n := f^n(0) = (f \circ f \circ \cdots \circ f)(0)$. Compute and draw the set of all $c \in \mathbb{C}$ for which the sequence a_n converges. Draw (with the help of a computer) the set of all $c \in \mathbb{C}$ for which the sequence a_n stays bounded (Mandelbrot Set). What is a Julia Set? What is a Fractal? Give examples and brief, informal descriptions.

[2] Why?

1.2 Analytic Functions

It had taken more than two and half centuries for mathematicians to come to terms with complex numbers, but the development of the powerful mathematical theory of how to do *calculus* with functions of such numbers (what we call now *complex analysis*) was astonishingly rapid. Most of the fundamental results were obtained by Cauchy, Dirichlet, Riemann, Weierstrass, and others between 1814 and 1873 - a span of sixty years that changed the face of mathematics forever. Before going into some of the details, let us try a preliminary answer to the question *"What is complex analysis?"*. It is clear that any short answer must be incomplete and highly subjective. In these lecture notes we take the position that the core of complex analysis is the study of power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ and of the characteristic properties of those functions f which can be represented locally as such a power series.[3] As we will see below, one characteristic property of such functions is *analyticity*.

Definition 1.2.1. *Let $D \subset \mathbb{C}$ be open, $f : D \to \mathbb{C}$, $z = x + iy$, $f = u + iv$.*

a) f is complex differentiable in $a \in D$ if $\lim_{z\to a} \frac{f(z)-f(a)}{z-a}$ exists for each sequence z in $D \setminus \{a\}$ converging to a.

b) f is analytic (holomorphic, regular) on D if it is complex differentiable for all $a \in D$. If f is analytic an $\mathbb{C}$, then it is called an entire function.

c) If f is complex differentiable in $a \in D$, then $f'(a) = f^{(1)}(a) = \frac{df}{dz}(a) = \lim_{z\to a} \frac{f(z)-f(a)}{z-a}$.

Example 1.2.2. (a) A function $f(z) = u(x, y)+iv(x, y)$ is continuous if its real part u and its imaginary part v is continuous at $(x_0.y_0)$.[4] It follows that the monomials $f(z) = z^n$ $(n \in \mathbb{N}_0)$ are complex differentiable and $f'(a) = \lim_{z\to a} \frac{z^n-a^n}{z-a} = \lim_{z\to a} z^{n-1} + az^{n-2} + \cdots + a^{n-2}z + a^{n-1} = na^{n-1}$. Thus, any complex polynomial $\sum_{n=0}^{N} a_n z^n$ is an entire function with derivative $\sum_{n=1}^{N} na_n z^{n-1}$.

(b) A convergent power series $\sum_{n=0}^{\infty} a_n z^n$ represents an analytic function inside its circle of convergence. To see this let $f(z) := \sum_{n=0}^{\infty} a_n z^n$ be convergent for $|z| < R$. Then $g(z) := \sum_{n=1}^{\infty} na_n z^{n-1}$ is convergent for $|z| < R^5$. Let $|z| < r < R$ and $|h| < r - |z|$. Then there exists $K > 0$ such that $|a_n r^n| \le K$ for all $n \in \mathbb{N}_0$. It follows from

$$\frac{f(z+h) - f(z)}{h} - g(z) = \sum_{n=0}^{\infty} a_n \left(\frac{(z+h)^n - z^n}{h} - nz^{n-1} \right)$$

and $\left|\frac{(z+h)^n-z^n}{h} - nz^{n-1}\right| = |\sum_{i=0}^{n-2} \binom{n}{i} z^i h^{n-i-1}| \le \sum_{i=0}^{n-2} \binom{n}{i} |z|^i |h|^{n-i-1} = \frac{(|z|+|h|)^n-|z|^n}{|h|} - n|z|^{n-1}$ that

$$\begin{aligned}
|\frac{f(z+h) - f(z)}{h} - g(z)| &\le K \sum_{n=0}^{\infty} \frac{1}{r^n} \left(\frac{(|z| + |h|)^n - |z|^n}{|h|} - n|z|^{n-1} \right) \\
&= K \left\{ \frac{1}{|h|} \left(\frac{1}{1 - \frac{|z|+|h|}{r}} - \frac{1}{1 - \frac{|z|}{r}} \right) - \frac{1}{r} \frac{1}{(1 - \frac{|z|}{r})^2} \right\} \\
&= \frac{Kr|h|}{(r - |z| - |h|)(r - |z|)^2},
\end{aligned}$$

[3] In a power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ the variable z is always a complex number, and the coefficients a_n are either complex numbers (*classical complex analysis*) or elements of an arbitrary Banach space X (*modern complex analysis*).

[4] Why?

[5] For a proof, see Section 2.7

which tends to zero with h. Hence, $f'(z) = g(z)$ if $|z| < R$.

It is one of the main results of complex analysis that all analytic functions f defined on a region $\Omega \subset \mathbb{C}$ can be represented as a convergent power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$, where z_0 is the center of the largest disc $S \subset \Omega$, and $z \in S$. Moreover, as we will see in the section on asymptotic complex analysis, not only the convergent power series but all formal power series can be uniquely represented by equivalence classes of analytic functions.

(c) The function $f(z) = \overline{z}$ is continuous, infinitely often real differentiable, but nowhere complex differentiable since $\frac{f(z)-f(a)}{z-a} = 1$ for $z = a + \frac{1}{n}$ and $\frac{f(z)-f(a)}{z-a} = -1$ for $z = a + \frac{i}{n}$. $\square$

Lemma 1.2.3. *Let $a \in D \subset \mathbb{C}$, D open, and $f : D \to \mathbb{C}$. Tfae:*

(i) *$f'(a)$ exists.*

(ii) *There exists a function $\Delta : D \to \mathbb{C}$ which is continuous in a and satisfies $f(z) = f(a) + \Delta(z)(z - a)$.*

Proof. Obvious. Also, notice that the differentiability of f in a implies the continuity of f in a. $\square$

Remark 1.2.4. The usual of differentiation from calculus can be used when differentiating holomorphic functions. The proofs extend without any change beyond replacing the real variables $x, y, ...$ by complex variables $z, v, ...$ As an example we discuss the chain rule:

> Let $D, U \subset \mathbb{C}$ be open, $f : D \to U$ and $h : U \to \mathbb{C}$. If f is complex differentiable in $a \in D$ and h is complex differentiable in $f(a)$, then $h \circ f$ is complex differentiable in a and $(h \circ f)'(a) = h'(f(a))f'(a)$.

Proof. Let $F := h \circ f$, $f(z) = f(a) + \Delta_1(z)(z - a)$, $b = f(a)$, and $h(w) = h(b) + \Delta_2(w)(w - b)$. Then $F(z) = h(f(z)) = h(b)+(\Delta_2 \circ f)(z)(f(z)-b) = h(f(a))+[(\Delta_2 \circ f)(z)](\Delta_1(z))(z-a) = F(a)+\Delta_3(z)(z-a)$, where $\Delta_3(z) = [(\Delta_2 \circ f)(z)]\Delta_1(z)$. Thus, F is complex differentiable in a and $F'(a) = (\Delta_2 \circ f)(a)\Delta_1(a) = h'(f(a))f'(a)$. $\square$

1.3 The Complex Exponential

In this section we introduce the complex exponential function and list some of its properties. We need the following crucial lemma, whose proof from advanced calculus carries over to complex series.

Lemma 1.3.1 (Cauchy Product of Series). *Let $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ be absolute convergent series. Then $\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) = \sum_{n=0}^{\infty} c_n$, where $c_n = \sum_{i=0}^{n} a_i b_{n-i}$ and $\sum_{n=0}^{\infty} c_n$ is absolute convergent.*

The complex exponential

$$\exp(z) := \sum_{i=0}^{\infty} \frac{1}{n!} z^n$$

has the following properties:

(a) The sum $\sum_{n=0}^{\infty} \frac{1}{n!} z^n$ converges absolutely and $|\exp(z)| \leq e^{|z|}$ for all $z \in \mathbb{C}$.

(b) The function exp maps addition into multiplication; i.e., $\exp(z)\exp(w) = \exp(z + w)$ for all $z, w \in \mathbb{C}$. This can be seen by using the Cauchy product for series: $\exp(z)\exp(w) = \left(\sum_{n=0}^{\infty} \frac{1}{n!} z^n\right)\left(\sum_{i=0}^{\infty} \frac{1}{i!} w^i\right) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{p=0}^{n} \frac{n!}{p!(n-p)!} z^p w^{n-p} = \sum_{n=0}^{\infty} \frac{1}{n!}(z + w)^n = \exp(z + w)$.

(c) The function exp is entire and $\exp' = \exp$. To see this let $a \in \mathbb{C}$. Then $\frac{\exp(z)-\exp(a)}{z-a} = \exp(a)\frac{\exp(z-a)-1}{z-a}$ $= \exp(a)\sum_{n=1}^{\infty}\frac{1}{n!}(z-a)^{n-1} = \exp(a) + \exp(a)\ (z-a)\sum_{n=2}^{\infty}\frac{1}{n!}(z-a)^{n-2} \to \exp(a)$ as $z \to a$ since $\sum_{n=2}^{\infty}\frac{1}{n!}(z-a)^{n-2}$ is bounded if $|z-a| < 1$.

(d) It follows from the Taylor expansion of the real exponential function e that $\exp(x) = e^x$ for all $x \in \mathbb{R}$. Therefore, we write often e^z instead of $\exp(z)$. Moreover, for all $y \in \mathbb{R}$, we have the *Euler formula*

$$e^{iy} = cos(y) + isin(y)$$

since, by using the Taylor expansions of cos and sin, $e^{iy} = \sum_{n=0}^{\infty}\frac{1}{n!}i^n y^n = \sum_{n=0}^{\infty}\frac{1}{(2n)!}(-1)^n y^{2n} + i\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}(-1)^n y^{2n+1} = \cos(y) + i\sin(y)$. Therefore, for $z = (x, y) \in \mathbb{C}$,

$$e^z = e^x e^{iy} = e^x(\cos(x) + i\sin(x)).$$

(e) Let $y \in \mathbb{R}$. Then $e^{\pm iy} = \cos(y) \pm i\sin(y)$. It follows that $\cos(y) = \frac{1}{2}(e^{iy} + e^{-iy})$ and $\sin(y) = \frac{1}{2i}(e^{iy} - e^{-iy})$. Therefore, if we define

$$\cos(z) := \frac{1}{2}(e^{iz} + e^{-iz}) \text{ and } \sin(z) := \frac{1}{2i}(e^{iz} - e^{-iz}),$$

then we obtain the famous *Euler Formula*

$$e^{iz} = \cos(z) + i\sin(z) \quad (z \in \mathbb{C}).$$

(f) It is often convenient to write complex numbers in their exponential or polar coordinate representation

$$z = |z|e^{i\arg(z)} = r(\cos(\theta) + i\sin(\theta),$$

where $z = (x, y)$, $\theta = \arg(z) \in (-\pi, \pi]$ is an angle such that $\tan(\theta) = \frac{y}{x}$, and $r = |z| = \sqrt{x^2 + y^2}$. In particular, if $z = r(\cos(\theta) + i\sin(\theta)$ and n is a positive integer, then $z^n = r^n(\cos(n\theta) + i\sin(n\theta)$ and the n n-th roots of z are given by $z_k = \sqrt[n]{r}(\cos(\frac{\theta}{n} + \frac{2\pi k}{n}) + i\sin(\cos(\frac{\theta}{n} + \frac{2\pi k}{n})$, $k = 0, 1, \cdots, n-1$.

Problems. (1) Show that for all $z \in \mathbb{C}$ such that $\sin(\frac{1}{2}z) \neq 0$ and for all $n \in \mathbb{N}$ the *trigonometric summation formula* $\frac{1}{2} + \cos(z) + \cos(2z) + \cdots + \cos(nz) = \frac{1}{\sin(\frac{1}{2}z)}\sin(n + \frac{1}{2}z)$ holds.

(2) Show that exp is periodic, $\exp(\mathbb{C}) = \mathbb{C}\setminus\{0\}$, and that cos and sin assume every value in $\mathbb{C}$ countably often.

1.4 The Cauchy-Riemann Theorem

Recall that a function $f = u + iv : D \to \mathbb{R}^2$ is called differentiable (in the real sense) in $z_0 \in D \subset \mathbb{R}^2$ if there exists a $\mathbb{R}$-linear map $A : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\frac{1}{|z-z_0|}(f(z) - f(z_0) - A(z - z_0)) \to 0$ as $z \to z_0$. If f is differentiable at z_0, then[6]

$$f'(z_0) := A = \begin{pmatrix} u_x(z_0) & u_y(z_0) \\ v_x(z_0) & v_y(z_0) \end{pmatrix}.$$

Moreover, if the partial derivatives u_x, u_y, v_x, v_y exist and are continuous in z_0, then f is differentiable at z_0.

[6] Why?

Theorem 1.4.1 (Cauchy-Riemann). *Let $D \subset \mathbb{C}$ be open, $z_0 \in D$, and $f : D \to \mathbb{C}$, $f = u + iv$. Tfae:*

(i) *f is complex differentiable in z_0.*

(ii) *f is differentiable in z_0 and $u_x = v_y$, $u_y = -v_x$ at z_0.*

In this case, $f'(z_0) = \frac{\delta f}{\delta x}(z_0) := u_x(z_0) + iv_x(z_0) = \frac{1}{i}\frac{\delta f}{\delta y}(z_0) := -iu_y(z_0) + v_y(z_0)$.

Proof. Let $\mu = a+ib \simeq A$, where $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, and $z = x+iy$. Then $\mu z = A(z)$. Thus, f is complex differentiable in $z_0 \iff \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} = \mu := f'(z_0) \iff \lim_{z\to z_0} \frac{1}{z-z_0}\left(f(z) - f(z_0) - \mu(z-z_0)\right) = 0 \iff \lim_{z\to z_0} \frac{1}{z-z_0}\left(f(z) - f(z_0) - A(z-z_0)\right) = 0 \iff f$ is differentiable in z_0 and $f'(z_0) = A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \begin{pmatrix} u_x(z_0) & u_y(z_0) \\ v_x(z_0) & v_y(z_0) \end{pmatrix}$. □

Example 1.4.2. (a) Let $z = x + iy$ and consider $f(z) = \overline{z} = (x, -y)$. Then f is differentiable and $f'(x, y) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ for all x, y. Thus, f is nowhere complex differentiable.

(b) Consider $f(z) = |z|^2 = (x^2 + y^2, 0)$. Then f is differentiable and $f'(x, y) = \begin{pmatrix} 2x & 2y \\ 0 & 0 \end{pmatrix}$ for all x, y. Thus, f is complex differentiable in z_0 iff $z_0 = 0$. □

Next we will collect some further properties of analytic functions without going into greater detail. A twice continuously differentiable function $u : D \to \mathbb{R}$ on a region $D \subset \mathbb{R}^2$ is called *harmonic* if u is in the kernel of the *Laplace operator* Δ; i.e., $\Delta u = u_{xx} + u_{yy} = 0$ on D. We will show in Section ??? below that any analytic function $f = u+iv : D \to \mathbb{C}$ is infinitely often differentiable. Thus, $u_{xx} = v_{yx} = v_{xy} = -u_{yy}$ and $v_{xx} = -u_{yx} = -u_{xy} = -v_{yy}$. This yields the following statement.[7]

The real- and imaginary parts of an analytic function are harmonic.

Assume that $f = u+iv$ is analytic on a region D and that the equations $u(x, y) = c_1$ and $v(x, y) = c_2$ define smooth curves in D for some $c_1, c_2 \in \mathbb{R}$; i.e., there exist smooth functions $\gamma, \sigma : \mathbb{R} \to D$, $\gamma(t) = (x(t), y(t))$, $\sigma(t) = (\tilde{x}(t), \tilde{y}(t))$ such that $u(\gamma(t)) = c_1$ and $v(\sigma(t)) = c_2$ for all t. Then $(u_x, u_y)\cdot(x', y') = u_x x' + u_y y' = \frac{d}{dt}u(x, y) = 0$ and $(u_{\tilde{x}}, u_{\tilde{y}}) \cdot (\tilde{x}', \tilde{y}') = u_{\tilde{x}}\tilde{x}' + u_{\tilde{y}}\tilde{y}' = \frac{d}{dt}u(\tilde{x}, \tilde{y}) = 0$ for all $t \geq 0$. Thus, for all t,

$$(u_x, u_y) \perp (x', y') \text{ and } (u_{\tilde{x}}, u_{\tilde{y}}) \perp (\tilde{x}', \tilde{y}').$$

Furthermore, since $0 = u_x v_x - v_x u_x = u_x v_x + u_y v_y = (u_x, u_y) \cdot (v_x, v_y)$, it follows that

$$(u_x, u_y) \perp (v_x, v_y)$$

for all $z \in \mathbb{C}$. Assume that the two curves γ and σ intersect; i.e., $z_0 = (x_0, y_0) := \gamma(t_1) = \sigma(t_2)$ for some $t_1, t_2 \in \mathbb{R}$. If $f'(z_0) \neq 0$, then $\gamma'(t_1) \perp \sigma'(t_2)$, since $(u_x(z_0), u_y(z_0)) \perp (v_x(z_0), v_y(z_0))$. This proves the following.

> If $f = u+iv$ is analytic and $z_0 = (x_0, y_0)$ is point common to two particular curves $u(x, y) = c_1$ and $v(x, y) = c_2$ and if $f'(z_0) \neq 0$, then the lines tangent to those curves at (x_0, y_0) are perpendicular.

[7] It will be shown later that any harmonic function u on a simply connected region D is the real part of an analytic function.

Let f be complex differentiable at z_0 with $f'(z_0) = \mu \neq 0$. Then $f(z_0 + h) \approx f(z_0) + \mu h$ if $h \approx 0$. Thus, locally at z_0, f is approximately a rotation by $\arg(\mu)$ with a magnification by $|\mu|$. It is therefore not surprising that analytic functions with $f' \neq 0$ are *conformal*; i.e.:

Analytic functions are locally *angle- and orientation preserving.*

Since we will not discuss this (very important) concept any further in this class, we refer to R. Remmert's *Theory of Complex Functions* [1991] for further reading.

Recall from your advanced calculus course that a continuously differentiable function $f : D \to \mathbb{R}^2$ has a differentiable inverse f^{-1} with $(f^{-1})'(f(z)) = (f'(z))^{-1}$ in some neighborhood of a point z_0 where $\det f'(z_0) \neq 0$ (*Inverse Function Theorem*). If $f = u + iv$ is also assumed to be analytic, then the Jacobian matrices $f'(z)$ can be represented by nonzero complex numbers in a neighborhood of z_0. It follows that the Jacobian matrices $(f^{-1})'(w)$, $w = f(z)$, can also be identified with complex numbers. Thus, the partial derivatives of $(f^{-1})'$ satisfy the Cauchy-Riemann equations in a neighborhood of $f(z_0)$ and we obtain the following result.

Corollary 1.4.3. *Let f be analytic in a region D (with f' continuous).*[8] *If $f'(z_0) \neq 0$ for some $z_0 \in D$, then there exists a neighborhood U of z_0 and a neighborhood V of $w_0 = f(z_0)$ such that $f : U \to V$ is one-to-one, onto, f^{-1} is analytic, and $(f^{-1})'(w) = \frac{1}{f'(z)}$ for $w = f(z) \in V$.*

Example 1.4.4 (The Complex Logarithm). Let U be a horizontal strip $\{(x, y) : y_0 < y < y_0 + 2\pi)\}$ and $f(z) = e^z = e^x(\cos(y), \sin(y))$. Then f is analytic, f' is continuous, and $f'(z) \neq 0$ for all $z \in U$. Let V be the sliced plane $\{z \neq 0 : \arg(z) \neq y_0 \mod 2\Pi\}$. Then $f : U \to V$ is one-to-one and onto. By the corollary above, $g := f^{-1}$ is locally analytic and satisfies $g'(w) = (f^{-1})'(w) = \frac{1}{f'(z)} = \frac{1}{f(z)} = \frac{1}{w}$ for $w = f(z) \in V$. Since $g'(w) = \frac{1}{w}$ for all $w \in V$ we call g the branch of the complex logarithm on V and write $\log_V(w) := g(w)$. $\square$

Problems. (1) If $f = u + iv$ is analytic, then $f(z) = 2u(\frac{z+\overline{z}_0}{2}, \frac{z-\overline{z}_0}{2i}) - \overline{f(z_0)} = 2iv(\frac{z+\overline{z}_0}{2}, \frac{z-\overline{z}_0}{2i}) + \overline{f(z_0)}$

(2) Study the Cauchy-Riemann equations and the complex differentiability of $f(z) = \frac{z^5}{|z|^4}$.

1.5 Contour Integrals

Let f be a continuous function on $D \subset \mathbb{C}$ and let $\gamma : [a, b] \to D$ be continuously differentiable. Then the integral of f along γ is defined as

$$\int_\gamma f(z)\, dz := \int_a^b f(\gamma(t))\gamma'(t)\, dt.$$

Let $a = a_0 < a_1 \cdots < a_n = b$ be a partition of the interval $[a, b]$. If $\gamma : [a, b] \to D$ is continuously differentiable on the subintervals $[a_{k-1}, a_k]$, $1 \leq k \leq n$, then γ is called piecewise smooth and

$$\int_\gamma f(z)\, dz := \sum_{k=1}^{n} \int_{a_{k-1}}^{a_k} f(\gamma(t))\gamma'(t)\, dt.$$

[8] We will see below that any analytic function is infinitely often continuously differentiable. Thus, the continuity assumption is always fulfilled.

It is obvious that contour integration is linear (i.e., $\int_\gamma cf + g = c\int_\gamma f + \int_\gamma g$ for all analytic functions $f, g : D \to \mathbb{C}$ and all $c \in \mathbb{C}$) and that

$$|\int_\gamma f(z)\,dz| \leq ML(\gamma),$$

where $L(\gamma) := \int_a^b |\gamma'(t)|\,dt$ denotes the arclength of γ and $M = \max_{z\in\gamma} |f(z)|$.

Example 1.5.1. Let $\gamma : [0, 2\pi] \to \mathbb{C}$ be given by $\gamma(t) = e^{it}$ and $f(z) = z^n$ for some $n \in \mathbb{Z}$. Then

$$\int_\gamma z^n dz = i\int_0^{2\pi} e^{(n+1)it}\,dt = \begin{cases} 2\pi i & n = -1 \\ 0 & \text{else} \end{cases}$$

In particular,

$$\frac{1}{2\pi i}\int_\gamma \frac{1}{z}\,dz = 1.$$

A function $\tilde{\gamma} : [\tilde{a}, \tilde{b}] \to \mathbb{C}$ is called a reparametrization of a piecewise smooth curve γ if there exists a strictly monotone C^1-function α that maps $[a, b]$ onto $[\tilde{a}, \tilde{b}]$ such that $\gamma(t) = \tilde{\gamma}(\alpha(t)))$ for all $t \in [a, b]$.

Lemma 1.5.2. *$\int_\gamma f(z)\,dz = \epsilon\int_{\tilde{\gamma}} f(z)\,dz$, where $\epsilon = 1$ if $\alpha' > 0$ and else $\epsilon = -1$.*

Proof. $\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt = \int_a^b f(\tilde{\gamma}(\alpha(t))\tilde{\gamma}'(\alpha(t))\alpha'(t)\,dt = \int_{\alpha(a)}^{\alpha(b)} f(\tilde{\gamma}(s))\tilde{\gamma}'(s)\,ds = \epsilon\int_{\tilde{a}}^{\tilde{b}} f(\tilde{\gamma}(s))\tilde{\gamma}'(s)\,ds = \epsilon\int_{\tilde{\gamma}} f(z)\,dz$. □

Let $K \subset \mathbb{C}$ be compact. A set $\Gamma \subset K$ is called a smooth boundary of K if there exists a bijective C^1-map $\gamma : [a, b] \to \Gamma$ such that $\gamma'(t) \neq 0$ for all $t \in [a, b]$ and for all $t \in [a, b]$ there exists $\epsilon > 0$ such that for all $s \in [-\epsilon, \epsilon]$ one has

$$\gamma(t) + is\gamma'(t) \in K \Longleftrightarrow s \geq 0.$$

If K is a compact set with a smooth boundary $\Gamma = \gamma([a, b]$, then we define

$$\int_{\delta K} f(z)\,dz := \int_\Gamma f(z)\,dz := \int_\gamma f(z)\,dz,$$

where the definition is independent of the parametrization γ. If a compact set K has a piecewise smooth boundary $\delta K = \cup_{k=1}^n \Gamma_k$, where Γ_k is smooth and $\Gamma_i \cap \Gamma_j$ is finite, then

$$\int_{\delta K} f(z)\,dz := \sum_{k=1}^n \int_{\Gamma_k} f(z)\,dz.$$

Example 1.5.3.
(a)

(b)

Chapter 2

The Works

2.1 Antiderivatives

Definition 2.1.1. *Let $D \subset \mathbb{C}$ a region and let $f : D \to \mathbb{C}$ be continuous. The function f is called Cauchy integrable if*

$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz$$

for all piecewise smooth curves $\gamma_i : [a,b] \to D$ with the same starting- and endpoints.

One should notice that the notion of Cauchy integrability has little to do with the usual definitions of Riemann or Lebesgue integrability. If f is continuous, then the contour integral along γ exists. Thus, the important element in the definition of *Cauchy integrability* is not the existence of the integral, but the fact that the value of the contour integral depends only on the starting- and endpoint of the path γ (*path-idependence*, see statement (iii) below).

Theorem 2.1.2. *Let $f : D \to \mathbb{C}$ be continuous. Tfae.*

(i) *f is Cauchy integrable.*

(ii) *$\int_\gamma f(z)\,dz = 0$ for all piecewise smooth, closed curves in D.*

(iii) *There exists a function $F : D \to \mathbb{C}$ such that $\int_\gamma f(z)\,dz = F(\gamma(b)) - F(\gamma(a))$ for all piecewise smooth curves $\gamma : [a,b] \to D$.*

(iv) *f has an antiderivative; i.e., there exists an analytic function $F : D \to \mathbb{C}$ with $F' = f$ on D.*

Moreover, $F : D \to \mathbb{C}$ is an antiderivative of f if and only if $\int_\gamma f(z)\,dz = F(\gamma(b)) - F(\gamma(a))$ for all piecewise smooth curves $\gamma : [a,b] \to D$.

Proof. We show first that (iv) implies (iii). We may assume that $\gamma : [a,b] \to D$ is smooth. Then $h(t) := F(\gamma(t))$ is continuously differentiable and $h'(t) = f(\gamma(t))\gamma'(t)$ for all $t \in [a,b]$. It follows that $F(\gamma(b)) - F(\gamma(a)) = h(b) - h(a) = \int_a^b h'(t)\,dt = \int_a^b f(\gamma(t))\gamma'(t)\,dt = \int_\gamma f(z)\,dz$. The implication $(iii) \implies (ii)$ is obvious. To show $(ii) \implies (i)$ we may assume that $\gamma_1, \gamma_2 : [0,1] \to D$ (possibly after a reparametrization). Define $\gamma : [0,2] \to D$ by $\gamma(t) = \gamma_1(t)\chi_{[0,1]}(t) + \gamma_2(2-t)\chi_{[1,2]}(t)$ Then γ is piecewise smooth and closed and therefore, $0 = \int_\gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz - \int_{\gamma_2} f(z)\,dz$. It remains to be shown that $(i) \implies (iv)$. Fix

$z_0 \in D$. Then, for all $w \in D$, there exists a smooth curve $\gamma : [a, b] \to D$ such that $\gamma(a) = z_0$ and $\gamma(b) = w$. Define

$$F(w) := \int_\gamma f(z)\,dz.$$

It follows from (iii) that the definition of $F(w)$ is independent of the choice of γ. Now let $\sigma : [c, d] \to D$ be an arbitrary, piecewise smooth path in D. Then we extend the path σ to a path $\tilde{\sigma} : [\tilde{c}, d] \to D$, $\tilde{c} < c$, $\tilde{\sigma}_{/[c,d]} = \sigma$ such that $\tilde{\sigma}(\tilde{c}) = z_0$ and obtain

$$\begin{aligned}\int_\sigma f(z)\,dz &= \int_{\tilde{\sigma}} f(z)\,dz - \int_{\tilde{\sigma}_{/[\tilde{c},c]}} f(z)\,dz \\ &= F(\tilde{\sigma}(d)) - F(\tilde{\sigma}(c)) = F(\sigma(d)) - F(\sigma(c)).\end{aligned}$$

Now let $w_0 \in D$ and $K_\epsilon(w_0) := \{w : |w - w_0| < \epsilon\} \subset D$. For $w \in K_\epsilon(z_0)$ define $\gamma(t) = w_0 + t(w - w_0)$ for $t \in [0, 1]$. Then $F(w) - F(w_0) = \int_\gamma f(z)\,dz = \int_0^1 f(w_0 + t(w - w_0))\,dt(w - w_0)$. Thus, $\frac{F(w)-F(w_0)}{w-w_0} \to f(w_0)$ as $w \to w_0$. □

Lemma 2.1.3. *Let $D \subset \mathbb{C}$ be an open sphere and $f : D \to \mathbb{C}$ continuous. If $\int_{\delta R} f(z)\,dz = 0$ for all rectangles $R \subset D$ with sides parallel to the axes, then f is Cauchy integrable on the sphere D*

Proof. Let $a \in D$ be fixed. For $w \in D$ define $F(w) := \int_{\gamma_w} f(z)\,dz$, where

It follows from

that $F(z_0 + h) - F(z_0) = \int_{\gamma_{z_0+h}} f(z)\,dz - \int_{\gamma_{z_0}} f(z)\,dz = \int_{\tilde{\gamma}} f(z)\,dz = \int_0^1 f(z_0 + th)\,dth$. Thus, $\frac{\delta F}{\delta x}(z_0) = f(z_0)$. Moreover, $F(z_0 + ih) - F(z_0) = \int_0^1 f(z_0 + ith)\,dtih$. Thus, $\frac{\delta F}{\delta y}(z_0) = if(z_0)$. Let $F = u + iv$. Then $\frac{\delta F}{\delta x} = u_x + iv_x = -i(\frac{\delta F}{\delta y} = -iu_y + v_y$, or $u_x = v_y$ and $v_x = -u_y$. Since the Cauchy-Riemann equations hold, F is analytic and $F' = u_x + iv_x = f$ on D. It follows from the previous theorem that F is an antiderivative of f on D; i.e., f is Cauchy integrable on D. □

It follows from Theorem 2.1.2 (iv) that every polynomial is Cauchy integrable on $\mathbb{C}$. Also, by Example 1.2.2 (b), every function f which has a power series representation, is Cauchy integrable on its circle of convergence.[1] As motivation for the next definition, consider $D = \mathbb{C} \setminus \{0\}$ and $f(z) = \frac{1}{z}$. If S denotes the unit sphere, then $\int_{\delta S} f(z)\,dz = 2\pi i$. This shows that f is not Cauchy integrable on $\mathbb{C} \setminus \{0\}$, although f is analytic there. However, as we will see next, f is locally Cauchy integrable on $\mathbb{C} \setminus \{0\}$.

Definition 2.1.4. *Let $D \subset \mathbb{C}$ be open and $f \in C(D)$. Then f is called locally Cauchy integrable if for all $a \in D$ there exists an open neighborhood $U \subset D$ of a such that $f_{/U}$ is Cauchy integrable.*

Theorem 2.1.5. *Let $D \subset \mathbb{C}$ be open. If $f : D \to \mathbb{C}$ is continuous and analytic on $D \setminus \{a\}$, then f is locally Cauchy integrable.*

[1] It follows from the standard comparison theorems for infinite series, that $\sum_n^\infty a_n z^n$ converges absolutely for $|z| < R$ if and only if t $\sum_n^\infty \frac{a_n}{n+1} z^{n+1}$ converges absolutely for $|z| < R$.

Proof We assume first that f is analytic on D. We show that $\int_{\delta R} f(z)\,dz = 0$ for all rectangles $R \subset D$ with sides parallel to the axes. Let R be such a rectangle, set $R_0 := R$, and $d := L(\delta R)$. Divide R_0 into four congruent rectangles R^1, R^2, R^3, R^4,

Then $L(\delta R^k) = \frac{1}{2}L(\delta R_0) = \frac{1}{d}$ and$\int_{\delta R} f(z)\,dz = \sum_{k=1}^{n} \int_{\delta R^k} f(z)\,dz$. Choose k_0 such that $|\int_{\delta R} f(z)\,dz| \leq 4|\int_{\delta R^{k_0}} f(z)\,dz|$ and define $R_1 := R_0^{k_0}$. Repeating this process (induction) yields a nested sequence of rectangles R_n such that

$$|\int_{\delta R_n} f(z)\,dz| \leq 4|\int_{\delta R_{n+1}} f(z)\,dz| \text{ and } L(\delta R_{n+1}) = \frac{1}{2}L(\delta R_n).$$

Thus,

$$|\int_{\delta R} f(z)\,dz| \leq 4^n|\int_{\delta R_n} f(z)\,dz| \text{ and } L(\delta R_n) = \frac{d}{2^n}.$$

Choose $a \in \cap_{n\in\mathbb{N}_0} R_n \neq \emptyset$.[2] Since f is analytic there exists a continuous function $g : D \to \mathbb{C}$ with $g(a) = 0$ such that, for all $z \in D$,

$$f(z) = f(a) + f'(a)(z-a) + g(z)(z-a).$$

For $\epsilon > 0$ choose $n \in \mathbb{N}$ such that $|g(z)\| \leq \frac{\epsilon}{d^2}$ for all $z \in R_n$. Then[3]

$$\begin{aligned} |\int_{\delta R} f(z)\,dz| &\leq 4^n|\int_{\delta R_n} f(a) + f'(a)(z-a)\,dz + \int_{\delta R_n} g(z)(z-a)\,dz| \\ &= 4^n|\int_{\delta R_n} g(z)(z-a)\,dz| \leq 4^n \frac{d}{2^n}\frac{\epsilon}{d^2}\frac{d}{2^n} = \epsilon. \end{aligned}$$

Thus, $\int_{\delta R} f(z)\,dz = 0$ for all analytic functions $f : D \to \mathbb{C}$.

Now assume that f is continuous on D and analytic on $D \setminus \{a\}$. Again, we show that $\int_{\delta R} f(z)\,dz = 0$ for all rectangles $R \subset D$ with sides parallel to the axes and consider the following three cases.
(1): If $a \notin R$, then $\int_{\delta R} f(z)\,dz = 0$ by the first part of the proof.
(2): If $a \in \delta R$, then we consider a sequence of rectangles $R_i \subset R$ with sides parallel to the axes such that $R_i \to R$ as $i \to \infty$. By continuity, $\lim_{i\to\infty} \int_{\delta R_i} f(z)\,dz = \int_{\delta R} f(z)\,dz$. By the first part of the proof, $\int_{\delta R_i} f(z)\,dz = 0$ for all i. Thus, $\int_{\delta R} f(z)\,dz = 0$.
(3): If $a \in R \setminus \delta R$, then we devide R into two recangles $R^1, R^2 \subset R$ with sides parallel to the axes, $R = R^1 \cup R^2$, and $a \in \delta R^1 \cap \delta R^2$. Then, as shown in case (2), $\int_{\delta R} f(z)\,dz = \int_{\delta R^1} f(z)\,dz + \int_{\delta R^2} f(z)\,dz = 0$. $\square$

Remark 2.1.6. Notice that the proof works also if $f : D \to \mathbb{C}$ is continuous and analytic on $D \setminus \{\text{finite number of vertical or horizontal, compact line segments}\}$.

[2] Assume $\cap R_n = \emptyset$. Then $R_0 \subset \cup R_n^c = \mathbb{C}$. Since R_0 is compact and the complements R_n^c are open, there exists $m \in \mathbb{N}$ such that $R_0 \subset \cup_{n=0}^m R_n^c$. But then $R_0^c \supset \left(\cup_{n=0}^m R_n^c\right)^c = \cap_{n=0}^m R_n = R_m$ which contradicts the fact thet $R_m \subset R_0$. Thus $\cap R_n \neq \emptyset$.

[3] Show that $\int_\gamma z\,dz = 0$ for all absolutely continuous, closed paths γ.

2.2 Cauchy's Theorem

Definition 2.2.1. *Let $D \subset \mathbb{C}$ be open, let $\gamma_0, \gamma_1 : [0,1] \to D$ be piecewise smooth paths (curves), and let $\delta : [0,1] \times [0,1] \to D$ be continuous.Then*

(a) γ_0, γ_1 are homotopic in D with fixed endpoints if $\gamma_0(t) = \delta(t,0)$, $\gamma_1(t) = \delta(t,1)$ for all $t \in [0,1]$ and $\gamma_0(0) = \delta(0,s) = \gamma_1(0)$, $\gamma_0(1) = \delta(1,s) = \gamma_1(1)$ for all $s \in [0,1]$.

(b) If γ_0, γ_1 are closed, then they are homotopic as closed curves if $t \to \delta(t,s)$ is a closed curve for each $s \in [0,1]$ and $\gamma_0(t) = \delta(t,0)$, $\gamma_1(t) = \delta(t,1)$ for all $t \in [0,1]$.

(c) A curve γ_0 is null-homotopic in D if there exists a constant curve $\gamma_1(t) = a \in D$ such that γ_0 and γ_1 are homotopic as closed curves.

(d) The open set D is simply connected if every closed curve in D is null-homotopic in D.

Example 2.2.2. *(a) D simply conected, not connected:*

(b) D connected, not simply connected:

(c) If D is convex, then D is simply connected since $\delta(t,s) := s\gamma_1(t) + (1 - s\gamma_0(t)$ connects any two closed curves continuously in$D$.
(d) An open set D is called star-shaped if there exists $a \in D$ such that for all $z \in D$ the line-segment between a and z is contained in D. It is easy to see that star-shaped sets are simply connected.
(e) A sliced half-plane $D := \mathbb{C} \setminus \mathbb{R}_+ e^{i\alpha}$ is simply connected. To see this consider $D = \mathbb{C} \setminus \mathbb{R}_-$. Let γ be a closed curve in D. Then γ has a polar-coordinates representation $\gamma(t) = r(t)e^{i\alpha(t)}$ with $-\pi < \alpha(t) < \pi$ for all $t \in [0,1]$. Define $\delta(t,s) := r((1-s)t)e^{i\alpha(t)(1-s)}$. Then δ folds back the curve γ and contracts it towards the constant curve $\gamma_1(t) = \gamma(0) \in D$. Thus, D is simply connected.

Theorem 2.2.3 (Cauchy's Theorem). *Let $f : D \to \mathbb{C}$ be continuous on on open set D and analytic on $D \setminus \{a\}$.*

(i) *If γ_0, γ_1 are homotopic as closed curves in D, then $\int_{\gamma_0} f(z)\,dz = \int_{\gamma_1} f(z)\,dz$.*

(ii) *If γ is a null-homotopic closed curve in D, then $\int_\gamma f(z)\,dz = 0$.*

(iii) *If γ_0, γ_1 are homotopic in D with fixed endpoints, then $\int_{\gamma_0} f(z)\,dz = \int_{\gamma_1} f(z)\,dz$.*

(iv) *If D is simply connected, then f is Cauchy integrable.*

Proof. With the help of a few technical lemmas, the first statement will be shown over the next few pages. It is clear that (i) implies (ii) if one sets $\gamma = \gamma_0$ and $\gamma_1(t) = a$, where $a \in D$ is as in Definition 2.2.1. If γ_0, γ_1 are homotopic in D with fixed endpoints, then $\tilde{\gamma}_0 := \gamma_0 + (-\gamma_0)$, $\tilde{\gamma}_1 := \gamma_1 + (-\gamma_0)$ are homotopic as closed curves.[4]Thus, if (i) holds then $0 = \int_{\tilde{\gamma}_0} f(z)\,dz = \int_{\tilde{\gamma}_1} f(z)\,dz = \int_{\gamma_1} f(z)\,dz - \int_{\gamma_0} f(z)\,dz$. This shows that (i) implies (iii). It follows immediately from Theorem 2.1.2 that (ii) implies (iv). It remains to be shown that (i) holds.

We need the following three lemmas, and for those the notion of *antiderivatives along continuous curves*. Let $Q \subset \mathbb{C}$, $\rho : Q \to D$ continuous, and $f : D \to \mathbb{C}$ continuous. A continuous function $F : Q \to \mathbb{C}$ is called an *antiderivative of f along ρ* if for all $q \in Q$ there exist

(1) a neighborhood U of $q \in Q$,

(2) a neighborhood V of $\rho(q) \in D$, and

(3) an antiderivative G of $f_{/V}$

such that $\rho(U) \subset V$ and $F(u) = G(\rho(u))$ for all $u \in U$.

Lemma A. *Let $F : Q \to \mathbb{C}$ be an antiderivative of f along ρ. Then*

(i) *If $\sigma : M \to Q$ is continuous, then $F \circ \sigma$ is an antiderivative of f along $\rho \circ \sigma$.*

(ii) *If Q is connected and if $\tilde{F}$ is another antiderivative of F along ρ, then $\tilde{F} = F + c$ for some $c \in \mathbb{C}$.*

Proof. The first statement follows from the continuity of σ. To prove the second statement, fix $a \in Q$, define $c := \tilde{F}(a) - F(a)$ and

$$M := \{q \in Q : \tilde{F}(q) - F(q) = c\}.$$

Since M is nonempty ($a \in M$), closed ($M = (\tilde{F} - F)^{-1}\{c\}$), and Q is connected, we have to show that M is open. Let $b \in M$; i.e., $\tilde{F}(b) = F(b) + c$. Then there exist neighborhoods $U, \tilde{U}$ of $b \in Q$, neighborhoods $V, \tilde{V}$ of $\rho(b) \in D$, and antiderivatives $G, \tilde{G}$ of $f_{/V}$, $f_{/\tilde{V}}$ such that $\rho(U) \subset V$, $\rho(\tilde{U}) \subset \tilde{V}$, $F(u) = G(\rho(u))$ and $\tilde{F}(u) = \tilde{G}(\rho(u))$ for all $u \in U$. Choosig a small enough neighborhood of b, one may assume that $U = \tilde{U}$, $V = \tilde{V}$ and that U, V are connected. Then

$$\tilde{G}(\rho(b)) = \tilde{F}(b) = F(b) + c = G(\rho(b)) + c$$

and $\tilde{G}' - G' = f_{/\tilde{V}} - f_{/V} = 0$ on V. This shows that $\tilde{F}(u) - F(u) = \tilde{G}(\rho(u)) - G(u) = c$ for all $u \in U$. Thus, $U \subset M$. This shows that M is open. $\square$

[4]If $\gamma : [0,1] \to D$, then we denote by $\gamma + (-\gamma)$ the curve $t \to \gamma(2t)$ if $t \in [0, 1/2)$ and $t \to \gamma(2 - 2t)$ if $t \in [1/2, 1]$

Lemma B. *If f is analytic and $Q = \{(x,y) : 0 \leq x, y \leq 1\}$, then there exists an antiderivative of f along $\rho : Q \to D$.*

Proof. Let T denote the set of all $t > 0$ which have the property that for all rectangles $R \subset Q$ with perimeter $L(\partial R) \leq t$ there exists an antiderivative of f along $\rho_{/_R}$. We have to show that $4 \in T$.

Step 1. We show that T is nonempty. Assume that T is empty. Then, for all $n \in \mathbb{N}$, there exists a rectangle R_n in Q with perimeter less than $1/n$ such that f has no antiderivative along $\rho_{/_{R_n}}$. Let $r_n \in R_n \subset Q$. Since Q is compact, there exists a subsequence $\tilde{r}_n \to r \in Q$. Since f is continuous on D and analytic on $D \setminus \{a\}$, it follows from the local Cauchy Theorem 2.1.5 that f is locally Cauchy integrable on D. Thus, there exist neighborhoods U of r and V of $\rho(r) \in D$ such that $f_{/_V}$ has an antiderivative G and $\rho(U) \subset V$. Thus, $F(u) := G(\rho(U))$ is an antiderivative of f along $\rho_{/_U}$. Since $R_n \subset U$ for n large, there exists an antiderivative of f along $\rho_{/_{R_n}}$,which is a contradiction.

Step 2. Let $t \in T$ and let $R \subset Q$ be a rectangle with $L(\partial R) \leq \frac{4}{3}t$. Then R is the union of two rectangles $R_1, R_2 \subset Q$ whose perimeter is at most t. Let F_k be the antiderivative of f along $\rho_{/_{R_k}}$ $(k = 1, 2)$. Since $R_1 \cap R_2$ is connected, it follows from Lemma A that $F_{1/_{R_1 \cap R_2}} = F_{2/_{R_1 \cap R_2}} + c$ for some constant c. Define F as F_1 on R_1 and as $F_2 + c$ on R_2. Then F is an antiderivative of f along $\rho_{/_R}$. Thus, if $t \in T$, then $\frac{4}{3}t \in T$. This shows that $4 \in T$. $\square$

Lemma C. *Let $\gamma : [0,1] \to D$ be a piecewise smooth curve and let F be an antiderivative of f along γ. Then $\int_\gamma f(z)\,dz = F(1) - F(0)$.*

Proof. Since $\gamma([0,1])$ is compact in D and f locally Cauchy integrable (by Theorem 2.1.5), there exist n open subsets $V_k \subset D$, a partition $0 = t_0 < t_1 < \cdots < t_n = 1$, and analytic function F_k such that $\gamma([t_{k-1}, t_k]) \subset V_k$ and $F_k' = f_{/_{V_k}}$ for all $1 \leq k \leq n$. It follows that $F(t) = F_k(\gamma(t)) + c_k$ for $t \in [t_{k-1}, t_k]$. Thus, $\int_g ammaf(z)\,dz = \sum_{k=1}^n \int_{\gamma_k} f(z)\,dz = \sum_{k=1}^n F_k(\gamma(t_k)) - F_k(\gamma(t_{k-1})) = \sum_{k=1}^n F(t_k) - f(t_{k-1}) = F(1) - F(0)$. $\square$

Proof of (i). Let $k = 0, 1$, $\sigma_k(t) := (k, t)$, $\tau_k(t) := (t, k)$. Since γ_0, γ_1 are homotopic as closed curves in D there exists a continuous $\delta : [0,1] \times [0,1] \to D$ such that $t \to \delta(t, s)$ is a closed curve for each $s \in [0,1]$ and $\gamma_0(t) = \delta(t, 0) = \delta(\tau_0(t))$, $\gamma_1(t) = \delta(t, 1) = \delta(\tau_1(t))$ for all $t \in [0,1]$. It follows from Lemma C that there exists an antiderivative F of f along δ. By Lemma A (i), there exist antiderivatives $F_k = F \circ \tau_k$ along $\delta \circ \tau_k = \gamma_k$. By Lemma C, $\int_{\gamma_k} f(z)\,dz = F_k(1) - F_k(0) = F(\tau_k(1)) - F(\tau_k(0)) = F(1,k) - F(0,k)$. By Lemma A (i), the functions $F \circ \sigma_k$ are antiderivatives of f along $\delta \circ \sigma_k$. Since $\delta \circ \sigma_0(t) = \delta(0,t) = \delta(1,t) = \delta \circ \sigma_1(t)$ it follows from Lemma A (ii) that $F \circ \sigma_1 - F \circ \sigma_0 = const$. Thus, $\int_{\gamma_1} f(z)\,dz = F(1,1) - F(0,1) = F \circ \sigma_1(1) - F \circ \sigma_0(1) = F \circ \sigma_1(0) - F \circ \sigma_0(0) = F(1,0) - F(0,0) = \int_{\gamma_0} f(z)\,dz$. $\square$

2.3 Cauchy's Integral Formula

Let γ be a null-homotopic closed curve in an open subset D of $\mathbb{C}$ and $f : D \to \mathbb{C}$ analytic. Then $g(z) = \frac{f(z) - f(a)}{z - a}$ with $g(a) := f'(a)$ is continuous on D and analytic on $D \setminus \{a\}$. Applying Cauchy's Theorem 2.2.3 to the function g, we obtain the following statement.

Theorem 2.3.1 (Cauchy's Integral Formula). *If $D \subset \mathbb{C}$ is open, $\gamma : [\alpha, \beta] \to D$ a null-homotopic closed curve in D, $f : D \to \mathbb{C}$ analytic, and $a \in D \setminus \gamma[\alpha, \beta]$, then*

$$Ind_\gamma(a) f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - a}\,dz,$$

where $Ind_\gamma(a) = \frac{1}{2\pi i}\int_\gamma \frac{1}{z-a}\,dz$ *denotes the index of the closed curve* γ *with respect to* a.

In the following we will collect some properties of the index $Ind_\gamma(a)$. Let $\gamma_0(t) := a + Re^{\pm 2\pi i t}$ with $t \in [0,n]$ denote a circle of radius R that winds n-times counterclockwise (resp. clockwise) around the point a. Then

$$(A) \qquad Ind_{\gamma_0}(a) = \frac{1}{2\pi i}\int_{\gamma_0} \frac{1}{z-a}\,dz = \pm\int_0^n 1\,dt = \pm n$$

is the *winding number* of the closed curve γ_0 around a. It follows from Cauchy's Theorem 2.2.3 that $Ind_\gamma(a) = Ind_{\gamma_0}(a)$ for all closed curves γ that are homotopic to γ_0 as closed curves in $\mathbb{C}\setminus\{a\}$. Thus, $Ind_\gamma(a)$ is also called the winding number of a closed curve γ around a.

Proposition 2.3.2. *Let* $\gamma : [a,b] \to \mathbb{C}$ *be a closed curve and* $W = \mathbb{C}\setminus\gamma[a,b]$.

(i) $Ind_\gamma : W \to \mathbb{Z}$ *is continuous and therefore locally constant.*

(ii) $Ind_{\gamma+\tilde\gamma} = Ind_\gamma + Ind_{\tilde\gamma}$ *for all closed curves* $\tilde\gamma$ *with the same endpoints as* γ.

(iii) $Ind_{-\gamma} = -Ind_\gamma$.

(iv) *The interior* $Int(\gamma) := \{a \in W : Ind_\gamma(a) \neq 0\}$ *is open and bounded, the exterior* $Ext(\gamma) := \{a \in W : Ind_\gamma(a) = 0\}$ *is open, and* $\mathbb{C}\setminus\gamma[a,b]$ *is the disjoint union of the intirior and exterior of the closed curve* γ.

Proof. We show below that Ind_γ maps W into $\mathbb{Z}$. If $a_n \to a$ in W, then $\frac{1}{z-a_n} \to \frac{1}{z-a}$ uniformly in $z \in \gamma[a,b]$. Thus, $Ind_\gamma(a_n) = \int_\gamma \frac{1}{z-a_n}\,dz \to \int_\gamma \frac{1}{z-a}\,dz = Ind_\gamma(a)$. This proves (i). The statements (ii) and (iii) are obvious. The openess of $Int(\gamma)$ and $Ext(gamma)$ follows from the continuity of γ as a function with values in $\mathbb{Z}$. Let $M_r := \{a \in \mathbb{C} : |a| > r\} \subset W$. By (i), Ind_γ is constant on M_r. Since $\int_\gamma \frac{1}{z-a}\,dz = 0$ for $|a|$ sufficiently large, it follows that $M_r \subset Ext(\gamma)$. This shows that $Int(gamma)$ is bounded. It remains to be shown that $Ind_\gamma(a) \in \mathbb{Z}$ for all $a \in W$. This follows from the following discussion of the complex logarithm. $\square$

Let $D \subset \mathbb{C}\setminus\{0\}$ be simply connected. By Cauchy's Theorem 2.2.3, there exists an analytic function $\tilde F$ such that $\tilde F'(z) = \frac{1}{z}$ for all $z \in D$. By the quotient rule, $\frac{d}{dz}\frac{e^{\tilde F(z)}}{z} = 0$. Therefore, $e^{\tilde F(z)} = \tilde c z$ for some $\tilde c$ and all $z \in D$. Choose c such that $e^{-c} = \tilde c$ and define $F(z) = \tilde F(z) + c$. Then F is an antiderivative of $z \to \frac{1}{z}$ on D and $e^{F(z)} = z$ for all $z \in D$. We call F a *branch of the logarithm on* D.

Notice that if F is a branch of the logarithm on D, then $F + 2\pi n i$ ($n \in \mathbb{Z}$) is another branch. If $1 \in D$ and $F(1) = 0$, then F is called the main branch of the logarithm on D.

If $D = \mathbb{C}\setminus(-\infty, 0]$ and $F(1) = 0$, then F is called *the main branch of the logarithm* and is denoted by "ln". In particular, if $x > 0$, then $F(x) = \int_1^x \frac{1}{t}\,dt = \ln x$.

Now let $\gamma : [\alpha,\beta] \to \mathbb{C}$ be a piecewise smooth curve and $a \in \mathbb{C}\setminus\gamma[\alpha,\beta]$. Then

$$(*) \qquad e^{\int_\gamma \frac{1}{z-a}\,dz} = \frac{\gamma(\beta)-a}{\gamma(\alpha)-a}.$$

In particular, if γ is closed, then $Ind_\gamma(a) = \frac{1}{2\pi i}\int_\gamma \frac{1}{z-a}\,dz \in \mathbb{Z}$. To show $(*)$ we observe first that $\int_\gamma \frac{1}{z-a}\,dz = \int_{\gamma-a}\frac{1}{z}\,dz$, where $(\gamma-a)(t) := \gamma(t) - a$. Hence, we may assume that $a = 0 \in \mathbb{C}\setminus\gamma[\alpha,\beta]$.

Then there exists a partition $\alpha = t_0 < \cdots < t_n = \beta$ and disks $V_1, \cdots, V_n \subset \mathbb{C} \setminus \{0\}$ such that $\gamma[t_{k-1}, t_k] \subset V_k$, $(1 \leq k \leq n)$. Let F_k be a branch of the logarithm on V_k $(1 \leq k \leq n)$. Then $\int_{\gamma_k} \frac{1}{z}\, dz = F_k(\gamma(t_k)) - F_k(\gamma(t_{k-1}))$, where $\gamma_k := \gamma_{/[t_{k-1}, t_k]}$. Therefore, $e^{\int_{\gamma_k} \frac{1}{z-a}\, dz} = \frac{\gamma(t_k)}{\gamma(t_{k-1})}$ and $e^{\int_{\gamma} \frac{1}{z-a}\, dz} = \prod_{k=1}^{n} \frac{\gamma(t_k)}{\gamma(t_{k-1})} = \frac{\gamma(\beta)}{\gamma(\alpha)}$.

Example 2.3.3. *Since $Ind_\gamma = Ind_{\gamma_1} + Ind_{\gamma_2} + Ind_{\gamma_3}$, it follows from (A) that the Ind_γ is given by*

Theorem 2.3.4 (Cauchy's Integral Formula for Derivatives). *Let $D \subset \mathbb{C}$ be open and $f : D \to \mathbb{C}$ analytic. Then f is infinitely often complex differentiable on D and, for every null-homotopic closed curve $\gamma : [\alpha, \beta] \to D$,*

$$Ind_\gamma(a) f^{(n)}(a) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)^{n+1}}\, dz$$

for all $a \in D \setminus \gamma[\alpha, \beta]$ and $n \in \mathbb{N}_0$.

Proof. (1) Let $g : \gamma[\alpha, \beta] \to \mathbb{C}$ be continuous. Define $H_n : \mathbb{C} \setminus \gamma[\alpha, \beta] \to \mathbb{C}$ by $H_n(a) := \int_\gamma \frac{g(z)}{(z-a)^n}\, dz$. Since $\frac{g(z)}{(z-a_k)^n} \to \frac{g(z)}{(z-a)^n}$ uniformly for $z \in \gamma[\alpha, \beta]$ as $a_k \to a$ in $\mathbb{C} \setminus \gamma[\alpha, \beta]$, it follows that H_n is continuous. Moreover, H_n is analytic and $H_n' = nH_{n+1}$. This follows from

$$\begin{aligned}
\frac{H_n(a) - H_n(a_0)}{a - a_0} &= \int_\gamma \frac{1}{a - a_0} \left(\frac{1}{(z-a)^n} - \frac{1}{(z-a_0)^n} \right) g(z)\, dz \\
&= \int_\gamma \frac{1}{a - a_0} \left(\frac{1}{(z-a)} - \frac{1}{(z-a_0)} \right) \left(\sum_{k=0}^{n-1} \frac{1}{(z-a)^{n-1-k}} \frac{1}{(z-a_0)^k} \right) g(z)\, dz \\
&= \sum_{k=0}^{n-1} \int_\gamma \frac{g(z)}{(z-a_0)^{k+1}} \frac{1}{(z-a)^{n-k}}\, dz \\
&= \sum_{k=0}^{n-1} \int_\gamma \frac{\tilde{g}(z)}{(z-a)^{n-k}}\, dz \;\to\; \sum_{k=0}^{n-1} \int_\gamma \frac{g(z)}{(z-a_0)^{n+1}}\, dz \;=\; nH_{n+1}(a_0)
\end{aligned}$$

as $a \to a_0$, where we set $\tilde{g}(z) := \frac{g(z)}{(z-a_0)^{k+1}}$ and use that $a \to \int_\gamma \frac{\tilde{g}(z)}{(z-a)^{n-k}}\, dz$ is continuous, as shown above.
(2) By Cauchy's Integral formula 2.3.1, $Ind_\gamma(a) f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)}\, dz$. Now the statement of the theorem follows inductively from (1) and the fact that $a \to Ind_\gamma(a)$ is piecewise constant. $\square$

Corollary 2.3.5 (Morera's Theorem). *Let $D \subset \mathbb{C}$ be open, $a \in D$ and $f : D \to \mathbb{C}$. The following are equivalent.*

(i) *f is analytic on D.*

(ii) *f is continuous on D and analytic on $D \setminus \{a\}$.*

(iii) *f is continuous and locally Cauchy integrable on D.*

Proof. Clearly, (i) implies (ii), and (ii) implies (iii) by Theorem 2.1.5. If (iii) holds, then there exists a local antiderivatice on some neighborhood U for each point in D. By Theorem 2.3.4, $f_{/U} = F'_{/U}$ is infinitely often complex differentiable. Thus, f is analytic. $\square$

2.4 Cauchy's Theorem for Chains

Let $D \subset \mathbb{C}$ be open, $\gamma_k : [\alpha, \beta] \to D$ $(1 \le k \le n)$ piecewise smooth closed curves, and $m_k \in \mathbb{Z}$. Then

$$\gamma := \sum_{k=1}^{n} m_k \gamma_k$$

is called a chain in D. The integral along the chain γ is defined as

$$\int_\gamma f(z)\, dz := \sum_{k=1}^{n} m_k \int_{\gamma_k} f(z)\, dz.$$

For $a \notin \gamma[\alpha, \beta]$ the winding number of γ with respect to a is $W(\gamma, a) := \frac{1}{2\pi i} \int_\gamma \frac{1}{z-w}\, dz$, and the interior of the chain γ is defined as the set $Int(\gamma) := \{a \in \mathbb{C} \setminus \gamma[\alpha, \beta] : W(\gamma, a) \neq 0\}$. A chain γ is called null-homologous in D if $Int(\gamma) \subset D$.

Example 2.4.1. (a) The chain $\gamma = 2\gamma_1 - 3\gamma_2 - \gamma_3$ is null-homologous in an open set D if D contains the interior of the three circles.

(b) The chain $\gamma := \gamma_1 + 2\gamma_2 + 2\gamma_3 - \gamma_4$ is null-homologous in $\mathbb{C} \setminus \{z_1, z_2, z_3\}$.

(c) Let $D := \mathbb{C} \setminus \{0\}$, $\gamma_1(t) := e^{2\pi t}$, and $\gamma_2(t) := 2e^{2\pi t}$ for $t \in [0,1]$. Then γ_1, γ_2, and $\gamma := \gamma_1 + \gamma_2$ are not null-homologous in D, but the chain $\tilde{\gamma} := \gamma_1 - \gamma_2$ is null- homologous in D.

(d) Let $D := \mathbb{C} \setminus \{\text{two disjoint discs}\}$. The path γ is null-homologous, but not null-homotopic.

We observe that every null-homotopic closed curve γ in D is null-homologous in D. In fact, let $a \in \mathbb{C} \setminus D$. Then $z \to \frac{1}{z-a}$ is analytic on D, and therefore $W(\gamma, a) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z-a}\, dz = 0$. Thus $a \notin Int(\gamma)$, $Int(\gamma) \subset D$, and therefore γ is null-homologous.

Theorem 2.4.2 (Cauchy's Theorem for Chains). *If $D \subset \mathbb{C}$ is open, $\gamma : [\alpha, \beta] \to D$ a null-homologous chain in D, $f : D \to \mathbb{C}$ analytic, and $a \in D \setminus \gamma[\alpha, \beta]$, then*

(i) $\int_\gamma f(z)\, dz = 0$,

(ii) $W(\gamma, a) f^{(n)}(a) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)^{n+1}}\, dz \quad (n \in \mathbb{N}_0)$.

Proof. We can assume without loss of generality that D is a bounded open set containing γ (if necessary, we can replace D by the intersection of D with a large disc containing the chain γ). We show that

$$(*) \qquad W(\gamma, a) f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a}\, dz.$$

Then (ii) folows with differentiation, and (i) follows by applying $(*)$ to the analytic function $g(z) := (z-a)f(z)$. This yields $\int_\gamma f(z)\, dz = \int_\gamma \frac{g(z)}{z-a}\, dz = 2\pi i W(\gamma, a) g(a) = 0$. To prove $(*)$ we define $g : D \times D \to \mathbb{C}$ by

$$g(v, z) := \begin{cases} \frac{f(v)-f(z)}{v-z} & v \neq z \\ f'(z) & v = z \end{cases}$$

The function g is continuous (see below) and the function $h : \mathbb{C} \to \mathbb{C}$ defined by

$$h(z) := \begin{cases} \int_\gamma g(v,z)\,dv & z \in D \\ \int_\gamma \frac{f(v)}{v-z}\,dv & z \notin D \end{cases}$$

is entire and bounded (see below). By Liouville's Theorem **??**, $h = 0$ and therefore $(*)$ holds. To prove the continuity of g, let $(v_0, z_0) \in D$. If $v_0 \neq z_0$, then g is obviously continuous. If $v_0 = z_0$, let U be a disc in D around z_0. For all $v, z \in U$ we have that

$$g(v,z) = \int_0^1 f'(z + t(v-z))\,dt.$$

Thus, by the uniform continuity of f' we obtain the continuity of g. If $v = z$, then the integral representation of g is obvious; if $v \neq z$, then $\int_0^1 f'(z+t(v-z))\,dt = \frac{1}{v-z}\int_\gamma f'(w)\,dw = \frac{1}{v-z}[f(\gamma(1)) - f(\gamma(0))] = \frac{1}{v-z}[f(v) - f(z)] = g(v,z)$.

For $v \in D$ we have that $z \to g(v,z)$ is continuous on D and analytic in $D \setminus \{v\}$. By Morera's Theorem 2.3.5, the function $z \to g(v,z)$ is analytic on D for all $v \in D$. If $\tilde{\gamma}$ is a null-homotopic path in D, then $\int_{\tilde{\gamma}} h(z)\,dz = \int_{\tilde{\gamma}}\int_\gamma g(v,z)\,dv\,dz = \int_\gamma \int_{\tilde{\gamma}} g(v,z)\,dz\,dw = 0$. Thus, h is analytic on D.

If $v \in D$ and $z \in \mathbb{C} \setminus \overline{D}$, then $z \to \frac{f(v)}{v-z}$ is analytic. As above, it follows that h is analytic on $\mathbb{C} \setminus \overline{D}$.

Ir remains to be shown that h is analytic in $a \in \partial D$. Since $a \notin D$, it follows that $a \notin Int(\gamma)$ or $W(\gamma, a) = 0$. There exists a disc U around a which is contained in $\mathbb{C} \setminus \gamma[\alpha, \beta]$. Since U is connected it follows that $W(\gamma, z) = 0$ for all $z \in U$. Thus, for all $z \in U \cap D$,

$$\begin{aligned} \int_\gamma \frac{f(v)}{v-z}\,dv &= \int_\gamma \frac{f(v) - f(z)}{v-z}\,dv + \int_\gamma \frac{f(z)}{v-z}\,dv \\ &= \int_\gamma g(v,z)\,dv + f(z) 2\pi i W(\gamma, z) = \int_\gamma g(v,z)\,dv. \end{aligned}$$

Since $h(z) = \int_\gamma g(v,z)\,dv = \int_\gamma \frac{f(v)}{v-z}\,dv$ for all $z \in U \cap D$ and, by definition, $h(z) = \int_\gamma \frac{f(v)}{v-z}\,dv$ for all $z \in U$ which are not in D, it follows that

$$h(z) = \int_\gamma \frac{f(v)}{v-z}\,dv$$

for all $z \in U$. For $v \in \gamma[\alpha,\beta]$, the function $z \to \frac{f(v)}{v-z}$ is analytic on U. As above it follows that h is analytic on U. Thus, h is entire. Since $h(z) = \int_\gamma \frac{f(v)}{v-z}\,dv$ for all z with $|z| > R$ (R large) it follows that $h(z)$ is bounded if $|z| > R$. Since $h(z)$ is bounded if $|z| \leq R$ (by continuity), it follows that h is bounded. □

Example 2.4.3. Let f be analytic on $D := \mathbb{C} \setminus \{z_1, z_2\}$ and consider the closed curves

Then $\tilde{\gamma} := \gamma - \gamma_1 - \gamma_2$ is null-homologous in D. This implies that $\int_{\tilde{\gamma}} f(z)\,dz = 0$, or

$$\int_\gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz.$$

2.5 Principles of Linear Analysis

Let X be a vector space over $\mathbb{C}$ with norm $\|\cdot\| : X \to \mathbb{R}_+$ satisfying, for all $x, y \in X$ and $\lambda \in \mathbb{C}$,

$$\|\lambda x\| = |\lambda|\|x\|, \ \|x+y\| \le \|x\| + \|y\| \text{ and } \|x\| = 0 \text{ iff } x = 0.$$

A sequence (x_n) $(n \in \mathbb{N})$ is called a Cauchy sequence in X if for all $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\|x_n - x_m\| \le \epsilon$ for all $n, m \ge n_0$. The normed linear space X is called a *Banach space* if it is complete; i.e., if every Cauchy sequence converges in X. Since every normed vector space X is a metric space with metric $d(x,y) := \|x-y\|$, it can always be enlarged to become complete; i.e., for all normed vector spaces X, it is possible to find a Banach space $\overline{X}^c$ in which X is isometrically embedded as a dense subset. The Banach space $\overline{X}^c$ is unique up to isometric isomorphisms and can be obtained, as ususal, as the vector space of all Cauchy sequences modulo the zero-sequences with norm $\|(x_n)\|_c := \lim_{n\to\infty} \|x_n\|$.

Example 2.5.1. (a) Every finite dimensional, normed vector space $\mathbb{C}^n$ is a Banach space and all norms on $\mathbb{C}^n$ are equivalent; i.e., $m\|\cdot\|_1 \le \|\cdot\| \le M\|\cdot\|_1$ for some $m, M > 0$.

(b) Let s be the vector space of all finite sequences (s_n) (i.e., $s_n = 0$ for all but finitely many $n \in \mathbb{N}$). The completion of s under the norm $\|(s_n)\|_1 := \sum_{n=0}^\infty |s_n|$ is the space of all absolutely summable sequences and is denoted by l_1. The completion of s under the norm $\|(s_n)\|_\infty := \sup_n |s_n|$ is the space of all zero sequences and is denoted by c_0.

(c) Let $P[0,1]$ be the vector space of all polynomials $p : [0,1] \to \mathbb{C}$. If the norm is $\|p\|_\infty := \sup_{t\in[0,1]} |p(t)|$, then the completion of $P[0,1]$ is $C[0,1]$, the space of all continuous functions $f : [0,1] \to \mathbb{C}$. If the norm is $\|p\|_1 := \int_0^1 |p(t)|\,dt$, then the completion is $L^1[0,1]$, the space of all (equivalence classes of) Lebesgue integrable functions. If the norm is $\|p\|_{1,\infty} := \|p\|_\infty + \|p'\|_\infty$, then the completion is $C^1[0,1]$, the space of all continuously differentiable functions. And finally, if the norm is $\|p\|_{-1,\infty} := \sup_{x\in[0,1]} \left|\int_0^x p(t)\,dt\right|$, then the completion is $C^{-1}[0,1]$, the space of all (equivalence classes of) generalized, Riemann integrable functions. Since $\|p\|_{-1,\infty} \le \|p\|_1 \le \|p\|_\infty \le \|p\|_{1,\infty}$ for all polynomials p, it follows from the construction of the completions via Cauchy sequences that $C^1[0,1] \subset C[0,1] \subset L^1[0,1] \subset C^{-1}[0,1]$.

Definition 2.5.2. *Let X,Y be normed vector spaces. A linear operator $T : X \to Y$ is called bounded if there exists $M > 0$ such that $\|Tx\|_Y \le M\|x\|_X$ for all $x \in X$.*

Proposition 2.5.3. *Let X,Y be normed vector spaces and $T : X \to Y$ linear. The following are equivalent:*

(i) *T is bounded* **(ii)** *T is continuous*

(iii) *T is continuous in 0* **(iv)** $\sup_{\|x\|\le 1} \|Tx\| < \infty$

Proof. Statement (i) implies (ii) since $\|Tx - Ty\| \le M\,\|x-y\|$. Clearly, (ii) implies (iii). If (iii) holds, then there exists $\delta > 0$ such that $\|Tx\| \le 1$ if $\|x\| \le \delta$. If $\|x\| \le 1$, then $\|T(\delta x)\| \le 1$ or $\sup_{\|x\|\le 1} \|Tx\| \le \frac{1}{\delta}$. If (iv) holds, choose $M > \sup_{\|x\|\le 1} \|Tx\|$. Then $\left\|T\frac{x}{\|x\|}\right\| \le M$ or $\|Tx\| \le M\,\|x\|$ for all $x \in X$. □

Proposition 2.5.4. *Let X be a normed space and Y be Banach space. Then the space $L(X,Y)$ of all bounded linear operators $T : X \to Y$ is a Banach space with norm $\|T\| := \sup_{\|x\|\leq 1} \|Tx\|$.*

Proof. It is easy to see that $L(X,Y)$ is a normed vector space. If (T_n) is a Cauchy sequence in $L(X,Y)$, then $\|T_n\| \leq M$ for some $M > 0$ and all $n \in \mathbb{N}$ and $(T_n x)$ is a Cauchy sequence in Y for all $x \in X$. Since Y is complete, we can define $Tx := \lim_n T_n x$. It is easy to see that T is linear and bounded. Let $\epsilon > O$. Then there exists n_0 such that $\|T_n - T_m\| \leq \epsilon$ for all $n, m \geq n_0$. Thus, $\|T_n x - T_m x\| \leq \epsilon \|x\|$ for all x and $n, m \geq n_0$. Taking the limit as $m \to \infty$, we obtain $\|T_n x - Tx\| \leq \epsilon \|x\|$ for all x and $n \geq n_0$. Thus, there exists n_0 such that $\|T_n - T\| \leq \epsilon$ for all $n \geq n_0$. This shows that $T_n \to T$ in $L(X,Y)$. $\square$

Corollary 2.5.5. *The dual space $X^* := L(X,\mathbb{C})$ of a normed vector space X is a Banach space.*

Example 2.5.6. *(a) $(\mathbb{C}^n)^* = \mathbb{C}^n$ and $\langle x, x^* \rangle := x^*(x) = \sum_{i=1}^n x_i^* x_i$ for all $x^* \in \mathbb{C}^n$ and $x \in \mathbb{C}^n$.*
(b) $c_0^* = l_1$ and $\langle x, x^* \rangle := x^*(x) = \sum_{i=1}^\infty x_i^* x_i$ for all $x^* \in l_1$ and $x \in c_0$.
(c) $l_1^* = l_\infty$ and $\langle x, x^* \rangle := x^*(x) = \sum_{i=1}^\infty x_i^* x_i$ for all $x^* \in l_\infty$ and $x \in l_1$.
(d) $L^1[0,1]^* = L^\infty[0,1]$ and $\langle x, x^* \rangle := x^*(x) = \int_0^1 x^*(t)x(t)\,dt$ for all $x^* \in L^\infty[0,1]$ and $x \in L^1[0,1]$.
(e) $C[0,1]^* = BV_0[0,1]$ and $\langle x, x^* \rangle := x^*(x) = \int_0^1 x(t)\,dx^*(t)$ for all $x^* \in BV_0[0,1]$ and $x \in C[0,1]$.

Theorem 2.5.7 (Principle of Uniform Boundedness). *Let X be a Banach space, Y a normed space and $\mathcal{F} \subset L(X,Y)$. If, for all $x \in X$, there exists $M_x > 0$ such that $\|Tx\| \leq M_x$ for all $T \in \mathcal{F}$, then there exists $M > 0$ s.t. $\|T\| \leq M$ for all $T \in \mathcal{F}$.*

Proof. Recall that a set $S \subset X$ is called nowhere dense if the interior of its closure $\overline{S}$ is empty and that Baire's Theorem states that in a complete metric space X no non-empty open subset of X is the countable union of nowhere dense subsets. Now, let $X_n = \{x \in X : \|Tx\| \leq n \text{ for all } T \in \mathcal{F}\} = \cap_{T\in\mathcal{F}} \{x \in X : \|Tx\| \leq n\}$. Then X_n is closed and $\bigcup X_n = X$. By Baire's Theorem, the interior of some X_{n_0} is nonempty. Let $x_0 \in X_{n_0}$ and $\epsilon > 0$ such that the spere $K(x_0, \epsilon) \subset X_{n_0}$. Then $\|T(x_0 + \epsilon z)\| \leq n_0$ for all $T \in \mathcal{F}$ and all $\|z\| \leq 1$. But then $\|T(z)\| \leq \frac{1}{\epsilon}(n_0 + \|T(x_0)\|) \leq \frac{2n_0}{\epsilon}$ for all $T \in \mathcal{F}$and $\|z\| \leq 1$. $\square$

Theorem 2.5.8 (Banach). [5] *Let X be a normed space, Y a Banach space, $T_n \in L(X,Y)$, and $\|T_n\| \leq M$ for all n. The following are equivalent :*

(i) *There exists $T \in L(X,Y)$ such that $T_n x \to Tx$ for all $x \in X$.*

(ii) *$(T_n x)$ is a Cauchy sequence for all x in a total subset of X.*[6]

(iii) *There exists $T \in L(X,Y) : T_n\cdot \to T\cdot$ uniformly on compact subsets of X.*

Proof. Clearly, (iii) implies (i) and (i) implies (ii). We show that (ii) implies (iii). If (ii) holds, then the sequences $T_n w$ converge for all $w \in D$, where D is the dense linear span of the total subset. Let $\epsilon > 0$, $x \in C$, C compact, $\delta = \frac{\epsilon}{5M}$. There exists a finite number of $y_i \in C$ such that all $x \in C$ can be written as $x = y_i + z_x$ where $\|z_x\| \leq \delta$. Also, there exists $w_i \in D$ such that $y_i = w_i + z_{y_i}$, where $\|z_{y_i}\| \leq \delta$. Thus, for all $x \in C$,

$$\|T_n x - T_m x\| \leq \|T_n w_i - T_m w_i\| + \|T_n z_x\| + \|T_m z_x\| + \|T_n z_{y_i}\| + \|T_m z_{y_i}\| \leq \|T_n w_i - T_m w_i\| + \frac{4\epsilon}{5}.$$

Now, choose n_0 such that $\|T_n w_i - T_m w_i\| \leq \frac{\epsilon}{5}$ for all $n, m \geq n_0$ and all of the finitely many w_i. Then, $\|T_n x - T_m x\| \leq \epsilon$ for all $n, m \geq n_0$ and all $x \in C$. Now, define $Tx := \lim T_n x$. Then T is linear, $\|T\| \leq M$ and $T_n \to T$ uniformly on C. $\square$

[5] This very useful statement is often referred to as "Schaefer Drei-Vier-Fünf"; see III.4.5 in H.H. Schaefer's *Topological Vector Spaces*, Springer Verlag

[6] A subset is called total if its linear span is dense in X

Theorem 2.5.9 (Hahn-Banach). [7] *Let M be a subspace of a complex vector space X, p a seminorm on X, $F : M \to \mathbb{C}$ linear, and $|F(x)| \leq p(x)$ for all $x \in M$. Then there exists $\mu : X \to \mathbb{C}$ linear such that $\mu_{/M} = F$ and $|\mu(x)| \leq p(x)$ for all $x \in X$.*

Corollary 2.5.10. *Every bounded linear map from a linear subspace M of a normed space X into $\mathbb{C}$ has a bounded linear extension to X with the same norm.*

Proof. Let $F \in L(M, \mathbb{C})$. Then there exists $M > 0$ such that $\|F(x)\| \leq M\|x\| =: p(x)$ for all $x \in M$. By the Hahn-Banach Theorem, there exists $x_0^* \in X^*$ such that $x^*_{0/M} = F$ and $|\langle x, x_0^*\rangle| \leq M\|x\|$ for all $x \in X$. Thus, $\|x_0^*\| \leq \|F\|$. Since x_0^* is an extension of F, we also have $\|F\| \leq \|x_0^*\|$. Thus $\|F\| = \|x_0^*\|$. $\square$

Corollary 2.5.11. *For all x_0 in a normed space X there exists $x_0^* \in X^*$ such that $\langle x_0, x_0^*\rangle = \|x_0\|$ and $\|x_0^*\| = 1$.*

Proof. Let $M = \{\alpha x_0 : \alpha \in \mathbb{C}\}$, $F : M \to \mathbb{C}$, $F(\alpha x_0) := \alpha\|x_0\|$. Then $F \in L(M, \mathbb{C})$, $\|F\| = 1$, and $F(x_0) = \|x_0\|$. Now the statement follows from the corollary above. $\square$

Corollary 2.5.12. *Every $x_0 \in X$ can be viewed as an element in X^{**}; i.e., a vector $x_0 \in X$ can also be viewed as a linear map $x_0 : X^* \to \mathbb{C}$, defined by $x_0(x^*) := x^*(x_0) = \langle x_0, x^*\rangle$ with the same norm; i.e., the Banach space norm of x_0 is the same as the operator norm of x_0 as an element of X^{**}.*

Proof. Let $x_0 \in X$. Then $\mu_{x_0} : X^* \to \mathbb{C}$ defined by $\mu_{x_0}(x^*) := x^*(x_0)$ is linear and $|\mu_{x_0}(x^*)| \leq \|x^*\|\,\|x_0\|$ for all $x^* \in X^*$. Thus, $\|\mu_{x_0}\| \leq \|x_0\|$. Since there exists $x^* \in X^*$ with $\|x^*\| = 1$ and $x^*(x_0) = \|x_0\|$ we get that $\|\mu_{x_0}\| = \sup_{\|x^*\|\leq 1} |\mu_{x_0}(x^*)| = \|x_0\|$. $\square$

Corollary 2.5.13. *Every normed vector space X can be embedded in the Banach space X^{**} and the completion of of the normed space X coincides with the closure of X in its bidual X^{**}.*

Example 2.5.14. Let s be the vector space of all finite sequences (s_n) with the sup-norm. Then the dual of s is l_1 and its bidual is l_∞. Clearly, the completion of s is c_0 which is also the closure of s in l_∞.

Definition 2.5.15. *Let X be a normed space. A subset Γ of X^* is called a norming set for X if $\|x\| = \sup_{x^*\in\Gamma} |\langle x, x^*\rangle|$ for all $x \in X$.*

Example 2.5.16. It follows from Corollary 2.5.12 that the dual unit sphere $S^* := \{x^* \in X^* : \|x^*\| = 1\}$ is a norming set for X. Let $x \in l_\infty$ and Γ be the set of all unit sequences $(\partial_{i,n})_{i\in\mathbb{N}}$, where $\partial_{i,n}$ denotes the Kronecker symbol. Then $\Gamma \subset l_\infty^*$ is a norming set for l_∞ since $\|x\| = \sup_i |x_i| = \sup_{x^*\in\Gamma} |\langle x, x^*\rangle|$.

[7] For a proof see, for example, H.L. Royden's *Real Analysis*, Macmillan.

2.6 Cauchy's Theorem for Vector-Valued Analytic Functions

Definition 2.6.1. *Let $D \subset \mathbb{C}$ be open, X a Banach space, and $\Gamma \subset X^*$ a norming set for X. A function $f : D \to X$ is called analytic on D if*

$$f'(a) := \lim_{z \to a} \frac{f(z) - f(a)}{z - a}$$

exists for all $a \in D$. The function f is called weakly analytic with respect to Γ if the functions $\langle f, x^* \rangle : D \to \mathbb{C}$ defined by $z \to \langle f(z), x^* \rangle$ are analytic on D for all $x^* \in \Gamma$. The function f is called weakly analytic if $\langle f, x^* \rangle : D \to \mathbb{C}$ is analytic on D for all $x^* \in X^*$. The function f is locally bounded if $\sup_{z \in C} \|f(z)\| < \infty$ *for all compact sets $C \subset D$.*

Theorem 2.6.2 (Dunford). *Let $D \subset \mathbb{C}$ be open, X a Banach space, and $f : D \to X$. The following are equivalent.*

(i) *f is analytic on D.*

(ii) *f is weakly analytic on D.*

(iii) *f is locally bounded and weakly analytic on D with respect to a norming set Γ of X.*

Proof. Clearly, (i) implies (ii). Assume that (ii) holds. Let C be a compact subset of D and $x^* \in X^*$. Then there exists $M_{x^*} > 0$ such that $|\langle f(z), x^* \rangle| \leq M_{x^*}$ for all $z \in C$. Considering the vectors $f(z)$ as linear maps in X^{**} (see Corollary 2.5.12) and using the Principle of Uniform Boundedness 2.5.7 it follows that there exists $M > 0$ such that $\|f(z)\| \leq M$ for all $z \in C$. Thus, (ii) implies (iii). We show that (iii) implies (i). To show the existence of $\lim_{z \to a} \frac{f(z)-f(a)}{z-a}$ for $a \in D$ we may assume that $a = 0$. For $h, k \in \mathbb{C} \setminus \{0\}$ let

$$u(h, k) := \frac{f(h) - f(0)}{h} - \frac{f(k) - f(0)}{k}.$$

We have to show that for $\epsilon > 0$ there exists $\delta > 0$ such that $\|u(h, k)\| \leq \epsilon$ whenever $|h| \leq \delta$ and $|k| \leq \delta$. Let $r > 0$ such that the sphere $S := K_{2r}(0) \subset D$ and let $M := \sup_{z \in S} \|f(z)\| = \sup_{z \in S, x^* \in \Gamma} |\langle f(z), x^* \rangle|$. By Cauchy's integral formula $\langle f(w), x^* \rangle = \frac{1}{2\pi i} \int_{|z|=2r} \frac{\langle f(z), x^* \rangle}{z-w} \, dz$ for all $w \in K_r(0)$ and $x^* \in \Gamma$. Thus, for $h, k \in K_r(0) \setminus \{0\}$ and $x^* \in \Gamma$

$$\begin{aligned} \langle u(h, k), x^* \rangle &= \frac{1}{2\pi i} \int_{|z|=2r} \langle f(z), x^* \rangle \left(\frac{1}{h} \left[\frac{1}{z-h} - \frac{1}{z} \right] - \frac{1}{k} \left[\frac{1}{z-k} - \frac{1}{z} \right] \right) dt \\ &= (h - k) \frac{1}{2\pi i} \int_{|z|=2r} \frac{\langle f(z), x^* \rangle}{z(z-h)(z-k)} \, dt. \end{aligned}$$

Thus, $|\langle u(h, k), x^* \rangle| \leq |h - k| \frac{M}{r^2}$ for all $x^* \in \Gamma$. Since Γ is a norming set for X we obtain that $\|u(h, k)\| \leq |h - k| \frac{M}{r^2}$. This proves the claim. ☐

Theorem 2.6.3 (Cauchy's Theorem). *If $D \subset \mathbb{C}$ is open, X a Banach space, $\gamma : [\alpha, \beta] \to D$ a null-homologous chain in D, $f : D \to X$ analytic, and $a \in D \setminus \gamma[\alpha, \beta]$, then*

(i) $\int_\gamma f(z) \, dz = 0,$

(ii) $W(\gamma, a) f^{(n)}(a) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)^{n+1}} \, dz \quad (n \in \mathbb{N}_0).$

Proof. Since f is continuous one can define the path-integrals. Now the statement follows from Cauchy's Theorem 2.4.2 for complex-valued functions; i.e., since $\langle \int_\gamma f(z)\,dz, x^*\rangle = \int_\gamma \langle f(z), x^*\rangle\,dz = 0$ for all $x^* \in X^*$, we obtain that $\int_\gamma f(z)\,dz = 0$. Similarly, one proves (ii). $\square$

Corollary 2.6.4 (Liouville's Theorem). *Let X be a Banach space and $f : \mathbb{C} \to X$. The following are equivalent.*

(i) *f is bounded and entire.*

(ii) *f is constant.*

Proof. If f is bounded and entire, then $f'(a) = \frac{1}{2\pi i}\int_{|z-a|=R} \frac{f(z)}{(z-a)^2}\,dz = \frac{1}{2\pi}\int_0^{2\pi} \frac{f(w+Re^{it})}{R^2e^{2it}} Re^{it}\,dt$. Thus, $\|f'(a)\| \le \frac{const}{R}$ for all $R > 0$. Thus, $f' = 0$ and therefore, f is constant. $\square$

Corollary 2.6.5 (The Fundamental Theorem of Algebra). *Every non-constant complex polynomial has at least one zero in $\mathbb{C}$.*

Proof. Let p be a polynomial of degree $n \ge 1$. Then $\frac{p(z)}{z^n} \to c_n$ as $|z| \to \infty$. Thus $|p(z)| \ge \frac{1}{2}c_n|z^n|$ for $|z| > R$. Suppose $p(z) \ne 0$ for all $z \in \mathbb{C}$. Then $f := 1/p$ is entire and bounded. By Liouville's Theorem above, f is constant (which is a contradiction). $\square$

Corollary 2.6.6 (Weierstrass' Convergence Theorem). *Let $D \subset \mathbb{C}$ be open and X a Banach space. If $f_n : D \to X$ is a sequence of analytic functions which converges uniformly on each compact subset of D towards a function f, then f is analytic and $f_n' \to f'$ uniformly on compact subsets of D.*

Proof. Clearly, f is continuous and for each rectangle $R \subset D$ we have that $0 = \int_R f_n(z)\,dz \to \int_R f(z)\,dz$. Thus, $z \to \langle f(z), x^*\rangle$ is locally Cauchy integrable for all $x^* \in X^*$. By Morera's Theorem 2.3.5, f is weakly analytic. By Dunford's Theorem 2.6.2, f is analytic. Let $c \in D$ and choose $\tilde\epsilon > \epsilon > 0$ such that the sphere $K_{\tilde\epsilon}(c) \subset D$. Since $\frac{f_n(z)}{(z-a)^2} \to \frac{f(z)}{(z-a)^2}$ uniformly for those z with $|z-c| = \tilde\epsilon$ and $a \in K_\epsilon(c)$, it follows from Cauchy's Integral Formula that

$$f_n'(a) = \frac{1}{2\pi i}\int_{|z-a|=\tilde\epsilon} \frac{f_n(z)}{(z-a)^2}\,dz \to \frac{1}{2\pi i}\int_{|z-a|=\tilde\epsilon} \frac{f(z)}{(z-a)^2}\,dz = f'(a)$$

uniformly for $a \in K_\epsilon(c)$. Now, if $K \subset D$ is compact, then $K = \cup_{i=1}^n K_{\epsilon_i}(c_i)$ and therefore $f_n' \to f'$ uniformly on compact subsets of D. $\square$

Example 2.6.7. Since $z \to \frac{1}{n^z} = e^{-z\ln n}$ is entire and $|\frac{1}{n^z}| \le \frac{1}{n^c}$ for all z with $Rez \ge c > 1$, it follows from the Weierstrass Convergence Theorem that *the Riemann-Zeta function* $\zeta(z) := \sum_{n+1}^\infty \frac{1}{n^z}$ is analytic and $\zeta'(z) := \sum_{n+1}^\infty \frac{\ln n}{n^z}$ for all z with $Rez > 1$.

Corollary 2.6.8 (Mazur). *A complex normed algebra which is a field is isomorphic to the complex field.*

Proof. Let e denote the unit in X. Since X is a normed algebra, $\|xy\| \le \|x\|\,\|y\|$ for all $x, y \in X$. In particular, $\|e\| \ge 1$. Assume that there exists $x \in X$ such that $x - \lambda e \ne 0$ for all $\lambda \in \mathbb{C}$. Then the resolvent $f : \lambda \to (x-\lambda e)^{-1}$ is well-defined for all $\lambda \in \mathbb{C}$. It is easy to see that f satisfies the resolvent equation

$$\frac{f(\lambda+h) - f(\lambda)}{h} = f(\lambda+h)f(h)$$

for all $\lambda, h \in \mathbb{C}$. Thus, f is entire if and only if f is continuous. Since

$$\|f(\lambda+h) - f(\lambda)\| \le |h|\,\|f(\lambda+h)\|\,\|f(\lambda)\|,$$

it follows that f is continuous if $\|f(\lambda+h)\|$ is bounded for small h. Since

$$f(\lambda+h) = f(\lambda)[e - hef(\lambda)]^{-1} = f(\lambda)\sum_{n=0}^{\infty} h^n ef(\lambda)^n$$

for $|h| < \frac{1}{\|f(\lambda)\|}$, it follows that $\|f(\lambda+h)\| \leq 2\,\|f(\lambda\|$ for $|h| < \frac{1}{2\|f(\lambda)\|}$. Thus, f is entire and $f'(\lambda) = f^2(\lambda)$ for all $\lambda \in \mathbb{C}$. Since $\|f(\lambda\| = \left\|-\frac{1}{\lambda}[e - \frac{1}{\lambda}x]^{-1}\right\| = \left\|-\frac{1}{\lambda}\sum_{n=0}^{\infty}(\frac{x}{\lambda})^n\right\| \leq \frac{2}{|\lambda|}$ if $2\,\|x\| \leq |\lambda|$, it follows from the continuity of f that f is bounded. Bt Louiville's Theorem, f is constant. But then $x^{-1} = f(0) = f(\lambda) = \lim_{\lambda\to\infty} f(\lambda) = 0$ and therefore $\|e\| = \left\|xx^{-1}\right\| \leq \|x\|\left\|x^{-1}\right\| = 0$, which is a contradiction. This shows that the map $\lambda \to \lambda e$ defines an isomorphism between $\mathbb{C}$ and X.

$\square$

2.7 Power Series

Proposition 2.7.1. *Let $a = (a_n)_{n\geq 0}$ be a sequences in a Banach space X and*

$$R(a) := \frac{1}{\limsup_{n\to\infty} \sqrt[n]{\|a_n\|}}.$$

Then $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely for $|z| < R(a)$ and diverges for all $|z| > R(a)$. Moreover, if $b = (b_n)_{n\geq 0}$ is another sequence in X, then $R(a+b) \geq \min(R(a), R(b))$. If b is a complex sequence and $b * a := (\sum_{i=0}^{n} b_{n-i}a_i)_{n\geq 0}$ denotes the Cauchy product, then $R(b*a) \geq \min(R(a), R(b))$.

Proof. If $|z| < R := R(a)$ or $\limsup_{n\to\infty} \sqrt[n]{\|a_n\|} = \frac{1}{R} < \frac{1}{|z|}$, then there exists $p < 1$ such that $\sqrt[n]{\|a_n\|} \leq \frac{p}{|z|}$ for all $n \geq n_0$. But then the power series converges absolutely since $\|a_n z^n\| \leq p^n$. To prove that the power series diverges for $|z| > R$, assume that the power series converges for some $z \in \mathbb{C}\setminus\{0\}$. Then $\|a_n z^n\| \leq 1$ for all $n \geq n_0$ or $\sqrt[n]{\|a_n\|} \leq \frac{1}{|z|}$ for all $n \geq n_0$. This shows that $|z| \leq R$. To show that $R(a+b) \geq \min(R(a), R(b))$ let $|z| \leq r < \min(R(a), R(b))$. Then $\sum_{n=0}^{\infty}(a_n + b_n)z^n$ exists and thus, $R(a+b) \geq r$. If b is a complex sequence and $|z| \leq r < \min(R(a), R(b))$, then $\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n} b_{n-i}a_i\right)z^n$ exists since $\sum_n \|\sum_{j+k=n} b_k a_j\| r^n \leq \sum_n \left(\sum_{j+k=n} |b_k|\,\|a_j\|\right) r^n = \left(\sum_j \|a_j\|\, r^j\right)\left(\sum_k |b_k| r^k\right)$. $\square$

Theorem 2.7.2. *Let $a = (a_n)_{n\geq 0}$ be a sequences in a Banach space X with $R(a) > 0$. Then $f(z) := \sum_{n=0}^{\infty} a_n z^n$ is analytic for $|z| < R(a)$, $f'(z) := \sum_{n=1}^{\infty} n a_n z^{n-1}$, and $a_n = \frac{1}{n!}f^{(n)}(0)$.*

Proof. Let C be a compact subset of the sphere $K_R(0)$ around the origin with radius R. Then $C \subset K_r(0)$ for some $r < R$. It follows that $f_N(z) := \sum_{n=0}^{N} a_n z^n$ converges uniformly on C. Now the claim follows from the Weierstrass Convergence Theorem 2.6.6, and $a_n = \frac{1}{n!}f^{(n)}(0)$ is proved by induction. $\square$

Remark 2.7.3. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with radius of convergence $R = R(a)$. Assume that the power series converges uniformly on the sphere $K_R(0)$. If $R < \infty$, then the power series converges for all $|z| \leq R$ and f is continuous for $|z| \leq R$. If $R = +\infty$, then f is a polynomial.*

Proof. Assume $R < \infty$. Then, for all $\epsilon > 0$ there exists n_0 such that $\|\sum_{k=n}^{m} a_k z^k\| \leq \epsilon$ for all $m > n \geq n_0$ and all $|z| < R$. By continuity, the estimate holds for all $|z| \leq R$. This proves the first statement. If $R = +\infty$, then $\|\sum_{k=n_0}^{m} a_k z^k\| \leq 1$ for all $m \geq n_0$ and all $z \in \mathbb{C}$. Thus $\|\sum_{k=n_0}^{\infty} a_k z^k\| \leq 1$ for all $z \in \mathbb{C}$. By Liouville's Theorem 2.6.4, $\sum_{k=n_0}^{\infty} a_k z^k = 0$. $\square$

Theorem 2.7.4 (Abel's Continuity Theorem). *If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ converges for some $z \in \mathbb{C}$, then $h(t) := f(tz)$ is continuous on $[0, 1]$.*

Proof. Define $b_n := a_n z^n$. Then $h(w) := \sum_{n=0}^{\infty} b_n w^n$ converges for $w = 1$. Thus, h is analytic for $|w| < 1$. We show that $h(1) = \lim_{t \nearrow 1} h(t)$ and assume that $h(1) = \sum_{n=0}^{\infty} b_n = 0$ (otherwise consider $\tilde{h}(t) := h(t) - h(1)$). Let $\epsilon > 0$ and define $c_n := \sum_{i=0}^{n} b_i$. Then $\|c_n\| \leq \epsilon/2$ for all $n \geq n_0$. Using the Cauchy product we obtain for all $0 \leq t < 1$, $\frac{1}{1-t} h(t) = \left(\sum_{k=0}^{\infty} t^k\right)\left(\sum_{j=0}^{\infty} b_j t^j\right) = \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n} b_j\right) t^n = \sum_{n=0}^{\infty} c_n t^n$. Thus, $\|h(t)\| \leq \|(1-t)\sum_{n=0}^{n_0} c_n t^n\| + (1-t)\sum_{n=n_0+1}^{\infty} \|c_n\| t^n \leq \epsilon$ for all $t_0 \leq t < 1$. □

Theorem 2.7.5. *Let $D \subset \mathbb{C}$ be open, X Banach space, and $f : D \to X$. The following are equivalent.*

(i) *f is analytic on D.*

(ii) *For all $z_0 \in D$ exist $a_n \in X$ such that $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ whenever z is in a sphere $K_\epsilon(z_0) \subset D$.*

(iii) *For all $z_0 \in D$ exist $\epsilon > 0$ and $a_n \in X$ such that $K_\epsilon(z_0) \subset D$ and $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ for all $z \in K_\epsilon(z_0)$.*

Moreover, if one of the statements holds, then

$$a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i}\int_{|z-z_0|=\epsilon} \frac{f(z)}{(z-z_0)^{n+1}}\, dz \quad (n \geq 0).$$

Proof. Clearly, (ii) implies (iii). If (iii) holds, then f is analytic by Theorem 2.7.2. It remains to be shown that (i) implies (ii). Let $z \in K_\epsilon(z_0) \subset D$. By Cauchy's Theorem 2.6.3 and the uniform convergence of $\sum_{n=0}^{\infty}\left(\frac{z-z_0}{w-z_0}\right)^n$ for $w \in \mathbb{C}$ with $|w - z_0| = \epsilon$,

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_{|w-z_0|=\epsilon} \frac{f(w)}{w-z}\, dw = \frac{1}{2\pi i}\int_{|w-z_0|=\epsilon} \frac{1}{w-z_0}\frac{f(w)}{1-\frac{(z-z_0)}{w-w_0}}\, dw \\ &= \frac{1}{2\pi i}\int_{|w-z_0|=\epsilon} \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(w-z_0)^{n+1}} f(w)\, dw \\ &= \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_{|w-z_0|=\epsilon} \frac{f(w)}{(w-z_0)^{n+1}}\, dw\right)(z-z_0)^n. \end{aligned}$$

□

Corollary 2.7.6. *Let $D \subset \mathbb{C}$ be open, $z_0 \in D$, X Banach space, and $f : D \to X$ analytic. Then there exists a neighborhood V of z_0 in D such that either $f_{/V} = 0$ or $f(z) \neq 0$ for all $z_0 \neq z \in V$.*

Proof. Let $z \in K_\epsilon(z_0) \subset D$. Then $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$. If $a_n = 0$ for all $n \geq 0$, then $f = 0$ on $K_\epsilon(z_0)$. Otherwise let a_m be the first non-zero coefficient. Then $f(z) = \sum_{n=m}^{\infty} a_n (z - z_0)^n = (z - z_0)^m \sum_{n=0}^{\infty} a_{n+m}(z - z_0)^n = (z - z_0)^m g(z)$. Since g is analytic and $g(z_0) = a_m \neq 0$, it follows that there exists a neighborhood V of z_0 in $K_\epsilon(z_0)$ such that $g(z) \neq 0$ for all $z \in V$. □

Corollary 2.7.7. *Let $D \subset \mathbb{C}$ be a region, X Banach space, and $0 \neq f : D \to X$ analytic. Then $N := \{z \in D : f(z) = 0\}$ has no accumulation point.*

Proof. Let Ω be the interior of the closed set N. Then Ω is open. Since $f \neq 0$, it follows that $\Omega \neq D$. Let $z_0 \in \partial\Omega \cap D$. Because $g := f_{/\Omega} = 0$ it follows that $g^{(n)} = 0$. Therefore, $f^{(n)}(z_0) = 0$ since $f^{(n)}$ is continuous on D and thus $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n = 0$ for $z \in K_\epsilon(z_0)$. This shows that z_0 is in Ω, the interior of N. Thus, Ω is closed. Since Ω is open and closed in D and $\omega \neq D$, it follows that N has empty interior. Now let $z_0 \in N$. By the previous corollary, there exists a neighborhood V of z_0 such that $V \cap N = \{z_0\}$. Thus N has no accumulation points. □

Corollary 2.7.8. *Let $D \subset \mathbb{C}$ be a region, X Banach space, and $f_1, f_2 : D \to X$ analytic. If f_1 and f_2 coincide on a subset of D which has an accumulation point in D, then $f_1 = f_2$ on D.*

Proof. The statement follows immediately from the previous corollary. □

2.8 Resolvents and the Dunford-Taylor Functional Calculus

A linear mapping $A : X \supset D(A) \to X$ (X Banach space), where the domain $D(A)$ is a linear subspace of X, is called a linear operator on X.

A closed linear operator is a linear operator whose graph $G(A) := \{(x, Ax) : x \in D(A)\}$ is closed in $X \times X$; i.e., if $x_n \in D(A)$, $x_n \to x$, and $Ax_n \to y$, then $x \in D(A)$ and $Ax = y$.

On $D(A)$ we consider the graph norm $\|x\|_A := \|x\| + \|Ax\|$. The mapping $\gamma : D(A) \to G(A), \gamma(x) := (x, Ax)$ is an isometric isomorphism between $[D(A)] := (D(A), \|\cdot\|_A)$ and $G(A) \subset X \times X$, where $X \times X$ is endowed with the 1-norm. It follows that A is closed if and only if $[D(A)]$ is complete (i.e., a Banach space).

Recall that a linear operator is called bounded if $D(A) = X$ and A is continuous; i.e., $\|A\| := \sup_{\|x\| \leq 1} \|Ax\| < \infty$. By the *Closed Graph Theorem*, a linear operator A is bounded if it is closed and $D(A) = X$.

Let A be a linear operator on X. By

$$\rho(A) := \{\lambda \in \mathbb{C} : \lambda I - A \text{ is bijective and } (\lambda I - A)^{-1} \in \mathcal{L}(X)\}$$

we denote the *resolvent set* of A and by

$$R : \rho(A) \to \mathcal{L}(X),\ \lambda \to R(\lambda, A) := (\lambda I - A)^{-1}$$

the *resolvent* of A. The set $\sigma(A) := \mathbb{C} \setminus \rho(A)$ is called the *spectrum* of A.

Proposition 2.8.1. *Let A be a linear operator on X. Then*

(a) If $\rho(A) \neq \emptyset$, then A is closed.

(b) If A is closed, then $\rho(A) := \{\lambda \in \mathbb{C} : \lambda I - A$ is bijective$\}$.

Proof. (a) Since A is closed if and only if $\lambda I - A$ is closed for some (all) $\lambda \in \mathbb{C}$, we can assume that $0 \in \rho(A)$. Let $x_n \in D(A)$, $x_n \to x$, and $Ax_n \to y$. Since A^{-1} is continuous, $x_n = A^{-1}Ax_n \to A^{-1}y$. Thus, $x = A^{-1}y \in D(A)$ and $Ax = y$.
(b) If a linear operator is closed and injective, then it is easy to see that its inverse is well defined and closed. Thus, if $\lambda I - A$ is bijective, then its inverse is closed and everywhere defined. By the closed graph theorem, $(\lambda I - A)^{-1} \in \mathcal{L}(X)$. □

Theorem 2.8.2. *Let A be a closed linear operator on X and $\lambda_0 \in \rho(A)$. If $|\lambda - \lambda_0| < 1/\|R(\lambda_0, A)\|$, then $\lambda \in \rho(A)$ and*

$$R(\lambda, A) = \sum_{n=0}^{\infty} (-1)^n R(\lambda_0, A)^{n+1} (\lambda - \lambda_0)^n .$$

In particular, $\rho(A)$ is open, $\sigma(A)$ is closed, the resolvent $R : \lambda \to R(\lambda, A)$ is analytic on $\rho(A)$,

$$R^{(n)}(\lambda_0, A) = (-1)^n n! R(\lambda_0, A)^{n+1} \quad (n \in \mathbb{N}_0),$$

and $dist(\lambda_0, \sigma(A)) \geq 1/\|R(\lambda_0, A)\|$.

Proof. Recall that if $T \in \mathcal{L}(X)$ with $\|T\| < 1$, then $(I - T)^{-1}$ exists and $(I - T)^{-1} = \sum_{n=0}^{\infty} T^n$. One has $\lambda I - A = (\lambda - \lambda_0)I + \lambda_0 I - A = [I - (\lambda_0 - \lambda)R(\lambda_0, A)](\lambda_0 I - A)$. Hence $R(\lambda, A) = R(\lambda_0, A)[I - (\lambda_0 - \lambda)R(\lambda_0, A)]^{-1} = R(\lambda_0, A) \sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n R(\lambda_0, A)^n$ exists whenever $|\lambda - \lambda_0| \|R(\lambda_0, A)\| < 1$. Now the other statements follow from the results of the previous section. $\square$

As the first derivative operator on $C[0, 1]$ shows, there are closed operators with empty resolvent set. This can not happen if A is bounded.

Proposition 2.8.3. *If $A \in \mathcal{L}(X)$, then $\sigma(A)$ is non empty and compact. Moreover,*

$$\sup |\sigma(A)| = \lim_{n\to\infty} \|A^n\|^{1/n} \leq \|A\|$$

and $R(\lambda, A) = \sum_{n=0}^{\infty} \frac{1}{\lambda^{n+1}} A^n$ for $|\lambda| > \sup |\sigma(A)|$.

Proof. By Proposition 2.7.1, $R(\lambda) := \sum_{n=0}^{\infty} \frac{1}{\lambda^{n+1}} A^n$ converges for $|\lambda| > \limsup_{n\to\infty} \|A^n\|^{1/n}$. It can be easily verified that $R(\lambda)(\lambda I - A) = (\lambda I - A)R(\lambda) = I$. Thus, $R(\lambda) = R(\lambda, A)$, $D := \{\lambda \in \mathbb{C} : |\lambda| > \limsup_{n\to\infty} \|A^n\|^{1/n}\} \subset \rho(A)$, and $\sup |\sigma(A)| \leq \limsup_{n\to\infty} \|A^n\|^{1/n}$. By Theorem 2.7.5 it follows that $R(\lambda, A) = \sum_{n=0}^{\infty} \frac{1}{\lambda^{n+1}} A^n$ converges for all $|\lambda| > \sup |\sigma(A)|$. Thus the radius of convergence of $R(\lambda)$ is less or equal to $\sup |\sigma(A)|$; i.e., $\limsup_{n\to\infty} \|A^n\|^{1/n} \leq \sup |\sigma(A)|$. This shows that $\sup |\sigma(A)| = \limsup_{n\to\infty} \|A^n\|^{1/n} \leq \|A\|$. We will show next that $\sup |\sigma(A)| \leq \liminf_{n\to\infty} \|A^n\|^{1/n}$. The factorization $(\lambda^n I - A^n) = (\lambda I - A)P_n(A) = P_n(A)(\lambda I - A)$ shows that if $(\lambda^n I - A^n)$ has a bounded inverse, so will $(\lambda I - A)$. Thus, if $\lambda \in \sigma(A)$, then $\lambda^n \in \sigma(A^n)$ and therefore by the above considerations, $|\lambda|^n \leq \|A^n\|$. This shows that $\sup |\sigma(A)| \leq \|A^n\|^{1/n}$ for all $n \in \mathbb{N}$ or $\sup |\sigma(A)| \leq \liminf_{n\to\infty} \|A^n\|^{1/n}$.

Since $R(\lambda, A) = \sum_{n=0}^{\infty} \frac{1}{\lambda^{n+1}} A^n$ for $|\lambda| > 2\|A\|$, it follows that $\|R(\lambda, A)\| \leq \frac{2}{|\lambda|}$ for $|\lambda| \geq 2\|A\|$. Suppose that the spectrum of A were empty. Then the resolvent is an entire function which is bounded (by continuity) on the compact set $\{\lambda \in \mathbb{C} : |\lambda| \leq 2\|A\|\}$ and bounded on the complement $\{\lambda \in \mathbb{C} : |\lambda| > 2\|A\|\}$. By Liouville's Theorem, $R(\lambda, A) = const$ for all $\lambda \in \mathbb{C}$. Since $R(\lambda, A) \to 0$ as $\lambda| \to \infty$, it follows that $R(\lambda, A) = 0$. But then $I = (\lambda I - A)R(\lambda, A) = 0$, which is a contradiction. Thus, $\sigma(A) \neq \emptyset$. $\square$

If $A \in \mathcal{L}(X)$, then we denote bt $\mathcal{F}(A)$ the family of all complex valued functions f which are analytic on some neighborhood of $\sigma(A)$. The neighborhood need not be connected and can depend of $f \in \mathcal{F}(A)$. Let $f \in \mathcal{F}(A)$ and let U be a neighborhood of $\sigma(A)$ with a piecewise smooth boundary such that $U \cup \delta U$ is contained in the domain of analyticity of f. Then

$$f(A) := \frac{1}{2\pi i} \int_{\delta U} f(\lambda) R(\lambda, A)\, d\lambda \in \mathcal{L}(X)$$

and the definition of $f(A)$ depends only on f and not on the domain U (by Cauchy's theorem).

Theorem 2.8.4 (Dunford Functional Calculus, Spectral Mapping Theorem). *If $A \in \mathcal{L}(X)$, $f, g \in \mathcal{F}(A)$, and $\alpha, \beta \in \mathbb{C}$, then*

(a) $\alpha f + \beta g \in \mathcal{F}(A)$ and $\alpha f(A) + \beta g(A) = (\alpha f + \beta g)(A)$.

(b) $f \cdot g \in \mathcal{F}(A)$ and $f(A)g(A) = (fg)(A)$.

(c) If $f(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n$ for all λ in some neighborhood of $\sigma(A)$, then $f(A) = \sum_{n=0}^{\infty} a_n A^n$.

(d) $f(\sigma(A)) = \sigma(f(A))$.

2.9 The Maximum Principle

Let $D \subset \mathbb{C}$ be a region and $f : D \to \mathbb{C}$ analytic. If $K \subset D$ is a compact region with interior K_0, then Cauchy's integral formula

$$f(z) = \frac{1}{2\pi i} \int_{\partial K} \frac{f(w)}{w - z}\, dw \qquad (z \in K_0)$$

shows that the values of the function f on the boundary ∂K determine f in the interior K_0. In this section we collect some consequences of this crucial fact.

Lemma 2.9.1. *Let $f : D \to \mathbb{C}$ be analytic and $\overline{K_r(c)} \subset D$. If $\inf\{|f(z)| : z \in \partial K_r(c)\} > |f(c)|$, then there exists $z_0 \in K_r(c)$ such that $f(z_0) = 0$.*

Proof. Assume that $f(z) \neq 0$ for all $z \in K_r(c) := \{z : |z - c| < r\}$. It follows from $|f(z)| \geq |f(c)|$ for $z \in \partial K_r(c)$ that $f(z) \neq 0$ for all $z \in \overline{K_r(c)}$. By a standard compactness argument one obtains that $f(z) \neq 0$ for all $z \in K_{\tilde{r}}(c) \subset D$ with $\tilde{r} > r$. Define $g(z) := \frac{1}{f(z)}$ for $z \in K_{\tilde{r}}(c)$. Then g is analytic and

$$\frac{1}{|f(c)|} = |g(c)| = |\frac{1}{2\pi i} \int_{|z-c|=r} \frac{g(z)}{z - c}\, dz| \leq \sup_{\{z:|z-c|=r\}} |g(z)| = \frac{1}{\inf_{\{z:|z-c|=r\}} |f(z)|},$$

or, $\inf_{\{z:|z-c|=r\}} |f(z)| \leq |f(c)|$, which is a contradiction. □

Theorem 2.9.2. *Let $f : D \to \mathbb{C}$ be analytic and not constant. If D is a region, then $f(D)$ is a region.*

Proof. Since continuous images of connected sets are connected, one has to show that $f(D)$ is open. Let $c \in D$. Since f is not constant, there exists $K_r(c) \subset D$ such that $f(z) \neq f(c)$ for all $z \in \partial K_r(c)$ (otherwise there would exist a sequence $z_n \to c$ such that $f(z_n) = f(c)$ for all $n \in \mathbb{N}$, and this would imply that $f(z) = f(c)$ for all $z \in D$). Define

$$\delta := \inf_{z \in \partial K_r(c)} |f(z) - f(c)| > 0.$$

Let $|b - f(c)| < \delta/2$. Then

$$|f(z) - b| \geq |f(z) - f(c)| - |f(c) - b| > \delta/2$$

for all $z \in \partial K_r(c)$. Therefore,

$$\inf_{z \in \partial K_r(c)} |f(z) - b| \geq \delta/2 > |f(c) - b|.$$

It follows from the lemma above that there exists $z_0 \in K_r(c)$ such that $b = f(z_0)$; i.e., $K_{\delta/2}(f(c)) \subset f(K_r(c)) \subset f(D)$. Hence, $f(D)$ is open. □

Example 2.9.3. Let $f(z) := z^2$. Then $f(K_1(0)) = K_1(0)$ is open in $\mathbb{C}$, but $f(-1,1) = [0,1)$ is not open in $\mathbb{R}$.

Corollary 2.9.4. *Let $D \subset \mathbb{C}$ be a region and $f : D \to \mathbb{C}$ be a scalar-valued analytic function. If $|f|$ has a minimum at $c \in D$, then $f(c) = 0$.*

Proof. Suppose that $f(c) \neq 0$. Then there exists an open neighborhood around $f(c)$ which is in the region $f(D)$. Thus, there exists $z \in D$ such that $|f(z)| < |f(c)|$, which is a contradiction. $\square$

Example 2.9.5. It is standard folklore in mathematics that classical complex analysis extends from scalar-valued functions to Banach space valued functions. This handy statement is true and useful if one keeps in mind that there are many exceptions to the rule. Examples of such exceptions are the statements above. In general, they do not hold if $f : D \to X$, X a Banach space. To see this, let $X := \mathbb{C}^2$ with norm $\|(z_1, z_2)\| := \max(|z_1|, |z_2|)$ and consider $f(z) := (1, z)$. Then $\|f(z)\| = 1$ for $|z| \leq 1$ and $\|f(z)\| = |z|$ for $|z| \geq 1$. Also, the image of the unit sphere contains no open subset of $\mathbb{C} \times \mathbb{C}$; i.e., f does not map regions into regions.

Theorem 2.9.6 (Maximum Principle). *Let $D \subset \mathbb{C}$ be a region and $f : D \to X$ analytic. Then the analytic landscape $z \to \|f(z)\|$ has no maximum on D unless it is constant on D.*

Proof. (1) First, we prove the claim for $X = \mathbb{C}$. Assume that f is not constant and $K_r(c) \subset D$. Since $U := f(K_r(c))$ is a region in $\mathbb{C}$ and $f(c) \in U$, it follows that $f(c)$ is in the interior of U. Thus, there exists $z \in K_r(c)$ such that $|f(z)| > |f(c)|$.

(2) Suppose that there exists $c \in D$ such that $\|f(z)\| \leq \|f(c)\|$ for all $z \in D$. Choose $x_0^* \in X^*$ with $\|x_0^*\| = 1$ such that $\langle f(c), x_0^* \rangle = \|f(c)\|$. Then $h : z \to \langle f(z), x_0^* \rangle$ satisfies

$$|h(z)| \leq \|x_0^*\| \|f(z)\| = \|f(z)\| \leq \|f(c)\| = |h(c)|$$

for all $z \in D$. By (1), h is constant on D. Thus,

$$\|f(z)\| \leq \|f(c)\| = h(c) = h(z) \leq \|f(z)\|,$$

or $\|f(z)\| = \|f(c)\|$ for all $z \in D$. $\square$

We remark that if $D \subset \mathbb{C}$ is a region and $f : D \to \mathbb{C}$ is analytic, then f is constant if and only if $|f|$ is constant. This does not hold in Banach spaces as the example $f(z) := (1, z)$ $(z \in K_1(0))$ illustrates.

Corollary 2.9.7. *Let $f : D \to X$ be analytic on a bounded region $D \subset \mathbb{C}$ and continuous on $\overline{D}$. Let $M := \max\{\|f(z\| : z \in \partial D\}$. Then either $\|f(z)\| = M$ for all $z \in \overline{D}$, or $\|f(z)\| < M$ for all $z \in D$.*

Proof. Since continuous function attain there maximum on compact sets, there exists $a \in \overline{D}$ such that $\|f(z\| \leq \|f(a)\|$ for all $z \in \overline{D}$. If $a \in D$, then it follows from the Maximum Principle that $\|f(z)\| = \|f(a)\|$ for all $z \in D$ and thus for all $z \in \overline{D}$. In this case, $\|f(z)\| = M$ for all $z \in \overline{D}$. If $a \in \partial D$, then $M = \|f(a)\|$. If there exists $z_0 \in D$ such that $\|f(z_0)\| = \|f(a)\|$, then $\|f\|$ is constant on D and thus on $\overline{D}$. Otherwise $\|f(z)\| < M$ for all $z \in D$. $\square$