A New Analytical Approach for Strongly Nonlinear Vibration of a Microbeam Considering Structural Damping Effect

Mousa Rezaee[1], Masoud Minaei[2*]

[1]Professor, Department of Mechanical Engineering, University of Tabriz, Tabriz, Iran
[2]Assistant Professor, Department of Mechanical Engineering, Azarbaijan Shahid Madani University, Tabriz, Iran

Abstract:

In this study, strongly nonlinear free vibration behaviour of a microbeam considering the structural damping effect is investigated analytically on the basis of modified couple stress theory. Employing Von Karman's strain-displacement relations and implementing the Galerkin's method, the governing nonlinear partial differential equation is reduced to a nonlinear ordinary differential equation which is related to the size effect of the beam. Because of large coefficient of nonlinear term and due to existence of the damping effect, none of the traditional perturbation methods leads to a valid solution. Also, there are many difficulties encountered in applying homotopy techniques when the damping effect is taken in to account in the strongly nonlinear damped system. To overcome these limitations, here, a new analytical method is presented which is based on classical perturbation methods and fundamentals of Fourier expansion with an embedding nondimensional parameter. To solve the equation, the nonlinear frequency is assumed to be time dependent. The comparison between time responses of the system obtained by the presented approach and numerical method indicates the high accuracy of the new method. To validate the results of the presented method with those available in the literatures which are obtained for a special case of an undamped system, the damping coefficient is set to zero. The comparison shows a good agreement between the results for a wide range of vibration amplitudes.

Keywords:

Damped microbeam; Strongly nonlinear vibration; New analytical approach; Size effect; Perturbation method.

*Corresponding Author: Email: ac.minaei@azaruniv.ac.ir

1. Introduction

Nowadays micro-electromechanical systems (MEMS) are widely used in smart materials in various fields of technologies such as mechanical, civil, aerospace and bio-engineering [1]. Clamped–clamped microbeams are used numerously in MEMS as solo components in devices or as spring components to support and add stiffness to other microstructures [2, 3]. Vibration analysis of microbeams is an important issue in modern engineering applications such as arched beam structures, micro-machined mechanical resonators, vibration shock sensors, atomic force microscopes and many other industrial usages. As the amplitude of oscillation increases, these microstructures are subjected to nonlinear vibrations which often lead to material fatigue and structural damage. These effects become more significant around the resonance frequencies of the system [4]. Therefore, it is very important to provide an accurate method for investigating the nonlinear vibration behaviour of the microstructures. The nonlinear vibration of a microbeam is governed by a nonlinear partial differential equation in space and time. For this equation, it is very difficult to find an exact or closed form solution. The importance of nonlinear Duffing equation has been widely recognized by scientists, since it plays a key role in some important practical phenomena, such as periodic orbit extraction, non-uniformity caused by an infinite domain, nonlinear mechanical oscillators, prediction of disease and so on [5]. A full analytical solution has not been introduced so far for the damped Duffing equation with strong nonlinear coefficients. Therefore, considerable attention has been directed to study of the strongly nonlinear oscillators. Several methods have been used to find approximate analytical solutions for the Duffing equation, including the perturbation techniques [6, 7], homotopy analysis method [8], homotopy perturbation method [9-13], modified homotopy perturbation method [4, 14], frequency-amplitude formulation [15], harmonic balance method [16], modified variational approach [17], energy balance method [18], max-min approach [19], modified Lindstedt-Poincare method [20], variational iteration method [21, 22], and some other techniques [23, 24]. Through these methodologies, there are many difficulties encountered in the application of perturbation techniques to solve the strongly nonlinear equations [4]. For example, one of the most frustrating is the fact that all classical perturbation techniques strongly rely on the assumption of a small parameter into the equation which might be artificial, and subsequent expansion of the solution through the perturbation series around this parameter. However, the solutions obtained by these methods may not be uniform, restricting the applicability of such perturbation methods [25].

To overcome the above mentioned limitations of classical perturbation techniques, many novel techniques have been proposed in recent years. One of these new methods is the homotopy perturbation method which is applicable to strongly nonlinear systems. He [26] proposed a new perturbation technique to solve the nonlinear undamped Duffing equation in which the maximum relative error at the first order approximation is less than 7%. However, the homotopy technique is not usually employed to solve a damped nonlinear equation, because it leads to equations that are too complicate to be solved analytically. Some researchers considered the damping effect in their studies. Recently two new methods, Laplace decomposition [27] and homotopy perturbation transform [28] are introduced for the solution of nonlinear and non-homogeneous differential equations which are capable of solving the damped Duffing equation. In decomposition based methods, obtaining Adomian polynomials is too complicated. Nonetheless, in homotopy perturbation transform method, this limitation is resolved using He polynomials. Therefore, to reach a valid solution, by implementing the homotopy perturbation transform method or the modified differential transform method, one must increase the power of the polynomials and this requires too cumbersome mathematical calculations. Nourazar and Mirzabeigy [29] applied the modified differential transform method to solve the nonlinear Duffing oscillator with damping effect, approximately. Following these descriptions, when the above mentioned methods are used to solve the nonlinear damped equation, one can only obtain the approximate response of the system with no elucidation about the nonlinear frequency.

In this paper, strongly nonlinear free vibration behaviour of a microbeam considering the structural damping effect is investigated analytically on the basis of modified couple stress theory. Employing Von Karman's strain-displacement relations and using the Hamilton's principle, the beam governing equation of motion is derived. By implementing the Galerkin's method and assuming the immovable clamped-clamped boundary conditions, the partial differential equation is reduced to a nonlinear ODE which is related to the size effect of the beam. To solve this nonlinear equation, according to the mentioned justifications, neither the classical perturbation techniques nor the homotopy methods, are not suitable solution methods. Here, a new analytical approach is presented for solving the mentioned nonlinear damped equation. In this new approach using the basic concepts of the classical perturbation methods together with the fundamentals of Fourier expansion with an embedding parameter which is considered as a small parameter ($0 < \varepsilon << 1$), and assuming the time dependent relation for the frequency, the strongly nonlinear damped equation of motion is solve. This presented analytical approach provides a valid asymptotic solution for any positive coefficients of the nonlinear terms, and it is the main advantage of this method over the other mentioned methods. For this damped strongly nonlinear system, comparing time responses obtained by the first order approximate solution of the new method and those obtained by the numerical technique, i. e. RK45, indicates the high accuracy of the new method for a wide range of vibration amplitudes. Since the studies about the strongly nonlinear vibration of microbeams are restricted to undamped cases, here in order to make it possible to validate the method, the comparisons are made for such cases. The comparisons show a good agreement for a wide range of the vibration amplitudes.

2. Equation of motion

An Euler-Bernoulli microbeam with a length of L, cross-sectional area of A, density of ρ, cross sectional area moment of inertia of I, the elasticity modulus of E and the shear modulus of G is shown in Fig. 1.

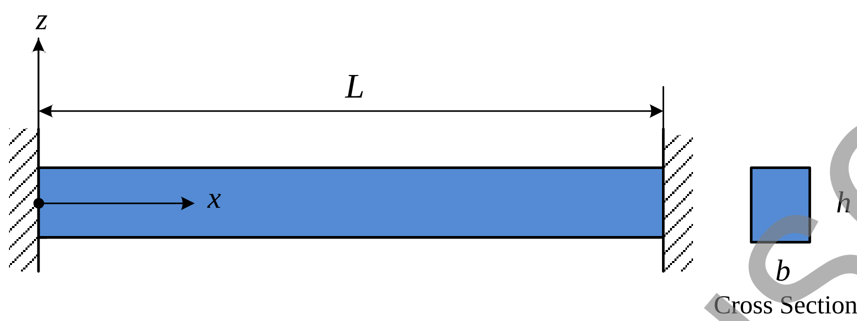

Fig. 1. A clamped-clamped microbeam.

The strain-displacement relations for a beam undergoing large deflections are as [30, 31]:

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \qquad \kappa_x = -\frac{\partial^2 w}{\partial x^2} \tag{1}$$

where ε_x is the axial strain at a generic point of the microbeam which is located at the mid-plane surface, κ_x is the curvature of the beam, u is the longitudinal displacement, w is the lateral displacement and x is the longitudinal coordinate,

Neglecting the axial inertia and using the modified couple stress theory was presented by Yang et al. in 2002, in which the strain energy density is a function of both strain tensor (conjugated with stress tensor) and curvature tensor (conjugated with couple stress tensor) [32, 33], the strain energy, U, and the kinetic energy, T, of the beam is given by [34]:

$$U = \frac{1}{2}\int_0^L \left(EA\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]^2 + \left(EI + GAl^2\right)\left(\frac{\partial^2 w}{\partial x^2}\right)^2 \right) \mathrm{d}x \tag{2}$$

$$T = \frac{1}{2}\int_0^L \rho A \left(\frac{\partial w}{\partial t}\right)^2 \mathrm{d}x \tag{3}$$

where l is a material length scale parameter and GAl^2 is related to the modified couple stress theory [35]. It should be mentioned that the current model based on the modified couple stress theory contains only one additional material constant besides two classical material parameters. The presence of l enables the incorporation of the material size features in the new model and renders it possible to explain the size effect. Furthermore, when the size effect is suppressed by letting $l=0$, the new model will reduce to the classical beam model [33].

Including the effects of the mid-plane stretching and employing the Hamilton's principle, one obtains the governing equations for Euler-Bernoulli microbeam as:

$$\frac{\partial}{\partial x}\left(EA\left[\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]\right)=0 \tag{4}$$

$$\left(EI+GAl^2\right)\frac{\partial^4 w}{\partial x^4}-\frac{\partial}{\partial x}\left(EA\left[\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]\frac{\partial w}{\partial x}\right)+\rho A\frac{\partial^2 w}{\partial t^2}=Q \tag{5}$$

where Q is the non-conservative force due to the internal damping and is obtained as:

$$Q=-\frac{\partial^2}{\partial x^2}\left(C_S I\frac{\partial^3 w}{\partial t\,\partial x^2}\right) \tag{6}$$

where C_S represents the beam internal damping coefficient.

Integrating Eq. (4) and substituting the result into Eq. (5), and using Eq. (6) leads to the following equation:

$$\left(EI+GAl^2\right)\frac{\partial^4 w}{\partial x^4}-N_0\frac{\partial^2 w}{\partial x^2}+\frac{EA}{2L}\left(\int_0^L\left(\frac{\partial w}{\partial x}\right)^2\mathrm{d}x\right)\frac{\partial^2 w}{\partial x^2}+\rho A\frac{\partial^2 w}{\partial t^2}+\frac{\partial^2}{\partial x^2}\left(C_S I\frac{\partial^3 w}{\partial t\,\partial x^2}\right)=0 \tag{7}$$

where N_0 is the pretension force.

It is more convenient to work with dimensionless parameters. Here, the dimensionless parameters are as:

$$\hat{t}=\omega t,\qquad \hat{x}=\frac{x}{L},\qquad \hat{w}=\frac{w}{L} \tag{8}$$

ω is the linear natural frequency of the microbeam with the value of:

$$\omega=\left(\lambda L\right)^2\sqrt{\frac{EI+GAl^2}{\rho AL^4}} \tag{9}$$

where λL is the eigenvalue of the microbeam with doubly clamped boundary conditions. In Eq. (9) by letting $l=0$, the system natural frequency computed by the new model reduces to that obtained by the classical beam model.

Substituting Eqs. (7) and (8) into Eq. (6) and using the chain rule for differentiation, one obtains:

$$\begin{aligned}&\frac{\partial^4\hat{w}}{\partial\hat{x}^4}-\frac{N_0L^2}{EI+GAl^2}\frac{\partial^2\hat{w}}{\partial\hat{x}^2}-\\&\frac{1}{2}\left(\frac{EL^2}{Er^2+Gl^2}\right)\left(\int_0^1\left(\frac{\partial\hat{w}}{\partial\hat{x}}\right)^2\mathrm{d}\hat{x}\right)\frac{\partial^2\hat{w}}{\partial\hat{x}^2}+\left(\lambda L\right)^4\frac{\partial^2\hat{w}}{\partial\hat{t}^2}+\frac{C_S r^2\omega}{Er^2+Gl^2}\frac{\partial^5\hat{w}}{\partial\hat{t}\,\partial\hat{x}^4}=0\end{aligned} \tag{10}$$

where $r=\sqrt{I/A}$ is the radius of gyration of the beam cross section.

The solution of Eq. (10) can be assumed as $\hat{w}(\hat{x},\hat{t})=\phi(\hat{x})q(\hat{t})$ [36] where $\phi(\hat{x})$ is the first mode shape of the beam. For the clamped-clamped beam, $\phi(\hat{x})$ is as follows [37]:

$$\phi(\hat{x})=\cosh(\lambda_n L\hat{x})-\cos(\lambda_n L\hat{x})-\frac{\cosh(\lambda_n L)-\cos(\lambda_n L)}{\sinh(\lambda_n L)-\sin(\lambda_n L)}\left(\sinh(\lambda_n L\hat{x})-\sin(\lambda_n L\hat{x})\right) \tag{11}$$

Using the Galerkin's method and multiplying both sides of Eq. (10) by $\phi(\hat{x})$ and integrating over the interval of [0, 1] one reaches:

$$\begin{aligned}&(\lambda L)^4\left(\int_0^1\phi^2\mathrm{d}\hat{x}\right)\ddot{q}+\frac{C_S r^2\omega}{Er^2+Gl^2}\left(\int_0^1\phi\phi^{(4)}\mathrm{d}\hat{x}\right)\dot{q}+\\&\left[\left(\int_0^1\phi\phi^{(4)}\mathrm{d}\hat{x}\right)-\frac{N_0L^2}{EI+GAl^2}\left(\int_0^1\phi\phi''\mathrm{d}\hat{x}\right)\right]q+\left[-\frac{1}{2}\left(\frac{EL^2}{Er^2+Gl^2}\right)\left(\int_0^1\phi'^2\mathrm{d}\hat{x}\right)\left(\int_0^1\phi\phi''\mathrm{d}\hat{x}\right)\right]q^3=0\end{aligned} \tag{12}$$

After some mathematical manipulations we have:

$$\ddot{q}(\hat{t})+\omega_0^2 q(\hat{t})+\hat{\beta}q(\hat{t})^3+\hat{\alpha}\dot{q}(\hat{t})=0 \tag{13}$$

where ω_0^2 , $\hat{\beta}$ and $\hat{\alpha}$ are defined as:

$$\omega_0^2=\frac{f_4\left(EI+GAl^2\right)-f_2N_0L^2}{f_1\left(EI+GAl^2\right)\lambda^4L^4},\quad \hat{\beta}=\frac{-EAf_2f_3}{2f_1\left(EI+GAl^2\right)\lambda^4L^2},\quad \hat{\alpha}=\frac{f_4C_SI}{f_1\rho A\omega L^4} \tag{14}$$

and

$$f_1=\int_0^1\phi^2\mathrm{d}\hat{x},\qquad f_2=\int_0^1\phi\phi''\mathrm{d}\hat{x},\quad f_3=\int_0^1\phi'^2\mathrm{d}\hat{x},\qquad f_4=\int_0^1\phi\phi^{(4)}\mathrm{d}\hat{x} \tag{15}$$

Eq. (13) is the differential equation of motion governing the nonlinear vibration of the microbeam. The initial conditions are assumed as:

$$q(0)=a_0=\frac{W_{max}}{L},\qquad \dot{q}(0)=0 \tag{16}$$

where W_{max} is the microbeam maximum deflection.

Using the change of variable $t=\omega_0\hat{t}$ and applying in Eq. (13) one reaches:

$$\ddot{q}(t)+q(t)+\beta q(t)^3+\alpha\dot{q}(t)=0 \tag{17}$$

where:

$$\beta=\frac{\hat{\beta}}{\omega_0^2},\qquad \alpha=\frac{\hat{\alpha}}{\omega_0} \tag{18}$$

There are different methods to solve Eq. (17). However, most of these methods do not result in a valid solution for the strongly nonlinear cases $(\beta>1)$. The coefficient of the nonlinear term, β , is dependent on the beam parameters as well as the boundary conditions [4]. The value of β for microbeams is very large compared with unity. As a case study, for the doubly microbeam, $\beta = 1941.3$. Therefore, the classical perturbation approaches do not lead to a valid expansion for the solution.

3. Introducing the new analytical method

As mentioned before, the perturbation methods have many limitations for solving and analyzing the behavior of strongly nonlinear systems, i.e. to use perturbation techniques, the coefficient of the nonlinear term should be smaller than unity [4]. Recently, some methods such as homotopy techniques are proposed to overcome this limitation. However, the homotopy methods have some limitations, too. For instance, as the coefficient of the nonlinear term increases, the homotopy techniques need more iterations to obtain accurate results, and this procedure leads to complicated equations which cannot be simply solved. Here, a new analytical approach is presented for solving the governing strongly nonlinear damped equation. In this new approach using the basic concepts of the classical perturbation methods together with the fundamentals of Fourier expansion with an embedding parameter which is considered as a small parameter ($0 < \varepsilon << 1$), and assuming the time dependent relation for the frequency, the strongly nonlinear damped equation is solved. This presented analytical approach provides a valid asymptotic solution for any positive coefficient of the nonlinear term, and this is the main advantage of the new method over the other available methods in the literature.

By making the change of variable, $\beta = \varepsilon\gamma$, Eq. (17) can be rewritten as below:

$$\ddot{q}(t) + q(t) + \varepsilon\gamma q(t)^3 + \alpha\dot{q}(t) = 0 \tag{19}$$

where ε is the small embedding parameter ($0 < \varepsilon << 1$). Just like the classical perturbation methods, the solution of Eq. (19) is considered as:

$$q(t) = q_0(t) + \varepsilon q_1(t) + O(\varepsilon^2) \tag{20}$$

Substituting Eq. (20) into Eq. (19) and collecting coefficients of equal powers of ε and setting each of the coefficients of like powers of ε equal to zero, the differential equations for q_i 's , $i = 0,1,2,\ldots$ become:

$$\varepsilon^0 : \ddot{q}_0(t) + \alpha\dot{q}_0(t) + q_0(t) = 0 \tag{21}$$

$$\varepsilon^1 : \ddot{q}_1(t) + \alpha\dot{q}_1(t) + q_1(t) = -\gamma q_0(t)^3 \tag{22}$$

$$\vdots$$

By solving Eq. (21), the main part of solution can be obtained as:

$$q_0(t) = A\cos(\Omega t + \theta) \tag{23}$$

Neglecting the homogenous solution of Eq. (22) and using truncated Fourier expansion, one reaches the following solution [38]:

$$q_1(t) = A_0 + \sum_{n=2}^{N} A_n \cos n(\Omega t) + B_n \sin n(\Omega t) \tag{24}$$

In Eq. (19) the term $q(t) + \beta q(t)^3$ is the restoring force, where β is always positive for a clamped-clamped microbeam and the nonlinear term acts as a hardening spring. The assumed form for $q(t)$ can be simplified by considering the symmetry of the nonlinear restoring force. First, Hayashi [39] pointed out that under circumstances when the nonlinearity is symmetric, i.e. when the restoring force is odd, A_0 can be discarded. Second, it was demonstrated by Urabe [40], numerically and theoretically, that the even harmonic components in Eq. (24) are zero. Therefore, the approximate solution is simplified to:

$$q_1(t) = \sum_{n=3,5,\ldots} C_n \cos n(\Omega t + \theta) \tag{25}$$

Substituting Eqs. (23) and (24) into Eq. (20) and making the change of variable, $\tau = \Omega t$, result in:

$$q(t) = A\cos(\tau + \theta) + \varepsilon\left[\sum_{n=3,5,\ldots} \{C_n \cos n(\tau + \theta)\}\right] + O(\varepsilon^2) \tag{26}$$

where θ is a constant and $0 \le \alpha \le O(\varepsilon)$. Since α is a small parameter, then the amplitude and the amplitude-dependent frequency will vary slowly with the time. Thus, the coefficients A and C_n in Eq. (26), are functions of the time.

Considering $n = 3$, one reaches:

$$q = A\cos(\tau + \theta) + \varepsilon\left[C_3 \cos 3(\tau + \theta) + \cdots\right] + O(\varepsilon^2) \tag{27}$$

As mentioned before, due to the damping effect, the nonlinear frequency will be time dependent. So, as a first approximation, let:

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = \Omega_0 + \varepsilon\Omega_1 + O(\varepsilon^2) \tag{28}$$

Using the change of variables as below:

$$\psi = \alpha t \tag{29}$$

$$Q = \frac{3}{4}\beta \tag{30}$$

the first time derivative of q , will be:

$$\dot{q} = \alpha \frac{\mathrm{d}A}{\mathrm{d}\psi}\cos(\tau+\theta) - (\Omega_0 + \varepsilon\Omega_1)A\sin(\tau+\theta) - \varepsilon\left[\alpha\frac{\mathrm{d}C_3}{\mathrm{d}\psi}\cos 3(\tau+\theta) - 3(\Omega_0+\varepsilon\Omega_1)C_3\sin 3(\tau+\theta)\right] + O(\varepsilon^2) \tag{31}$$

Also, for the second derivative:

$$\begin{aligned}\ddot{q} &= \alpha^2\frac{\mathrm{d}^2A}{\mathrm{d}\psi^2}\cos(\tau+\theta) - 2\alpha(\Omega_0+\varepsilon\Omega_1)\frac{\mathrm{d}A}{\mathrm{d}\psi}\sin(\tau+\theta) - \alpha\left(\frac{\mathrm{d}\Omega_0}{\mathrm{d}\psi} + \varepsilon\frac{\mathrm{d}\Omega_1}{\mathrm{d}\psi}\right)A\sin(\tau+\theta) - \\ &A(\Omega_0+\varepsilon\Omega_1)^2\cos(\tau+\theta) + \varepsilon\left[\alpha^2\frac{\mathrm{d}^2C_3}{\mathrm{d}\psi^2}\cos 3(\tau+\theta) - 6\alpha\frac{\mathrm{d}C_3}{\mathrm{d}\psi}(\Omega_0+\varepsilon\Omega_1)\sin 3(\tau+\theta) - \right.\\ &\left. 3\alpha\left(\frac{\mathrm{d}\Omega_0}{\mathrm{d}\psi} + \varepsilon\frac{\mathrm{d}\Omega_1}{\mathrm{d}\psi}\right)C_3\sin 3(\tau+\theta) - 9(\Omega_0+\varepsilon\Omega_1)^2C_3\cos 3(\tau+\theta)\right] + O(\varepsilon^2)\end{aligned} \tag{32}$$

Since the embedding parameter ($0<\varepsilon<<1$) and the damping coefficient ($0\le\alpha\le O(\varepsilon)$) are small parameters, the terms including $\varepsilon\alpha$, α^2 and $O(\varepsilon^2)$ will be negligible in comparison with the other terms. So:

$$q = A\cos(\tau+\theta) + \varepsilon C_3\cos 3(\tau+\theta) \tag{33}$$

$$\dot{q} = -\Omega_0 A\sin(\tau+\theta) + \alpha\frac{\mathrm{d}A}{\mathrm{d}\psi}\cos(\tau+\theta) - \varepsilon\left[\Omega_1 A\sin(\tau+\theta) + 3\Omega_0 C_3\sin 3(\tau+\theta)\right] \tag{34}$$

$$\ddot{q} = -\Omega_0^2 A\cos(\tau+\theta) - \alpha\left(2\Omega_0\frac{\mathrm{d}A}{\mathrm{d}\psi} + A\frac{\mathrm{d}\Omega_0}{\mathrm{d}\psi}\right)\sin(\tau+\theta) - \varepsilon\left[2\Omega_0\Omega_1 A\cos(\tau+\theta) + 9\Omega_0^2 C_3\cos 3(\tau+\theta)\right] \tag{35}$$

and

$$\beta q^3 = QA^3\cos(\tau+\theta) + \frac{1}{3}QA^3\cos 3(\tau+\theta) + \varepsilon\left[QA^2C_3\cos(\tau+\theta) + 2QA^2C_3\cos 3(\tau+\theta)\right] + \varepsilon\left[\sum_{n\ge 5}\left\{E_n\cos n(\tau+\theta)\right\}\right] \tag{36}$$

Substituting Eqs. (33) to (36) into Eq. (17) and setting the coefficients of $\cos(\tau+\theta)$, $\sin(\tau+\theta)$, $\varepsilon\cos(\tau+\theta)$, and $\cos 3(\tau+\theta)$ to zero, a system of perturbed equations is obtained as:

$$\cos(\tau+\theta):\quad \left(1-\Omega_0^2\right)A + QA^3 = 0 \tag{37}$$

$$\sin(\tau+\theta):\quad 2\Omega_0\frac{\mathrm{d}A}{\mathrm{d}\psi} + A\frac{\mathrm{d}\Omega_0}{\mathrm{d}\psi} + \Omega_0 A = 0 \tag{38}$$

$$\varepsilon\cos(\tau+\theta):\quad -2\Omega_0\Omega_1 A + QA^2C_3 = 0 \tag{39}$$

$$\cos 3(\tau+\theta):\quad \varepsilon\left(1-9\Omega_0^2+2QA^2\right)C_3 + \frac{1}{3}QA^3 = 0 \tag{40}$$

Eq. (37), leads to:

$$\Omega_0^2 = 1 + QA^2 \tag{41}$$

From Eqs. (38) and (41), one obtains:

$$\left(\frac{2}{A} + \frac{QA}{1+QA^2}\right)\frac{\mathrm{d}A}{\mathrm{d}\psi} = -1 \tag{42}$$

By making the change of variable $\Gamma = A^2$, and then integrating Eq. (42) yields:

$$\Gamma^3 + \frac{1}{Q}\Gamma^2 = Me^{-2\psi} > 0 \tag{43}$$

where M is a constant to be determined from the initial conditions.

Defining $\eta = \frac{27}{2}Q^3 Me^{-2\psi} - 1$, it can be easily verified that Eq. (43) has one real root for $1 < \eta < \infty$ and one positive and two negative roots for $-1 \le \eta \le 1$. For the case of $1 < \eta < \infty$, i. e. when $0 \le t \le \frac{1}{2\alpha}\ln(\frac{27}{4}MQ^3)$, the real root is given by:

$$\Gamma = \frac{1}{3Q}\left[\sqrt[3]{\eta + \sqrt{\eta^2 - 1}} + \sqrt[3]{\eta - \sqrt{\eta^2 - 1}} - 1\right] \tag{44}$$

Also, for the case of $-1 \le \eta \le 1$, i. e. when $\frac{1}{2\alpha}\ln(\frac{27}{4}MQ^3) \le t < \infty$, the positive root is given by:

$$\Gamma = \frac{1}{3Q}\left[2\cos(\frac{\cos^{-1}\eta}{3}) - 1\right] \tag{45}$$

It is worth mentioning that when $\alpha \to 0$, Eq. (44) is valid for $t \ge 0$.

Eqs. (39), (40) and (41) lead to:

$$\varepsilon\Omega_1 = \frac{Q^2\Gamma^2}{6\Omega_0\left(7\Omega_0^2 + 1\right)} = \frac{\left(\Omega_0^2 - 1\right)^2}{6\Omega_0\left(7\Omega_0^2 + 1\right)} \tag{46}$$

and from Eqs. (41) and (42) one obtains:

$$\frac{\mathrm{d}\psi}{\mathrm{d}\Omega_0} = -\frac{3\Omega_0^2 - 1}{\Omega_0\left(\Omega_0^2 - 1\right)} \tag{47}$$

Hence:

$$
\begin{aligned}
\tau &= \int_0^{\tau} \left(\Omega_0 + \varepsilon\Omega_1\right) \mathrm{dt} = \frac{1}{\alpha} \int_{\Omega_0(0)}^{\Omega_0} \left(\Omega_0 + \varepsilon\Omega_1\right) \frac{\mathrm{d}\psi}{\mathrm{d}\Omega_0} \mathrm{d}\Omega_0 \\
&= -\frac{1}{\alpha} \int_{\Omega_0(0)}^{\Omega_0} \left[\frac{3\Omega_0^2 - 1}{\Omega_0^2 - 1} + \frac{\left(3\Omega_0^2 - 1\right)\left(\Omega_0^2 - 1\right)}{6\Omega_0^2 \left(7\Omega_0^2 + 1\right)} \right] \mathrm{d}\Omega_0 \\
&= -\frac{1}{\alpha} \int_{\Omega_0(0)}^{\Omega_0} \left[\frac{43}{14} + \frac{1}{6\Omega_0^2} + \frac{2}{\Omega_0^2 - 1} - \frac{40}{21\left(7\Omega_0^2 + 1\right)} \right] \mathrm{d}\Omega_0
\end{aligned} \tag{48}
$$

therefore,

$$
\begin{aligned}
\tau = \frac{1}{\alpha} \Bigg\{ & \frac{43}{14} (\Omega_0(0) - \Omega_0) - \frac{1}{6} \left(\frac{1}{\Omega_0(0)} - \frac{1}{\Omega_0} \right) - \\
& \ln \left[\frac{\left(\Omega_0 - 1\right)\left(\Omega_0(0) + 1\right)}{\left(\Omega_0 + 1\right)\left(\Omega_0(0) - 1\right)} \right] + \frac{40}{21\sqrt{7}} \left[\arctan(\sqrt{7}\Omega_0) - \arctan(\sqrt{7}\Omega_0(0)) \right] \Bigg\}
\end{aligned} \tag{49}
$$

From Eq. (42) we have:

$$
\Omega_0^2(0) = 1 + Q\Gamma(0) = 1 + \frac{3}{4} \beta A^2(0) \tag{50}
$$

Integrating Eq. (21), neglecting $O(\varepsilon^2)$ in this equation and using Eq. (46), it should be noted that when $\alpha \to 0$, then:

$$
\tau \to \left[\Omega_0(0) + \frac{\left(\Omega_0^2(0) - 1\right)^2}{6\Omega_0(0)\left(7\Omega_0^2(0) + 1\right)} \right] t \tag{51}
$$

To determine the constants $A(0)$ and θ, considering the initial conditions $q(0) = a_0$ and $\dot{q}(0) = 0$, and using the Eqs. (33) and (34), one reaches:

$$
A(0)\cos\theta + \varepsilon C_3(0)\cos 3\theta = q(0) = a_0 \tag{52}
$$

$$
-\Omega_0(0)A(0)\sin\theta + \alpha \frac{\mathrm{d}A}{\mathrm{d}\psi}\Big|_{\psi=0} \cos\theta - \varepsilon A(0)\Omega_1(0)\sin\theta - 3\varepsilon C_3(0)\Omega_0(0)\sin 3\theta = \dot{q}(0) = 0 \tag{53}
$$

Where, $\frac{\mathrm{d}A}{\mathrm{d}\psi}\Big|_{\psi=0}$ and M may be obtained through Eqs. (42) and (43), respectively, as:

$$
\frac{\mathrm{d}A}{\mathrm{d}\psi}\Big|_{\psi=0} = -\frac{A(0)\left(1 + QA^2(0)\right)}{2 + 3QA^2(0)} \tag{54}
$$

$$
M = A^6(0) + \frac{4A^4(0)}{3\beta} \tag{55}
$$

So the first approximation will be:

$$q(t) = A\cos(\tau+\theta) + \frac{\beta A^3}{32+27\beta A^2}\cos 3(\tau+\theta) + O(\varepsilon^2) \tag{56}$$

where A is obtained from Eq. (44) and/or Eq. (45), and τ is obtained from Eq. (49).
According to Eq. (49), the nonlinear natural frequency is a function of vibration amplitude and the damping of the system.

4. Results and discussion

For demonstrating the accuracy of the new approach, as a case study, the method is applied to the doubly clamped microbeam. The maximum value of βa_0^2 for which the beam does not exceed the linear elastic limit, depends on the beam characteristics, the boundary conditions, the mode shape and the initial conditions. It is worth noting that the bigger values of βa_0^2 may be occurred in some other nonlinear equations [4].
Since the studies about the strongly nonlinear vibration of microbeams are restricted to undamped cases, here in order to make it possible to validate the method, the comparison is made for such a case. Table 1 summarizes the comparison between the results for the nondimensional nonlinear resonance frequency obtained through the new approach and the other methods reported in the literature for a wide range of vibration amplitudes. Table 1 shows an excellent agreement between the results obtained through the presented method and the exact solution. It is worth mentioning that by qualitative analysis of conservative systems and integrating the level curve in phase plane for a given total energy level, Nayfeh obtained the exact value for the system period. After some manipulations he reached the system period as [4, 7]:

$$T_{ex} = \frac{4}{\sqrt{\omega_0^2+\beta a_0^2}} F(\frac{\pi}{2}, k) \tag{57}$$

where $F(\frac{\pi}{2}, k)$ is called a complete elliptic integral of the first kind and its value is [4, 7]:

$$F(\frac{\pi}{2}, k) = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}x}{\sqrt{1-k^2\sin^2 x}} \tag{58}$$

and

$$k = \sqrt{\frac{\beta a_0^2}{2(\omega_0^2+\beta a_0^2)}} \tag{59}$$

So, the exact frequency of the free oscillations reads:

$$\Omega_{ex} = \frac{2\pi}{T_{ex}} \tag{60}$$

Table 1. Comparison of nonlinear frequencies obtained via different methods for small values of βa_0^2 ; (= 0.25, 1, 2.25)

The Solution Method	βa_0^2		
	0.25	1	2.25
The exact solution	1.0892	1.3178	1.6257
The first order approximation of the new method	1.0889	1.3164	1.6240
First order HPM	1.0897	1.3229	1.6394
First order VIM	1.0903	1.3278	1.6519
Azrar-Second order [41]	1.0897	1.3229	1.6394
HAM [42]	1.0897	1.3228	1.6393
Qaisi [43]	1.0897	1.3229	1.6394
Ritz method [44]	1.0897	1.3229	1.6394

Table 2 indicates the nonlinear frequencies obtained by the new presented approach, the VIM, the HPM and the exact solution for different values of nondimensional vibration amplitudes. From Table 2 it can be seen that there is a good agreement between the results obtained from the new method and the exact solution for larger values of βa_0^2 ,.

Table 2. Nonlinear frequencies obtained by the new method, the VIM, the HPM and the exact solution for some large values of βa_0^2 .

βa_0^2	Nondimensional Nonlinear Frequency			
	Exact solution	First Order VIM	First order HPM	First order of the new method
1	1.317776065	1.327715663	1.322875656	1.316378979
10	2.866640137	2.957903561	2.915475948	2.868818098
100	8.533586191	8.873915594	8.717797888	8.553506457
1000	26.81073847	27.90602205	27.40437921	26.87909003
10000	84.72747996	88.19719974	86.60831369	84.94536879

The maximum relative error of the system period, $RE\,(\%)$, is defined as [4, 45]:

$$RE\,(\%) = \left| \lim_{\beta \to \infty} \frac{T - T_{ex}}{T_{ex}} \right| \times 100 \tag{57}$$

For an undamped system, as β approaches infinity, the maximum relative error for the first order approximation through the new approach and the first order approximation via VIM and HPM are obtained 0.26%, 4.1% and 2.22%, respectively. Consequently, the first order of the new method is more accurate than the others.

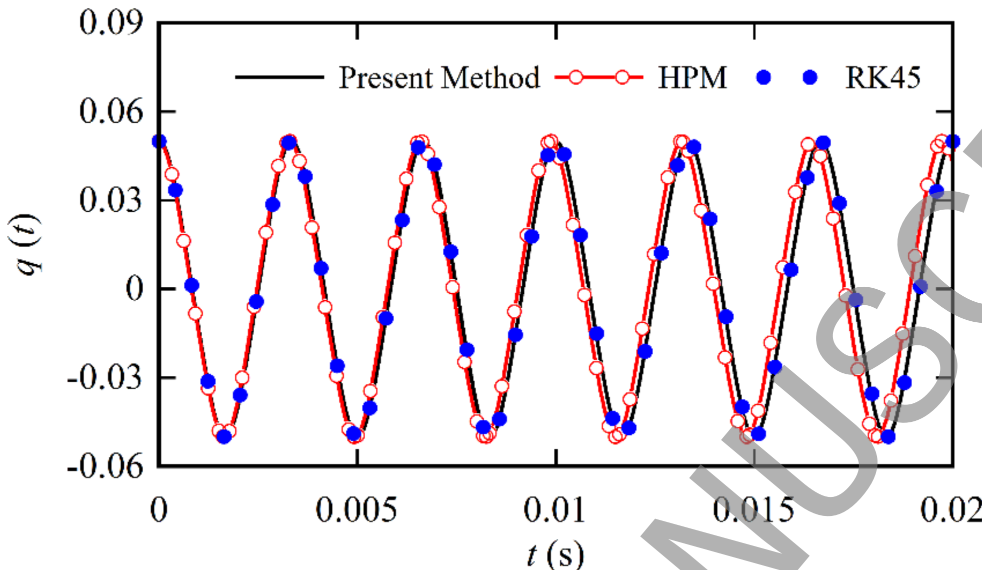

Fig. 2. Comparison among the responses of the clamped-clamped microbeam for $a_0 = 0.05$ obtained by the new method, HPM and the numerical method.

As mentioned before, the homotopy technique is not usually employed to solve a damped nonlinear equation, because it leads to equations that are too complicate to be solved analytically. So, in order to demonstrate the accuracy and the effectiveness of the new approach, the damping coefficient of the beam is set to zero and the beam response at its mid-span is obtained by the first order of the new method. Then the result is compared with that obtained by the first order of HPM and the fourth-order Runge–Kutta method. Fig. 2 illustrates the response of a clamped-clamped microbeam with a nondimensional amplitude of 0.05 and the pretension load of $N_0 = 0$. For this case, $\omega_0^2 = 1$ and $\beta = 1941.3$, ($\beta >> 1$). It can be readily deduced from Fig. 2 that the response obtained by the first order of the new method is more accurate than that obtained by the first order of HPM. Moreover, the figure reveals that the first order of the new method can follow the RK45 more accurate than the first order of HPM.

According to Eqs. (28) and (49), the new method provides a relation which relates the system nonlinear frequency to the vibration amplitude and the time. Due to the damping effect, the vibration amplitude decreases by the time, so one can illustrate the variations of the nonlinear frequency, the vibration amplitude and the time as a three dimensional plot. Fig. 3 shows the variation of the frequency ratio, Ω/ω_0, against the nondimensional amplitude and time. As this figure indicates, for small values of t, due to the larger vibration amplitudes, the frequency ratio is high. As the time goes on, the vibration amplitude decreases and therefore, the nonlinear frequency tends to the linear one. In other words, at the larger amplitudes, the effect of nonlinearity has a significant influence on the vibration behaviour of the microbeam.

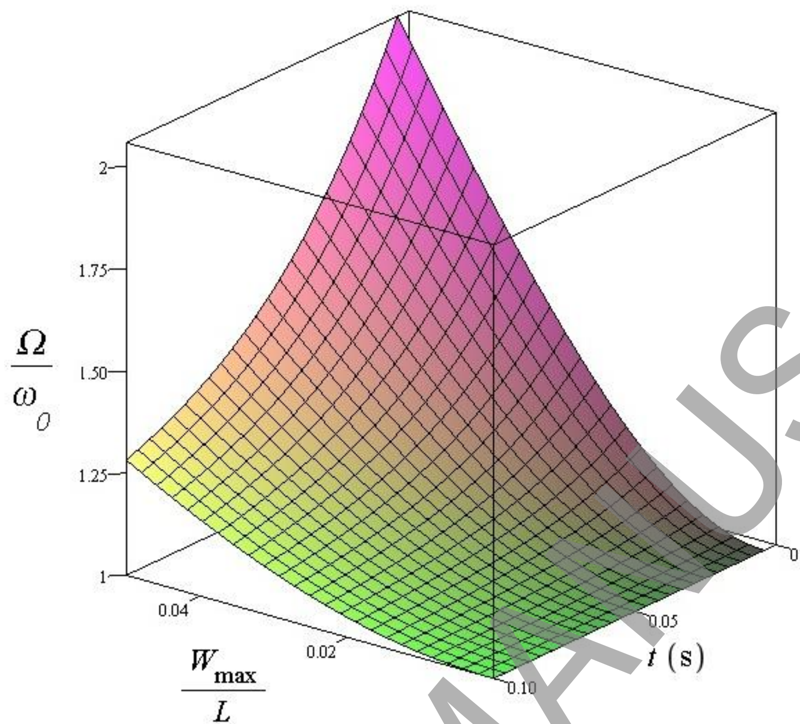

Fig. 3. Variation of the frequency ratio versus the nondimensional vibration amplitude and time.

To show both accuracy and effectiveness of the first order of the new approach, the damped responses of the system at the mid-span of the microbeam is obtained by the new method and the results are compared with those obtained by the fourth-order Runge–Kutta method. Figs. 4 (a) to 4 (c) show the free responses of a doubly clamped microbeam for various values of nondimensional amplitude in the absence of N_0. As it can be seen from these figures, for a wide range of vibration amplitudes, the responses obtained by the first order of the new method follow the responses obtained by the RK45 with a good accuracy. Also, these figures reveal that even the first order approximation of the presented approach is in excellent agreement with the numerical solution.

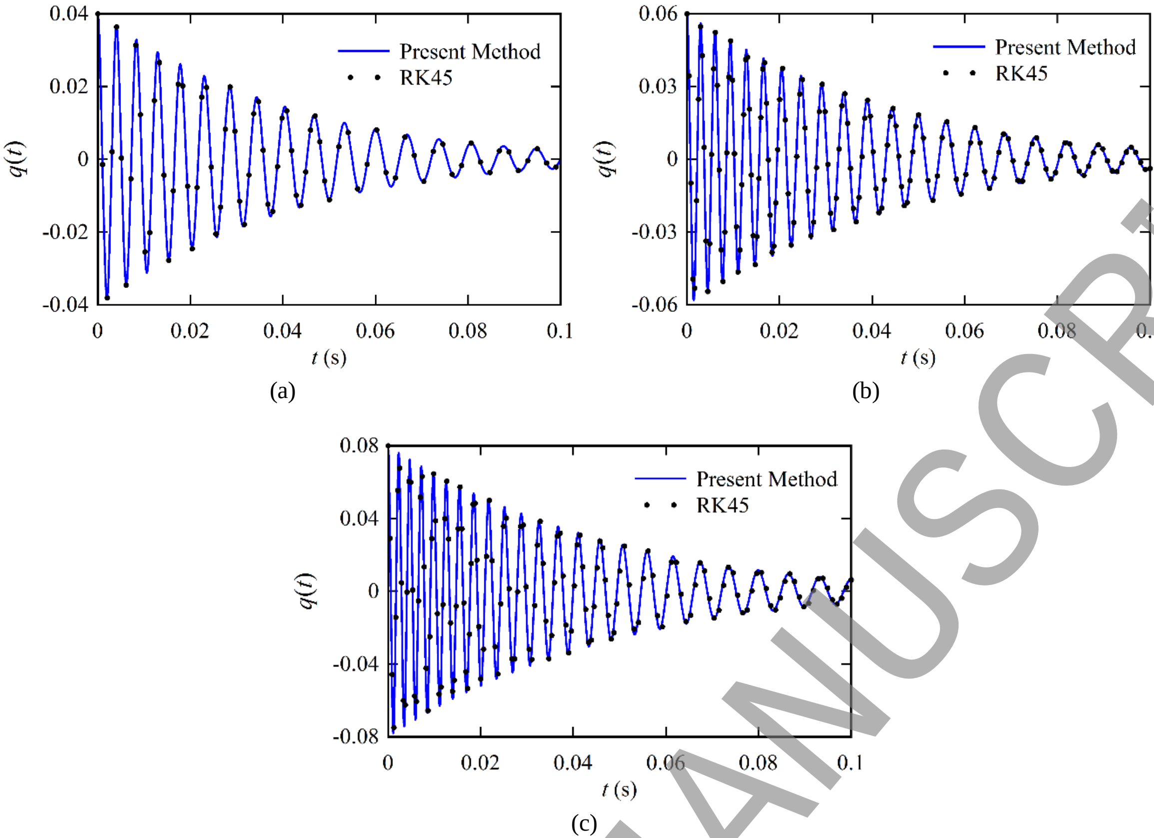

Fig. 4. Comparison between the responses of the clamped-clamped microbeam obtained by the first order of the new method and the numerical method for nondimensional vibration amplitudes of: (a) $a_0 = 0.04$ (b) $a_0 = 0.06$ (c) $a_0 = 0.08$.

Fig. 5 illustrates the effect of the beam internal damping coefficient, C_S , on the frequency ratio of the clamped-clamped beam for various values of vibration amplitudes. It can be seen that by increasing the vibration amplitude, the frequency ratio, Ω/ω_0, increases. Also, at the larger vibration amplitudes, the effect of the beam internal damping coefficient on the frequency ratio becomes more significant. Moreover, for a given vibration amplitude, all the curves start from a common point and as the time goes over, they move away from each other due to the damping effect.

Fig. 6 shows the effect of the microbeam material length scale parameter, l , on the frequency ratio of the clamped-clamped beam for various values of vibration amplitudes. The figure reveals that by increasing the vibration amplitude, the frequency ratio, Ω/ω_0, increases. Moreover, the effect of the microbeam material length scale parameter on the frequency ratio is very considerable at the larger vibration amplitudes.

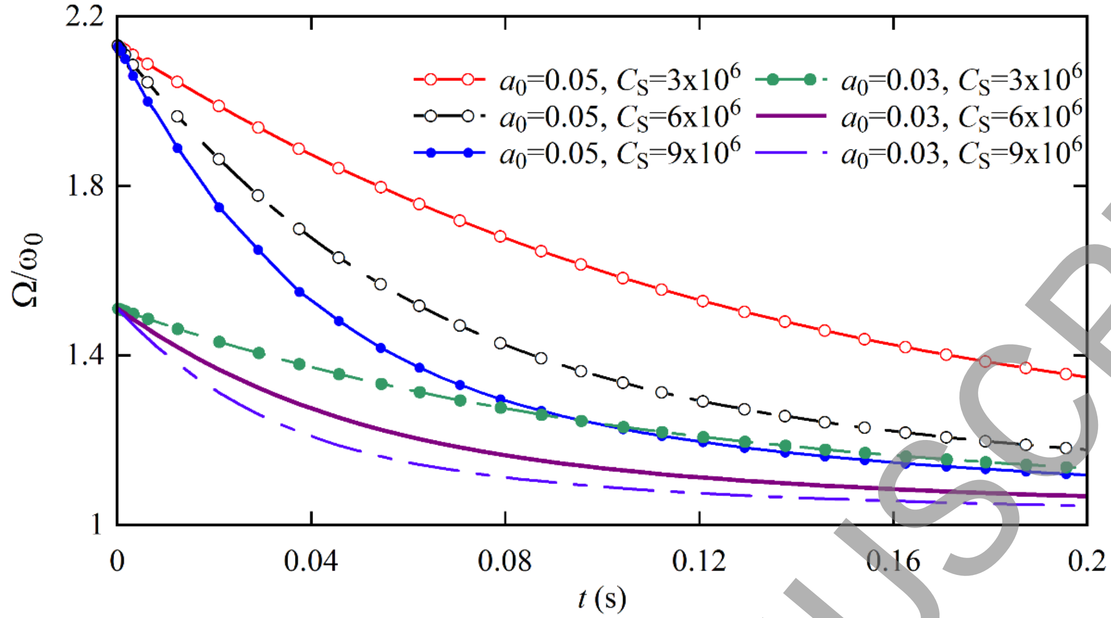

Fig. 5. Variation of the frequency ratio, Ω/ω_0 , versus time for various values of the microbeam structural damping, C_S .

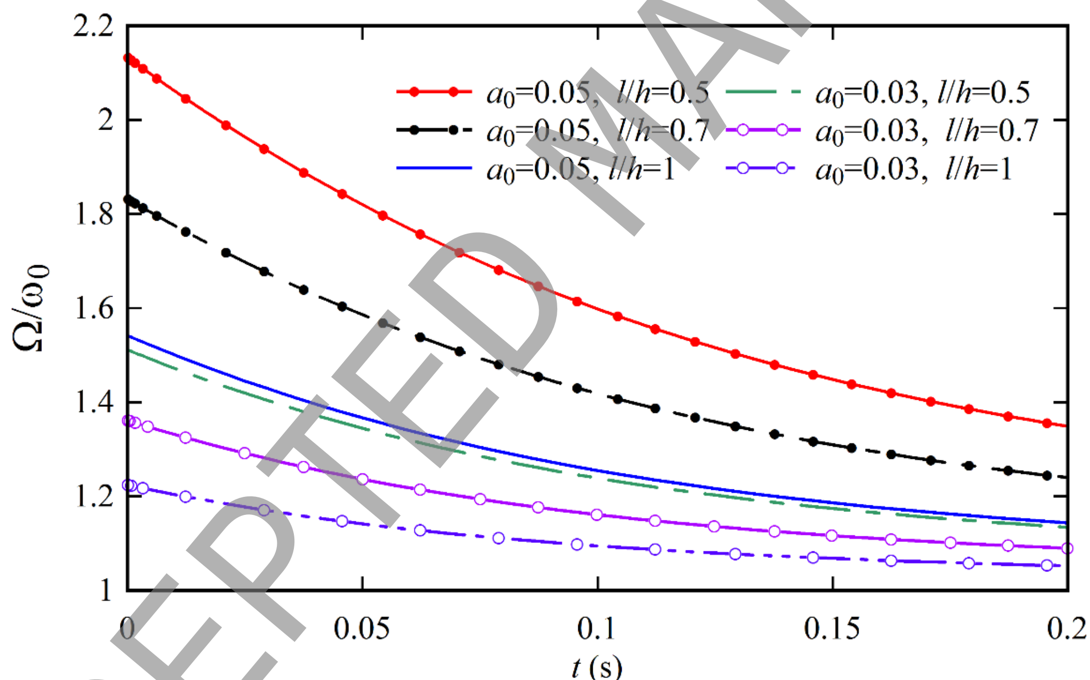

Fig. 6. Variation of the frequency ratio, Ω/ω_0 , versus time for various values of the microbeam material length scale parameter, l .

5. Conclusion

In this paper, strongly nonlinear free vibration behaviour of a microbeam considering the structural damping effect has been studied. The effect of mid-plane stretching of the microbeam on its nonlinear vibrational behaviour is considered on the basis of modified couple stress theory. Because of the large coefficient of the nonlinear term and due to existence of the damping effect, none of the traditional perturbation methods leads to a valid solution. Also, there are many difficulties encountered in the application of the homotopy techniques when the damping effect is taken in to account in the strongly nonlinear damped systems. To overcome these limitations, a new analytical approach is presented for solving the strongly nonlinear damped system. This new method is based on the classical perturbation methods and the fundamentals of the Fourier expansion with an embedding nondimensional parameter. To apply the method, it is assumed that the nonlinear frequency is time dependent. Despite the classical perturbation methods, the new approach does not depend upon the assumption of small parameter and it is applicable to a damped system for a wide range of vibration amplitudes.

The comparison between the time responses of the system obtained by the first order approximate solution of the new method and the numerical technique demonstrates the high accuracy of the new method. Moreover, in order to demonstrate the capability of the method, the results are also compared with those obtained by the other recently introduced methods, e.g. HPM and VIM, as well as the numerical method. Finally, it is worth stating that the presented approach can help handle the situations of high nonlinearity occurring in the damped nonlinear systems.

6. References

[1] M.I. Younis, MEMS linear and nonlinear statics and dynamics, Springer Science & Business Media, 2011.

[2] M. Ghadiri, N. Shafiei, S.A. Mousavi, Vibration analysis of a rotating functionally graded tapered microbeam based on the modified couple stress theory by DQEM, Applied Physics A, 122(9) (2016) 837.

[3] A.M. Bataineh, M.I. Younis, Dynamics of a clamped–clamped microbeam resonator considering fabrication imperfections, Microsystem Technologies, 21(11) (2015) 2425-2434.

[4] M. Rezaee, M. Minaei, A theoretical and experimental investigation on large amplitude free vibration behavior of a pretensioned beam with clamped–clamped ends using modified homotopy perturbation method, Proceedings of the Institution of Mechanical Engineers, Part C: Journal of Mechanical Engineering Science, 230(10) (2016) 1615-1625.

[5] P. Roura, J.A. Gonzalez, Towards a more realistic description of swing pumping due to the exchange of angular momentum, European Journal of Physics, 31(5) (2010) 1195.

[6] A.H. Nayfeh, D.T. Mook, Nonlinear oscillations, 1979.

[7] A.H. Nayfeh, Introduction to perturbation techniques, 1981.

[8] S.J. Liao, A.T. Chwang, Application of homotopy analysis method in nonlinear oscillations, ASME Journal of Applied Mechanics 65 (1998) 914.

[9] J.H. He, An elementary introduction to the homotopy perturbation method, Computers and Mathematics with Applications, 57 (2009) 410–412.

[10] J.H. He, Homotopy perturbation method with an auxiliary term, Hindawi Publishing Corporation-Abstract and Applied Analysis, (2012).
[11] J.H. He, New interpretation of homotopy perturbation method, International Journal of Modern Physics B, 20(18) (2006) 2561–2568.
[12] J.H. He, Homotopy perturbation technique, Computer Methods in Applied Mechanical Engineering, 178 (1999) 257–262.
[13] J.H. He, A coupling method of homotopy technique and perturbation technique for nonlinear problems, International Journal of Nonlinear Mechanics 35 (1999) 37–43.
[14] S. Momani, G.H. Erjaee, M.H. Alnasr, The modified homotopy perturbation method for solving strongly nonlinear oscillators, Computers and Mathematics with Applications, 58 (2009) 2209–2220.
[15] J.H. He, Comment on He's frequency formulation for nonlinear oscillators, European Journal of Physics, 29 (2008) 19–22.
[16] R.E. Mickens, Mathematical and numerical study of the Duffing-harmonic oscillator, Journal of Sound and Vibration, 244 (2001) 563–567.
[17] M.K. Yazdi, A. Mirzabeigy, H. Abdollahi, Nonlinear oscillators with non-polynomial and discontinuous elastic restoring forces, Nonlinear Science Letters A, 3 (2012) 48–53.
[18] J.H. He, Preliminary report on the energy balance for nonlinear oscillations, Mechanics Research Communications, 29 (2002) 107-111.
[19] J.H. He, Max-min approach to nonlinear oscillators, International Journal of Nonlinear Sciences and Numerical Simulation, 9 (2008) 207–210.
[20] J.H. He, Modified Lindstedt-Poincare methods for some strongly non-linear oscillations Part I: expansion of a constant, International Journal of Non-Linear Mechanics, 37 (2002) 309-314.
[21] J.H. He, A variational iteration method to Duffing equation (in Chinese), Chinese Journal of Computational Physics, 16 (1999) 121–133.
[22] J.H. He, Variational iteration method: A kind of nonlinear analytical technique, International Journal of Nonlinear Mechanics, 34 (1999) 699–708.
[23] M.A. Noor, W.A. Khan, New iterative methods for solving nonlinear equation by using homotopy perturbation method, Applied Mathematics and Computation, 219 (2012) 3565–3574.
[24] J.H. He, A review on some new recently developed nonlinear analytical techniques, International Journal of Nonlinear Science and Numerical Simulation 1(1) (2000) 51–70.
[25] A.H. Nayfeh, Problems in pertubation, 1985.
[26] J.H. He, A new perturbation technique which is also valid for large parameters, Journal of Sound and Vibration, 229(5) (2000) 1257-1263.
[27] Y. Khan, F. Austin, Application of the Laplace decomposition methodto nonlinear homogeneous and non-homogenous advection equations, Zeitschriftfür Naturforschung A, 65a (2010) 849–853.
[28] Y. Khan, Q. Wu, Homotopy perturbation transform method for nonlinear equations using He's polynomials, Computers and Mathematics with Applications, 61 (2011) 1963–1967.
[29] S. Nourazar, A. Mirzabeigy, Approximate solution for nonlinear Duffing oscillator with damping effect using the modified differential transform method, Scientia Iranica, 20(2) (2013) 364–368.
[30] M. Akbarzadea, Y. Khan, Dynamic model of large amplitude non-linear oscillations arising in the structural engineering: Analytical solutions, Mathematical and Computer Modelling, 55 (2012) 480–489.
[31] M. Ghadiri, N. Shafiei, Nonlinear bending vibration of a rotating nanobeam based on nonlocal Eringen's theory using differential quadrature method, Microsystem Technologies, 22(12) (2016) 2853-2867.
[32] F. Yang, A. Chong, D.C.C. Lam, P. Tong, Couple stress based strain gradient theory for elasticity, International Journal of Solids and Structures, 39(10) (2002) 2731-2743.
[33] S. Kong, S. Zhou, Z. Nie, K. Wang, The size-dependent natural frequency of Bernoulli–Euler micro-beams, International Journal of Engineering Science, 46(5) (2008) 427-437.
[34] M.H. Ghayesh, H. Farokhi, M. Amabili, Coupled nonlinear size-dependent behaviour of microbeams, Applied Physics A, 112(2) (2013) 329-338.

[35] R. Sourki, S. Hoseini, Free vibration analysis of size-dependent cracked microbeam based on the modified couple stress theory, Applied Physics A, 122(4) (2016) 413.
[36] A.E. Abouelregal, A.M. Zenkour, Generalized thermoelastic vibration of a microbeam with an axial force, Microsystem Technologies, 21(7) (2015) 1427-1435.
[37] H.M. Sedighi, A. Reza, J. Zare, Study on the frequency-amplitude relation of beam vibration, International Journal of the Physical Sciences, 6(36) (2011) 8051-8056.
[38] Z.K. Peng, Z.Q. Lang, S.A. Billings, G.R. Tomlinson, Comparisons between harmonic balance and nonlinear output frequency response function in nonlinear system analysis, Journal of Sound and Vibration, 311(1) (2008) 56–73.
[39] C. Hayashi, Subharmonic oscillations in nonlinear systems, Journal of Applied Physics, 24(5) (1953) 521–529.
[40] M. Urabe, Numerical investigation of subharmonic solutions to Duffing's equation, Publications of the Research Institute for Mathematical Sciences, 5(1) (1969) 79–112.
[41] L. Azrar, R. Benamar, R.G. White, A semi-analytical approach to the nonlinear dynamic response problem of S–S and C–C beams at large vibration amplitudes part I: General theory and application to the single mode approach to free and forced vibration analysis, Journal of Sound and Vibration 224 (1999) 183–207.
[42] T. Pirbodaghi, M.T. Ahmadian, M. Fesanghary, On the homotopy analysis method for nonlinear vibration of beams, Mechanics Research Communications 36 (2009) 143–148.
[43] M.I. Qaisi, Application of the harmonic balance principle to the nonlinear free vibration of beams, Applied Acoustics 40 (1993) 141–151.
[44] R. Lewandowski, Application of the Ritz method to the analysis of nonlinear free vibrations of beams, Journal of Sound and Vibration 224 (1987) 183–207.
[45] J.H. He, Some asymptotic methods for strongly nonlinear equations, International Journal of Modern Physics B, 20(10) (2006) 1141–1199.