5.1 The Unit Circle

Objectives

- The Unit Circle
- Terminal Points on the Unit Circle
- The Reference Number

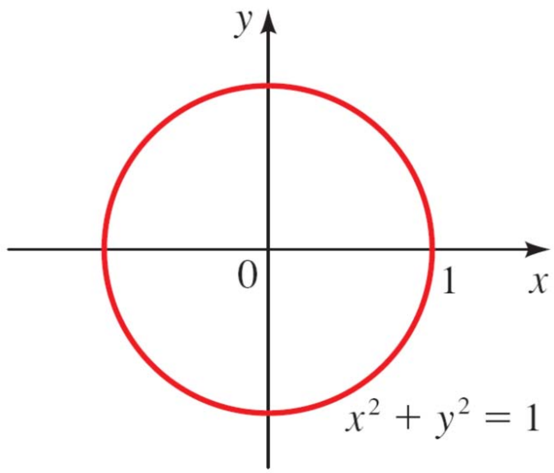
The Unit Circle

In this section we explore some properties of the circle of radius 1 centered at the origin.

The Unit Circle

The Unit Circle

The set of points at a distance 1 from the origin is a circle of radius 1 (see Figure 1).



The unit circle

Figure 1

The Unit Circle

The equation of this circle is $x^2 + y^2 = 1$.

THE UNIT CIRCLE

The **unit circle** is the circle of radius 1 centered at the origin in the *xy*-plane. Its equation is

$$x^2 + y^2 = 1$$

Example 1 – *A Point on the Unit Circle*

Show that the point *P* $\left(\frac{\sqrt{3}}{3}, \frac{\sqrt{6}}{3}\right)$ is on the unit circle.

Solution:

We need to show that this point satisfies the equation of the unit circle, that is, $x^2 + y^2 = 1$.

Since

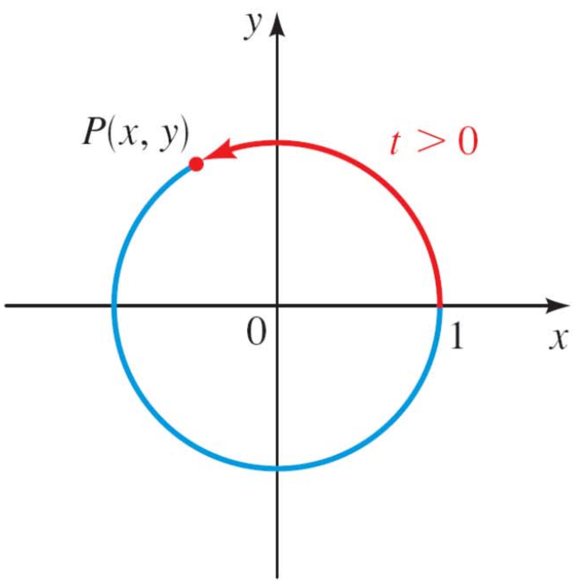
$$\left(\frac{\sqrt{3}}{3}\right)^2 + \left(\frac{\sqrt{6}}{3}\right)^2 = \frac{3}{9} + \frac{6}{9} = 1$$

P is on the unit circle.

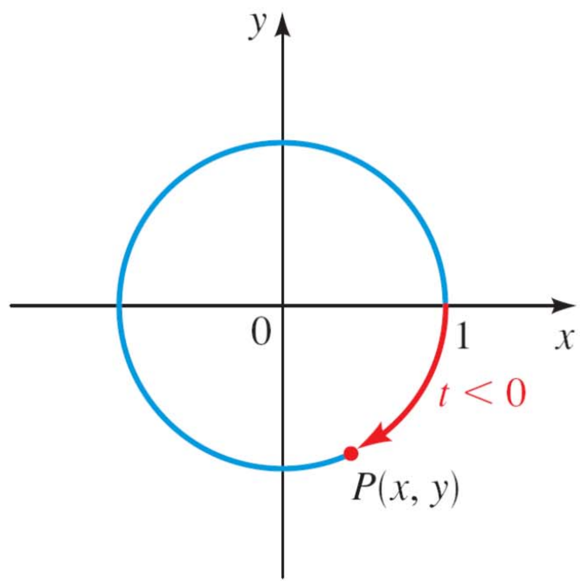
Terminal Points on the Unit Circle

Terminal Points on the Unit Circle

Suppose t is a real number. Let's mark off a distance t along the unit circle, starting at the point (1, 0) and moving in a counterclockwise direction if t is positive or in a clockwise direction if t is negative (Figure 2).



(a) Terminal point $P(x, y)$ determined by $t > 0$



(b) Terminal point $P(x, y)$ determined by $t < 0$

Figure 2

Terminal Points on the Unit Circle

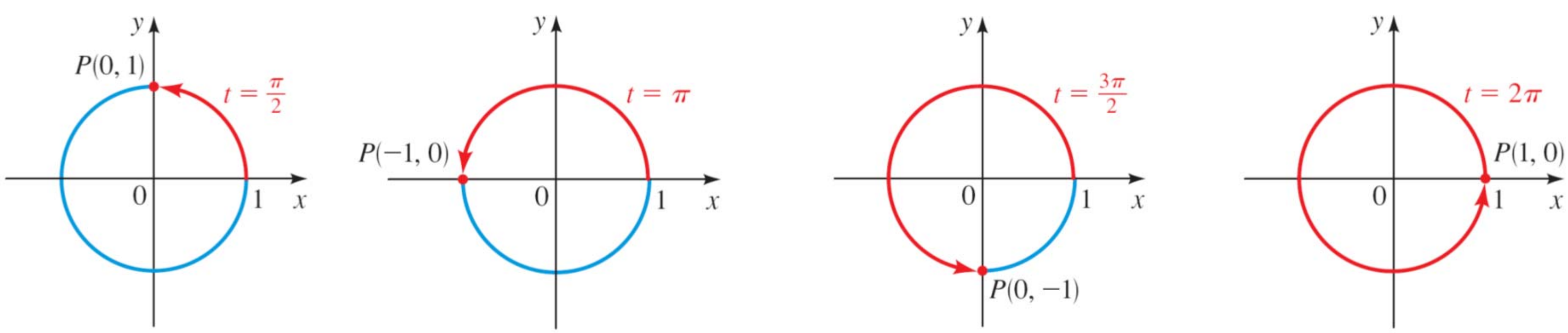
In this way we arrive at a point $P(x, y)$ on the unit circle. The point $P(x, y)$ obtained in this way is called the **terminal point** determined by the real number t.

The circumference of the unit circle is $C = 2\pi(1) = 2\pi$. So if a point starts at (1, 0) and moves counterclockwise all the way around the unit circle and returns to (1, 0), it travels a distance of 2π.

To move halfway around the circle, it travels a distance of $\frac{1}{2}(2\pi) = \pi$. To move a quarter of the distance around the circle, it travels a distance of $\frac{1}{4}(2\pi) = \pi/2$.

Terminal Points on the Unit Circle

Where does the point end up when it travels these distances along the circle? From Figure 3 we see, for example, that when it travels a distance of π starting at (1, 0), its terminal point is (–1, 0).



Terminal points determined by $t = \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, and 2π

Figure 3

Example 3 – *Finding Terminal Points*

Find the terminal point on the unit circle determined by each real number *t*.

(a) $t = 3\pi$ **(b)** $t = -\pi$ **(c)** $t = -\frac{\pi}{2}$

Solution:

From Figure 4 we get the following:

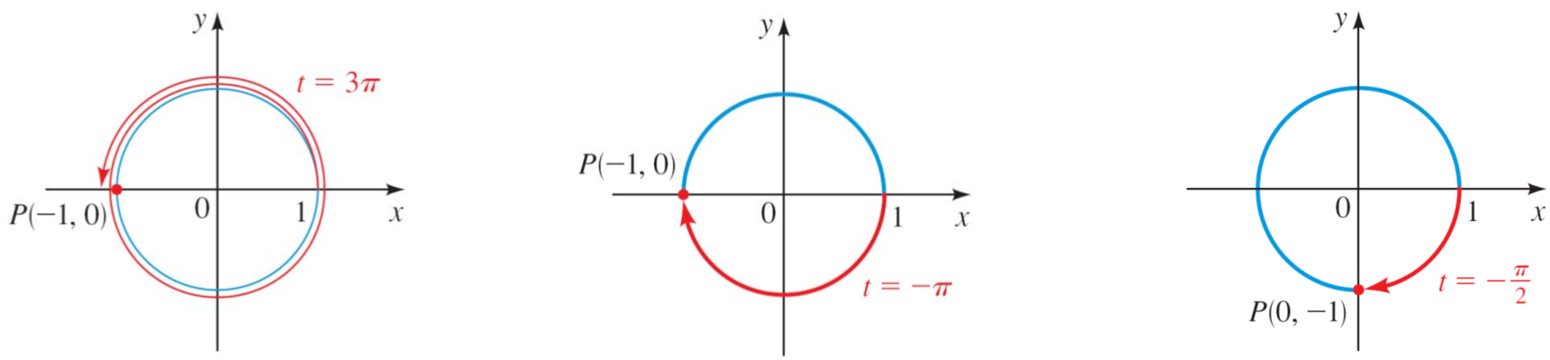


Figure 4

Example 3 – *Solution*

cont'd

(a) The terminal point determined by 3π is (–1, 0).

(b) The terminal point determined by $-\pi$ is (–1, 0).

(c) The terminal point determined by $-\pi/2$ is (0, –1).

Notice that different values of *t* can determine the same terminal point.

Terminal Points on the Unit Circle

The terminal point $P(x, y)$ determined by $t = \pi/4$ is the same distance from (1, 0) as (0, 1) from along the unit circle (see Figure 5).

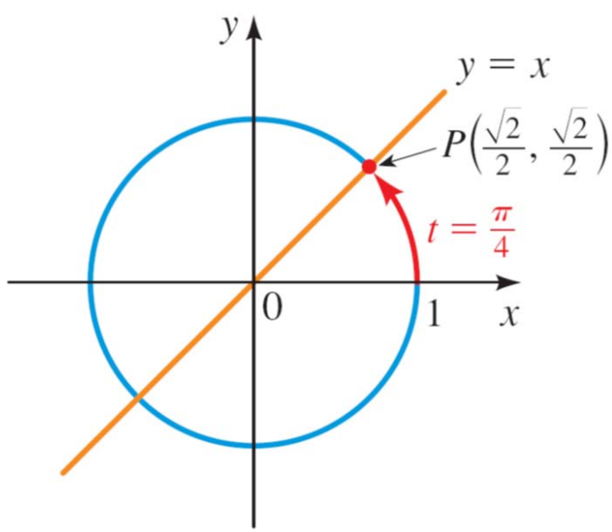


Figure 5

Terminal Points on the Unit Circle

Since the unit circle is symmetric with respect to the line $y = x$, it follows that P lies on the line $y = x$.

So P is the point of intersection (in the first quadrant) of the circle $x^2 + y^2 = 1$ and the line $y = x$.

Substituting x for y in the equation of the circle, we get

$$x^2 + x^2 = 1$$

$$2x^2 = 1 \qquad \text{Combine like terms}$$

Terminal Points on the Unit Circle

$$x^2 = \frac{1}{2}$$ Divide by 2

$$x = \pm\frac{1}{\sqrt{2}}$$ Take square roots

Since *P* is in the first quadrant $x = 1/\sqrt{2}$, and since $y = x$, we have $y = 1/\sqrt{2}$ also.

Thus, the terminal point determined by $\pi/4$ is

$$P\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = P\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

Terminal Points on the Unit Circle

Similar methods can be used to find the terminal points determined by $t = \pi/6$ and $t = \pi/3$. Table 1 and Figure 6 give the terminal points for some special values of t.

t	Terminal point determined by t
0	$(1, 0)$
$\frac{\pi}{6}$	$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
$\frac{\pi}{4}$	$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
$\frac{\pi}{3}$	$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
$\frac{\pi}{2}$	$(0, 1)$

Table 1

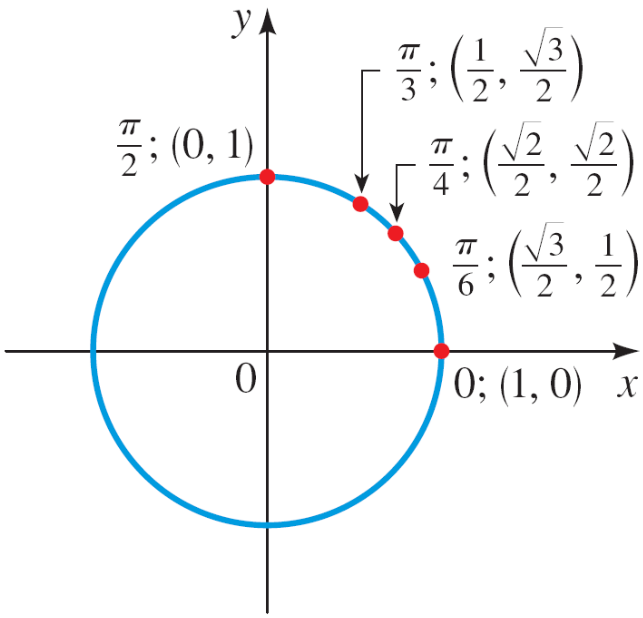


Figure 6

Example 4 – *Finding Terminal Points*

Find the terminal point determined by each given real number *t*.

(a) $t = -\dfrac{\pi}{4}$ **(b)** $t = \dfrac{3\pi}{4}$ **(c)** $t = -\dfrac{5\pi}{6}$

Solution:

(a) Let *P* be the terminal point determined by $-\pi/4$, and let *Q* be the terminal point determined by $\pi/4$.

Example 4 – *Solution*

cont'd

From Figure 7(a) we see that the point *P* has the same coordinates as *Q* except for sign.

Since *P* is in quadrant IV, its *x*-coordinate is positive and its *y*-coordinate is negative. Thus, the terminal point is $P(\sqrt{2}/2, -\sqrt{2}/2)$.

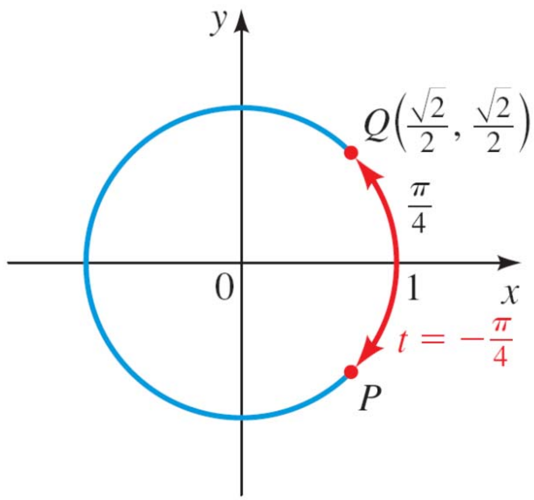


Figure 7(a)

Example 4 – *Solution*

cont'd

(b) Let P be the terminal point determined by $3\pi/4$, and let Q be the terminal point determined by $\pi/4$.

From Figure 7(b) we see that the point P has the same coordinates as Q except for sign. Since P is in quadrant II, its x-coordinate is negative and its y-coordinate is positive.

Thus, the terminal point is $P(-\sqrt{2}/2, \sqrt{2}/2)$.

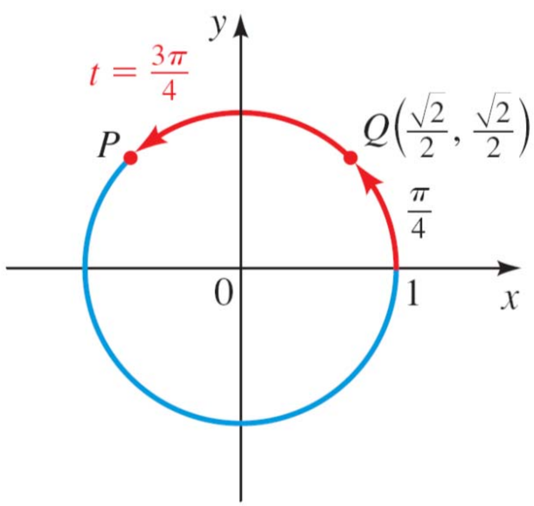


Figure 7(b)

Example 4 – *Solution*

cont'd

(c) Let P be the terminal point determined by $-5\pi/6$, and let Q be the terminal point determined by $\pi/6$.

From Figure 7(c) we see that the point P has the same coordinates as Q except for sign. Since P is in quadrant III, its coordinates are both negative.

Thus, the terminal point is $P\left(-\sqrt{3}/2, -\frac{1}{2}\right)$.

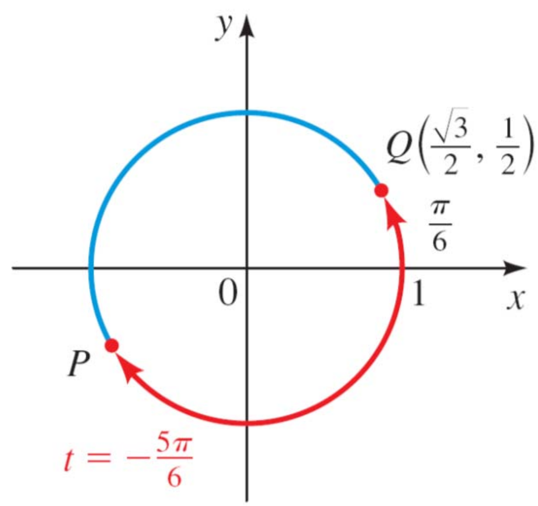


Figure 7(c)

The Reference Number

The Reference Number

From Examples 3 and 4 we see that to find a terminal point in any quadrant we need only know the “corresponding” terminal point in the first quadrant.

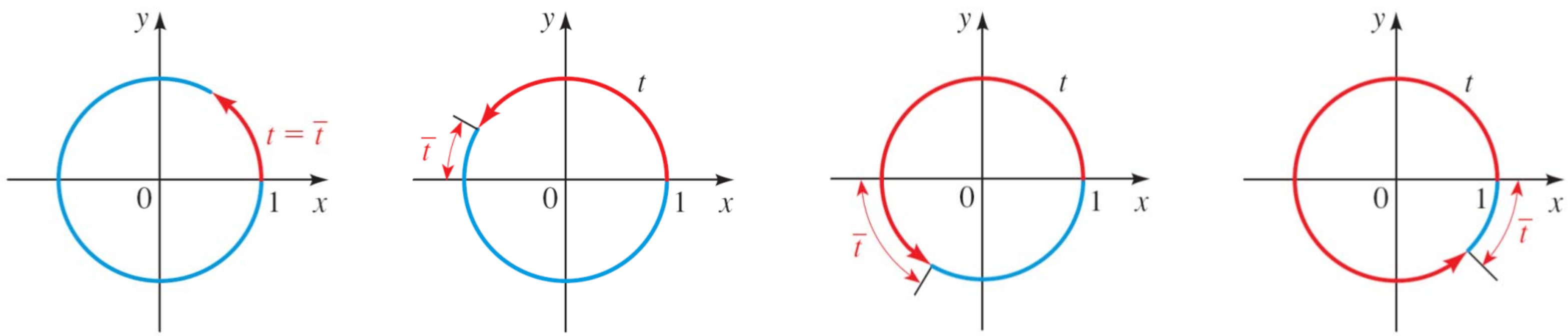
We use the idea of the *reference number* to help us find terminal points.

REFERENCE NUMBER

Let t be a real number. The **reference number** $\bar{t}$ associated with t is the shortest distance along the unit circle between the terminal point determined by t and the x-axis.

The Reference Number

Figure 8 shows that to find the reference number $\bar{t}$, it's helpful to know the quadrant in which the terminal point determined by *t* lies.



The reference number $\bar{t}$ for *t*

Figure 8

The Reference Number

If the terminal point lies in quadrants I or IV, where *x* is positive, we find $\bar{t}$ by moving along the circle to the *positive* *x*-axis.

If it lies in quadrants II or III, where *x* is negative, we find $\bar{t}$ by moving along the circle to the *negative* *x*-axis.

Example 5 – *Finding Reference Numbers*

Find the reference number for each value of *t*.

(a) $t = \frac{5\pi}{6}$ **(b)** $t = \frac{7\pi}{4}$ **(c)** $t = -\frac{2\pi}{3}$ **(d)** $t = 5.80$

Solution:

From Figure 9 we find the reference numbers as follows:

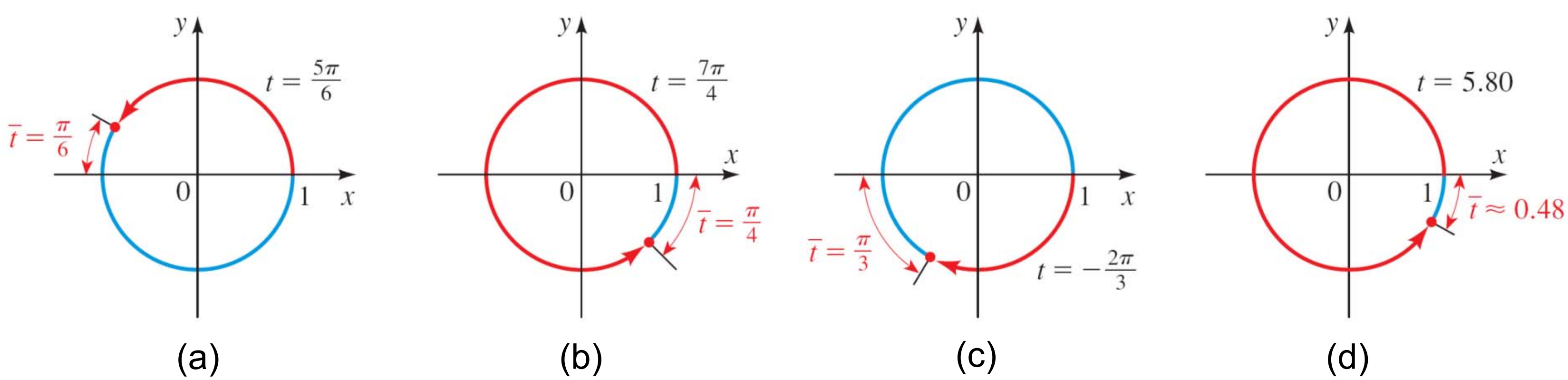


Figure 9

Example 5 – *Solution*

cont'd

(a) $\bar{t} = \pi - \frac{5\pi}{6} = \frac{\pi}{6}$

(b) $\bar{t} = 2\pi - \frac{7\pi}{4} = \frac{\pi}{4}$

(c) $\bar{t} = \pi - \frac{2\pi}{3} = \frac{\pi}{3}$

(d) $\bar{t} = 2\pi - 5.80 \approx 0.48$

The Reference Number

USING REFERENCE NUMBERS TO FIND TERMINAL POINTS

To find the terminal point P determined by any value of t, we use the following steps:

1. Find the reference number $\bar{t}$.
2. Find the terminal point $Q(a, b)$ determined by $\bar{t}$.
3. The terminal point determined by t is $P(\pm a, \pm b)$, where the signs are chosen according to the quadrant in which this terminal point lies.

Example 6 – *Using Reference Numbers to Find Terminal Points*

Find the terminal point determined by each given real number *t*.

(a) $t = \frac{5\pi}{6}$ **(b)** $t = \frac{7\pi}{4}$ **(c)** $t = -\frac{2\pi}{3}$

Solution:

The reference numbers associated with these values of *t* were found in Example 5.

Example 6 – *Solution*

cont'd

(a) The reference number is $\bar{t} = \pi/6$, which determines the terminal point $\left(\sqrt{3}/2, \frac{1}{2}\right)$ from Table 1.

TABLE 1

t	Terminal point determined by t
0	$(1, 0)$
$\frac{\pi}{6}$	$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
$\frac{\pi}{4}$	$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
$\frac{\pi}{3}$	$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
$\frac{\pi}{2}$	$(0, 1)$

Example 6 – *Solution*

cont'd

Since the terminal point determined by *t* is in Quadrant II, its *x*-coordinate is negative and its *y*-coordinate is positive.

Thus, the desired terminal point is

$$\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$$

(b) The reference number is $\bar{t} = \pi/4$, which determines the terminal point ($\sqrt{2}/2$, $\sqrt{2}/2$) from Table 1.

Since the terminal point is in Quadrant IV, its *x*-coordinate is positive and its *y*-coordinate is negative.

Example 6 – *Solution*

cont'd

Thus, the desired terminal point is

$$\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$$

(c) The reference number is $\bar{t} = \pi/3$, which determines the terminal point $\left(\frac{1}{2}, \sqrt{3}/2\right)$ from Table 1.

Since the terminal point determined by *t* is in Quadrant III, its coordinates are both negative. Thus, the desired terminal point is

$$\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$$

The Reference Number

Since the circumference of the unit circle is 2π, the terminal point determined by t is the same as that determined by $t + 2\pi$ or $t - 2\pi$.

In general, we can add or subtract 2π any number of times without changing the terminal point determined by t.

We use this observation in the next example to find terminal points for large t.

Example 7 – *Finding the Terminal Point for Large t*

Find the terminal point determined by $t = \dfrac{29\pi}{6}$.

Solution:

Since

$$t = \frac{29\pi}{6} = 4\pi + \frac{5\pi}{6}$$

we see that the terminal point of *t* is the same as that of $5\pi/6$ (that is, we subtract 4π).

Example 7 – *Solution*

cont'd

So by Example 6(a) the terminal point is $\left(-\sqrt{3}/2, \frac{1}{2}\right)$.
(See Figure 10.)

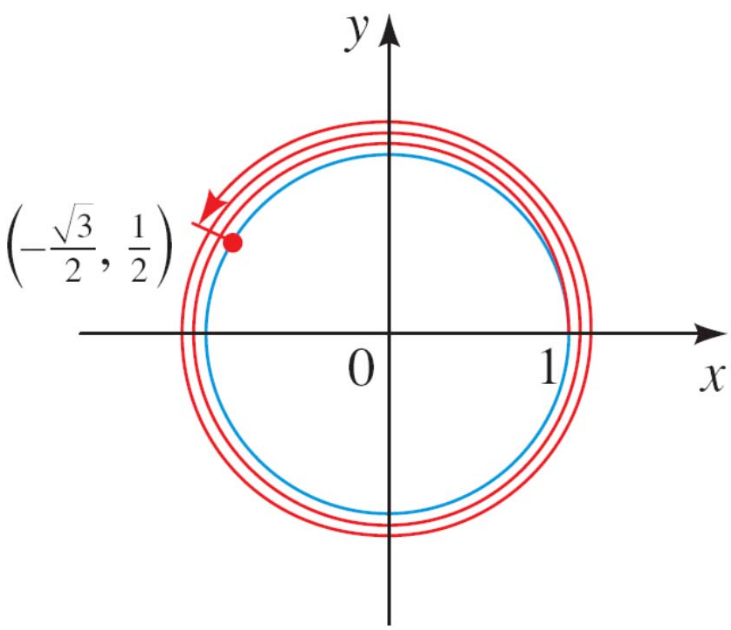


Figure 10