

Conic sections

In this unit we study the conic sections. These are the curves obtained when a cone is cut by a plane. We find the equations of one of these curves, the parabola, by using an alternative description in terms of points whose distances from a fixed point and a fixed line are equal. We also find the equation of a tangent to a parabola using techniques from calculus, and we use this to prove the reflective property of the parabola.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- decide, when given the eccentricity of a conic, whether the conic is an ellipse, a parabola or a hyperbola;
- write down the equation of a parabola passing through the origin with its focus on the positive x -axis;

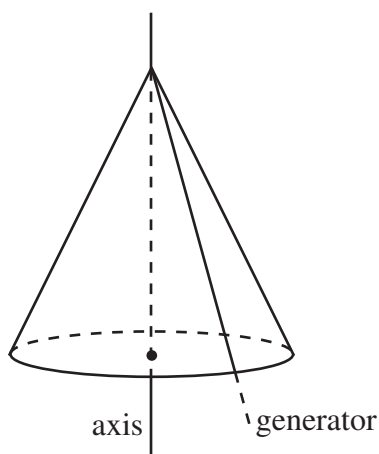
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1. Introduction

The conic sections, or *conics*, are curves obtained by making sections, or cuts, at particular angles through a cone. These curves were known to the ancient Greeks, who first explored their properties. But, ancient as they are, these curves have exceedingly modern applications — for example, the television dish aerial that brings sports and films into our homes, and the giant radio telescopes such as the one at Jodrell Bank in Cheshire, which look deep into space, all depend upon what is known as the ‘reflective property’ of one of the conic sections.

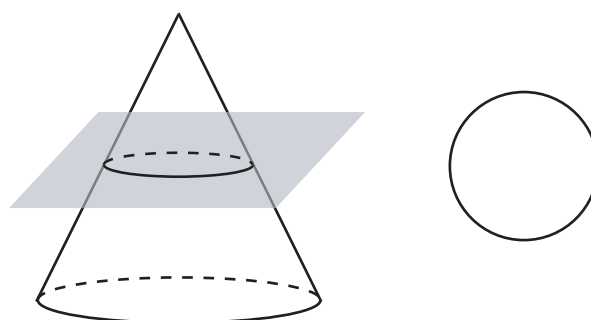
In the description below, we shall use diagrams showing just a part of a cone, but you should imagine that the cone extends infinitely far. We shall also talk about the *axis* of the cone, and a *generator*. The axis is the central line about which the cone is symmetric. A generator is a line which, when rotated about the axis, sweeps out the cone.



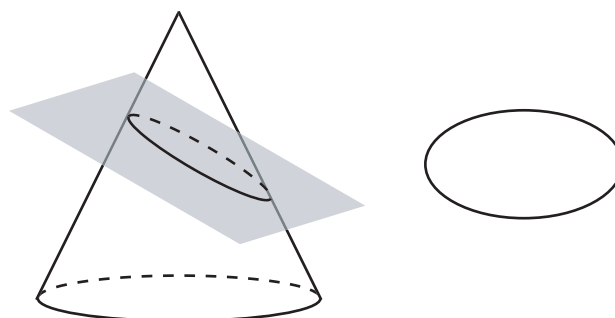
2. The sections of a cone

If we cut a cone at different angles, then we will obtain different types of conic section. There are four different types we can obtain.

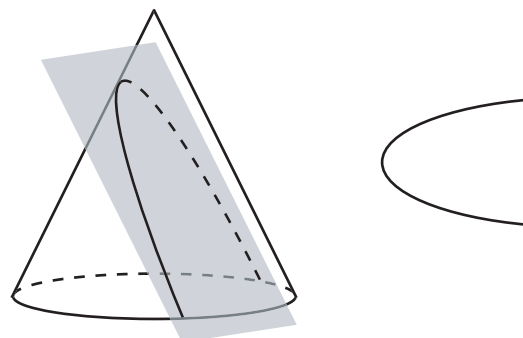
First, we can make the obvious cut, or section, perpendicular to the axis of the cone. This gives us a circle.



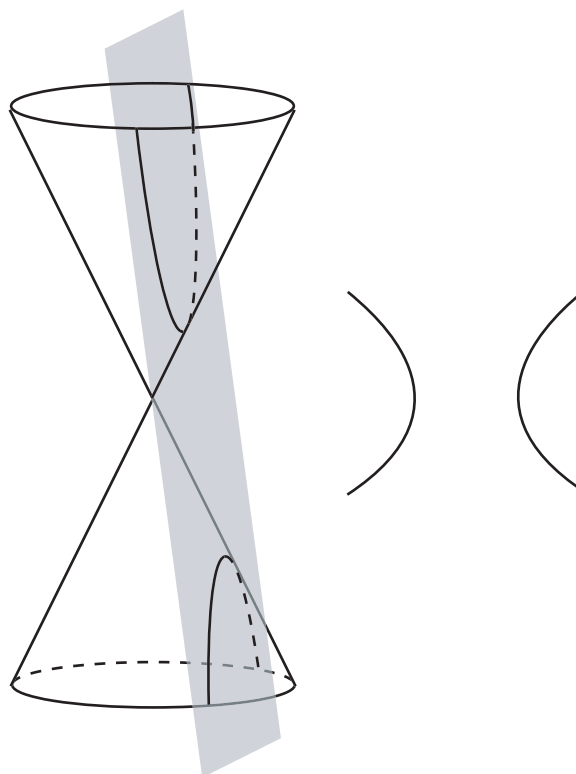
Next, we can make the cut at an angle to the axis of the cone, so that we still get a closed curve which is no longer a circle. This curve is an ellipse.



If we now make the cut parallel to the generator of the cone, we obtain an open curve. This is a parabola.



Finally, we make the cut at an even steeper angle. If we imagine that we have a double cone, that is, two cones vertex to vertex, then we obtain the two branches of a hyperbola.



The conic sections were first identified by Menaechmus in about 350 BC, but he used three different types of cone, taking the same section in each, to produce the three conic sections, ellipse, parabola and hyperbola. It was Apollonius of Perga, (c. 255–170 BC) who gave us the conic sections using just one cone.



Key Point

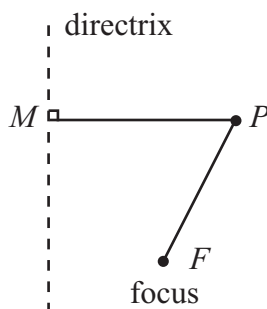
The four different types of conic section are:

- the circle, where the cone is cut at right-angles to its axis;
- the ellipse, where the cone is cut at an oblique angle shallower than a generator;
- the parabola, where the cone is cut parallel to a generator;
- the hyperbola, where a double cone is cut at an angle steeper than a generator.

3. Eccentricity

In order to be able to obtain the equations of conics, we need definitions that are given in terms of algebra as well as geometry.

Suppose that we are given a fixed straight line, called the *directrix*, and a fixed point, called the *focus*. If we have another point P , then we can consider the perpendicular distance of P from the line, and also the distance of P from the focus. What happens if one of these distances is a fixed multiple of the other?



Suppose that, for some constant e , the equation

$$PF = ePM$$

is always true. All the points P satisfying this equation lie on a curve called the *locus*. We shall see that we get curves of particular types, depending upon the value of the constant e :

- if $0 < e < 1$ then the curve is an ellipse;
- if $e = 1$ then the curve is a parabola; and
- if $e > 1$ then the curve is a hyperbola.

The constant e is called the *eccentricity* of the conic. It can be shown that these definitions are equivalent to the definitions given in terms of sections of a cone.

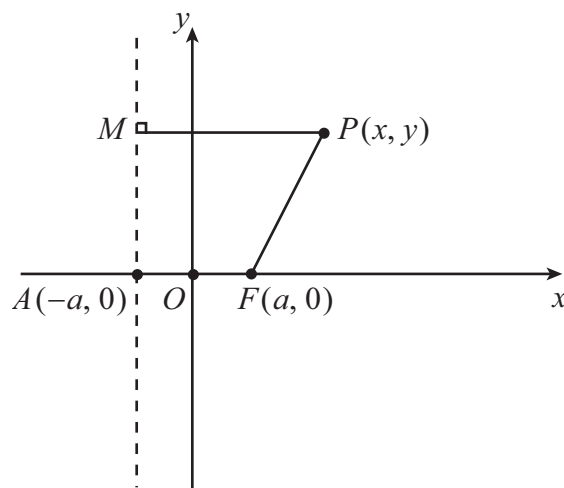
Exercise 1

In each case below, the given point lies on a conic with focus $(2, 0)$ and directrix $x = -2$. Find the type of conic, and its eccentricity e .

- (a) $(1, 0)$, (b) $(0, 0)$, (c) $(-1, 0)$, (d) $(8, 0)$, (e) $(2, 5)$,
(f) $(5, -4)$, (g) $(2, 4)$, (h) $(-6, 6)$.

4. The parabola

We will look in some detail at the parabola. We start by obtaining the equation of a parabola in Cartesian co-ordinates.

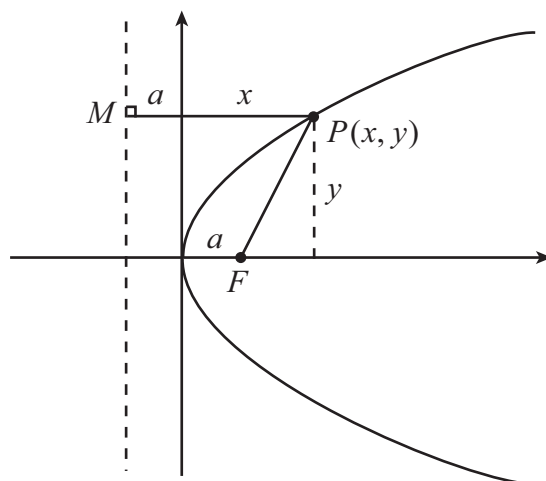


First, we must position the two axes on our diagram. We shall take the x -axis to pass through F and be perpendicular to the directrix. We shall let A be the point where the x -axis cuts the directrix.

For the parabola, $e = 1$ so that $PF = PM$. Thus each point on the curve is equidistant from the focus and the directrix, and so the curve will pass through the mid-point of AF . We shall choose to place the origin at O , the mid-point of AF , and so this gives us the position of the y -axis.

In order to have a scale in the space where we are placing the curve, we shall take OF to be a distance $a > 0$, giving F as the point $(a, 0)$, and thus A is the point $(-a, 0)$.

Now as P moves away from O we can see that PF and PM can become as large as we wish, provided they remain equal in length. Furthermore, P can be below the x -axis, and as the value of PM is unaffected by this switch, the magnitude of PF is unchanged. So we can sketch an open curve that is entirely to the right of the y -axis, and symmetrical about the x -axis.



Now we can find the equation of the parabola. To do this, we need to write down some of the lengths on our diagram. If P is the point (x, y) then the length PM is the distance a from the directrix to the y -axis, plus the x co-ordinate of P , so that

$$PM = x + a .$$

We also know, from Pythagoras' Theorem in the right-angled triangle, that

$$PF = \sqrt{(x - a)^2 + y^2}.$$

But for a parabola, we have $PF = PM$, so that

$$x + a = \sqrt{(x - a)^2 + y^2}.$$

Squaring both sides, we obtain

$$(x + a)^2 = (x - a)^2 + y^2,$$

and expanding the brackets gives us

$$x^2 + 2ax + a^2 = x^2 - 2ax + a^2 + y^2$$

which simplifies to

$$y^2 = 4ax.$$

This is the standard Cartesian equation of the parabola.

We can see how the equation confirms our sketch of the curve. First, y^2 must be greater than or equal to zero, and since a is defined to be positive, x must also be greater than or equal to zero. Similarly, for each value of x , to establish a value of y we must take a square root, giving plus and minus values of y , equal in magnitude. This explains the symmetry of the parabola about the x -axis.



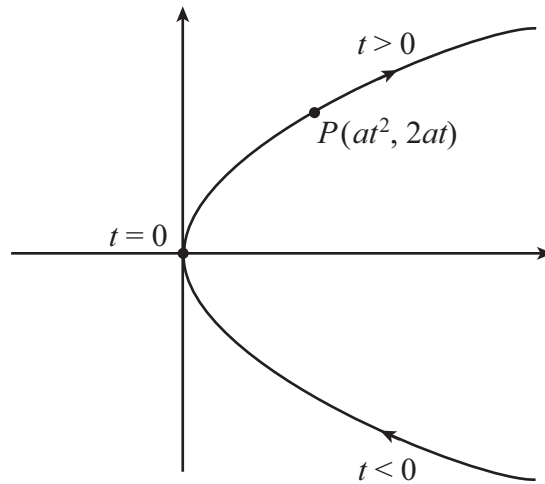
Key Point

The equation of a parabola with directrix $x = -a$ and focus $(a, 0)$ is

$$y^2 = 4ax.$$

It is sometimes convenient to express both x and y values on the parabola in terms of a third variable t . This variable is called a *parameter*, and the equations we obtain are called the parametric equations of the parabola.

It makes sense to try to choose how we express x and y in terms of t so that we avoid a square root, and we will do this if we can make $4ax$ a complete square. Let us take $x = at^2$. Then $y^2 = 4ax = 4a^2t^2$, so that $y = 2at$, where t can be either positive or negative. As t increases from a large negative value to a large positive value, the point moves along the parabola, passing through the origin when $t = 0$.

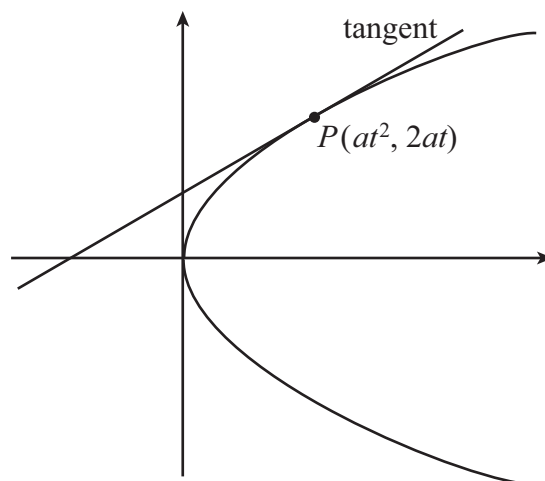


Key Point

The parametric equation of a parabola with directrix $x = -a$ and focus $(a, 0)$ is

$$x = at^2, \quad y = 2at.$$

We can use the parametric equation of the parabola to find the equation of the tangent at the point P .



We shall use the formula for the equation of a straight line with a given gradient, passing through a given point. We know that the tangent passes through the point P , but what is its gradient? We can work it out using calculus, because the gradient is just the derivative dy/dx . If we use the formula for differentiating a function of a function, we get

$$\frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}.$$

Now $y = 2at$ and $x = at^2$, so that

$$\frac{dy}{dt} = 2a, \quad \frac{dx}{dt} = 2at$$

giving

$$\frac{dy}{dx} = \frac{2a}{2at} = \frac{1}{t}.$$

Now the equation of a straight line with gradient m passing through the point (x_1, y_1) is

$$y - y_1 = m(x - x_1),$$

so the equation of the tangent at P is

$$y - 2at = \frac{1}{t}(x - at^2).$$

We can multiply by t to give

$$t(y - 2at) = x - at^2,$$

and then multiplying out the bracket and collecting together like terms we get

$$ty = x + at^2$$

as the equation of the tangent to the parabola at P .



Key Point

The tangent to a parabola at the point $P(at^2, 2at)$ is given by

$$ty = x + at^2.$$

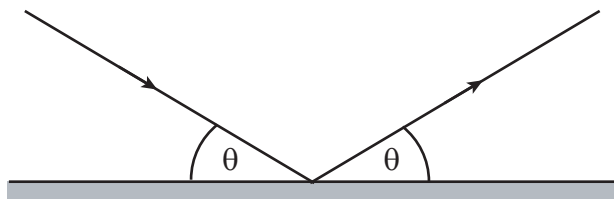
Exercise 2

In each case below, the given point lies on the parabola $y^2 = 4ax$ for some value of a . Find a , and give the equation of the tangent to the parabola at that point.

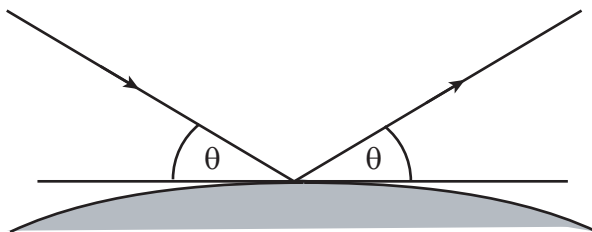
- (a) $(2, 4)$, (b) $(9, 6)$, (c) $(4, -8)$.

5. The reflective property of the parabola

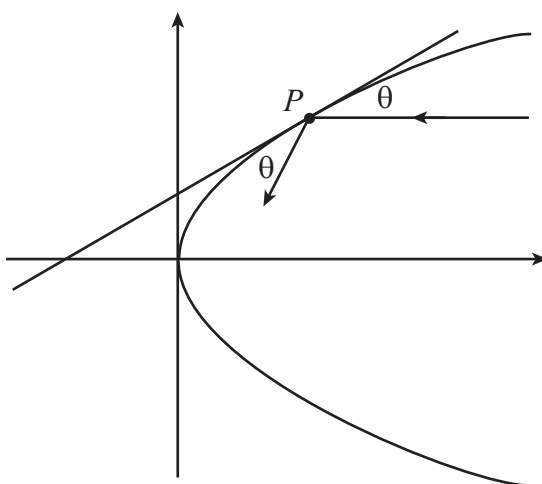
The law of reflection states that the angle made by an incident ray of light with a flat mirror is the same as the angle made by the reflected ray.



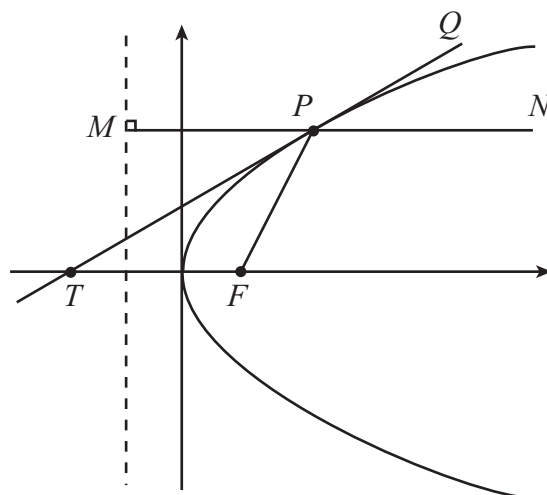
For a curved mirror, we must extend the law to say that the angles made by the incident and reflected rays with the tangent to the mirror must be equal.



Now suppose that we have a parabola, and that a ray of light is reflected at the point P . If the incident ray is parallel to the axis of the parabola, can we describe the reflected ray in terms of the geometry of the parabola?



To give a geometrical description of the reflected ray, let us return to our diagram of the parabola with its focus and directrix, together with the incident ray NP and the tangent line at P . We shall let T denote the point where the tangent cuts the x -axis.



Now the equation of the tangent TP is

$$ty = x + at^2,$$

so the point T has co-ordinates $y = 0$, $x = -at^2$. Therefore the distance TF from T to the focus F is given by

$$TF = a + at^2.$$

But $PF = PM$, the distance from the directrix, by the definition of a parabola. So

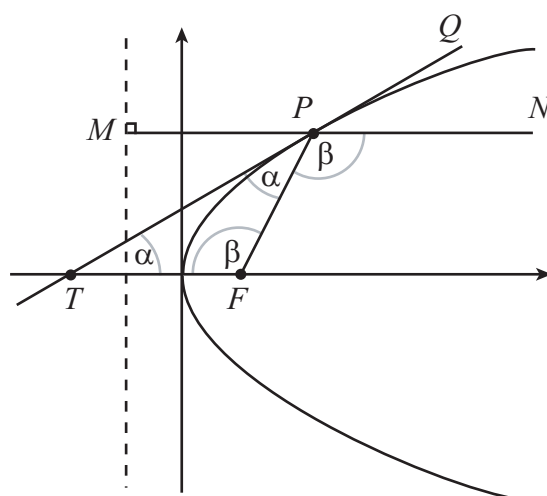
$$PF = PM = a + at^2 = TF,$$

showing us that the triangle TPF is isosceles, so that

$$\angle FTP = \angle FPT = \alpha.$$

We also know that the line MPN is parallel to the x -axis and so

$$\angle TFP = \angle FPN = \beta.$$



Now in the triangle TPF the angles must sum to 180° , so that

$$2\alpha + \beta = 180^\circ;$$

but at the point P the three angles on the tangent line must also sum to 180° , so that

$$\alpha + \beta + \angle QPN = 180^\circ$$

and therefore that

$$\angle QPN = \alpha = \angle TPF.$$

So if NM is an incoming ray of light making an angle QPN with the tangent, it will be reflected so that it makes an equal angle with the tangent, namely the angle TPF . It will then pass through the focus F .

We can now see how a satellite dish receiver works. The dish may be obtained by spinning a parabola about its axis. The radio waves arrive parallel to the axis, and are reflected to be collected at the focus. The same argument applies for a searchlight, where the rays go in the opposite direction: if we have a light source at the focus of a parabolic mirror, the light will be reflected at the parabola to give a beam of light parallel to the axis.



Key Point

Rays parallel to the axis of a parabola are reflected so that they all pass through the focus.

Answers

1.

- (a) ellipse, $e = \frac{1}{3}$ (b) parabola, $e = 1$ (c) hyperbola, $e = 3$ (d) ellipse, $e = \frac{3}{5}$
 (e) hyperbola, $e = \frac{5}{4}$ (f) ellipse, $e = \frac{5}{7}$ (g) parabola, $e = 1$ (h) hyperbola, $e = 2$

2.

- (a) $a = 2$, tangent is $y = x + 2$ (b) $a = 1$, tangent is $3y = x + 9$
 (c) $a = 4$, tangent is $-y = x + 4$