Recurrences

CSC 1300 – Discrete Structures
Villanova University

Some sequences and summations we have seen so far…

$$1+2+3+4+5+\ldots+n = \sum_{j=1}^{n} j = \frac{n(n+1)}{2}$$

$$1+2+4+8+16+\ldots+2^n = \sum_{j=1}^{n} 2^j = 2^{n+1} - 1$$

In this chapter …

- **Recurrence**
  - A sequence in which the next value in the sequence is dependent on a preceding value
  - Example: **Fibonacci** sequence
  - Seek to find explicit formulas for recurrence
    - Intuition
    - Arithmetic expression (linear)
    - Geometric expression (exponential)

Defining a Numerical Sequence

Two ways:

- Formula for the generic term s_n involving index n only

  2, 4, 6, 8, 10, 12,… $s_n = 2n$ for $n \geq 1$
- Equation relating the generic term s_n to one or more preceding terms of the sequence

  2, 4, 6, 8, 10, 12,… $s_n = s_{n-1} + 2$ for $n > 1$, $s_1 = 2$

Recurrence Relations

Definition A ***recurrence relation*** (or simply ***recurrence***) is an equation that defines the generic term of a sequence s_n in terms of one or more of its predecessors.

Examples: $s_n = s_{n-1} + 2$ for $n > 1$

$t_n = 3t_{n-1}$ for $n > 0$

$f_n = f_{n-1} + f_{n-2}$ for $n > 1$

- To define a sequence uniquely, a recurrence relation needs an ***initial condition.***

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Examples: $s_n = s_{n-1} + 2$ for $n > 1$ $\quad$ $s_1 = 2$

$t_n = 3t_{n-1}$ for $n > 0$ $\quad$ $t_0 = 1$

$f_n = f_{n-1} + f_{n-2}$ for $n > 1$ $\quad$ $f_0 = 0$ $\quad$ $f_1 = 1$

- To define a sequence uniquely, a recurrence relation needs an ***initial condition.***

Solving Recurrence Relations

To ***solve a recurrence relation subject to an initial condition*** means to find a formula expressing its generic term as a function of n, the index of the sequence (referred to as the ***closed form***).

Example:

$s_n = s_{n-1} + 2$ & $s_1 = 2$ — recurrence relation with initial condition

$s_n = 2n$ — closed form

Method of forward substitutions

- Using the recurrence, generate a few terms:

  $x_0 \quad x_1 \quad x_2 \quad x_3 \quad x_4 \ldots$

- Try to discern a pattern
- [Prove the formula's validity (use mathematical induction, substituting it in the initial condition and recurrence).]
- Example: $x_n = x_{n-1} + 3$ for $n > 0$, $x_0 = 0$

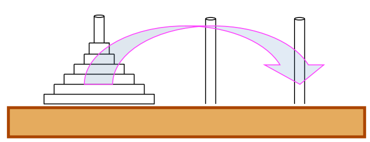
Method of backward substitutions

- Using the recurrence, substitute for previous terms (i.e., terms preceding x_n) in a hope to see a pattern.
- Consider what would happen if you keep substituting until the initial condition is reached.
- [Prove the formula's validity (use mathematical induction, substituting it in the initial condition and recurrence).]
- Example: $x_n = x_{n-1} + 3$ for $n > 0$, $x_0 = 0$

Method of finite differences

- This is useful if you have a recurrence of the form $x_n = x_{n-1} + p(n)$ for $x_1 = c$
- Compute differences of subsequent terms.
  - if difference is constant, you are done
  - otherwise, compute 2nd differences, etc.
- This method is described in some detail in Section 8.6
- Example: $x_n = x_{n-1} + 3$ for $n > 0$, $x_0 = 0$

Example: Tower of Hanoi Puzzle



H_n = number of moves needed to solve the n-disk puzzle

2nd order linear homogeneous recurrence with constant coefficients

A recurrence that can be written in the form:

- $ax_n + bx_{n-1} + cx_{n-2} = 0$

where a, b, c are real numbers (called the coefficients), $a \neq 0$.

- Examples:

2nd order linear homogeneous recurrence with constant coefficients

Unless $b = c = 0$, the recurrence

$$ax_n + bx_{n-1} + cx_{n-2} = 0$$

has infinitely many solutions (sequences), called the ***general solution*** to the recurrence. All of them can be obtained by a single formula; the type of this formula depends on the roots of the quadratic equation called the ***characteristic equation*** for the above recurrence:

$$ar^2 + br + c = 0.$$

Theorem If the characteristic equation has two distinct real roots r_1, r_2 then the solutions will be of the form: $x_n = q_1 r_1^n + q_2 r_2^n$

If the characteristic equation has two equal real roots $r_1 = r_2 = r$ then the solutions will be of the form: $x_n = q_1 r^n + q_2 n r^n$

(q_1 and q_2 are any two real numbers)

Example 1: Application of the theorem

Find the general solution to the recurrence $x_n = 5\,x_{n-1} - 6\,x_{n-2}$

Find the ***particular solution*** to this recurrence that satisfying the initial conditions $x_0 = 1$, $x_1 = 0$

Example 2: Application of the theorem

Find the general solution to the recurrence $x_n = 4\,x_{n-1} - 4\,x_{n-2}$

Find the ***particular solution*** to this recurrence satisfying the initial conditions $x_0 = 1$, $x_1 = 3$

Application to the Fibonacci numbers

$f_n = f_{n-1} + f_{n-2}$ for $n > 1$, $f_0 = 0$, $f_1 = 1$