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A double inequality for bounding Toader mean by the centroidal mean

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Abstract. In this paper, the authors find the best numbers α and β such that $\bar{C}(\alpha a+(1-\alpha)b, \alpha b+(1-\alpha)a) < T(a,b) < \bar{C}(\beta a+(1-\beta)b, \beta b+(1-\beta)a)$ for all $a, b > 0$ with $a \neq b$, where $\bar{C}(a,b) = \frac{2(a^2+ab+b^2)}{3(a+b)}$ and $T(a,b) = \frac{2}{\pi}\int_0^{\pi/2} \sqrt{a^2\cos^2\theta + b^2\sin^2\theta}\,\mathrm{d}\theta$ denote respectively the centroidal mean and Toader mean of two positive numbers a and b.

Keywords. Toader mean; centroidal mean; complete elliptic integral; double inequality.

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1. Introduction

In [13], Toader introduced a mean

$$T(a,b) = \frac{2}{\pi}\int_0^{\pi/2} \sqrt{a^2\cos^2\theta + b^2\sin^2\theta}\,\mathrm{d}\theta \tag{1.1}$$

$$= \begin{cases} \dfrac{2a}{\pi}\mathcal{E}\left(\sqrt{1-\left(\dfrac{b}{a}\right)^2}\right), & a > b, \\ \dfrac{2b}{\pi}\mathcal{E}\left(\sqrt{1-\left(\dfrac{a}{b}\right)^2}\right), & a < b, \\ a, & a = b, \end{cases} \tag{1.2}$$

where

$$\mathcal{E} = \mathcal{E}(r) = \int_0^{\pi/2} \sqrt{1 - r^2\sin^2\theta}\,\mathrm{d}\theta$$

for $r \in [0, 1]$ is the complete elliptic integral of the second kind.

In recent years, there have been plenty of literature dedicated to Toader mean [6, 7, 9–11, 15].

For $p \in \mathbb{R}$ and $a, b > 0$, the centroidal mean $\bar{C}(a, b)$ and the p-th power mean $M_p(a, b)$ are defined respectively by

$$\bar{C}(a, b) = \frac{2\left(a^2 + ab + b^2\right)}{3(a + b)} \tag{1.3}$$

and

$$M_p(a, b) = \begin{cases} \left(\dfrac{a^p + a^p}{2}\right)^{1/p}, & p \neq 0, \\ \sqrt{ab}, & p = 0. \end{cases} \tag{1.4}$$

In [14], Vuorinen conjectured that

$$M_{3/2}(a, b) < T(a, b) \tag{1.5}$$

for all $a, b > 0$ with $a \neq b$. This conjecture was verified by Qiu and Shen [12] and by Barnard *et al.* [3]. In [1], Alzer and Qiu presented that

$$T(a, b) < M_{(\ln 2)/\ln(\pi/2)}(a, b) \tag{1.6}$$

for all $a, b > 0$ with $a \neq b$, which gives a best possible upper bound for Toader mean in terms of the power mean.

Very recently, Chu *et al.* proved in [8] that the double inequality

$$C\left(\alpha a+(1-\alpha)b, \alpha b+(1-\alpha)a\right) < T(a, b) < C\left(\beta a+(1-\beta)b, \beta b+(1-\beta)a\right) \tag{1.7}$$

is valid for all $a, b > 0$ with $a \neq b$ if and only if $\alpha \leq \frac{3}{4}$ and $\beta \geq \frac{1}{2} + \frac{\sqrt{4\pi - \pi^2}}{2\pi}$, where $C(a, b) = \frac{a^2+b^2}{a+b}$ is the contraharmonic mean.

For positive numbers $a, b > 0$ with $a \neq b$, let

$$J(x) = \bar{C}\left(xa + (1 - x)b, xb + (1 - x)a\right) \tag{1.8}$$

on $[\frac{1}{2}, 1]$. It is easy to see that $J(x)$ is continuous and strictly increasing on $[\frac{1}{2}, 1]$. Now it is natural to ask the question: What are the best constants $\alpha \geq \frac{1}{2}$ and $\beta \leq 1$ such that the double inequality

$$\bar{C}\left(\alpha a+(1-\alpha)b, \alpha b+(1-\alpha)a\right) < T(a, b) < \bar{C}\left(\beta a+(1-\beta)b, \beta b+(1-\beta)a\right) \tag{1.9}$$

holds for $a, b > 0$ with $a \neq b$? This problem can be affirmatively answered by the following theorem which is the main result of this paper.

Theorem 1. *For positive numbers $a, b > 0$ with $a \neq b$, the double inequality (1.9) is valid if and only if $\alpha \leq \frac{1}{2}\left(1 + \frac{\sqrt{3}}{2}\right)$ and $\beta \geq \frac{1}{2} + \frac{1}{2}\sqrt{\frac{12}{\pi} - 3}$.*

2. Proof of Theorem 1

For $0 < r < 1$, denote $r' = \sqrt{1-r^2}$. It is known that Legendre's complete elliptic integrals of the first and second kinds are defined respectively by

$$\begin{cases} \mathcal{K} = \mathcal{K}(r) = \displaystyle\int_0^{\pi/2} \frac{1}{\sqrt{1-r^2\sin^2\theta}}\,\mathrm{d}\theta, \\ \mathcal{K}' = \mathcal{K}'(r) = \mathcal{K}(r'), \\ \mathcal{K}(0) = \dfrac{\pi}{2}, \\ \mathcal{K}(1^-) = \infty \end{cases}$$

and

$$\begin{cases} \mathcal{E} = \mathcal{E}(r) = \displaystyle\int_0^{\pi/2} \sqrt{1-r^2\sin^2\theta}\,\mathrm{d}\theta, \\ \mathcal{E}' = \mathcal{E}'(r) = \mathcal{E}(r'), \\ \mathcal{E}(0) = \dfrac{\pi}{2}, \\ \mathcal{E}(1^-) = 1, \end{cases}$$

(see [4, 5]). For $0 < r < 1$, the following formulas were presented in Appendix E, pp. 474–475 of [2]:

$$\frac{\mathrm{d}\mathcal{K}}{\mathrm{d}r} = \frac{\mathcal{E}-(r')^2\mathcal{K}}{r(r')^2}, \quad \frac{\mathrm{d}\mathcal{E}}{\mathrm{d}r} = \frac{\mathcal{E}-\mathcal{K}}{r}, \quad \frac{\mathrm{d}(\mathcal{E}-(r')^2\mathcal{K})}{\mathrm{d}r} = r\mathcal{K},$$
$$\frac{\mathrm{d}(\mathcal{K}-\mathcal{E})}{\mathrm{d}r} = \frac{r\mathcal{E}}{(r')^2}, \quad \mathcal{E}\left(\frac{2\sqrt{r}}{1+r}\right) = \frac{2\mathcal{E}-(r')^2\mathcal{K}}{1+r}.$$

For simplicity, denote

$$\lambda = \frac{1}{2}\left(1+\frac{\sqrt{3}}{2}\right) \quad \text{and} \quad \mu = \frac{1}{2}+\frac{1}{2}\sqrt{\frac{12}{\pi}-3}\,.$$

It is clear that, in order to prove the double inequality (1.9), it suffices to show that

$$T(a,b) > \bar{C}\left(\lambda a+(1-\lambda)b, \lambda b+(1-\lambda)a\right) \tag{2.1}$$

and

$$T(a,b) < \bar{C}\left(\mu a+(1-\mu)b, \mu b+(1-\mu)a\right). \tag{2.2}$$

From (1.1) and (1.3) we see that both $T(a,b)$ and $\bar{C}(a,b)$ are symmetric and homogenous of degree 1. Hence, without loss of generality, we assume that $a > b$. Let $t = \frac{b}{a} \in (0,1)$ and $r = \frac{1-t}{1+t} \in (0,1)$ and let $p \in \left(\frac{1}{2}, 1\right)$. Then

$$T(a,b) - \bar{C}\left(pa+(1-p)b, pb+(1-p)a\right) = \frac{2a}{\pi}\mathcal{E}\left(\sqrt{1-\left(\frac{b}{a}\right)^2}\right)$$

$$
\begin{aligned}
&-2a\frac{[p+(1-p)b/a]^2+[p+(1-p)b/a](pb/a+1-p)+(pb/a+1-p)^2}{3(1+b/a)}\\
&=\frac{2a}{\pi}\mathcal{E}(\sqrt{1-t^2})-2a\frac{[p+(1-p)t]^2+[p+(1-p)t](pt+1-p)+(pt+1-p)^2}{3(1+t)}\\
&=\frac{2a}{\pi}\frac{2\mathcal{E}-(1-r^2)\mathcal{K}}{1+r}-a\frac{(1-2p)^2r^2+3}{3(1+r)}\\
&=\frac{a}{1+r}\left\{\frac{2}{\pi}[2\mathcal{E}-(1-r^2)\mathcal{K}]-\frac{1}{3}(1-2p)^2r^2-1\right\}.
\end{aligned}
\tag{2.3}
$$

Let

$$
f(r)=\frac{2}{\pi}[2\mathcal{E}-(1-r^2)\mathcal{K}]-\frac{1}{3}(1-2p)^2r^2-1, \tag{2.4}
$$

and let $f_1(r)=rf'(r)$ and $f_2(r)=\frac{f_1'(r)}{r}$. Then, by standard argument, we have

$$
\begin{aligned}
&f(0)=0,\quad f_1(0)=0,\quad f_2(0)=1-\frac{4}{3}(1-2p)^2,\\
&f(1^-)=\frac{4}{\pi}-1-\frac{1}{3}(1-2p)^2,\quad f_1(1^-)=\frac{2}{\pi}-\frac{2}{3}(1-2p)^2,\quad f_2(1^-)=+\infty,\\
&f_1(r)=\frac{2}{\pi}[\mathcal{E}-(1-r^2)\mathcal{K}]-\frac{2}{3}(1-2p)^2r^2,\quad f_2(r)=\frac{2}{\pi}\mathcal{K}-\frac{4}{3}(1-2p)^2,
\end{aligned}
$$

When $p=\lambda=\frac{1}{2}(1+\frac{\sqrt{3}}{2})$, it follows that $f_2(0)=0$. An easy argument leads to $f(r)>0$ for $r\in(0,1)$. Together with this, the inequality (2.1) follows from (2.3) and (2.4).

When $p=\mu=\frac{1}{2}+\frac{1}{2}\sqrt{\frac{12}{\pi}-3}$, it is simple to derive that

$$
f(1^-)=0,\quad f_1(1^-)=\frac{2(\pi-3)}{\pi}>0,\quad f_2(0)=\frac{5\pi-16}{\pi}<0.
$$

Consequently, considering the monotonicity of $f_2(r)$, it is deduced that there exists $r_0\in(0,1)$ such that $f_2(r)<0$ on $(0,r_0)$ and $f_2(r)>0$ on $(r_0,1)$. Hence, the function $f_1(r)$ is strictly decreasing on $(0,r_0)$ and strictly increasing on $(r_0,1)$. Similarly, there exists $r_1\in(0,1)$ such that $f_1(r)<0$ on $(0,r_1)$ and $f_1(r)>0$ on $(r_1,1)$. Thus, the function $f(r)$ is strictly decreasing on $(0,r_1)$ and strictly increasing on $(r_1,1)$. As a result, inequality (2.2) follows.

If $p>\lambda$, then $f_2(r)<0$. From the continuity of $f(r)$, $f_1(r)$ and $f_2(r)$, it follows that there exists $\delta_1=\delta_1(p)>0$ such that $f(r)<0$ on $(0,\delta_1)$. Combining this with (2.3) and (2.4) yields $T(a,b)<\bar{C}(pa+(1-p)b,pb+(1-p)a)$ for $\frac{b}{a}\in(\frac{1-\delta_1}{1+\delta_1},1)$. If $p<\mu$, then $f(1^-)>0$. Hence, there exists $\delta_2=\delta_2(p)\in(0,1)$ such that $f(r)>0$ on $(1-\delta_2,1)$. Combining this with (2.3) and (2.4) reveals that $T(a,b)>\bar{C}(pa+(1-p)b,pb+(1-p)a)$ for $\frac{b}{a}\in(0,\delta_2/(2-\delta_2))$. These imply that the constants λ and μ are the best possible. The proof of Theorem 1 is complete.

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