# Complex Analysis

## 1 The field of complex numbers

The set of complex numbers is denoted by $\mathbb{C}$. The cartesian representation of $z \in \mathbb{C}$ is $z = x+iy$ with $x, y \in \mathbb{R}$ and $i^2 = -1$. The real and imaginary parts of $z$ are $\mathrm{Re}\,(z) = x$ and $\mathrm{Im}\,(z) = y$, respectively. Addition and multiplication of complex numbers (defined in a predictable way) satisfy all the properties we would have expected — meaning that $\mathbb{C}$ is a field. The polar representation of $z \in \mathbb{C}$ is $z = re^{i\theta}$ with $r \geq 0$ and $\theta \in \mathbb{R}$. We call $r = |z|$ the modulus of $z$ and $\theta = \arg(z)$ — not necessarily unique — an argument of $z$. We have $r = \sqrt{x^2+y^2}$ and $\tan(\theta) = y/x$. De Moivre's theorem states that $(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)$, or in simplified form, that $(e^{i\theta})^n = e^{in\theta}$ — this uses Euler formula $e^{i\varphi} = \cos(\varphi) + i\sin(\varphi)$. Note also the identities $\cos(\theta) = (e^{i\theta} + e^{-i\theta})/2$ and $\sin(\theta) = (e^{i\theta} - e^{-i\theta})/(2i)$. In general, one has $\mathrm{Re}\,(z) = (z+\bar{z})/2$, $\mathrm{Im}\,(z) = (z-\bar{z})/(2i)$, and $|z|^2 = z\bar{z}$. Here $\bar{z} = x - iy = re^{-i\theta}$ is the complex conjugate of $z$. The fundamental theorem of algebra ensures that every nonconstant polynomial $p(z) = a_n z^n + \cdots + a_1 z + a_0$ has a complex roots (in turn, that every polynomial with complex coefficients has all its roots in $\mathbb{C}$, i.e., $\mathbb{C}$ is algebraically closed).
A possible argument goes along those lines: pick $z_0 \in \mathbb{C}$ such that $|p(z_0)| = \min_{z\in\mathbb{C}} |p(z)|$ and suppose $|p(z_0)| > 0$; write that $p$ equals its Taylor polynomial at $z_0$, i.e., $p(z_0) + \sum_{j=k}^n b_j (z-z_0)^j$ where $b_k \neq 0$; note that $\sum_{j=k+1}^n |b_j|\rho^j < |b_k|\rho^k < |p(z_0)|$ for $\rho > 0$ sufficiently small; observe that $p(z_0) + b_k(z-z_0)^k$ describes $k$ times the circle $\{|\zeta - p(z_0)| = |b_k|\rho^k\}$ when $z$ describes the circle $\{|z - z_0| = \rho\}$, hence there exists $z_1$ with $|z_1 - z_0| = \rho$ such that $p(z_0) + b_k(z_1 - z_0)^k$ lies between $0$ and $p(z_0)$, so that $|p(z_0) + b_k(z_1-z_0)^k| = |p(z_0)| - |b_k|\rho^k$; derive a contradiction from

$$|p(z_1)| \leq |p(z_0) + b_k(z_1 - z_0)^k| + |\sum_{j=k+1}^n b_j (z_1 - z_0)^j| \leq |p(z_0)| - |b_k|\rho^k + \sum_{j=k+1}^n |b_j|\rho^j < |p(z_0)|.$$

Another possible argument involves Cauchy formula for holomorphic functions (see below): suppose that $p$ does not vanish on $\mathbb{C}$, so that $q = 1/p$ is holomorphic on $\mathbb{C}$; for $R > 0$ sufficiently large to have $|p(z)| \geq (|a_n| - |a_{n-1}|/|z| - \cdots |a_0|/|z|^n)|z|^n \geq |a_n||z|^n/2$ whenever $|z| = R$, a contradiction follows from

$$0 < |q(0)| = \left| \frac{1}{2\pi i} \oint_{|z|=R} \frac{q(z)dz}{z} \right| \leq \frac{1}{2\pi} \oint_{|z|=R} \frac{dz}{|z||p(z)|} \leq \frac{1}{2\pi} \oint_{|z|=R} \frac{2\,dz}{|a_n|R^{n+1}} = \frac{2}{|a_n|R^n} \underset{R\to\infty}{\longrightarrow} 0.$$

## 2 Holomorphic functions

A function $f$ defined on an open subset of $\mathbb{C}$ is differentiable at $z_0$ if one can make sense of

$$f'(z) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

In particular, the limit is independent of how $z_0$ is approached. If the function $f$ of the variable $z = x + iy$ is differentiable at $z_0 = x_0 + iy_0$, then it satisfies the Cauchy–Riemann equations

$$\frac{\partial \mathrm{Re}\, f}{\partial x}(x_0, y_0) = \frac{\partial \mathrm{Im}\, f}{\partial y}(x_0, y_0) \qquad \text{and} \qquad \frac{\partial \mathrm{Re}\, f}{\partial y}(x_0, y_0) = -\frac{\partial \mathrm{Im}\, f}{\partial x}(x_0, y_0).$$

A converse holds provided the first-order partial derivatives are continuous.

A function $f$ is called holomorphic at $z_0$ if it is differentiable in some neighborhood of $z_0$ (i.e., whenever $|z - z_0| < r$ for some $r > 0$). Every power series $\sum_{n=0}^{\infty} c_n(z - z_0)^n$ with radius of convergence $R > 0$ defines a holomorphic function on $\{|z - z_0| < R\}$. Conversely, every holomorphic function is analytic, i.e., locally representable by powers series (hence holomorphic and analytic are synonymous terms for complex functions). This fact shows that holomorphic functions are infinitely differentiable and that their zeros are isolated (unless the function vanishes everywhere).

Let $G$ be a simply connected open region, let $\gamma$ be a simple closed path oriented counterclockwise and contained in $G$, and let $z_0 \in \mathbb{C}$ be inside $\gamma$. If $f$ is holomorphic in $G$, then it satisfies Cauchy integral formulas

$$\text{(1)} \quad \oint_\gamma f(z)dz = 0, \qquad \oint_\gamma \frac{f(z)dz}{z - z_0} = 2\pi i f(z_0), \qquad \oint_\gamma \frac{f(z)dz}{(z - z_0)^n} = 2\pi i f^{(n)}(z_0) \text{ for all integer } n \geq 0.$$

Cauchy formula implies Liouville's theorem, which states that a function $f$ holomorphic and bounded on $\mathbb{C}$ is constant. Indeed, if $\gamma$ is the circular contour oriented counterclockwise with center $0$ and radius $R$ large enough so that $|z - z_0|, |z - z_1| \geq R/2$, then, for all $z_0, z_1 \in \mathbb{C}$,

$$\begin{aligned} |f(z_0) - f(z_1)| &= \left| \frac{1}{2\pi i} \oint_\gamma f(z) \left( \frac{1}{z - z_0} - \frac{1}{z - z_1} \right) dz \right| = \left| \frac{z_0 - z_1}{2\pi i} \oint_\gamma \frac{f(z)}{(z - z_0)(z - z_1)} dz \right| \\ &\leq \frac{|z_0 - z_1|}{2\pi} \oint_\gamma \frac{\max(|f|)}{(R/2)^2} dz = \frac{4|z_0 - z_1| \max(|f|)}{R} \underset{R \to \infty}{\longrightarrow} 0. \end{aligned}$$

Cauchy formula also implies the maximum principle, which sates that, if $f$ is homomorphic on $\{|z - z_0| \leq r\}$, then

$$\max_{|z - z_0| \leq r} |f(z)| = \max_{|z - z_0| = r} |f(z)|.$$

## 3 Meromorphic functions

If a function is holomorphic on an annulus $A = \{r < |z - z_0| < R\}$ for some $R > r \geq 0$, then $f$ has a unique Laurent expansion at $z_0$ of the form

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n, \qquad z \in A.$$

A function $f$ holomorphic in some punctured neighborhood of $z_0$ (i.e., an annulus where $r = 0$) but not at $z_0$ is said to have an isolated singularity at $z_0$. These can be of three different kinds: removable singularity if $c_n = 0$ for all $n < 0$ (for instance $\sin(z)/z$ at $z_0 = 0$), poles if $c_{-m} \neq 0$ and $c_n = 0$ for all $n < -m$, in which case $m$ is called the order of the pole (for instance rational functions at $z_0$ equal to a zero of the denominator), and essential singularities if $\inf\{n : c_n \neq 0\} = -\infty$. A function which is holomorphic in an open subset $G$ of $\mathbb{C}$ except possibly for poles is said to be meromorphic in $G$.

Let $G$ be a a simply connected open region and let $\gamma$ be a simple closed path oriented counterclockwise and contained in $G$. Cauchy residue theorem states that, if $f$ is meromorphic in $G$ with all its poles $z_1, \ldots, z_N$ inside $\gamma$, then

$$\oint_\gamma f(z)dz = 2\pi i \sum_{k=1}^{N} \mathrm{Res}(f, z_k),$$

where the residue $\mathrm{Res}(f, z_k)$ of $f$ at $z_k$ is defined as the coefficient $c_{-1}$ of $(z - z_k)^{-1}$ in the Laurent expansion of $f$ at $z_k$. It follows that, if $f$ is holomorphic on $G$ and does not vanish on $\gamma$, then the number of zeros of $f$ inside $\gamma$ equals $\frac{1}{2\pi i}\oint_\gamma \frac{f'(z)}{f(z)}dz$. From here, we can deduce Rouché's theorem which states that, if $f$ and $g$ are holomorphic in $G$ and if $|f(z)| > |g(z)|$ on $\gamma$, then $f$ and $f + g$ have the same number of zeros (counting multiplicity) inside $\gamma$.

## 4 Exercises

Ex.1: Find the set of all $z \in \mathbb{C}^n$ such that $|z| + |z + 1| = 2$.

Ex.2: Prove the identity

$$\cos^n(\theta) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\cos((n-2k)\theta).$$

Ex.3: Prove the necessity of the Cauchy–Riemann equations.

Ex.4: Establish the fundamental integral

$$\oint_{\gamma(z_0,r)} (z - z_0)^n dz = \begin{cases} 0 & \text{if } n \neq -1, \\ 2\pi i & \text{if } n = -1, \end{cases}$$

where $\gamma(z_0, r)$ denotes the circular contour oriented counterclockwise with center $z_0$ and radius $r$. Derive (informally) formulas (1) with $\gamma = \gamma(z_0, r)$ for analytic functions.

Ex.5: Use the maximum principle to prove Schwarz lemma: if $f$ is holomorphic on $\{|z| = 1\}$, if $M := \max_{|\zeta|=1} |f(\zeta)|$, and if $f(0) = 0$, then $|f(z)| \leq M\,|z|$ whenever $|z| \leq 1$.

Ex.6: Use Cauchy residue theorem to evaluate

$$\oint_\gamma \frac{dz}{1+z^4},$$

where $\gamma$ is the semicircle $\{|z| = R, \mathrm{Im}\,(z) \geq 0\} \cup [-R, R]$ oriented counterclockwise. Deduce the value of the integral

$$\int_0^\infty \frac{dx}{1+x^4}.$$