Reasoning about the Pythagorean Theorem

Goals

- Explore the Pythagorean theorem to move from level 1 to level 3 in Waring's (2000) developmental framework of understanding mathematical proof
- Gain a well-reasoned understanding of the Pythagorean theorem by understanding a general proof of the theorem

Materials and Equipment

For each student—

- A copy of each of the following activity sheets:
  - "Take a Look at Those Squares"
  - "Sets of Right Triangles for 'Can You Prove It?'"
  - "Can You Prove It?"
- Three or four sheets of centimeter grid paper

For each group of two to four students—

- (Optional) A ruler calibrated in centimeters and millimeters
- A calculator
- A pair of scissors
- Access to the applet Squaring the Triangle (available on the CD-ROM and on the Web at http://www.shodor.org/interactivate/activities/pyth/index.html)

For the teacher—

- (Optional) A sheet of chart paper or one or two overhead transparencies

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The template "Centimeter Grid Paper" on the CD-ROM allows you to print out sheets of grid paper for your students' use in the investigation.

Prior Knowledge

The students should know how to find the areas of a square and a triangle and have had previous experience in decomposing shapes into other familiar shapes (see chapter 2). Students will also find it helpful to have worked with area models for multiplication and the distributive property of multiplication over addition.

Learning Environment

The students complete their own activity sheets while working together in pairs, with the teacher encouraging them to formulate conjectures, evaluate them, and eventually attempt to justify those that they suppose to be true. The teacher serves as a facilitator and questioner throughout, supporting students as they reason about the relationship among the squares constructed on the sides of a right triangle.

If your classroom has only one computer but you can project the screen image for everyone to see, you can use the applet in a whole-class demonstration instead of having the students work with it in small groups. (The investigation can also be effective without the applet.)

Discussion

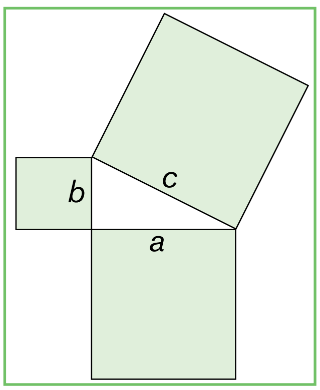
The Pythagorean theorem states that the sum of the areas of squares constructed on the legs of a right triangle is equal to the area of a square constructed on the triangle's hypotenuse. Figure 6.1 illustrates the theorem and shows an algebraic expression of the relationship between the area of the square on the hypotenuse and the areas of the squares on the legs of the triangle.

An exploration of this theorem can move students toward a general proof, taking them from level 1 to level 3 in their thinking, as described in Waring's (2000) framework:

Level 1 — Students are aware of the need to prove a conjecture but think that finding a few examples that support it is enough for a proof.

Level 2 — Students are aware that finding a few examples that support a conjecture does not prove it, but they think that examples that are more varied or selected more randomly provide proof or that a generic example forms a proof for a class.

Level 3 — Students are aware of the need for a generalized proof of a conjecture, although they are as yet unable to construct such a proof on their own. Nevertheless, they are likely to understand simple proofs and follow explanations of their construction.



If—

- c is the length of the hypotenuse of a right triangle;
- a and b are the lengths of the legs of the triangle;

then $a^2 + b^2 = c^2$, with—

- c^2 as the area of a square with a side length of c;
- a^2 as the area of a square with side length a; and
- b^2 as the area of a square with side length b.

Fig. 6.1.

According to the Pythagorean theorem, if the hypotenuse of a right triangle has a length of c and the triangle's legs have lengths of a and b, then $a^2 + b^2 = c^2$.

Engage

Ask, "What is a right triangle?" Review the fact that a right triangle has a right angle, which measures 90 degrees. Remind the students that the side that is opposite the right angle is called the hypotenuse. Elicit the fact that the triangle's other two angles are acute—their measurements are less than 90 degrees—and the sides that are opposite these angles are called the legs of the right triangle.

On the board, draw a right triangle with squares on each of its sides. Ask, "Do you suppose that any special relationship exists among the areas of these three squares?" Many students will suspect (or have already learned) that the area of the square on the hypotenuse is equal to the sum of the areas of the squares on the legs (see fig. 6.1). In this investigation, the students will explore this relationship.

Explore

Give each student a copy of the activity sheet "Take a Look at Those Squares." Assign the students in groups of two (up to four, if necessary) to work at classroom computers with the applet Squaring the Triangle, which comes into play in step 5. If you have only one computer in your classroom but can project a screen image for everyone to see, you can gather your students together for a whole-class consideration of this step. If you have no classroom computers at all, your students can still complete the activity sheet successfully by making right triangles on grid paper.

In steps 1–4, the students inspect different right triangles with squares constructed on their sides. They determine the areas of the squares (as well as their side lengths) and make conjectures about a relationship among the areas. To get your students started, you may want to walk them through step 1. Depending on their level of mathematical understanding, they may need some guidance at the outset.

Note that all the right triangles in steps 1–4 appear on grids, with all vertices at the intersections of grid lines. In steps 1 and 2, the legs of the triangles lie on grid lines as well (see fig. 6.2). Thus, in these steps, the sides of the squares on the triangles' legs lie on grid lines, making the areas of these squares very easy to determine. The area of the square on the hypotenuse is another matter, however. Although this square's vertices are at the intersections of grid lines, its sides do not lie on grid lines, and thus finding the square's area is more problematic.

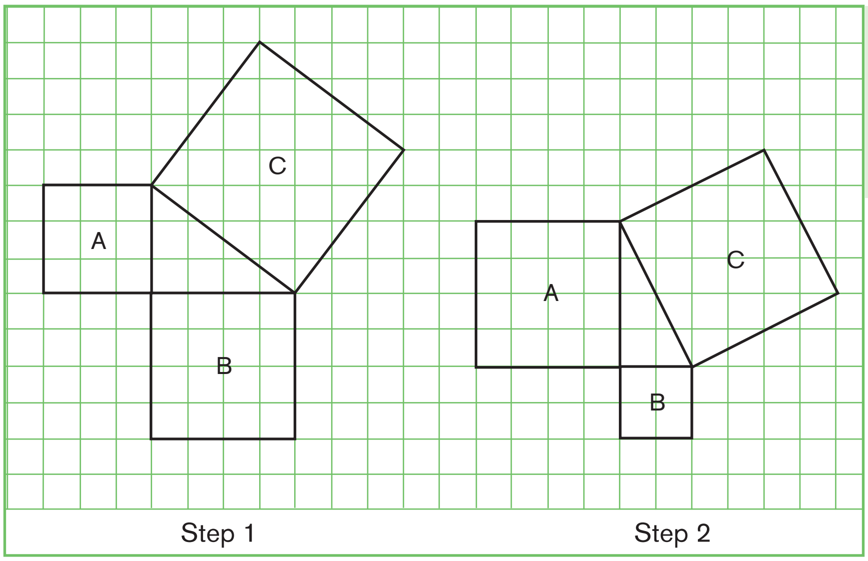


Fig. 6.2.

In both step 1 and step 2, the sides of the squares on the legs of the right triangle lie on grid lines, but the sides of the square on the hypotenuse do not.

You can guide your students in decomposing this square along grid lines into four congruent right triangles and an enclosed square that is bounded on all four sides by grid lines (see fig. 6.3). Then they can easily find the area of the square on the hypotenuse by finding the areas of the component shapes that they have defined. To find the area of one of the triangles, they can apply the formula

$$A = \frac{1}{2}bh.$$

After the students have found the total area of the square on the hypotenuse by adding the areas of the four congruent right triangles

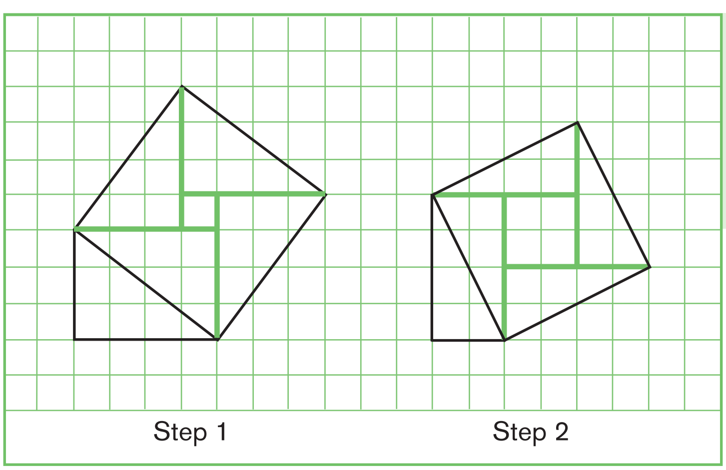
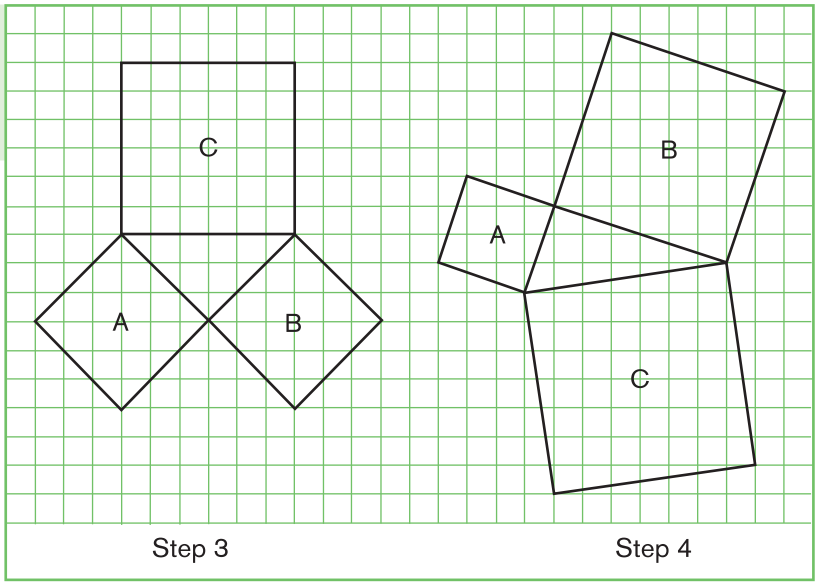


Fig. 6.3.

The square on the hypotenuse of the right triangle in step 1 and step 2, decomposed along grid lines into four congruent triangles and an enclosed square, with all vertices at the intersections of grid lines and the enclosed square, they can find the side length of the larger square as the square root of this total area.

Fig. 6.4.

Only one side of the right triangle in step 3 lies on a grid line, and no side of the right triangle in step 4 lies on a grid line.



If you don't want to have your students decompose the square on the hypotenuse to find its area, you can have them count the squares, combining and accounting for partial squares. This can be a tedious process, however, with opportunities for error.

Another possibility is to have the students work the other way around—from side length to area instead of from area to side length. If you wish, your students can simply measure the side of the square by using a ruler calibrated in centimeters and millimeters, determining measures to the nearest tenth of a centimeter. (The figures on the activity page appear on centimeter grids for this purpose.) Then the students can square the side length to calculate the area of the square.

Select the method that suits your purposes, and let your students complete steps 2–4 on their own. These steps have the same form as step 1: the students find the areas and side lengths of all three squares on the sides of a right triangle, and then they speculate about the relationship among the areas of the squares. The right triangle in step 1 is a neat 3-4-5 triangle, but it is the only right triangle with all whole-number side lengths that the students encounter in steps 1–4. (For example, the length of the hypotenuse of the right triangle in step 2 is equal to $\sqrt{20}$ linear grid units. Students who decompose the square on the hypotenuse as shown in figure 6.3 and use a calculator to estimate $\sqrt{20}$ might give the length as approximately 4.472 units; students who are measuring with a ruler might estimate it as 4.4 or 4.5 centimeters.)

Steps 3 and 4 show triangles that are oriented in different ways on the grid (see fig. 6.4). In each case, however, the triangle's vertices continue to be at the intersections of grid lines, and as a result, the students can use the same method of decomposition as before. However, depending on the right triangle's orientation with respect to grid lines, the students may have to decompose two squares on the triangle's sides, or all three squares, instead of just one, as in steps 1 and 2 (see fig. 6.5).

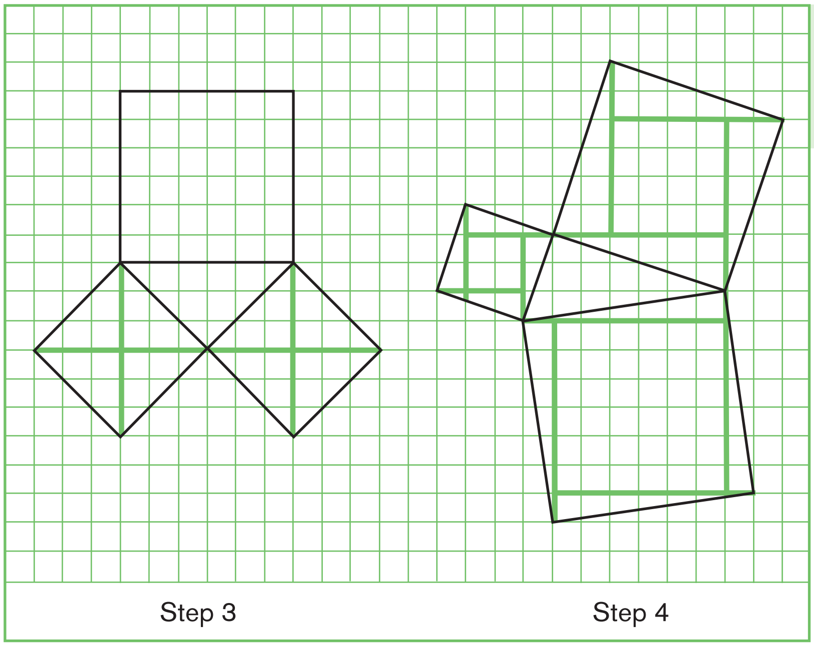


Fig. 6.5.

Decomposing squares on the sides of right triangles in steps 3 and 4 to find their areas easily

Moreover, as figure 6.5 illustrates, the students will observe in step 3 that they can decompose each of the squares on the legs of the right triangle into the four congruent triangles shown without creating an enclosed, smaller square. (Alternatively, they can decompose each of these squares into two congruent triangles.) The orientation of the legs of this right triangle with respect to the grid—each leg forms a 45-degree angle with the grid lines—eliminates the small square from the decomposition. Despite the slight variations in steps 3 and 4, however, partitioning continues to be a relatively easy way for the students to determine the areas of the squares constructed on the sides of the triangles, as figure 6.5 shows.

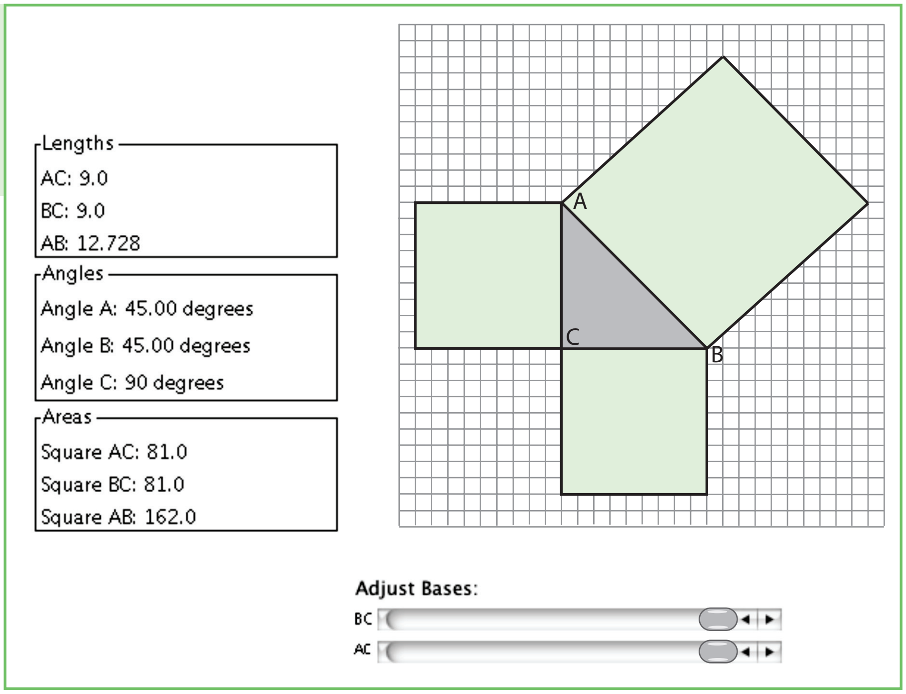
The students' work in steps 1–4 will probably quickly suggest to them—if they weren't already aware—that the area of the square on the hypotenuse is equal to the sum of the areas of the squares on the legs. Students whose understanding of proof is at level 1 will recognize the need to prove this idea, but on the basis of their work with a few specific examples, they may want to claim that the relationship that they have observed in the examples always holds, without being able to construct an argument that goes beyond an assertion such as, "I think it's true, because that's what happened in each case."

The activity continues, however, leading the students gently to level 2 by helping them recognize the insufficiency of their four examples to prove their conjecture. Step 5 invites them to test more examples—the natural impulse of those who have attained level 2. If they have access to the applet Squaring the Triangle (see fig. 6.6), they can use it in this step to find and check more varied examples.

The applet displays a right triangle ABC with a square on each of its three sides. The interactive Pythagorean construction appears on a grid and looks very much like the constructions that the students have worked with so far—especially in steps 1 and 2, where the legs of the right triangle appear on grid lines. The left side of the applet screen displays data on the triangle's side lengths and angles, as well as the

Fig. 6.6.

The applet Squaring the Triangle lets users adjust the lengths of the legs of a right triangle and see the resulting Pythagorean construction and its numerical data.



areas of the squares constructed on its sides. The students can use two slides below the figure to adjust AC and BC—the lengths of the triangle's legs. Any adjustments that they make register instantly in the figure and the accompanying data.

Encourage your students to explore different conditions as they work with the applet and complete the chart in step 5, entering data on squares constructed on the sides of five new triangles. For example, they can investigate cases in which leg AC of the right triangle is shorter than leg BC, cases in which leg BC is shorter than leg AC, and cases in which legs AC and BC are equal. In each instance, they should make observations about the relationship among the areas of the squares. Figure 6.7 shows the chart with sample values from the applet.

Fig. 6.7.

The chart in step 5, with sample values obtained from the applet Squaring the Triangle

Lengths (linear grid units)			Areas (square grid units		
AC	BC	AB	$(AC)^2$	$(BC)^2$	$(AB)^2$
5.0	4.0	6.403	25.0	16.0	41.0
3.0	8.0	8.544	9.0	64.0	73.0
2.0	8.0	8.246	4.0	64.0	68.0
9.0	1.0	9.055	81.0	1.0	82.0
5.0	5.0	7.071	25.0	25.0	50.0

If you are doing step 5 as a whole-class demonstration on one computer, or if you do not have a computer in your classroom, give your students grid paper and have them make other right triangles, draw the squares on the sides, and find the squares' areas and side lengths. They can fill in the chart in step 5 with the data from their own explorations. If you are demonstrating the applet on a single computer, you may be able to recreate their examples and confirm their results. Step 6 wraps up this part of the investigation by asking studetns to make a clear statement of their conjecture on the basis of all the work that they have done so far.

Evaluate

One way to review the results of your students' work is to make a large chart based on the one that the students themselves complete in step 5 (see fig. 6.7). Use a sheet of chart paper, the board, or an overhead transparency for your chart, and display the solutions for the problems in steps 1–4 as well as a sampling of data on different triangles and constructed squares in step 5.

Spend some time discussing how the students determined the length of the hypotenuse in each of these problems. Although many students will relate their reasoning to area of a square or a direct measurement of the length, some may already be applying the Pythagorean theorem informally on the basis of "hearsay" or a few examples, without being able to say for sure that the theorem is true. Students at this level need to continue to collect examples in order to solidify their conjecture; thus, it is important to make sure that the class verifies the areas of the squares in one fashion or another.

Be sure to talk about side lengths that are irrational numbers. Discuss each case to probe students' understanding of the particular estimates that they made.

Call several students to the board to write the general statements that they made in step 6 about the relationship among the areas of the squares. Let the class examine each statement, and discuss the need for clear and precise wording. Name the theorem that they have developed, and ask them if they think that they could prove it for any right triangle.

Extend

Your students are now set to think about a general proof of the Pythagorean theorem. Their work with right triangles with squares constructed on the sides has led them to a conjecture, which they have tested with multiple examples. They have also probably begun to recognize that one successful example is much like another in its ability or inability to prove the conjecture. Moreover, they may have discovered that amassing successful examples is something like sitting in a car that is spinning its wheels. Everything under the hood is functioning as well as anyone could wish, but the car is nevertheless churning in the same spot without making any headway. Is the conjecture true in every case? At this point, the students think that it is, but they would like to know for sure, and they are persuaded that having this certainty would be valuable. Yet, they probably do not know how to show that the conjecture is true in every case.

The students have gone from level 1 to level 2 to level 3 in their thinking about the Pythagorean theorem. At level 3, students are aware of the need for a generalized proof, and, although they are unable to construct a valid proof unaided, they can understand and follow a proof at an appropriate level of difficulty. Indeed, by using geometric and algebraic methods, students can explore a more general proof of the

Pythagorean theorem. The proof considered here depends on some knowledge of and skill with symbolic algebraic representations.

Give each student one or two sheets of centimeter grid paper as well as copies of the activity sheets “Can You Prove It?” and “Sets of Right Triangles for ‘Can You Prove It?’” Again have the students work in pairs (or in small groups of up to four members) so that they can discuss their ideas as they work. Write a clear and complete statement of the theorem on the board: The sum of the areas of squares constructed on the legs of a right triangle is equal to the area of a square constructed on the hypotenuse.

In step 1, the students cut out three sets of four congruent right triangles. Leg a is longer than leg b in one set, equal to leg b in another set, and shorter than leg b in a third set. In step 2, the students work with the sets one at a time on grid paper, using each set of four triangles to make a square that has a side length of a + b and circumscribes a second square with a side length of c (see fig. 6.8). (The students assume, but do not prove that the inscribed figure is a square. They could make the proof by considering the straight angle on the side of the larger square. This angle is formed by the acute angles of a right triangle together with an angle of the inscribed figure, so this angle must measure 90 degrees.) In step 3, the students draw the three sets of circumscribed and inscribed squares on grid paper, as they appear in figure 6.8, with the sides of the circumscribed squares on grid lines.

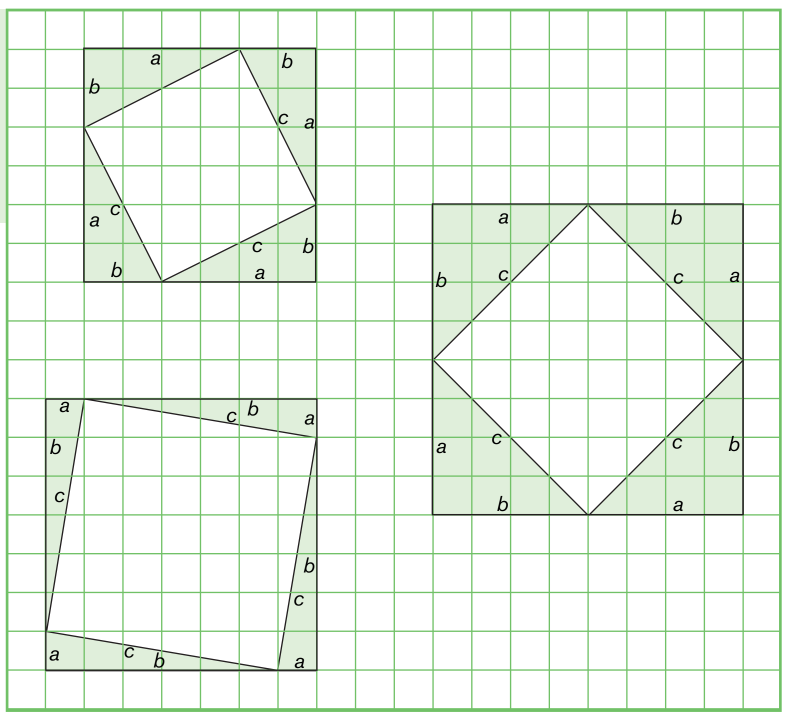


Fig. 6.8.

A set of four congruent right triangles with legs of length a and b and a hypotenuse of length c can form a square that has a side length of a + b and circumscribes a square with a side length of c.

Steps 4, 5, and 6 all have the same form. For each square with an inscribed square, the students consider that they can express the area of the circumscribed square in two ways: (1) as the side length squared and (2) as the sum of the areas of the component regions. If they think of the area of the circumscribed square in the first way, they write the expression

$$(a + b)^2, \text{ or } a^2 + 2ab + b^2,$$

for its area. If, however, they think of the area of this square in the second way, they write the expression

$$4\times\left(\frac{a\times b}{2}\right)+c^2, \text{ or } 2ab+c^2$$

for the area. In the case of each square with an inscribed square, the students consider that these expressions are equivalent:

$$a^2 + 2ab + b^2 = 2ab + c^2.$$

Thus, $a^2 + b^2 = c^2$. The sum of the areas of the squares on the legs of the triangle is equal to the area of the square on the hypotenuse.

Students in grades 6–8 may not have had sufficient experience in manipulating algebraic expressions to follow this proof with ease. The binomial expansion of $(a + b)^2$ may be particularly challenging to those who have not begun to study algebra formally. If your students have difficulty with these expressions and manipulations, approach the proof in a different way.

To do so, have your students draw on grid paper three new copies of the circumscribed (larger, or "outer") squares that they created with the sets of right triangles. After they have drawn these three squares with side length a + b, have them use their sets of cut-out triangles one at a time, again partitioning the respective areas of these squares, but in a new way, as shown in figure 6.9. This time, let them use each set of four triangles to form two a-by-b rectangles. Direct them to join the rectangles in each pair at a vertex in such a way that the long sides of the rectangles form a right angle. In each case, have them then locate this construction inside the corresponding larger square, as shown.

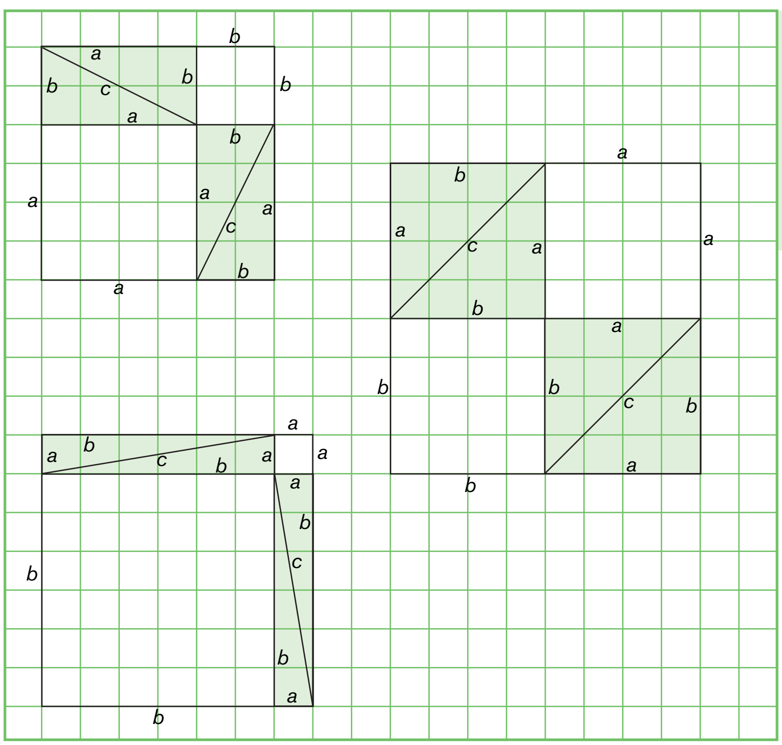


Fig. 6.9.

A set of four congruent right triangles with legs of length a and b and a hypotenuse of length c can partition a square with side length a + b in a different way to help prove that prove that $a^2 + b^2 = c^2$.

The students can compare their new drawing with their original drawing in each case. They can see that when they arranged the triangles in the original way, making the inscribed square with side length c, they naturally represented the area of the circumscribed square as $2ab + c^2$. However, when they arrange the triangles in the new way, as two a 3 b rectangles, they just as naturally represent the area of the same square as $a^2 + 2ab + b^2$. They can also see clearly that taking away the four congruent triangles from the original square leaves c^2, and taking them away from the new square leaves $a^2 + b^2$. But since the areas of the squares are equal, $a^2 + b^2 = c^2$.

This work also gives the students a geometric representation of the algebraic expansion of $(a + b)^2$ presented earlier. The new partitioning offers an area model for the multiplication of (a + b) times (a + b) that shows the product, $a^2 + 2ab + b^2$, very concretely. Use your students' understanding of the distributive property of multiplication over addition to help them understand this result. Point out that the product applies the distributive property twice:

$$\begin{aligned}(a+b)\times(a+b) &= \big((a+b)\times a\big)+\big((a+b)\times b\big) \quad \text{(first application of the distributive property)}\\ &= (a\times a)+(b\times a)+(a\times b)+(b\times b) \quad \text{(second application of the property)}\\ &= a^2+ba+ab+b^2\\ &= a^2+2ab+b^2.\end{aligned}$$

As a final extension of the activity, you might want to give your students an opportunity to experience the usefulness of the Pythagorean theorem. Present several right triangles, giving two side measurements and asking them to use the theorem to determine the missing side length. Have them find the length of a leg in some cases and the length of the hypotenuse in others. Emphasize that the certainty that the relationship holds in every case—certainty that the generalized proof gives—allows them to make these determinations. Stress the fact that this important theorem is indispensable to the study of geometry.

Conclusion

Being able to reason is essential to understanding mathematics. Students need regular opportunities to develop ideas and make, explore, and justify conjectures. In this way, they learn to expect that they will always be able to make sense of mathematics. Initially, students may rely more on inductive reasoning, but as they work with more sophisticated and abstract concepts, they can begin to reason deductively as well. Although no one expects students in the middle grades to construct formal proofs routinely or without assistance, you can begin to prepare your students for the rigorous thinking of proof by giving them opportunities to justify their reasoning and, whenever possible, to engage in activities that make them aware of the need for, and the methods of, formal proof.