A SIMPLIFIED FORMULA IN REDUCING THE POWERS OF TANGENT AND COTANGENT USING SQUARES OF TANGENT THEOREM

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Abstract

The objective of this paper is to determine a simplified formula in reducing the powers of tangent and cotangent from exponent 4 to 1 with any value of n. The squares of tangent theorem will be used as the initial approach to attain the formula desired. The traditional way of reducing the powers of tangent and cotangent will be shown and compared to the empirical one. Results showed that using the new formula developed the simplification of $Tan^4 n\theta$ and $Cot^4 n\theta$ using squares of tangent theorem made the procedure fast and reliable. This formula offers a variety of use in the field of mathematics especially in Trigonometry. From the squares of tangent theorem the two new formulas will emerge.

Keywords**:** Squares of Tangent Theorem, Half Angle identity, multiple angle formula, half angle formula, and trigonometric identities.

1. INTRODUCTION

The process of solving mathematical problems is a burden to some students especially if math is their waterloo. Dr. Euler a well- known mathematician discovered a formula that will cure mathematical illnesses. The process of reducing the powers of sine and cosine was earlier studied that lead to the development of some formula gearing to a simplified approach of integration techniques. Half angle formulas play a vital role in the development of mathematical formula in trigonometry and calculus. It is the initial step commonly used in evaluating multiple angle formulas. The other formulas used are the other multiple angle formulas like the double angle formula and the sum to product and product to sum formula. Tangent function usually used the identity involving sine and cosine to simplify the expression. Trigonometric identities are widely used also to simplify expressions (Fehribach, 2006; Suello, 2015).

The use of Half angle formula for sine and cosine functions is an initial guide on how to evaluate and simplify different functions. This is manifested in study of some mathematicians wherein they developed a simplified approach to teaching mathematics. With the continuous development of mathematical formula a complex equations becomes an easy one (Dampil, 2014).

The objective of this paper is to develop a new formula that will simplify the solution in expressing $Tan^4 n\theta$ and $Cot^4 n\theta$ in terms of cosine function with exponent 1. Some examples are given using the old method and the new one. The results of the new formula show that the new one saves time therefore easier to use. Square theorem for tangent is also known as half angle identity (Proof Week, 2014).

Half angle identities for sine, cosine and tangent (Kuang & Kase, 2012, p. 101; Swokowski & Cole, 2012, p. 574; Bible, n.d.).

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$$\sin^2 u = \frac{1 - \cos 2u}{2}$$

$$\cos^2 u = \frac{1 + \cos 2u}{2}$$

$$\tan^2 u = \frac{1 - \cos 2u}{1 + \cos 2u}$$

2. PROCEDURE OF THE OLD METHOD

To reduce the exponents of tangent function the square theorem or half angle identity for tangent is used. The formula for tangent was derived using the formula of square of sine and cosine written above. Normally it is being factored to reduce the exponent. All functions with equivalent identity are substituted before manipulating and simplifying the equation.

If you are to express tan^4u in terms of the cosine function with exponent 1 the procedure will be as follows:

Factor $\tan^4 u$

$$\tan^4 u = (\tan^2 u)(\tan^2 u)$$

Substitute

$\tan^2 u$ by $(\frac{1 - \cos 2u}{1 + \cos 2u})$

$$\tan^4 u = (\frac{1 - \cos 2u}{1 + \cos 2u})\ (\frac{1 - \cos 2u}{1 + \cos 2u})$$

Multiply

$$\tan^4 u = \frac{1 - 2\cos 2u + \cos^2 2u}{1 + 2\cos 2u + \cos^2 2u}$$

But $\cos^2 2u = \frac{1 + \cos 4u}{2}$

Substitute

$$\tan^4 u = \frac{1 - 2\cos 2u + \frac{1 + \cos 4u}{2}}{1 + 2\cos 2u + \frac{1 + \cos 4u}{2}}$$

Simplify

$$\tan^4 u = \frac{1 - 2\cos 2u + \frac{1}{2} + \frac{\cos 4u}{2}}{1 + 2\cos 2u + \frac{1}{2} + \frac{\cos 4u}{2}}$$

$$\tan^4 u = \frac{\frac{3}{2} - 2\cos 2u + \frac{\cos 4u}{2}}{\frac{3}{2} + 2\cos 2u + \frac{\cos 4u}{2}}$$

$$\tan^4 u = \frac{3 - 4\cos 2u + \cos 4u}{3 + 4\cos 2u + \cos 4u}$$

Similarly, if you are to express $tan^4 2u$ in terms of the cosine function with exponent 1 the procedure will be:

Factor $\tan^4 2u$

$\tan^4 2u = (\tan^2 2u)(\tan^2 2u)$

Substitute

$\tan^2 2u$ by $(\frac{1 - \cos 4u}{1 + \cos 4u})$

$\tan^4 2u = (\frac{1 - \cos 4u}{1 + \cos 4u})(\frac{1 - \cos 4u}{1 + \cos 4u})$

Multiply

$\tan^4 2u = \frac{1 - 2\cos 4u + \cos^2 4u}{1 + 2\cos 4u + \cos^2 4u}$

But $\cos^2 4u = \frac{1 + \cos 8u}{2}$

Substitute

$$\tan^4 2u = \frac{1 - 2\cos 4u + \frac{1 + \cos 8u}{2}}{1 + 2\cos 4u + \frac{1 + \cos 8u}{2}}$$

Simplify

$$\tan^4 2u = \frac{1 - 2\cos 4u + \frac{1}{2} + \frac{\cos 8u}{2}}{1 + 2\cos 4u + \frac{1}{2} + \frac{\cos 8u}{2}}$$

$$\tan^4 2u = \frac{\frac{3}{2} - 2\cos 4u + \frac{\cos 8u}{2}}{\frac{3}{2} + 2\cos 4u + \frac{\cos 8u}{2}}$$

$$\tan^4 2u = \frac{3 - 4\cos 4u + \cos 8u}{3 + 4\cos 4u + \cos 8u}$$

Another example is express $tan^4 3u$ in terms of cosine function with exponent 1. The same procedure will be followed:

Factor $\tan^4 3u$

$\tan^4 3u = (\tan^2 3u)(\tan^2 3u)$

Substitute

$\tan^2 3u$ by $(\frac{1 - \cos 6u}{1 + \cos 6u})$

$\tan^4 3u = (\frac{1 - \cos 6u}{1 + \cos 6u})(\frac{1 - \cos 6u}{1 + \cos 6u})$

Multiply

$\tan^4 3u = \frac{1 - 2\cos 6u + \cos^2 6u}{1 + 2\cos 6u + \cos^2 6u}$

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But $\cos^2 6u = \frac{1+\cos 12u}{2}$

Substitute

$$\tan^4 3u = \frac{1 - 2\cos 6u + \frac{1+\cos 12u}{2}}{1 + 2\cos 6u + \frac{1+\cos 12u}{2}}$$

Simplify

$$\tan^4 3u = \frac{1 - 2\cos 6u + \frac{1}{2} + \frac{\cos 12u}{2}}{1 + 2\cos 6u + \frac{1}{2} + \frac{\cos 12u}{2}}$$

$$\tan^4 3u = \frac{\frac{3}{2} - 2\cos 6u + \frac{\cos 12u}{2}}{\frac{3}{2} + 2\cos 6u + \frac{\cos 12u}{2}}$$

$$\tan^4 3u = \frac{3 - 4\cos 6u + \cos 12u}{3 + 4\cos 6u + \cos 12u}$$

Generalizing the results for the three examples given

$$\tan^4 u = \frac{3 - 4\cos 2u + \cos 4u}{3 + 4\cos 2u + \cos 4u}$$

$$\tan^4 2u = \frac{3 - 4\cos 4u + \cos 8u}{3 + 4\cos 4u + \cos 8u}$$

$$\tan^4 3u = \frac{3 - 4\cos 6u + \cos 12u}{3 + 4\cos 6u + \cos 12u}$$

We can say that for every function tan^4nu the simplified expression in terms of cosine function with exponent 1 is:

$$\tan^4 nu = \frac{3 - 4\cos 2nu + \cos 4nu}{3 + 4\cos 2nu + \cos 4nu}$$

In the case of cotangent function with exponent 4, the same method will be applied:

If you are to express cot^4u in terms of the cosine function with exponent 1 the procedure will be as followed:

Use reciprocal identity for $\cot^4 u$ then factor

$$\cot^4 u = \frac{1}{\tan^4 u}$$

$$\cot^4 u = \frac{1}{\frac{1-\cos 2u}{1+\cos 2u}}$$

$$\cot^4 u = \frac{1+\cos 2u}{1-\cos 2u}$$

$$\cot^4 u = (\cot^2 u)(\cot^2 u)$$

Substitute $\cot^2 u$ by $(\frac{1+\cos 2u}{1-\cos 2u})$

$$\cot^4 u = (\frac{1+\cos 2u}{1-\cos 2u})\,(\frac{1+\cos 2u}{1-\cos 2u})$$

Multiply

$$\cot^4 u = \frac{1+2\cos 2u\ +\cos^2 2u}{1-2\cos 2u\ +\cos^2 2u}$$

But $\cos^2 2u = \frac{1+\cos 4u}{2}$

Substitute

$$\cot^4 u = \frac{1+2\cos 2u\ +\frac{1+\cos 4u}{2}}{1-2\cos 2u\ +\frac{1+\cos 4u}{2}}$$

Simplify

$$\cot^4 u = \frac{1+2\cos 2u\ +\frac{1}{2}+\frac{\cos 4u}{2}}{1-2\cos 2u\ +\frac{1}{2}+\frac{\cos 4u}{2}}$$

$$\cot^4 u = \frac{\frac{3}{2}+2\cos 2u\ +\frac{\cos 4u}{2}}{\frac{3}{2}-2\cos 2u\ +\frac{\cos 4u}{2}}$$

$$\cot^4 u = \frac{3+4\cos 2u\ +\cos 4u}{3-4\cos 2u\ +\cos 4u}$$

Similarly, if you are to express $cot^4 2u$ in terms of the cosine function with exponent 1 the procedure will be:

Use reciprocal identity for $\cot^4 2u$ then factor

$$\cot^4 2u = \frac{1}{\tan^4 2u}$$

$$\cot^4 2u = \frac{1}{\frac{1-\cos 2u}{1+\cos 2u}}$$

$$\cot^4 2u = \frac{1+\cos 2u}{1-\cos 2u}$$

$$\cot^4 2u = (\cot^2 2u)(\cot^2 2u)$$

Substitute $\cot^2 2u$ by $(\frac{1+\cos 4u}{1-\cos 4u})$

$$\cot^4 2u = \left(\frac{1+\cos4u}{1-\cos4u}\right)\left(\frac{1+\cos4u}{1-\cos4u}\right)$$

Multiply

$$\cot^4 2u = \frac{1+2\cos4u + \cos^2 4u}{1-2\cos4u + \cos^2 4u}$$

But $\cos^2 4u = \frac{1+\cos8u}{2}$

Substitute

$$\cot^4 2u = \frac{1+2\cos4u + \frac{1+\cos8u}{2}}{1-2\cos4u + \frac{1+\cos8u}{2}}$$

Simplify

$$\cot^4 2u = \frac{1+2\cos4u + \frac{1}{2} + \frac{\cos8u}{2}}{1-2\cos4u + \frac{1}{2} + \frac{\cos8u}{2}}$$

$$\cot^4 2u = \frac{\frac{3}{2} + 2\cos4u + \frac{\cos8u}{2}}{\frac{3}{2} - 2\cos4u + \frac{\cos8u}{2}}$$

$$\cot^4 2u = \frac{3+4\cos4u + \cos 8u}{3-4\cos4u + \cos 8u}$$

Another example is express $cot^4 3u$ in terms of cosine function with exponent 1. The same procedure will be followed:

Use reciprocal identity for $\cot^4 3u$ then factor

$$\cot^4 3u = \frac{1}{\tan^4 3u}$$

$$\cot^4 3u = \frac{1}{\frac{1-\cos6u}{1+\cos6u}}$$

$$\cot^4 3u = \frac{1+\cos6u}{1-\cos6u}$$

$$\cot^4 3u = (\cot^2 3u)(\cot^2 3u)$$

Substitute $\cot^2 3u$ by $\left(\frac{1+\cos6u}{1-\cos6u}\right)$

$$\cot^4 3u = \left(\frac{1+\cos6u}{1-\cos6u}\right)\left(\frac{1+\cos6u}{1-\cos u}\right)$$

Multiply

$$\cot^4 3u = \frac{1+2\cos6u + \cos^2 6u}{1-2\cos6u + \cos^2 6u}$$

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But $\cos^2 6u = \frac{1+\cos 12u}{2}$

Substitute

$$\cot^4 3u = \frac{1 + 2\cos 6u + \frac{1+\cos 12u}{2}}{1 - 2\cos 6u + \frac{1+\cos 12u}{2}}$$

Simplify

$$\cot^4 3u = \frac{1 + 2\cos 6u + \frac{1}{2} + \frac{\cos 12u}{2}}{1 - 2\cos 6u + \frac{1}{2} + \frac{\cos 12u}{2}}$$

$$\cot^4 3u = \frac{\frac{3}{2} + 2\cos 6u + \frac{\cos 12u}{2}}{\frac{3}{2} - 2\cos 6u + \frac{\cos 12u}{2}}$$

$$\cot^4 3u = \frac{3 + 4\cos 6u + \cos 12u}{3 - 4\cos 6u + \cos 12u}$$

Generalizing the results for the three examples given

$$\cot^4 u = \frac{3 + 4\cos 2u + \cos 4u}{3 - 4\cos 2u + \cos 4u}$$

$$\cot^4 2u = \frac{3 + 4\cos 4u + \cos 8u}{3 - 4\cos 4u + \cos 8u}$$

$$\cot^4 3u = \frac{3 + 4\cos 6u + \cos 12u}{3 - 4\cos 6u + \cos 12u}$$

We can say that for every function cot^4nu the simplified expression in terms of cosine function with exponent 1 is:

$$\cot^4 nu = \frac{3 + 4\cos 2nu + \cos 4nu}{3 - 4\cos 2nu + \cos 4nu}$$

Therefore, instead of simplifying the function tan^4nu and cot^4nu into a function with exponent 1 using the long method we can use the simplified formula which is:

$$\tan^4 nu = \frac{3 - 4\cos 2nu + \cos 4nu}{3 + 4\cos 2nu + \cos 4nu}$$

$$\cot^4 nu = \frac{3 + 4\cos 2nu + \cos 4nu}{3 - 4\cos 2nu + \cos 4nu}$$

3. PROCEDURE OF THE NEW METHOD

With the new formula we can solve the same problem with different values of n. This makes the procedure easy and solution shorter.

EXAMPLE:

Express the following example in terms of cosine function with exponent 1 using the formula written below

$$\tan^4 nu = \frac{3 - 4\cos 2nu + \cos 4nu}{3 + 4\cos 2nu + \cos 4nu}$$

$$\cot^4 nu = \frac{3 + 4\cos 2nu + \cos 4nu}{3 - 4\cos 2nu + \cos 4nu}$$

1. $\tan^4 4u = \dfrac{3 - 4\cos 2nu + \cos 4nu}{3 + 4\cos 2nu + \cos 4nu}$
$= \dfrac{3 - 4\cos 2(4)u + \cos 4(4)u}{3 + 4\cos 2(4)u + \cos 4(4)u}$
$= \dfrac{3 - 4\cos 8u + \cos 16u}{3 + 4\cos 8u + \cos 16u}$

2. $\tan^4 5u = \dfrac{3 - 4\cos 2nu + \cos 4nu}{3 + 4\cos 2nu + \cos 4nu}$
$= \dfrac{3 - 4\cos 2(5)u + \cos 4(5)u}{3 + 4\cos 2(5)u + \cos 4(5)u}$
$= \dfrac{3 - 4\cos 10u + \cos 20u}{3 + 4\cos 10u + \cos 20u}$

3. $\tan^4 6u = \dfrac{3 - 4\cos 2nu + \cos 4nu}{3 + 4\cos 2nu + \cos 4nu}$
$= \dfrac{3 - 4\cos 2(6)u + \cos 4(6)u}{3 + 4\cos 2(6)u + \cos 4(6)u}$
$= \dfrac{3 - 4\cos 12u + \cos 24u}{3 + 4\cos 12u + \cos 24u}$

4. $\cot^4 4u = \dfrac{3 + 4\cos 2nu + \cos 4nu}{3 - 4\cos 2nu + \cos 4nu}$
$= \dfrac{3 + 4\cos 2(4)u + \cos 4(4)u}{3 - 4\cos 2(4)u + \cos 4(4)u}$
$= \dfrac{3 + 4\cos 8u + \cos 16u}{3 - 4\cos 8u + \cos 16u}$

5. $\cot^4 5u = \dfrac{3 + 4\cos 2nu + \cos 4nu}{3 - 4\cos 2nu + \cos 4nu}$
$= \dfrac{3 + 4\cos 2(5)u + \cos 4(5)u}{3 - 4\cos 2(5)u + \cos 4(5)u}$
$= \dfrac{3 + 4\cos 10u + \cos 20u}{3 - 4\cos 10u + \cos 20u}$

6. $\cot^4 6u = \dfrac{3 + 4\cos 2nu + \cos 4nu}{3 - 4\cos 2nu + \cos 4nu}$
$= \dfrac{3 + 4\cos 2(6)u + \cos 4(6)u}{3 - 4\cos 2(6)u + \cos 4(6)u}$
$= \dfrac{3 + 4\cos 12u + \cos 24u}{3 - 4\cos 12u + \cos 24u}$

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4. CONCLUSION

The traditional way of reducing the powers of tangent and cotangent is usually done by factoring before applying trigonometric identities. The equivalent identities is substituted then simplified before the final answered is achieved. Simplifying this kind of form is a very long process. With the use of this new formula for tangent and cotangent coming from the squares of tangent theorem reducing the powers became easier. The process to solve this shorter which require only the formula itself. The formula presented is easy to since it is the generalized formula for tangent and cotangent with exponent 4 only. This concludes that reducing the exponents of the said function can be easily attained if the formula is sufficiently followed. This is very useful in trigonometry and in other areas of Mathematics.

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