FOUNDATIONS OF A GENERAL THEORY OF FUNCTIONS OF A VARIABLE COMPLEX MAGNITUDE

Bernhard Riemann

1851, Göttingen

Typesetter's preface – Caveat lector

This is the LaTeX-ed version of an English translation of Bernhard Riemann's 1851 thesis, which marked the beginning of the geometrical theory of complex analysis, titled *Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse*. The translation is obtained from
`http://science.larouchepac.com/riemann/page/31`,
and the original German version, available on
`http://www.emis.de/classics/Riemann/`,
served as a handy and authentic reference. Full collection of English translations of Riemann's papers is also published by Kendrick Press. It should be noted that this transcription is almost word-by-word copying the translation, and the typesetter DOES NOT understand what all the sentences mean. (Only very few minor changes are made and most of them are given explicit notes.) It also should be noted that one page is missing in the translation and the original German words are copied there for completeness. What's more, I did not include the notes appearing and the end of the translation because presumably they are not Riemann's original notes since they do not appear in the German version. For any correction and suggestion please email the typesetter, whose personal email address is `iamdelta7@gmail.com`.

Overview[1]

1

A variable complex magnitude $w = u + vi$ is called a function of another variable magnitude $z = x + yi$, if the function so changes that $\frac{\mathrm{d}w}{\mathrm{d}z}$ is independent of $\mathrm{d}z$. This definition is based on the definition that this always occurs when the slope (dependency) of magnitude $w(z)$ is given by an analytic expression.

[1] This overview is almost completely based on Riemann.

2

Points O and Q on planes A and B represent the values of variable complex magnitudes z and w, and an image of one plane projected on to the other represents their dependency (slope) on each other.

3

If the dependency is of such a kind (chapter 1) so that $\frac{\mathrm{d}w}{\mathrm{d}z}$ is independent of $\mathrm{d}z$, then the original and its image are similar down to their smallest segments.

4

The condition that $\frac{\mathrm{d}w}{\mathrm{d}z}$ is independent of $\mathrm{d}z$ is identical with the following:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad , \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

from it we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad . \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

5

For the location of point O, we will replace plane A with a bounded surface T extended through plane extended over plane A. Branch points of this surface.

6

On the cohesion (connectedness) of a plane.

7

The integral $\int \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} \right) \mathrm{d}T$ which extends across all of surface T, is equal to $-\int (X \cos \xi + Y \cos \eta) \, \mathrm{d}s$ throughout its entire boundary when X and Y are arbitrary, continuous functions in all points on T for x and y.

8

The introduction of coordinates s and p of point O in regard to an arbitrary line. We will established the mutual dependency (slope) of the signs $\mathrm{d}s$ and $\mathrm{d}p$ in such a way that $\frac{\partial x}{\partial s} = \frac{\partial y}{\partial p}$.

9

Application of the theorem in chapter 7, if $\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} = 0$ in all surface segments.

10

Those are the conditions, under which a function u, that is inside of a surface T which simply covers A, and that is generally satisfying the equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ is universally finite and continuous along with all of its differential quotients.

11

The characteristics of such a function.

12

The conditions under which a function $w(z)$, that is inside of a surface T is simply connected and that simply covers A, is universally finite and continuous together with its differential equations.

13

The discontinuities of such a function in an interior point.

14

The extension of the theorems in chapter 12 and 13 to the points in the interior of an arbitrary, level surface.

15

The general characteristics of the image of a surface T, which extends in plane A, onto a surface S which extends in plane B. We can geometrically represent the value of a function $w(z)$ through this.

16

The integral $\int \left[\left(\frac{\partial \alpha}{\partial x} - \frac{\partial \beta}{\partial y} \right)^2 + \left(\frac{\partial \alpha}{\partial y} + \frac{\partial \beta}{\partial x} \right)^2 \right] \mathrm{d}T$ which extends throughout all of plane T always has a minimum value for one function. This is caused by the changes in α around continuous function, or around functions which are only discontinuous in a couple of points, with these functions being equal to 0 at the margin. If we exclude the discontinuities in isolated points through modification, then we will get a minimum value for only one function.

17

This is the foundation, using the boundary method, of a theorem that was presupposed in the previous chapter.

18

Assume that a function $\alpha+\beta i$ for x, y, is given in a level surface T, T being arbitrarily connected and broken down into a simply connected T^* through cuts. Then the function is finite, and has $\int\left[\left(\frac{\partial\alpha}{\partial x}-\frac{\partial\beta}{\partial y}\right)^2+\left(\frac{\partial\alpha}{\partial y}+\frac{\partial\beta}{\partial x}\right)^2\right]\mathrm{d}T$ extending throughout the whole plane. Then we can turn this function into a function of z only and always through one method, through adding a function of $\mu+\nu i$ of x, y, which is conditioned in the following ways

1. μ equals 0 at the margin, and ν is given for one point.
2. The changes μ undergoes are in T, and the changes ν undergoes are in T^*, are only in isolated points, and are only so discontinuous that $\int\left[\left(\frac{\partial\mu}{\partial x}\right)^2+\left(\frac{\partial\mu}{\partial y}\right)^2\right]\mathrm{d}T$ and $\int\left[\left(\frac{\partial\nu}{\partial x}\right)^2+\left(\frac{\partial\nu}{\partial y}\right)^2\right]\mathrm{d}T$ remain finite throughout the whole surface, and the latter expression remains equal on both sides of the cut.

19

A rough calculation about the conditions that are necessary and sufficient to define a function of a complex argument inside a given numerical domain.

20

The previous method of defining a function by numerical operations contains superfluous elements. As a result of the observations that we have carried out here, we can trace the range of the parts that define a function back to the necessary standard.

21

Two simple connected surfaces can relate to each other so continuously that every point in one surface corresponds to the point that is continuously progressing with it in the other surface, in addition to their being similar down to their smallest parts. Naturally, any one inteior point and any one boundary point can be arbitrarily given a corresponding point. This is what defines the relation for all points.

22

Final remarks.

1

If we consider z to be a variable magnitude which can gradually assume all possible real values, then we call w a function of z, when each of its real values corresponds to a single value of undetermined magnitude such as w. If w also constantly changes while z continuously goes through all of the values lying between two fixed values, then we call this function within these intervals a continuous or a *continuirlich* function.[2]

Obviously, this definition does not set up any absolute law between the individual values of the function, because when we assign a determinate value to this function, the way in which it continues outside of this interval remains totally arbitrary.

We can express the slope function (dependence) of magnitude $w(z)$ by a mathematical law so that we can find the corresponding value of w for every value of z through determinate numerical operations (*Grössenoperationen*). Previously, people have only considered a certain kind of function (functiones continuae according to Euler's usage) as having the ability of being able to determine all the values of z lying between a given interval by using that same slope function law; however. in the meantime, new research has shown that there are analytic expressions that can represent each and every continuous function for a given interval. This holds, regardless of whether the slope function of magnitude w (magnitude z) is conditionally defined as an arbitrary given numerical operation, or as an determinate numerical operation. As a result of the theorems mentioned above, both concepts are congruent.

But the situation is different when we do not limit the variability of magnitude z to real values, but instead allow complex values of the form $x+yi$ (where $i=\sqrt{-1}$).

Assume that $x+yi$ and $x+yi+\mathrm{d}x+\mathrm{d}y\,i$ are two infinitesimally slightly different values for magnitude z, which correspond to the values $u+vi$ and $u+vi+\mathrm{d}u+\mathrm{d}v\,i$ for magnitude w. So then, if the slope function of magnitude $w(z)$ is an arbitrary given one, then generally speaking, the ratio $\dfrac{\mathrm{d}u+\mathrm{d}v\,i}{\mathrm{d}x+\mathrm{d}y\,i}$ changes for the values for $\mathrm{d}x$ and $\mathrm{d}y$, because when we have $\mathrm{d}x+\mathrm{d}y\,i=\varepsilon e^{\varphi i}$,

[2] Riemann's original German for *continuous* is *stetig* or its variations, and the translation on Internet reads *constant* here and occasionally after. The typesetter considers this translation to be inappropriate, and changed the word here and after. Yet sometimes this change can not be adequately made, as the typesetter does not have the time to proofread every single sentence with understanding, and I'm sorry for that. – Typesetter

then

$$\begin{aligned}\frac{\mathrm{d}u+\mathrm{d}v\,i}{\mathrm{d}x+\mathrm{d}y\,i} &= \frac{1}{2}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)i \\ &\quad +\frac{1}{2}\left[\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}+\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)i\right]\frac{\mathrm{d}x-\mathrm{d}y\,i}{\mathrm{d}x+\mathrm{d}y\,i} \\ &= \frac{1}{2}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)i \\ &\quad +\frac{1}{2}\left[\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}+\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)i\right]e^{-2\varphi i}\end{aligned}$$

However, regardless of the manner in which we define w as a function of z through compounding these simple numerical operation, the value of the differential quotient $\frac{\mathrm{d}w}{\mathrm{d}z}$ is always independent of the special values of differential $\mathrm{d}z$.[3] Obviously, not every arbitrary slope function of complex magnitude w(complex magnitude z) can be expressed in this manner.

We will base our following investigations on the characteristic that we just emphasized and which belongs to all functions that are in a any way definable by numerical operations. We will consider such functions independently of their expressions, and will proceed from the following definition without proving for now its universal validity and its adequacy for the concept of a slope function that can be expressed by numerical operations.

We will call a variable complex magnitude w a function of another variable complex magnitude z, if the first function changes in such a way in connection with the second function, that the value of the differential quotient $\frac{\mathrm{d}w}{\mathrm{d}z}$ is independent of the value of the differential $\mathrm{d}z$.

2

We can consider magnitude w, as well as magnitude z, to be variable magnitudes, which can assume any complex value. Our comprehension of such variability, which extends itself into a connected field of two dimensions, can be substantially facilitated by acquaintance with spatial perception.

Assume that every value $x + yi$ of magnitude z is represented by point O on plane(*Ebene*) A, whose rectangular coordinates are x, y and that every value $u + vi$ of magnitude w is represented by point Q on plane B, whose rectangular coordinates are u and v. Each slope function of magnitude $w(z)$ will then show up as a slope function of point Q's position according to point O's position. Assuming that every value for z corresponds to a determinate value for w which in turn is continuously changing itself in conjunction with z, or in other words,

[3]This proposition is obviously justified in all cases where, by means of the rules of differentiation, expressing $\frac{\mathrm{d}w}{\mathrm{d}z}$ by z is permitted by expressing w by z. This proposition's rigorous universal validity is valid from now on in. – Riemann's original footnote.

that u and v are continuous functions of x and y, then every point on plane A becomes a point on plane B, and generally speaking, each line corresponds to one line, and each connected surface segment of the plane corresponds to one other connected surface segment. We can thus represent this slope function of magnitude $w(z)$ as an image of plane A projected on plane B.

3

We will now investigate what properties this image has when w is a function of complex magnitude z, i.e., when $\frac{\mathrm{d}w}{\mathrm{d}z}$ is independent from $\mathrm{d}z$.

We will designate an indeterminate point on plane A in the vicinity of O by o, and its image on plane B by q, in addition to designating the values of magnitudes z and w at these points by $x+yi+\mathrm{d}x+\mathrm{d}y\,i$ and $u+vi+\mathrm{d}u+\mathrm{d}v\,i$. We can then consider $\mathrm{d}x$, $\mathrm{d}y$ and $\mathrm{d}u$, $\mathrm{d}v$ to be rectangular coordinates for points o and q in reference to points O and Q as the points of origin. And when we have $\mathrm{d}x+\mathrm{d}y\,i=\varepsilon e^{\varphi i}$ and $\mathrm{d}u+\mathrm{d}v\,i=\eta e^{\psi i}$, then the magnitudes $\varepsilon,\varphi,\eta,\psi$ become polar coordinates for these points with the same points of origin. Now if o' and o'' are any two determinate positions of point o within an infinitesimal vicinity of O, and if we express the meaning of the remaining symbols that are dependent on o' and o'' by corresponding indices, then the postulate

$$\frac{\mathrm{d}u'+\mathrm{d}v'\,i}{\mathrm{d}x'+\mathrm{d}y'\,i}=\frac{\mathrm{d}u''+\mathrm{d}v''\,i}{\mathrm{d}x''+\mathrm{d}y''\,i}$$

and consequently

$$\frac{\mathrm{d}u'+\mathrm{d}v'\,i}{\mathrm{d}u''+\mathrm{d}v''\,i}=\frac{\eta'}{\eta''}e^{(\psi'-\psi'')i}=\frac{\mathrm{d}x'+\mathrm{d}y'\,i}{\mathrm{d}x''+\mathrm{d}y''\,i}=\frac{\varepsilon'}{\varepsilon''}e^{(\varphi'-\varphi'')i},$$

from which $\frac{\eta'}{\eta''}=\frac{\varepsilon'}{\varepsilon''}$ and $\psi'-\psi''=\varphi'-\varphi''$, i.e. the angles $o'Oo''$ and $q'Qq''$ are equal in the triangles $o'Oo''$ and $qQ'q''$ and the sides that include them are proportional to each other. Thus we find that two infinitely small and corresponding triangles are similar, and that this also universally holds for the smallest segments of surface A and their images on surface B. The only exception to this theorem occurs in those special cases when the mutually corresponding changes in magnitudes z and w do not occur in a finite relationship. This was tacitly assumed in the derivation.[4]

[4] One should see the following on this subject:
"Universal Solution to the Problem: Describing the segments of a given surface so that the images of what are described are similar down to their smallest parts" by C.F.Gauss. (This was published in *Astronomische Abhandlungen*, herausgegaben von Schumacher. Drittes Heft. Altona. 1825, as the answer to the question in the contest set uo by the Royal Society of the Sciences in Copenhagen of the year 1822.)(Gauss Werke Bd. IV, p. 189.) – Riemann's original footnote.

4

When we transform the differential quotients $\dfrac{\mathrm{d}u + \mathrm{d}y\, i}{\mathrm{d}x + \mathrm{d}y\, i}$ into the form

$$\frac{\left(\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial x} i\right) \mathrm{d}x + \left(\dfrac{\partial v}{\partial y} - \dfrac{\partial u}{\partial y} i\right) \mathrm{d}y\, i}{\mathrm{d}x + \mathrm{d}y\, i},$$

then it is evident that we will get the same values for any two values of $\mathrm{d}x$ and $\mathrm{d}y$ if

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

These conditions are also necessary and sufficient in order to have $w = u + vi$ become a function of $z = x + yi$, and the following individual terms of this function also comes from the conditions :

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

These two equations form the foundation for the investigation of the characteristics that an individual term in such a function is considered to have. We will allow the proof for the most important of these characteristics to be proceeded by a thorough consideration of the entire function. But first, however, we will discuss and define some points which belong to the universal domain in order to smooth out the ground for this investigation.

5

In the following observations, we will limit the variability of magnitudes x and y to a finite domain in which we consider the location of point O as no longer on plane A itself, but a surface T, which extends over plane A. We have chosen this wording, which makes it inoffensive to speak of planes lying on top of one another in order to leave open the possibility that the location of point O repeatedly occurs over the same segment of the plane. Nevertheless, we assume in such a case that surface segments which are lying on top of each other are not connected along a line, for this would cause convolutions in the planes, or a fissure in the segments lying on top of each other.

We can then fully determine the number of plane segments that are lying on top of each other in every surface segment when the boundary is given according to location and direction (i.e., the boundary's inner and outer side); however, the actual course of the boundary can still develop differently.

In reality, if we draw an arbitrary line through the segment of the plane covered by the surface, then the number of surface segments lying above one another will only change when we cross the boundary. Naturally, when we cross the boundary going from outside to the inside, the change is +1, while going in the opposite direction the change is -1, and this holds everywhere. Every bordering surface segment along the edge of this line continues to carry on in a totally determined manner so long as the line does not touch the boundary, because the only place any undeterminateness can occur at all is in an isolated point, either in an isolated point on the line itself, or in an isolated point a finite distance from the line. Therefore, when we limit our observations to a passing section of line L that is inside the surface, and limit it on both sides to a sufficiently small surface surface strip, we can speak of determinately contiguous surface segments, the amount of which is equal on every side, and which we can describe on the left as $a_1, a_2, \ldots a_n$, and on the right as $a'_1, a'_2, \ldots a'_n$, if we give the line a definite direction. Every surface segment a will then be continued in a surface segment a', and of course, this will be universally the same for the entire course of line L, even though it can change in a couple of its points for special positions of L. Let us assume that above a certain point σ, (i.e., along the anterior segment of L) the surface segments $a_1, a_2, \ldots a_n$ are connected in succession to surface segments $a'_1, a'_2, \ldots a'_n$, but that below the same point there are the surface segments $a_{\alpha_1}, a_{\alpha_2}, \ldots a_{\alpha_n}$, where $\alpha_1, \alpha_2, \ldots \alpha_n$ are only differentiated through the series $1, 2, \ldots n$. Then a point above σ that stands for a_1 in a'_1 will end up in surface segment a_{α_1} if it crosses over to the left side under σ. And when this same point encircles point σ going from left to right, then the index of the surface segment in which this point finds itself will run through the numbers $1, \alpha_1, \alpha_2, \ldots, \mu, \alpha_\mu, \ldots$ in series. In this series, as long as the term 1 does not repeat itself, it is inevitable that all the terms will differ from each other because with any given arbitrary average term α_μ it is imperative that μ and all other previous terms back to 1 precede it in direct succession. However, when the term 1 repeats itself in a series of terms which are evidently smaller than n and equal to m, then the remaining terms have to follow in that very same order. The point which was circling around σ will then, in conformity with m, revert back to circulating in the same surface segment, and is limited by m to the surface segments lying on top of each other, which are united to each other at a single point above σ. We will call this point a branch point of the $(m-1)$ order of surface T. By applying this same operation to the remaining $(n-m)$ surface segments, these surface segments, if they do not develop otherwise, will break down into a system of $m_1, m_2, \ldots$ surface segments. In this case, there would also be branch points of the $(m_1 - 1)$th, $(m_2 - 1)$th order in point σ.

[Wenn die Lage und der Sinn der Begrenzung von T und die Lage ihrer Windungspunkte gegeben ist, so ist T entweder vollkommen bestimmt oder doch auf eine endliche Anzahl verschiedener Gestalten beschränkt; Letzteres, in so fern sich diese Bestimmungsstücke auf verschiedene der auf einander liegenden Flächentheile beziehen können.

Eine veränderliche Grösse, die für jeden Punkt O der Fläche T, allgemein zu reden, d. h. ohne eine Ausnahme in einzelnen Linien und Punkten[5] auszuschliessen,

[5]Diese Beschränkung ist zwar nicht durch den Begriff einer Function an sich geboten, aber um Infinitesimalrechnung auf sie anwenden zu können erforderlich: eine Function, die in allen Punkten einer Fläche unstetig ist, wie z. B. eine Function, die für ein commensurables x und ein commensurables y den Werth 1, sonst aber den Werth 2 hat, kann weder einer Differen-

Einen bestimmten mit der Lage desselben stetig sich ändernden Werth annimmt, kann offenbar als eine Function von x, y, angesehen werden, und überall, wo in der Folge von Functionen von x, y die Rede sein wird, werden wir den Begriff derselben auf diese Art festlegen.

Ehe wir uns jedoch zur Betrachtung solcher Functionen wenden, schalten wir noch einige Erörterungen über den Zusammenhang einer Fläche ein. Wir beschränken uns dabei auf solche Flächen, die sich nicht längs einer Linie spalten.][6]

6

When this possibility does not exist, we will consider these surface segments to be separate.

Our investigation of the continuity (connectedness) of a surface is based on cutting up the plane into transverse segments, i.e., through lines which simply cut across the interior – not cutting one point more than once – going from boundary point to boundary point. The latter boundary point can also lie in a segment that is added to the boundary, or thus, in an earlier point on the transverse cut.

We say a surface is connected, when every transverse breaks it down into pieces so that they are either simply connected ot multiply connected.

Pedagogical Theorem 1 *A simply connected surface A is broken down by any cut ab into two simple connected pieces.*

Assume that one of these pieces is not partitioned by cut *cd*. We can then obviously see that although none of this piece's endpoints, nor endpoint c, nor both endpoints fall on line *ab*, we can get a connected surface by cutting A that is contrary to our postulate by establishing contact along all of line *ab*, or along part of *cb*, or along part of *cd*.

Pedagogical Theorem 2 *When we break surface T down into a system of T_1 of m_1 simply connected surface segments by using an amount n_1 of cuts* [7] *q_1 and when we break it down into a system T_2 of m_2 surface segments by using an amount n_2 of cuts q_2, then $n_2 - m_2$ cannot be larger than $n_1 - m_1$.*

If any line q_2 does not entirely fall into the q_1 cut system, then at the same time it also becomes one or more of the cuts q_2' across surface T_1. We can consider the endpoints of cut q_2' to be:

tiation, noch einer Integration, also (unmittelbar) der Infinitesimalrechnung überhaupt nicht unterworfen werden. Die für die Fläche T hier willkürlich gemachte Beschränking wird sich später (Art. 15) rechtfertigen.

[6] These paragraphs are missing in the translation, and the typesetter therefore has to put the original German text here for completeness. – Typesetter

[7] Dividing up a surface through various cuts always means a successive division, i.e., that kind of division, where the planes that result from a cut get partitioned again by a new cut. - Riemann's original footnote

1. the $2n_2$ endpoints of cut q_2, except when their ends coincide with a segment of the line system q_1,

2. any average point on cut q_2, where it joins up with any average point on line q_1, except when the former point is already on another line q_1, i.e., when it coincides with one end of cut q_1.

We shall now: define μ as how frequently lines from both systems meet or cross in their course (we will count a single common point twice), define ν_1 as how frequently an end section q_1 coincides with a middle section q_2, and define ν_2 as how frequently an end section q_2 coincides with a middle section q_1. Finally, we will define ν_3 as how frequently an end section q_1 coincides with an end section q_2. Given the above Nr. 1 $2n_2 - \nu_2 - \nu_3$, Nr.2 $\mu - \nu_1$ produce the endpoints for cut q_2'. But if we take both cases together, then they contain all the endpoints, and each endpoint only once. Therefore, the number of cuts is:

$$\frac{2n_2 - \nu_1 - \nu_3 + \mu - \nu_2}{2} = n_2 + s$$

We can get the number of cuts q_1' of surface T_2 by a totally similar deduction, which is based on the lines q_1,

$$\frac{2n_1 - \nu_1 - \nu_3 + \mu - \nu_2}{2}$$

thus $= n_1 + s$. Surface T has now been obviously transformed, through the $n_2 + s$ cuts q_2', into that very same surface in which T_2 is broken down by $n_1 + s$ cuts q_1'. However, what we get out of T_1 as a result of m_1 are simply connected pieces, which break down according to Theorem 1, and through $n_2 + s$ cuts, into $m_1 + n_2 + s$[8] surface segments. From this it would have to follow that if m_2 were smaller than $m_1 + n_2 - n_1$, the number of surface segments T_2 produced by the $n_1 + s$ cuts would have to be more than $n_1 + s$, which is absurd.

According to this theorem, if n does not define the number of cuts, then m describes the number of pieces, $n - m$ being constant for all partitions of a plane into simply connected pieces. For if we observe any two determinate partitions by n_1 cuts into m_1 pieces, and by n_2 into m_2 pieces, then if the former pieces are simply connected, $n_2 - m_2 \leq n_1 - m_1$, while if the later pieces are simply connected, then $n_1 - m_1 \leq n_2 - m_2$. When both conditions occur, $n_2 - m_2 = n_1 - m_1$.

We can appropriately call this number the "degree of connection" of a plane; it is

– according to definition – decreased by 1 with every cut,

not changed by a line simply cutting from an interior point through the interior to a boundary point or to an earlier point on the cut, and

increased by 1 through an interior cut that is universally simple and that has two endpoints,

[8] Here the English translation reads $m_1 + n_2$ and s, changed according to the German version. – Typesetter

because the first case can be changed by one cut, but the last case can only be changed by having two cuts in one cut.

And last of all, we can obtain a degree of connection from a surface consisting of several pieces if we add the degrees of connection of these various pieces together.

In the following section, we will generally limit ourselves to a surface consisting of one section (piece) and we will suit ourselves by using the artificial description of a simple, twofold, etc. connection for its connection, so that what we mean by an n-fold connected surface is one which is divisible by $n-1$ cuts into a simply connected surface.

When we consider the slope function of a boundary's connectedness in relation to the connectedness of a surface it is readily apparent that:

1. The boundary of a simply connected surface necessarily consists of one encircling line.
   If the boundary consisted of fragmented pieces, then cut q, which links a point in section (a region, piece) a with a point in another section b, would only be separating connected surface segments from each other. This would be so because inside the surface along a, a line would lead from one side of cut q to the opposite side, and therefore q would not partition the surface, which is contrary to the supposition.

2. Every cut either increases the number of sections in the boundary by 1, or decreases it by 1.

Cut q either connects a point on a boundary section a with a point on another boundary section b, – and in this case, all of these together form the series a, q, b, q forming a single boundary from one encircling piece –

or cut q connects two points on one boundary piece – and in this case the segment breaks down into two pieces through both of the end points of this cut. Both of these pieces now form, together with the cut, a section of the boundary that circles back into itself.

or finally, cut q ends at one of its earlier pints and we can consider it as composed of one line o that circles back into itself, and of another line L which connects a point on o with a point on boundary segment a, – in which case, o forms one part of, and a, L, o, L form another part of a boundary piece that circles back into itself.

So there are either – in the first place, only one boundary piece in place of two, – or in both of the latter examples two boundary portions in place of one, from which our proposition comes.

Therefore, the number of pieces comprising the boundary of an m-fold connected surface segment is either $= n$ or is smaller by a precise number.

We can even produce a corollary from this:

Corollary 3 *If the number of boundary pieces that an n-fold connected surface* *has is $= n$, then this surface breaks down into two separate pieces with every cut in the surface's interior that circles back into itself.*

The degree of connection is not changed as result of this, and the numbers of pieces in the boundary are increased by two; so if the surface were a connected one, it would have n-fold connectedness and $n + 2$ boundary pieces, which is impossible.

7

Assume that X and Y are two continuous functions of x, y which are in all points of the surface T, which in turn is extended over A. Then the integral that extends to all the elements $\mathrm{d}T$ in this surface

$$\int \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} \right) \mathrm{d}T = -\int (X \cos \xi + Y \cos \eta)\, \mathrm{d}s$$

if we describe ξ as the inclination against the x axis of a straight line drawn from the boundary towards the interior, for every point on the boundary, and if we describe η as the inclination against the y axis. And finally, this integral equals the other one if this integration covers collectively all of the elements $\mathrm{d}s$ that are on the boundary line.

In order to transform the integral $\int \frac{\partial X}{\partial x} \mathrm{d}T$ we will partition the segment of plane A that is covered by surface T into primary bands(*Elementarstreifen*) by means of a system of lines parallel to the x axis. And we will do this in such a way that every one of surface T's branch points falls on one of these line. As a result of this precondition, we get one or more differentiated trapezoidal shaped pieces developing from every one of these surface T segments that falls of one of the lines. Given then any undetermined primary band which segregates the element $\mathrm{d}y$ our of the y axis, this band's contribution to the value of $\int \frac{\partial X}{\partial x} \mathrm{d}T$ will obviously be $= \mathrm{d}y \int \frac{\partial X}{\partial x} \mathrm{d}x$, if this integral is extended through this or these straight line belonging to surface T, these straight lines falling on a normal proceeding from a point $\mathrm{d}y$. If we describe the lower endpoints of these lines (i.e., which correspond to the smallest values of x) as $O_{\prime}, O_{\prime\prime}, O_{\prime\prime\prime}$, the upper end points as O', O'', O''', the x-value in these points as $X_{\prime}, X_{\prime\prime}, \ldots\ X', X'', \ldots$, the matching elements which are segregated by the planar bands out of the boundary as $\mathrm{d}s_{\prime}, \mathrm{d}s_{\prime\prime}, \ldots\ \mathrm{d}s', \mathrm{d}s'', \ldots$, and the values of ξ in these as $\xi_{\prime}, \xi_{\prime\prime}, \ldots\ \xi', \xi'', \ldots$ then,

$$\int \frac{\partial X}{\partial x} \mathrm{d}x = -X_{\prime} - X_{\prime\prime} - X_{\prime\prime\prime} \ldots + X' + X'' + X''' \ldots$$

It is evident that angle ξ becomes acute at the lower end points, and obtuse at the higher endpoints. Therefore

$$\mathrm{d}y = \cos \xi_{\prime} \mathrm{d}s_{\prime} = \cos \xi_{\prime\prime} \mathrm{d}s_{\prime\prime} \ldots = -\cos \xi' \mathrm{d}s' = -\cos \xi'' \mathrm{d}s'' \ldots$$

Through substitution this value results in $\mathrm{d}s\int\frac{\partial X}{\partial x}\mathrm{d}x=-\sum X\cos\xi\mathrm{d}s$ where the summation relates to all the boundary elements which have $\mathrm{d}y$ as a projection in the y axis.

We can obviously exhaust all of the elements in surface T and all of the elements in the boundary by the integration of all $\mathrm{d}y$ that comes into consideration. Considering this environment, we get,

$$\int\frac{\partial X}{\partial x}\mathrm{d}T=-\int X\cos\xi\mathrm{d}s$$

And we get as a result of totally similar conclusions

$$\int\frac{\partial Y}{\partial y}\mathrm{d}T=-\int Y\cos\eta\mathrm{d}s$$

and consequently

$$\int\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}\right)\mathrm{d}T=-\int\left(X\cos\xi+Y\cos\eta\right)\mathrm{d}s$$

Q.E.D.

8

Consider a boundary line proceeding from an established starting point cut into a direction that will be defined later. We will describe the length of this boundary up to an undefined point O_0 by s. Next, consider the distance of a normal set up from point O_0 to an undefined point O which we will call p and which we will consider to be positive on the inside of the boundary. Then we can consider the values that x and y have in point O to be functions of s and p, and the partial differential quotients

$$\frac{\partial x}{\partial p}=\cos\xi,\ \frac{\partial y}{\partial p}=\cos\eta,\ \frac{\partial x}{\partial s}=\pm\cos\eta,\ \frac{\partial y}{\partial s}=\mp\cos\xi$$

in the points of the boundary line. Turn out so that in these differential quotients the upper notation shows in cases the direction, in which we consider magnitude s to be growing, includes an equal angle in with p, just as the x axis includes the angle with the y axis, when one is counter-posed to the other, the lower.[9] We will assume this direction to be such in all segments of the boundary so that

$$\frac{\partial x}{\partial s}=\frac{\partial y}{\partial p}\quad\text{and consequently}\quad\frac{\partial y}{\partial s}=-\frac{\partial x}{\partial p}$$

which does not at all essentially infringe upon our result' universality.

[9] The English translation here is incomprehensible, and the typesetter made some modification based on the original German version. – Typesetter

We can also expand these determinations to lines inside of T. And in order to determine the signs for dp and ds, if we want to continue their mutual dependency (slope function) as it was previously, we can add on a statement which will determine the signs for dp or ds. In creating such an encircling line, naturally we will indicate which of the surface segments separated by such a line also serves as this line's boundary. It is through this that we determine the sign for dp, not with an encircling line, but at its beginning point, i.e., at the endpoints where s assumes the smallest value.

When we introduce the values we got for $\cos\xi$ and $\cos\eta$ from the proven equations in the previous chapter we will then get, to the same extent as we got in the previous chapter,

$$\int\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}\right)\mathrm{d}T=-\int\left(X\frac{\partial x}{\partial p}+Y\frac{\partial y}{\partial p}\right)\mathrm{d}s=\int\left(X\frac{\partial y}{\partial s}-Y\frac{\partial x}{\partial s}\right)\mathrm{d}s$$

9

When we apply the theorem from the conclusion of the previous chapter to the situation where $\dfrac{\partial X}{\partial x}+\dfrac{\partial Y}{\partial y}=0$ in all the segments of the plane, then we get the following theorem:

1. If X and Y are finite and continuous for all the points in T, and if they provide satisfactory functions for the equation, $\dfrac{\partial X}{\partial x}+\dfrac{\partial Y}{\partial y}=0$ then if we expand through the whole boundary for T,

   $$\int\left(X\frac{\partial x}{\partial p}+Y\frac{\partial y}{\partial p}\right)\mathrm{d}s=0$$

   If we can now imagine an arbitrary surface T, that is stretched our over A, breaking down into pieces T_2 and T_3 in an arbitrary manner, then in relations to the boundaries for T_2, we can consider the integral $\int\left(X\dfrac{\partial x}{\partial p}+Y\dfrac{\partial y}{\partial p}\right)\mathrm{d}s$ to be the difference of the integral in relation to the boundary for T_1 and T_3, while in the case where T_3 runs right up to T_1's boundary, both integrals conceal each other out. However, all the remaining elements correspond to an element in the boundary of T_2.

   Through this transformation, we can get the following out of Theorem 1:

2. The value of the integral $\int\left(X\dfrac{\partial x}{\partial p}+Y\dfrac{\partial y}{\partial p}\right)\mathrm{d}s$, which covers the entire boundary of a surface that is extended over A, will remain constant during arbitrary expansions and contractions only if it does not gain of lose any surface segments as a result of this. If this were to happen, the preconditions for Theorem 1 would not be fulfilled.

   If the functions X and Y suffice for every surface segment of T in the differential equation that we just described, but if they are afflicted with discontinuity in isolated lines or points, then we can encapsulate every one of these lines and points in an arbitrarily small plane segment, like a seed pod. We then get the following by applying Theorem 2:

3. In reference to the entire boundary of T, the integral $\int \left(X\frac{\partial x}{\partial p} + Y\frac{\partial y}{\partial p} \right) \mathrm{d}s$ is equal to the sum of the integrals $\int \left(X\frac{\partial x}{\partial p} + Y\frac{\partial y}{\partial p} \right) \mathrm{d}s$ in relation to the encapsulation of all discontinuities. Naturally, this integral also has the same value for every one of these discontinuities, no matter how compact the boundaries are that encircle them.

   This value is necessarily equal to null for a simple discontinuous point, if the distance of point O from the discontinuous point ϱ becomes infinitely small at the same time that ϱX and ϱY do too. We can then introduce the polar coordinates ϱ, φ in reference to such a point as a starting point and in reference to an arbitrary initial direction. Finally, in order to encapsulate these polar coordinates, we can choose to draw a circle around them that has the radius ϱ, so that the integral that relates to this is

$$\int_0^{2\pi} \left(X\frac{\partial x}{\partial p} + Y\frac{\partial y}{\partial p} \right) \varrho \mathrm{d}\varphi$$

   Consequently, it cannot have a value for κ different than null, because just as we can always assume κ to be small, we can also assume ϱ to be too, so that irrespective of the symbol $\left(X\frac{\partial x}{\partial p} + Y\frac{\partial y}{\partial p} \right)$, ϱ can become smaller than $\frac{\kappa}{2\pi}$ for every value of φ. Consequently,

$$\int_0^{2\pi} \left(X\frac{\partial x}{\partial p} + Y\frac{\partial y}{\partial p} \right) \varrho \mathrm{d}\varphi < \kappa$$

4. Let us take a simple connected surface extended over A. If the integrals $\int \left(Y\frac{\partial x}{\partial s} - X\frac{\partial y}{\partial s} \right) \mathrm{d}s = 0$, and $\int \left(X\frac{\partial x}{\partial p} + Y\frac{\partial y}{\partial p} \right) \mathrm{d}s$ they being integrals that cover the whole boundary of every surface segment, then these integrals will have the same value for any two fixed point O_0 and O in relation to all lines going from O_0 to O in these integrals.

   The pair of lines s_1 and s_2 which connect the points O_0 and O form together a line s_3 that circles back into itself. This line in turn either has the property of being unable to cut across any point more than one, or the property of being capable of partition into several totally simple lines that circle back into themselves. It has this second property because when we want to go back to an earlier point, we can eliminate the segment which has become continuous in the meantime from an arbitrary point on these very same continuous lines. We can then consider what follows as a direct continuation of what went on before. However, every one of these lines partition the plane into a simply and into a twofold connected segment, and therefore it necessarily follows that one of these lines form the the entire boundary for one of these segments, while the integral $\int \left(Y \frac{\partial x}{\partial s} - X \frac{\partial y}{\partial s} \right) \mathrm{d}s$ extends through this plane equals zero in accordance with the proposition. This also holds true for the integral that extends through all of line s_3, if we consider magnitude s to be increasing everywhere in the same direction. Therefore, the integrals that extend through lines s_1 and s_2 must cancel each other out, if this direction remains unchanged, i.e., if it goes in one direction from O_0 to O, and in the other direction from O to O_0. So if the latter direction is changed, the integrals become equal.

If somewhere there is now an arbitrary surface T, in which, generally speaking, $\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} = 0$, when we can next exclude the inconsistencies if this is necessary, so that in the remaining surface sections

$$\int \left(Y \frac{\partial x}{\partial s} - X \frac{\partial y}{\partial s} \right) \mathrm{d}s = 0$$

for every surface segment. This is then partitioned by cuts into a simple connected surface T^*. ACCORDINGLY, our integral has the same value for every line that goes from a point O_0 to another O inside surface T^*. This value, for which the notation $\int_{O_0}^{O} \left(Y \frac{\partial x}{\partial s} - X \frac{\partial y}{\partial s} \right) \mathrm{d}s$ suffices as shorthand, holds O_0 to be fixed and O to be moving. We can also consider it to be a determinate function for every one of O's positions, regardless of the course of the connecting lines. Consequently, we can consider it to be a function of x, y.

We can express the change that occurs in this function by displacing O along an arbitrary linear element ds by $\left(Y \frac{\partial x}{\partial s} - X \frac{\partial y}{\partial s} \right) \mathrm{d}s$ and the change in this function in continuous for T^* everywhere, as well as being equal along both sides of a cut across T.

5. Therefore, when we consider O_0 to be fixed, the integral

$$Z = \int_{O_0}^{O} \left(Y \frac{\partial x}{\partial s} - X \frac{\partial y}{\partial s} \right) \mathrm{d}s$$

forms a function of x, y, which is continuous everywhere in T^*. However, when this function passes beyond the cuts in T, it changes around a function along the cut from being a branch point to being another constant magnitude. The partial differential quotient for this is

$$\frac{\partial Z}{\partial x} = Y \, , \frac{\partial Z}{\partial y} = -X$$

The changes which we brought about by passing beyond the cuts are dependent on having the same number of cuts as there are magnitudes that are independent of each other. For when we go this system of cuts backwards – doing the later segments first – this change is generally determined when its value is given at the beginning of every cut. However, the later values are independent of each other.

10

If we replace the functions that have been described by X up to now with $u\frac{\partial u'}{\partial x} - u'\frac{\partial u}{\partial x}$ and $u\frac{\partial u'}{\partial y} - u'\frac{\partial u}{\partial y}$ for Y, then $\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} = u\left(\frac{\partial^2 u'}{\partial x^2} + \frac{\partial^2 u'}{\partial y^2}\right) - u'\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$ and if the functions u and u' satisfy the equations

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \frac{\partial^2 u'}{\partial x^2} + \frac{\partial^2 u'}{\partial y^2} = 0$$

then

$$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} = 0$$

and we can find the application of the theorem in the previous chapter in the expression $\int\left(X\frac{\partial x}{\partial p} + Y\frac{\partial y}{\partial p}\right) \mathrm{d}s$ which is equal to $= \int\left(u\frac{\partial u'}{\partial p} - u'\frac{\partial u}{\partial p}\right) \mathrm{d}s$.

Now if in relation to function u we make the hypothesis that this function, together with its first differential quotient, does not tolerate any possible kind of discontinuity along a line, and if we also assume that function u becomes infinitely small for every discontinuous point as the distance ϱ of point O from those very same $\varrho\frac{\partial u}{\partial x}$ and $\varrho\frac{\partial u}{\partial y}$ does at the same time, then we can conclude from the notes to section III of the previous chapter that we can keep on disregarding the discontinuous in u.

Therefore,we can assume a value R of ϱ for every straight line that proceeds from an discontinuous point, so that

$$\varrho\frac{\partial u}{\partial \varrho} = \varrho\frac{\partial u}{\partial x}\frac{\partial x}{\partial \varrho} + \varrho\frac{\partial u}{\partial y}\frac{\partial y}{\partial \varrho}$$

$\varrho(u - U)$ and thus ϱu along with ϱ will become infinitely small at the same time. And according to the proposition, the same goes for which always finite at its lower end. We can also describe U as the value of u for $\varrho = R$, M, for every interval, regardless of the signs of the greatest value for the function $\varrho\frac{\partial u}{\partial \varrho}$. Then, following the same interpretation, $u-U$ will always be $< M(\log\varrho - \log R)$, and consequently, $\varrho\frac{\partial u}{\partial x}$ and $\varrho\frac{\partial u}{\partial y}$. Consequently, if u' is not burdened with any discontinuities the same also goes for

$$\varrho\left(u\frac{\partial u'}{\partial x}-u'\frac{\partial u}{\partial x}\right)\quad\text{and}\quad\varrho\left(u\frac{\partial u'}{\partial y}-u'\frac{\partial u}{\partial y}\right);$$

the cases discussed in the previous chapter also making their appearance in this.

We will even assume further, that surface T, which forms the site for point O, has extended over A everywhere, and that an arbitrary fixed point O_0, where u, x, y have the values of u_0, x_0, and y_0, is in this same extended surface. If we consider the magnitude $\frac{1}{2}\log\left((x-x_0)^2+(y-y_0)^2\right)=\log r$ to be a function of x, y, then it has the characteristic that

$$\frac{\partial^2\log r}{\partial x^2}+\frac{\partial^2\log r}{\partial y^2}=0$$

so that it is only subjected to a discontinuity when $x=x_0$, $y=y_0$. Thus in our case, this only occurs for one point on surface T.

According to Article 9, theorem III, when we replace u' with $\log r$, $\int\left(u\frac{\partial\log r}{\partial p}-\log r\frac{\partial u}{\partial p}\right)\mathrm{d}s$ the entire boundary around T is equal to this integral with regard to an arbitrary encirclement of point O_0. So when we want to select the periphery of a circle in this case, where R has a constant value, and where by starting out from one of the points on the periphery and proceeding in a fixed arbitrary direction, we can describe the arc up to O in terms of segments of the radius by then the integral directly above is equal to $-\int_0^{2\pi}u\frac{\partial\log r}{\partial r}r\mathrm{d}\varphi-\log r\int\frac{\partial u}{\partial p}\mathrm{d}s$ or therefore to $\int\frac{\partial u}{\partial p}\mathrm{d}s=0\,,=-\int_0^{2\pi}u\mathrm{d}\varphi$, whose value for an infinitely small r crosses over into $-u_0 2\pi$ when u is continuous in point O_0.

Therefore, in regard to the propositions we established for u and T, when we have an arbitrary point O_0 in which u is continuous inside the surface

$$u_0=\frac{1}{2\pi}\int\left(\log r\frac{\partial u}{\partial p}-u\frac{\partial\log r}{\partial p}\right)\mathrm{d}s$$

in relation to the entire boundary itself and

$$=\frac{1}{2\pi}\int_0^{2\pi}u\mathrm{d}\varphi$$

in relation to a circle drawn around O_0. We can draw the following conclusions from the first expression in this paragraph:

Pedagogical Theorem 4 *If a function u, which is inside of a surface T that itself simple covers plane A everywhere, generally satisfies the differential equation:* $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ *so that,*

1. *The point in which this differential equations is not fulfilled are not surface segments,*

2. *The points in which u, $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ become discontinuous, do not continuously satisfy any line,*

3. *The magnitudes $\varrho\frac{\partial u}{\partial x}$, $\varrho\frac{\partial u}{\partial y}$ become infinitely small for every discontinuous point as well as for the distance ϱ of point O from that same inconsistent point.*

4. *u excludes any discontinuity that can be cancelled out by changing its value in isolated point.*
   Then this function u is necessarily finite and continuous along with all of its differential quotients for all of the points inside this surface.

In reality, however, we will consider point O_0 to be movable so that in the expression, $\int\left(\log r\frac{\partial u}{\partial p} - u\frac{\partial \log r}{\partial p}\right)\mathrm{d}s$ only the values $\log r$, $\frac{\partial \log r}{\partial x}$, $\frac{\partial \log r}{\partial p}$ change. However, these magnitudes are also finite and continuous functions of x_0, y_0, for every element in the boundary, so long as O_0 remains inside of T, and in addition to all of their differential quotients. These finite and continuous functions can be expressed by the broken rational functions of these magnitudes, the functions that only have powers of r in their denominators. And this also holds for the value of our integral, and consequently for function u_0 itself, because under our earlier propositions, function u_0 could only have a value different from the value of our integral in those isolated points in which it would be discontinuous. And this possibility has been eliminated by proposition 4 of our theorems.

11

Using the same preconditions that we applied to u and T at the end of the last chapter, we get the following theorems:

I. When $u = 0$ along a line, and $\frac{\partial u}{\partial p} = 0$, then $u = 0$ everywhere.

Next we can prove that line λ, where $u = 0$ and $\frac{\partial u}{\partial p} = 0$, cannot form the boundary of surface segment a, where u is positive.

Given that this occurs, then we can cut a piece out of a. This piece's boundary is partially formed by λ, and partially by a circumferential line. In addition, this piece does not contain point O_0 which is centre for the circumferential line, and this whole construction is possible. Then when we describe O's polar coordinates in relation to O_0 by r, φ, we get

$$\int \log r\frac{\partial u}{\partial p}\mathrm{d}s - \int u\frac{\partial \log r}{\partial p}\mathrm{d}s = 0$$

expanding through this piece's entire boundary. As a consequence,

$$\int u \mathrm{d}\varphi + \log r \int \frac{\partial u}{\partial p} = 0,$$

our assumption for all of the arcs that also belong to the boundary, would be,

$$\int \frac{\partial u}{\partial p} \mathrm{d}s = 0 \qquad \text{or} \qquad \int u \mathrm{d}\varphi = 0$$

which is irreconcilable with our presumption, that u is positive in a's interior.

In a similar manner, we can also prove that equations $u = 0$ and $\frac{\partial u}{\partial p} = 0$ cannot occur in a boundary segment belonging to s surface piece b where u is negative.

So if $u = 0$ and $\frac{\partial u}{\partial p} = 0$ on a line in surface T, and if u were to be different from null in any one of surface T's segments, then such a surface segment would obviously have to be bounded either by this line itself, or by a surface segment where u would be equal to 0. So in any case it would be bounded by a line where u and $\frac{\partial u}{\partial p}$ would be equal to 0 and this would necessarily return us to one of the assumptions we negated a few lines back.

II. When we are given the values for u and $\frac{\partial u}{\partial p}$ along a line, then this defines u in all segments in T.

If u_1 and u_2 are any two determinate functions which satisfy the conditions that we imposed on function u, then these conditions also hold for their difference, $u_1 - u_2$, and we can show this right away by substituting this difference into these conditions. And if u_1 and u_2 as well as their first differential quotients, converge towards p when they are on a line, but they do not do so in another surface segment, then $u_1 - u_2 = 0$ and $\frac{\partial(u_1 - u_2)}{\partial p} = 0$ along this line, without being equal to 0 every else. This would then be contrary to Theorem I.

III. The points inside of T where u has a constant value necessarily form lines if u is not constant everywhere. These lines then divide those surface segments where u is larger from the surface segments where u is smaller.

This theorem is composed of the following conditions:

- u cannot have either a maximum or a minimum value in a point inside of T.
- u cannot be constant in only one section of the plane.
- the lines in which u equals a cannot bound both sides of the surface segment where $u - a$ has the same symbol.

As we can easily see, theorems which always have to lead to violating the equations we proved in the last chapter;

$$u_0 = \frac{1}{2\pi} \int_0^{2\pi} u \mathrm{d}\varphi \quad \text{or} \quad \int_0^{2\pi} (u - u_0) \mathrm{d}\varphi$$

are therefore impossible.

12

We will now return to considering a complex variable magnitude $w = u+vi$, and we will consider it generally (i.e., without excluding the exceptions in isolated lines and points.) This magnitude has a determinate value for every point O in surface T that continuously changes with point O's position, and in conformity with the equations

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial u}{\partial x}$$

we will characterise this property of w according to the way we did earlier and so we will call w a function of $z = x + yi$. In order to simplify what is coming, we will pre-establish that a discontinuity that can be eliminated by changing its value in an isolated point cannot occur in a function z.

First of all, we will attribute surface T with a simple connectedness and with simple expansion everywhere over plane A.

Pedagogical Theorem 5 *If function $w(z)$ does not have any break in its continuity anywhere along a line, and furthermore, if $w(z - z')$ becomes infinitely small as it approaches point O for any arbitrary point O' in the surface where $z = z'$, then this function is necessarily finite and continuous for all points inside the surface and for all of its differential quotients.*

The preconditions which we set up for the changes in magnitude w break down when we substitute $z - z' = \varrho e^{\varphi i}$ for u and v in

$$1) \quad \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0$$

and

$$2) \quad \frac{\partial u}{\partial y} + \frac{\partial x}{\partial x} = 0$$

for every segment in surface T; and when

3) function u and v are not discontinuous along a line;

4) ϱu and ϱv become infinitely small along with the distance from point O to O' for any point O'.

5) function u and v exclude any discontinuities that can be eliminated by changing their values in isolated points.

As a result of preconditions 2,3 and 4, integral $\int \left(u\frac{\partial x}{\partial s} - v\frac{\partial y}{\partial s} \right) \mathrm{d}s$ which extends to the boundaries of surface T, ends up being equal to 0 for every segment in surface T, according to chapter 9, III. According to chapter 9, IV, integral

$\int_{O_0}^{O}\left(u\frac{\partial x}{\partial s} - v\frac{\partial y}{\partial s}\right)\mathrm{d}s$ has the same value for every line going from O_0 to O. Additionally, when we consider O_0 as fixed, this integral forms function $U(x, y)$ which is necessarily continuous up to isolated point, and for which the differential quotient $\frac{\partial U}{\partial u} = u$ and $\frac{\partial U}{\partial y} = -v$ for every point, (according to 5). But by substituting these values for u and v, preconditions 1,2 and 3 change over into the conditions of the pedagogical theorem at the end of chapter 10. In this case therefore, function U, along with its differential quotients is finite and continuous for all points in T, and this also holds for the complex function $w = \frac{\partial U}{\partial x} - \frac{\partial U}{\partial y}i$ and its differential quotients according to z.

13

We will now investigate what happens when we assume, still retaining chapter 12's special preconditions, that $(z - z')w = \varrho e^{\varphi i}w$ no longer becomes infinitely small for a determinate point O' as we infinitely converge on point O. In this case, as point O converges infinitely close to point O', w becomes infinitely large. We can assume that when magnitude w does not remain with $\frac{1}{\varrho}$ in the same series, i.e., if both of their quotients approach a finite boundary, then at least the order of both magnitudes will be in such a finite ratio to each other, that a power of ϱ will result whose product in w for an infinitely small ϱ will be either infinitely small or remain finite. If μ is the exponent of such a power, and if n is the next largest whole number, then magnitude $(z - z')^n w = \varrho^n e^{n\varphi i}w$ will become infinitely small with ϱ, and therefore $(z - z')^{n-1}w$ is a function of z (because, $\mathrm{d}a\frac{\mathrm{d}(z - z')^{n-1}w}{\mathrm{d}s}$ is independent from $\mathrm{d}z$) which satisfies the preconditions in chapter 12 for these surface segments. Consequently, $(z - z')^{n-1}$ is also finite and continuous in point O'. If we describe its value in point O' by a_{n-1}, then $(z - z')^{n-1}w - a_{n-1}$ is a function which is continuously at this point, and which $= 0$. Therefore, it becomes infinitely small through. From this we can conclude according to chapter 12 that $(z - z')^{n-2}w - \frac{a_{n-1}}{z - z'}$ is a continuous function at point O'. By continuing this procedure we can see that w gets turned into a function which remains continuous and finite at point O' through subtracting an expression from the form $\frac{a_1}{z - z'} + \frac{a_2}{(z - z')^2} + \ldots + \frac{a_{n-1}}{(z - z')^{n-1}}$.

Therefore, when this change occurs according to the preconditions in chapter 12 so that function w becomes infinitely large as O converges infinitely on a point O' inside of surface T, then this infinite quality's order (when we consider a magnitude that is increasing in reverse relationship to the distance as an infinite magnitude of the first order) when it is infinite, will necessarily be a whole number. And if this number $= m$, then function w can be changed into a function that is continuous at this point O' by the addition of a function that contains $2m$ arbitrary constants.

Note: We consider a function as containing an arbitrary constant if the possible varieties that it agrees with encompass a continuous domain of one dimension.

14

The limitation which we established in chapter 12 & 13 for surface T are not essential for the validity of the results we achieved. It is plain that we can surround any point in the interior of an arbitrary surface with a piece of the same surface. This piece will have the same properties that were presupposed for that surface with the sole exception being the case where this point is a branch point in the surface.

In order to investigate this case, we will assume that we can draw surface T, or an arbitrary piece of it which contains a branch point of the $(n-1)$th order of O' where $z = z' = x' + y'\,i$, by means of the function $\zeta = (z - z')^{\frac{1}{n}}$, onto a different plane A. I.e., we can imagine the value of the function $\zeta = \xi + \eta\,i$ at point O by a point Θ, whose rectangular coordinates are ξ, η and which is represented in this latter plane. So we can consider Θ as the image of a point O. This means that we get a connected surface extended over A as an image of this segment of surface T. And as we will show very soon. this new surface which has the image of point O' in point Θ does not have any branch point.

In order to get a grasp of this mental image,we should think of a circle around point O with a radius R on plane A. We will also draw a chord parallel to the x-axis, where $z - z'$ becomes a real value. Then the piece of surface T which surrounds the branch point, and which we have cut out of the area by the circle, will then separate into scattered half circle shaped surface segments on both sides of the diameter in n, if R is kept sufficiently small. We will describe these surface segments by $a_1, a_2, \ldots, a_n$ on the side of the chord where $y - y'$ is positive, and those surface segments on the other side by $a'_1, a'_2, \ldots, a'_n$. We will also assume that $a_1, a_2, \ldots, a_n$ is the series associated with negative values of $z - z'$ and that $a'_1, a'_2 \ldots a'_n$ is the series associated with the positive values which is connected to $a'_n, a'_1, \ldots, a_{n-1}$. This way, a point that encircles point O' (in the required sense) runs through the series of surfaces $a_1, a'_1, a_2, a'_2, \ldots, a_n, a'_n$ and succeeds in getting back to a_1 through a'_n, which is an obvious assumption. Next, we will introduce polar coordinates for both planes by setting up $z = z' = \varrho e^{\varphi i}$, $\zeta = \sigma e^{\psi i}$ and we will select that value of $(z - z')^{\frac{1}{n}} = \varrho^{\frac{1}{n}} e^{\frac{\varphi}{n} i}$ for depicting surface segment a_1 whose expression comes under the assumption of $0 \leq \varphi \leq \pi$. So $\sigma \leq R^{\frac{1}{n}}$ and $0 \leq \psi \leq \dfrac{\pi}{n}$ will hold for all points in a_1 and the image of these points will all collectively be in plane A, in a sector stretching from $\psi = 0$ to $\psi = \dfrac{\pi}{n}$ of a circle drawn around Θ with a radius of $R^{\frac{1}{n}}$. Naturally, every point in a_1 immediately corresponds to a point in this sector that is constantly advancing along with it, and the reverse also holds. What follows then is that the image of surface a_1 is a simply connected surface extended over this sector. In a similar manner, the image for surface a'_1 is a sector stretching from $\psi = \dfrac{\pi}{n}$

to $\psi = \dfrac{2\pi}{n}$, the image for surface a_2 is a sector stretching from $\psi = \dfrac{2\pi}{n}$ to $\psi = \dfrac{3\pi}{n}$, and the image for surface a'_n is a sector stretching from $\psi = \dfrac{2n-1}{n}\pi$ to $\psi = 2\pi$ if we select φ for every point on this surface in the series between π and 2π, 2π and 3π ... $(2n-1)\pi$ and $2n\pi$ which is always possible, and which is only possible, in one way. These sectors also connect up with each other in the very same manner as do surfaces a and a' so that the points adjoining one another in one sector correspond to points adjoining one another on another sector. Therefore, we can combine these sectors into a connected image of one of the pieces of surface T that includes point O'. Obviously, this image is a surface that is simply extended over plane A.

A variable magnitude that has a determinate value for every point O also has a determinate value for every point Θ and the reverse also holds, because every O corresponds to only one Θ, and every Θ only corresponds to one O. Furthermore, if this variable magnitude is a function of z, then it is also a function of ζ, for when $\dfrac{\mathrm{d}w}{\mathrm{d}z}$ is independent of $\mathrm{d}z$, $\dfrac{\mathrm{d}w}{\mathrm{d}\zeta}$ is also independent of $\mathrm{d}\zeta$. The reverse also holds, and we get from this that we can apply the theorems from chapter 12 and 13 ro all function $w(z)$, even to the branch point O' if we consider them to be functions of $(z-z')^{\frac{1}{n}}$. This gives us the following theorem:

When function $w(z)$ becomes infinitely small through the finite convergence of O to a branch point $(n-1)$th order of O', then this infinite magnitude necessarily has the same order with a power of distance, as that whose exponent is a multiple of $\dfrac{1}{n}$. If this exponent is $= -\dfrac{m}{n}$, then this infinite magnitude can be changed into a function that is continuous at point O' through adding an expression of the form $\dfrac{a_1}{(z-z')^{\frac{1}{n}}} + \dfrac{a_2}{(z-z')^{\frac{2}{n}}} + \ldots + \dfrac{a_m}{(z-z')^{\frac{m}{n}}}$, where $a_1, a_2, \ldots, a_m$ are arbitrary complex magnitudes.

This theorem contains a corollary stating that function w is continuous at point O' when $(z-z')^{\frac{1}{n}} w$ becomes infinitely small as a result of the infinte convergence of point O towards O'.

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We will now consider a function of z, which has a determinate value for every point O on a surface T that arbitrarily extends over A, and which is not constant everywhere. Picture it geometrically to that is value $w = u + vi$ at point O is represented by a point Q on plane B, whose rectangular coordinates are u, v. We then get the following:

I. We can consider the totality of point Q as forming a surface S, in which every point corresponds to a determinate point O that continuously keeps advancing in T as the point in S does.

In order to prove this, it is obviously only necessary to prove that the position of point Q always (and of course, generally speaking, continuously) changes along with the point O. This is contained in the theorem:

A function $w = u + vi$ of z cannot be constant along a line unless it is constant everywhere.

Proof: if w were to have a constant value $a + bi$ along a line, then $u - a$ and $\frac{\partial(u-a)}{\partial p}$, which is equal to $= -\frac{\partial v}{\partial s}$ would be equal to zero for this line and for $\frac{\partial^2(u-a)}{\partial x^2} + \frac{\partial^2(u-a)}{\partial y^2}$ generally. And then according to chapter 11, I, $u - a$ and $v - b$ too, (because of $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$) would also have to be equal to 0 everywhere, which is contrary to our presuppositions.

II. As a result of the precondition we established in section I, there cannot be any connection between the segments of S without connection among the corresponding segments of T. The reverse is universal too, for where connection occurs in T and w is continuous, the surface S also has a corresponding connection.

If we presuppose this, then S's boundary corresponds on one hand to T's boundary, and on the other hand to discontinuous positions. However its inner segments, excluding isolated points, extends smoothly (*schilicht*) over B everywhere, i.e., there is neither a fissure in the segments lying on top of each other, nor is there a fold anywhere either.

Because T is correspondingly connected everywhere, the first condition could only occur if T underwent a fissure – which is contrary to our assumption. We can prove the second condition is the same way.

Next of all, we will prove that point Q', where $\frac{\mathrm{d}w}{\mathrm{d}z}$ is finite, cannot lie in a fold on surface T.

In reality, what we would do is surround point O', which corresponds to Q' with a piece of surface T' that is of arbitrary form and indeterminate dimensions. We could have to assume this piece's dimensions to be so small (according to Chapter 3) that the form (*Gestalt*) of the corresponding segment of S will deviate in an arbitrarily small way, so that its boundary will exclude a piece including Q' from plane B. But this is impossible if Q' lies in a fold on surface S.

So now if we consider $\frac{\mathrm{d}w}{\mathrm{d}z}$ as a function of z, according to I, it can only be equal to 0 in isolated points. And because w is continuous in the points of T that are under consideration, $\frac{\mathrm{d}w}{\mathrm{d}z}$ can only become infinite in the branch points of this surface. Therefore, Q.E.D.

III. Surface S therefore is a surface which satisfies the preconditions we established in chapter 5 for T, and the indeterminate magnitude z has one determinate value for every point Q on this surface. This one determinate value continuously changes with the position of Q in such a way that $\frac{\mathrm{d}z}{\mathrm{d}w}$ is independent of the change in location. Therefore, in the sense that was established earlier, what we get forming is a continuous function of the variable complex magnitude w for the entire magnitudinal field (*Gebiet*) presented by S.

What follows is :

Let O' and Q' be two corresponding interior points on surface T and S, and $z = z'$ and $w = w'$ in those same surface. Then if neither of these points are a branch point, $\frac{w - w'}{z - z'}$ will converge towards a finite limit, as O infinitely converges on O', and the image here will be similar down to the smallest segments. However, if Q' is a branch point of the $(n-1)$th order, and O' is a branch point of the $(n-1)$th order, then $\frac{(w-w')^{\frac{1}{n}}}{(z-z')^{\frac{1}{m}}}$ approaches a finite limit as O infinitely converges on O'. We can easily get a method of depicting the adjoining surface segments from chapter 14.

16

Pedagogical Theorem 6 *Let α and β be two arbitrary functions of x, y, for which the integral* $\int \left[\left(\frac{\partial \alpha}{\partial x} - \frac{\partial \beta}{\partial y} \right)^2 + \left(\frac{\partial \alpha}{\partial y} + \frac{\partial \beta}{\partial x} \right)^2 \right] \mathrm{d}T$ *has a finite value as it expands through all the segments of surface which is arbitrarily extended above A. Then when we alter around continuous functions, or around functions which are only discontinuous in isolated points, (both kinds of functions being $= 0$ at their margin) the integral will always have a minimum value for one of these functions. And if we exclude the discontinuities that occur by making changes in isolated points, then we would only get a minimum value for one function.*

We will define λ as being an indeterminate, continuous function or as a function that is only discontinuous in a couple of points. It will be $= 0$ at its margin and the integral $L = \int \left(\left(\frac{\partial \lambda}{\partial x} \right)^2 + \left(\frac{\partial \lambda}{\partial y} \right)^2 \right) \mathrm{d}T$ which extends over the entire surface will have a finite value for this function. Additionally we will define ω as an indeterminate of the function $\alpha + \lambda$ and we will define Ω as the integral $\int \left[\left(\frac{\partial \omega}{\partial x} - \frac{\partial \beta}{\partial y} \right)^2 + \left(\frac{\partial \omega}{\partial y} + \frac{\partial \beta}{\partial x} \right)^2 \right] \mathrm{d}T$, which extends over the entire surface. The totality of the λ functions form a cohesive, self-contained domain in which each one of these functions continuously change into others. However, these functions themselves cannot infinitely discontinuously converge on a line, without having L become infinite (chapter 17). This is so because when we assume $\omega = \alpha + \lambda$ for every λ, Ω becomes a finite value that becomes infinite along with L, and that continuously changes with the form (*Gestalt*) of λ, but that can never sink below null. Therefore it follows that Ω has a minimum for at least one form (*Gestalt*) of the function ω.

In order to prove the second part of our theorem, let u be one of the functions of ω which gives Ω a minimum value. Let h be a constant magnitude that is indeterminate on the entire surface,so that $u + h\lambda$ satisfies the preconditions set up for function ω. Then the value of Ω for $\omega = u + h\lambda$ which

$$\begin{aligned}
&\int \left[\left(\frac{\partial u}{\partial x} - \frac{\partial \beta}{\partial y} \right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial \beta}{\partial x} \right)^2 \right] \mathrm{d}T \\
&+ 2h \int \left[\left(\frac{\partial u}{\partial x} - \frac{\partial \beta}{\partial y} \right) \frac{\partial \lambda}{\partial x} + \left(\frac{\partial u}{\partial y} + \frac{\partial \beta}{\partial x} \right) \frac{\partial \lambda}{\partial y} \right] \mathrm{d}T \\
&+ h^2 \int \left(\left(\frac{\partial \lambda}{\partial x} \right)^2 + \left(\frac{\partial \lambda}{\partial y} \right)^2 \right) \mathrm{d}T \\
&= M + 2Nh + Lh^2
\end{aligned}$$

must therefore be greater than M fr every λ (according to the concept of the minimum), as long as we assume h to he sufficiently small. But this then requires that every $\lambda N = 0$ for otherwise $2Nh + Lh^2 = Lh^2(1 + \frac{2N}{Lh})$ would become negative when is counter-posed to N, irrespective of the signs $< \frac{2N}{L}$. Therefore the value of Ω for $\omega = u + \lambda$ which is the form that obviously contains all possible values for ω, becomes $= M + L$. Consequently, because L is essentially positive, Ω cannot have a smaller value for any form (*Gestalt*) of function ω than $\omega = u$.

Then if there is a minimum value M' of Ω for another u' of the functions ω, the same obviously holds for this. We will get $M' \leq M$ and $M \leq M'$, and consequently $M = M'$. But if we introduce u' into the form $u + \lambda'$ then we get the expression $M + L'$ for M', as long as L' describes the value of L for $\lambda = \lambda'$, and the equation $M = M'$ gives $L' = 0$. This is only possible when $\frac{\partial \lambda'}{\partial x} = 0$, $\frac{\partial \lambda'}{\partial y} = 0$ in all surface segments. Therefore, as long as λ' is continuous, this function is necessarily continuous. And because it is $= 0$ at its margin, and it is not discontinuous along a line it can only have a value different from null, at the most, in some isolated points. So then, two of the functions of ω, which give Ω a minimum value, can only be different from each other in isolated points. And if we put aside the discontinuities in function u that crop up by making changes in isolated points, then this function is totally determinate.

17

We will now supply the proof that λ cannot infinitely converge on a discontinuous γ located on a line without prejudicing L's infiniteness. I.e., if function λ is subjected to the condition of agreeing with γ outside of a surface segment T' that includes the line of discontinuities, then we can always assume T' to be so small that L must become larger than an arbitrarily given magnitude C.

Assuming s and p as having their usual relation in relation to the line of discontinuity, we will define κ as the curvature of an indeterminate s, a curvature which is convex on the side of the positive p, and which we will consider as positive. We will define p_1 as the value of p at the boundary of T' on the positive side, and on the negative side by p_2. We will define the corresponding values of γ as γ_1 and γ_2. So if we now consider a continuity curved segment of this line, and if the segment of T' that is contained between the normals in the endpoints does not reach to the middle point of the curvature, then this segment of T' contributes the following expression to L: $\int \mathrm{d}s \int_{\mathrm{d}} p{p_1}^{p_2}(1 - \kappa p)\left[\left(\frac{\partial\lambda}{\partial p}\right)^2 + \left(\frac{\partial\lambda}{\partial s}\right)\frac{1}{(1-\kappa p)^2}\right]$; however, we find the smallest value of the expression $\int_{p_1}^{p_2}\left(\frac{\partial\lambda}{\partial p}\right)^2(1-\kappa p)\mathrm{d}p$ at the fixed boundary values γ_1 and γ_2 of λ to be equal to, according to well-known rules, to $= \frac{(\gamma_1-\gamma_2)^2\kappa}{\log(1-\kappa p_2)-\log(1-\kappa p_1)}$.

Therefore, we will have to necessarily assume that every contribution, as well as λ inside T', to be $> \int \frac{(\gamma_1-\gamma_2)^2\kappa\mathrm{d}s}{\log(1-\kappa p_2)-\log(1-\kappa_1)}$. Function γ would be continuous for $p = 0$ if the greatest value which could contain $(\gamma_1-\gamma_2)^2$ for $\pi_1 > p_1 > 0$ and $\pi_2 < p_2 < 0$ were to become infinitely small through $\pi_1 - \pi_2$. Therefore we can assume that for every value of s there exists a finite magnitude m so that no matter how small $\pi_1 - \pi_2$ is assumed to be, m will always be contained inside the boundary value of p_1 and p_2, which are expressed by $\pi_1 > p_1 > 0$ and $\pi_2 < p_2 < 0$ (in which their equality is mutually excluded), and for which $(\gamma_1-\gamma_2)^2 > m$. Furthermore, if we arbitrarily assume a form (*Gestalt*) for T' in accordance with the earlier limitations, we will give p_1 and p_2 the determinate values of P_1 and P_2 and define a as the value of the integral $\int \frac{m\kappa\mathrm{d}s}{\log(1-\kappa P_2)-\log(1-\kappa P_1)}$ which extends through the segment of the line of discontinuities that we are considering. Then we can obviously make $\int \frac{(\gamma_1-\gamma_2)^2\kappa\mathrm{d}s}{\log(1-\kappa P_2)-\log(1-\kappa P_1)} > C$ to the extend that we so assume p_1 and p_2 for every value of s so that the inequalities $p_1 < \frac{1-(1-\kappa P_1)^{\frac{a}{C}}}{\kappa}$, $p_2 > \frac{1-(1-\kappa P_2)^{\frac{a}{C}}}{\kappa}$ and $(\gamma_1-\gamma_2)^2 > m$ will suffice. But this leads to the consequence that we assume that the segment of L that comes from the piece of T' that we are considering, and therefore even to a greater degree L itself, are larger than C, just as we would assume to be inside T. Q.E.D.

18

According to chapter 16, the function u which we established there is $= 0$, as are any of the functions $N = \int \left[\left(\frac{\partial u}{\partial x} - \frac{\partial\beta}{\partial y}\right)\frac{\partial\lambda}{\partial x} + \left(\frac{\partial u}{\partial y} + \frac{\partial\beta}{\partial x}\right)\frac{\partial\lambda}{\partial y}\right]\mathrm{d}T$ which extend throughout all of surface T. We will now draw some further conclusions from this equation.

Let us take a piece T', that includes the discontinuous points, u, β, λ and cut it out from surface Y. We can then find segment N, which is based on the remaining pieces T'', with the aid of chapter 7 and 8, if we replace $\left(\frac{\partial u}{\partial x}-\frac{\partial \beta}{\partial y}\right)\lambda$ for X and $\left(\frac{\partial u}{\partial y}+\frac{\partial \beta}{\partial x}\right)\lambda$ for Y,

$$= -\int \lambda \left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \mathrm{d}T - \int \left(\frac{\partial u}{\partial p}+\frac{\partial \beta}{\partial s}\right) \lambda \mathrm{d}s$$

As a consequence of the boundary conditions that have already been imposed on function λ, the segment of $\int \left(\frac{\partial u}{\partial p}+\frac{\partial \beta}{\partial s}\right)\lambda \mathrm{d}s$ relating to the joint boundary piece that T'' has with T is equal to 0. We can then consider N to be composed out of the integral $-\int \lambda \left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \mathrm{d}T$ relative to T'', and of $\int \left[\left(\frac{\partial u}{\partial x}-\frac{\partial \beta}{\partial y}\right)\frac{\partial \lambda}{\partial x}+\left(\frac{\partial u}{\partial y}+\frac{\partial \beta}{\partial x}\right)\frac{\partial \lambda}{\partial y}\right] \mathrm{d}T + \int \left(\frac{\partial u}{\partial p}+\frac{\partial \beta}{\partial s}\right)\lambda \mathrm{d}s$ relative to T'.

So now it is obvious, that if $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}$ were to be different from zero in any segment of surface T, N would likewise behave a value different from 0 so long as λ, which is free, is equal to zero inside of T', and so long as we choose λ inside T'' so that $\lambda\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$ would have the same sign everywhere. However, if $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}$ are $=0$ in all segments of T, then the component segment N which is based on T'' vanished for every λ. The result of the condition $N=0$ will then be that component segments relating to discontinuous points $=0$.

Concerning functions $\frac{\partial u}{\partial x}-\frac{\partial \beta}{\partial y}, \frac{\partial u}{\partial y}+\frac{\partial \beta}{\partial x}$ therefore, what we get when we have the first one $=X$ and the latter $=Y$, if we just do not want to speak generally, is the equation $\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}=0$. But, to extent that this equation really has a determinate value, $\int \left(X\frac{\partial x}{\partial p}+Y\frac{\partial y}{\partial p}\right)\mathrm{d}s=0$.

So (according to chapter 9,V) if surface T has the property of multiple connection, we divide it by cuts into a simply connected $T\#$. As a result, the integral $-\int_{O_0}^{O}\left(\frac{\partial u}{\partial p}+\frac{\partial \beta}{\partial s}\right)\mathrm{d}s$ has the same value for every line in $T\#$'s that goes from O_0 to O. And when we consider O_0 to be fixed, this integral will then also form a function for x and y that undergoes a continuous change and a change that is equal on both sides of a cut in $T\#$. When we add this function ν to β we get a function $v=\beta+\nu$ whose differential quotient is $\frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial y}=\frac{\partial u}{\partial x}$.[10]

Therefore we have the following:

[10]Here, again, the English translation doesn't agree with the original German, according to which a modification has been made. – Typesetter

Pedagogical Theorem 7 *Assume that a complex function is given a connected surface T which in turn is divided by cuts into a simple connected surface $T\#$. In terms of this function* $\int \left[\left(\frac{\partial\alpha}{\partial x} - \frac{\partial\beta}{\partial y}\right)^2 + \left(\frac{\partial\alpha}{\partial y} + \frac{\partial\beta}{\partial x}\right)^2\right] \mathrm{d}T$ *extends throughout the entire surface and has a finite value, so that it can always be changed and can only be changed into a function of z through the addition of a function which satisfies the following conditions:*

1. *is $= 0$ at the margin, or is only different from it in isolated points, while v is arbitrarily given for a point.*
2. *the changes made by in T, and changes made by v in $T\#$, only occur in isolated pints and are so discontinuous that* $\int \left[\left(\frac{\partial\mu}{\partial x}\right)^2 + \left(\frac{\partial\mu}{\partial y}\right)^2\right] \mathrm{d}T$ *and* $\int \left[\left(\frac{\partial\nu}{\partial x}\right)^2 + \left(\frac{\partial\nu}{\partial y}\right)^2\right] \mathrm{d}T$ *remain finite as they extend through the entire surface, and the latter expression remains equal on both sides along the cut.*

These conditions' adequacy in determining $\mu + \nu\, i$ stems from having μ, through which we determined ν up to an additive constant, always furnish a minimum for the integral Ω at the same time. This is so because given $u = a+\mu$, N will obviously be $= 0$ for every λ: a property which can only belong to one function, according to chapter 16.

19

The principles which are the basis for the pedagogical theorem at the conclusion of the previous chapter open up the path for investigating the determinate functions of a variable complex magnitude (independent of an expression for the same.)

A quick review of the range of the conditions that are necessary for the determination of such a function inside a given numerical domain (*Gr'́ossengebiets*) will serve us as an orientation to this field.

First of all, we will pause at a specific case: If the surface which is extended over A (which is how we will represent this numerical domain) is a simply connected surface, then the function $w = u+vi$ of z will be determined according to the following conditions:

1. a value is given for u in all the boundary points, and when this value undergoes an infinitely small change of position, it changes by an infinitely small magnitude of the same order. Otherwise, the value will change arbitrarily.[11]

[11]In themselves, the changes in this value are only subject to the limitation that they are not discontinuous along a part of the boundary. We have only imposed the additional limitation in order to avoid formal difficulties which are unnecessary here. – Riemann's original footnote

2. the value for v ar any point is arbitrarily given.

3. the function should be finite and continuous at all points. The function is totally determined by these conditions.

In reality, this does follow from the pedagogical theorem in the previous chapter, if we so define $\alpha + \beta i$ so that α at the margin is equal to the given value, and if the change in $\alpha + \beta i$ is infinitely small and of the same order for every infinitely small change of location in the entire surface. It is always possible for us to define $\alpha + \beta i$ this way.

Generally speaking, therefore, we can have u at the margin be as a totally arbitrary function of s, and we can also define v anywhere through this. We can assume the reverse, too, for if v is arbitrarily given for all boundary points, then the value for u follows from this. So the full range for the choice of values for a w at the margin encompasses a one-dimensional manifold for every boundary point. In order to totally define this manifold, what we need for every boundary point is an equation for which it is not essential that every one of the equations is solely based on the value of one term in one boundary point. Our definition can also turn out in such a way, so that what we get for every boundary point if an equation containing both terms that continuously changes its form (*Form*) as the position of this boundary point changes. Or, what can happen simultaneously to several segments of the boundary is that every point defined as an $(n-1)$point of this segment gets matched to one point in such a way that for every n amount of such points, we collectively get an n amount of equations that continuously change with their locations. However, these conditions, whose totality constitutes a continuous manifold, and which are expressed by comparison (equations) between arbitrary functions, generally speaking, still require either limitation of amplification by means of isolated conditional equations – equations for arbitrary constants – in order to get a reliable and adequate definition for a function that is continuous everywhere inside a numerical domain. These conditions require this, that is, to the extend that the accuracy we used in our evaluations does not reach up to this level.

Our observations will not have to undergo any essential modifications of the situation where magnitude z's domain of variability is represented by a multiply connected plane because the application of the theorem in chapter 18 creates a function constituted as before, excepting the changes that occur in overstepping the cuts – changes, which can be made $= 0$ if the boundary conditions contain an amount of disposable constants that are equal to the number of cuts.

The situation in the interior, where we have relinquished all claims for continuity along a line, organises itself like the previous situation if we consider this line to be a cut on the surface.

And finally, if we allow continuity to be violated at an isolated point, then, according to chapter 12, this is how a function becomes infinite. So, by retaining the special preconditions that we made in the first case for this point, function z can be arbitrarily given after its concurring function becomes continuous. However, as a result of this, function z becomes completely defined. For if we assume the magnitude which is in an arbitrarily small circle drawn around the discontinuous point to be equal to the given function, and, moreover, to also conform to the earlier formulae, then the integral $\int \left(\left(\frac{\partial \alpha}{\partial x} - \frac{\partial \beta}{\partial y} \right)^2 + \left(\frac{\partial \alpha}{\partial y} + \frac{\partial \beta}{\partial x} \right)^2 \right) \mathrm{d}T =$ 0 when it is extended over this circle, and equals a finite magnitude when it is extended over the remaining segment. And so we can apply the theorem from the previous chapter, through which we get a function with the desired properties. From this we can generally derive, with the aid of the theorems in chapter 13, that when a function can become infinitely large to the nth order in a discontinuous point, then a number of $2n$ constants become available.

According to chapter 15, let us assume a function w of a variable complex magnitude z that is inside of a given magnitude domain of two dimensions. Then when we represent this function geometrically, we will get an image S covering B that is similar down to its smallest segment to a surface T covering a given A. The only exception to this are isolated points. And, we will not base the value of the conditions that are necessary and sufficient for defining this function on either boundary points or on discontinuous points. Consequently, (according to chapter 15) the conditions that define this function all turn out to be the conditions for the position of S's boundary, and naturally, they give a conditional equation for every boundary point. So if every one of these conditional equations only relates to one boundary point, then we can represent them by a group of curves, each one of which forms the geometrical location for each boundary point. If we then jointly subject two boundary points that keep in step with each other continuously to two conditional equations, what we then get is such a dependency (slope) between the two boundary segments that when we arbitrarily assume a position for one point, we can derive the position of the other point from it. In like manner, we can also get something of geometrical importance out of the other forms of the conditional equations, but we do not want to pursue this further here.

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The origin and the immediate purpose for the introduction of complex number into mathematics is the theory of creating simpler[12] dependency laws (slope laws) between complex magnitudes by expressing these laws through numerical operations (*Grössenoperationen*). And, if we give these dependency laws an expanded range by assigning complex values to the variable magnitudes, on which the dependency laws are based, then what makes its appearance is a harmony and regularity which is especially indirect (*versteckt*) and lasting. Of course, up until now the situation in which this occurs have encompassed a small

[12]Here we will consider Addition, Subtraction, Multiplication and Division as elementary operation, and we will consider a dependency law (slope law) to be all the more simpler – the fewer are the elementary operations that determine the dependency. In reality, all of the functions that have been used up to now in this analysis can be defined by a finite number of these operations. – Riemann's original footnote

domain – we can almost totally trace these situations back to those very laws covering the dependency between two variable magnitudes, where one function is either an algebraic function of the other [13] or is that kind of function whose differential quotient is an algebraic function. – But in almost every step that we have taken here, we have not just simply given a simpler, more consistent *Gestalt* to our results without any help from complex magnitudes. Our steps have also pioneered the way for new discoveries, and the account of our examination of algebraic function, circular – or exponential function, elliptical and Abelian functions furnished the evidence for this.

We will now briefly indicate what the theory of these functions has gained through our examination.

The previous methods that were used to deal with these functions always has, as the basis of their definition, and expression of the function through which the function's value would be given for every value in the argument. Our examination has shown as a result of the general character of a function of a variable complex magnitude, what we get in a definition of this kind is that any one segment of the pieces making up the definition is a direct consequence of the remaining segments, and of course, we can trace the range of pieces making up the definition back to those pieces that are necessary for the definition, which essentially simplifies our treatment of the definition. For example, in order to prove that two expressions of the same functions are equal, we would have had to previously show that both agree for every value of the complex magnitude. But now, the evidence of their agreement in a considerable smaller range is sufficient.

A theory of these functions that is based on the foundations that we have supplied here would define the function's configuration (*Gestalting*)(i.e. its value for every value in the argument), independent of the method of determining this through numerical operations(*Grössenoperationen*). For in this new definition, we would only add the features that are necessary to define the function to the general conception of a function of a variable complex magnitude. And only then would we add these features to the various expressions which the function is capable of undergoing. We can then express the common characteristic of a species of function, which could be expressed in a similar manner by numerical operations, in the form of the boundary – and discontinuity conditions that are imposed on the functions.

Assume, for example, that magnitude z's domain of variability extends either simply, or multiply over all of infinite plane A, and that inside this same plane our function is discontinuous only in isolated points. We will also only tolerate a function that is becoming infinite and whose order is finite. As a result, we will consider this magnitude itself to be an infinite magnitude of the first order for an infinite z', but we will consider $\dfrac{1}{z - z'}$ to be an infinite magnitude of the first order for every finite value of z'. So, the function is necessarily algebraic, and conversely, every algebraic function fulfils this condition.

[13] I.e., where an algebraic equation occurs between both. – Riemann's original footnote

In our paper, we have abstained for now from realising this theory, because as we remarked, this realisation would be characterised by bringing simple dependency (slope) laws that are conditional on numerical operations out into the light of day. We have not done this so far because we have rule out considering the expression of such a function for the present.

And for these very same reason, we also did not concern ourselves here with our theorem's usefulness as the foundation of a general theory of these dependency (slope) laws. What we would need for this is a proof that the concept of a function of variable complex magnitude, which is our basis here, is in complete agreement with a dependency (slope) that is expressible by a numerical operation. [14]

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Nevertheless, a detailed example of its application can be of use in illustrating our general theory.

The application of our theory which was described in the previous chapter is only a special application, even though it was intended to be our first example. Assume a dependency is conditioned by a finite number of the numerical operations that we considered to be elementary operations in the previous chapter. Then its function contains only a finite number of parameters that succeed in having no arbitrary determinate conditions at all occur under them along a line at any point. This is so regardless what the form is of the system of mutually independent boundary – and discontinuity conditions that are adequate to define the function. Therefore it seems better suited for our present purpose if we do not select an example that comes from that situation, but if we instead take an example where the function of the complex variable is dependent on an arbitrary function.

In order to make an assessment, and to get a more comfortable framework we will give our example the geometrical form that we used at the end of chapter 19. What we will then appear to have is an investigation of the possibility of producing an analogous image, connected down to its smallest segments, of a given surface. The image's *Gestalt* is given in the form that was expressed above, where there is locational curve for every boundary point in the image, and where the locational curve is given for all these boundary points, with the exception of the boundary and branch point as given in chapter 5. We will limit ourselves to solving this problem for the situations where every point in one surface will only correspond to one point in the other surface, and where

[14]This includes every one of the dependencies that can be expressed by a finite or infinite number of the four simple methods of calculation: addition and subtraction, multiplication and division. In terms of magnitude operations (*Grössenoperationen, usually tran. previously as numerical operations. – Note from the translator*), in contrast to counting operations (*Zahlenoperationen*), their expression themselves should indicate those methods of calculation which do not bring these magnitudes' commensurability into question. – Riemann's original footnote

the surfaces are simple connected. This situation is contained in the following pedagogical theorem.

Pedagogical Theorem 8 *Two simply connected surfaces can always relate to each other in such a manner that every point on one surface corresponds to the point on the other surface that is steadily progressing with it, and so that their smallest corresponding segments are similar. Naturally, we can arbitrarily give corresponding points to the interior points in one surface, and to the boundary points on another, but this is what determines the relationship for all points.*

If two surfaces T and R are so related to a third surface S that their smallest corresponding segments are similar to S's, then a relation develops out of this between surface T and R which is obviously the same as the first relationship. We can trace our task, which consists of relating two arbitrary surfaces to each other so that they are similar in their smallest segments, back to portraying every arbitrary surface through another surface which we define as similar down to its smallest segments. According to this, when we draw a circle K with the radius 1 around the point in plane B where $w = 0$, all we have to prove in order to follow our pedagogical theorem is that: we can portray an arbitrarily, simply connected surface T that covers A, as a continuously connected surface, and one that is similar down to its smallest segment, by circle K in such a manner, and only in such a manner, that an arbitrarily given interior point O_0 corresponds to the circle's middle point, and an arbitrarily given boundary point O' on surface T corresponds to an arbitrarily given points on the circle's periphery.

We will describe the meanings defined for z, Q for point O_0, and O' by corresponding indices, and we can ascribe the middle point of an arbitrary circle Θ, which does not reach up to T's boundary, and which does not have any branch points, as being around O_0 in T. With the introduction of polar coordinate to the extent that we have $z - z_0 = re^{\varphi i}$ then the function $\log(z - z_0) = \log r + \varphi i$. As a result of this, all the real values in the entire circle change continuously, except for point O_0, whose value becomes infinite. But wherever we select the smallest possible value for φ among all possible values, the imaginary value has 0 value on the one side, and the value of 2π on the other side along the radius where $z - z_0$ assumes real positive values. In all other points, however, the imaginary numbers change continuously. Obviously, this radius can be replaced by a totally arbitrary line L drawn from the middle point to the periphery, so that function $\log(z - z_)$ undergoes a sudden diminution of about $2\pi i$ when point O crosses over from the negative side of this line (i.e., where p becomes negative according to chapter 8) to the positive side. Elsewhere, however, the function continuously changes with the position O has in circle Θ. If we also assume that in circle Θ complex function $\alpha + \beta\, i$ of x, $y = \log(z - z_0)$, except when we arbitrarily expand l up to the margin, then the function

1. will become totally imaginary at the margin of T, and on $\Theta = \log(z - z_0)$'s periphery,
2. will change by approximately $-2\pi i$ in crossing over from the negative to the positive side of L, and otherwise, it will change by an infinitely small

magnitude of the same order with every small change in location, all of which becomes increasingly more possible.

Therefore, integral $\int \left(\left(\frac{\partial \alpha}{\partial x} - \frac{\partial \beta}{\partial y} \right)^2 + \left(\frac{\partial \alpha}{\partial y} + \frac{\partial \beta}{\partial x} \right)^2 \right) \mathrm{d}T$, has a value of null when it expands across Θ, and when it extends across the remaining segments it has a finite value. Therefore, we can change $\alpha + \beta i$ into a function $t = m + ni$ of z through the addition of a continuous function of x, y which is continuously determinate with the exception of a totally imaginary constant remainder, and which is totally imaginary on the margin. The real segment m of this function will be $= 0$ on the margin, will be $= -\infty$ at point O_0, and will continuously change in all the rest of T. Therefore, for every value a of m that lies between 0 and $-\infty$, T, disintegrates as the result of a line where $m = a$, disintegrates into segments where m is smaller than a and where O_0 is contained on the inside, and disintegrates into segments on one side and the other side where $m > a$ and where these segments' boundaries are partially formed by T's margin, and partially through lines where $m = a$. As a result of this disintegration: either the order of surface T's connection does not change, or it is reduced. And so, because this order is equal to -1, the surface disintegrates into two pieces, or into more than two pieces. But this latter situation is impossible because then m would have to be finite and continuous everywhere in at least one of these pieces, and constant in all the segments of the boundary. As a result, either there would have to be one constant value in each surface segment, or there would have to be a minimum or a maximum value anywhere – in a point or along a line, which is contrary to article 11,III. So then the points where m is constant form simple, self-encircling lines everywhere, lines which bound a piece including 0, m will necessarily decrease going towards the interior, which results in n increasing continuously, as long as it is continuous in a positive range, (where s increases according to chapter 8). And if once again we disregard multiples of 2π then every value between 0 and 2π becomes equal because n only undergoes a quick change of about -2π[15] in crossing over from the negative side of line L to its positive side. If we then have $e^t = w$ then e^m and n will become the polar coordinates of point Q in relation to the middle point of circle K. The totality of point Q will then obviously form a surface s that extends simply over K everywhere; point O_0 itself will then be at the middle point of the circle, and point Q' can be backed into an arbitrarily given point on the periphery with the help of the constants that are still available in n.

Q.E.D.

For the case where point O_0 is a branch point of the $(n-1)$ order, and if we replace $\log(z - z_0)$ with $\frac{1}{n}\log(z - z_0)$, then we will use every similar conclusions

[15]Because line l leads from a point lying in the interior to one lying outside, then it must go one more time from the inside to the outside then it goes from the outside to the inside, if it crosses the boundary several times. Therefore, the sum of the sudden changes in n in a positive range will always be -2π.

to reach the goal whose further exploitation we can easily fill out from chapter 14.

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We will not completely carry out the investigation of the general case in the last chapter, where one point in one surface should correspond to several points in other surfaces, and where we do not make the prerequisite that these points just have simple connections. We will not carry this out completely because our entire investigation has had to lead to a general *Gestalt*, if we comprehend it from a geometrical viewpoint. For this reason, it was not essential that we limited ourselves to level, smooth (*schlicht*) surfaces with the exception of isolated points; rather, our task has been to portray one arbitrarily given surface onto another arbitrarily given one so that they are similar down to their smallest segments, or to give it very similar treatment. We will content ourselves here by referring to two Gaussian treatises which are cited in chapter 3 and the general inquiry about surfaces in chapter 13.