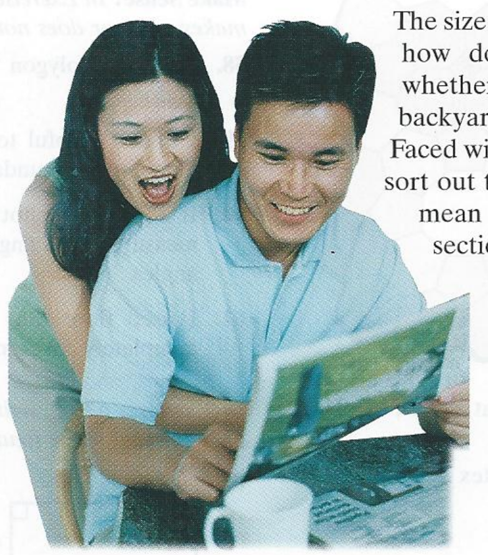
OBJECTIVES

1. Use area formulas to compute the areas of plane regions and solve applied problems.
2. Use formulas for a circle's circumference and area.

10.4 Area and Circumference



The size of a house is described in square feet. But how do you know from the real estate ad whether the 1200-square-foot home with the backyard pool is large enough to warrant a visit? Faced with hundreds of ads, you need some way to sort out the best bets. What does 1200 square feet mean and how is this area determined? In this section, we discuss how to compute the areas of plane regions.

1 Use area formulas to compute the areas of plane regions and solve applied problems.

Formulas for Area

In Section 9.2, we saw that the area of a two-dimensional figure is the number of square units, such as square inches or square miles, it takes to fill the interior of the figure. For example, **Figure 10.36** shows that there are 12 square units contained within the rectangular region. The area of the region is 12 square units. Notice that the area can be determined in the following manner:

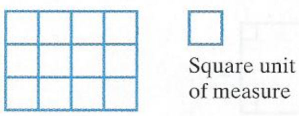


FIGURE 10.36 The area of the region on the left is 12 square units.

Distance across | Distance down

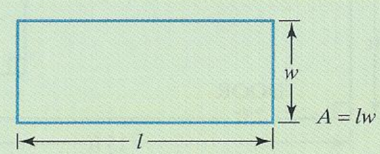
$$4 \text{ units} \times 3 \text{ units} = 4 \times 3 \times \text{units} \times \text{units}$$
$$= 12 \text{ square units.}$$

The area of a rectangular region, usually referred to as the area of a rectangle, is the product of the distance across (length) and the distance down (width).

AREA OF A RECTANGLE AND A SQUARE

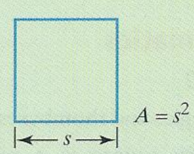
The area, A, of a rectangle with length l and width w is given by the formula

$$A = lw.$$



The area, A, of a square with one side measuring s linear units is given by the formula

$$A = s^2.$$



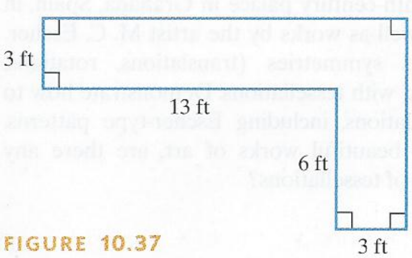


FIGURE 10.37

EXAMPLE 1 **Solving an Area Problem**

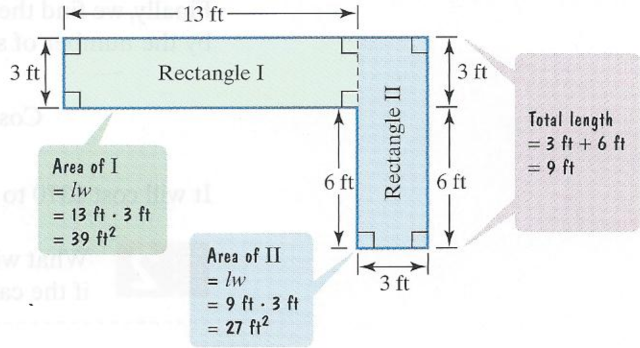
You decide to cover the path shown in **Figure 10.37** with bricks.

a. Find the area of the path.

b. If the path requires four bricks for every square foot, how many bricks are needed for the project?

Solution

a. Because we have a formula for the area of a rectangle, we begin by drawing a dashed line that divides the path into two rectangles. One way of doing this is shown at the right. We then use the length and width of each rectangle to find its area. The computations for area are shown in the green and blue voice balloons.



The area of the path is found by adding the areas of the two rectangles.

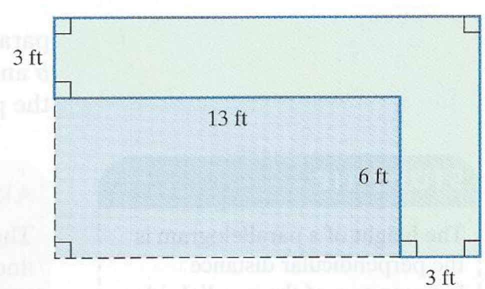
$$\text{Area of path} = 39 \text{ ft}^2 + 27 \text{ ft}^2 = 66 \text{ ft}^2$$

b. The path requires 4 bricks per square foot. The number of bricks needed for the project is the number of square feet in the path, its area, times 4.

$$\text{Number of bricks needed} = 66 \cancel{\text{ft}^2} \cdot \frac{4 \text{ bricks}}{\cancel{\text{ft}^2}} = 66 \cdot 4 \text{ bricks} = 264 \text{ bricks}$$

Thus, 264 bricks are needed for the project.

CHECK POINT 1 Find the area of the path described in Example 1, rendered on the right as a green region, by first measuring off a large rectangle as shown. The area of the path is the area of the large rectangle (the blue and green regions combined) minus the area of the blue rectangle. Do you get the same answer as we did in Example 1(a)?



BLITZER BONUS

APPRAISING A HOUSE

A house is measured by an appraiser hired by a bank to help establish its value. The appraiser works from the outside, measuring off a rectangle. Then the appraiser adds the living spaces that lie outside the rectangle and subtracts the empty areas inside the rectangle. The final figure, in square feet, includes all the finished floor space in the house. Not included are the garage, outside porches, decks, or an unfinished basement.

A 1000-square-foot house is considered small, one with 1500 square feet average, and one with more than 2000 square feet pleasantly large. If a 1200-square-foot house has three bedrooms, the individual rooms might seem snug and cozy. With only one bedroom, the space may feel palatial!

In Section 9.2, we saw that although there are 3 linear feet in a linear yard, there are 9 square feet in a square yard. If a problem requires measurement of area in square yards and the linear measures are given in feet, to avoid errors, first convert feet to yards. Then apply the area formula. This idea is illustrated in Example 2.

EXAMPLE 2 Solving an Area Problem

What will it cost to carpet a rectangular floor measuring 12 feet by 15 feet if the carpet costs $18.50 per square yard?

Solution We begin by converting the linear measures from feet to yards.

$$12 \text{ ft} = \frac{12 \cancel{\text{ft}}}{1} \cdot \frac{1 \text{ yd}}{3 \cancel{\text{ft}}} = \frac{12}{3} \text{ yd} = 4 \text{ yd}$$

$$15 \text{ ft} = \frac{15 \cancel{\text{ft}}}{1} \cdot \frac{1 \text{ yd}}{3 \cancel{\text{ft}}} = \frac{15}{3} \text{ yd} = 5 \text{ yd}$$

Next, we find the area of the rectangular floor in square yards.

$$A = lw = 5 \text{ yd} \cdot 4 \text{ yd} = 20 \text{ yd}^2$$

Finally, we find the cost of the carpet by multiplying the cost per square yard, \$18.50, by the number of square yards in the floor, 20.

$$\text{Cost of carpet} = \frac{\$18.50}{\cancel{\text{yd}^2}} \cdot \frac{20 \cancel{\text{yd}^2}}{1} = \$18.50(20) = \$370$$

It will cost \$370 to carpet the floor.

CHECK POINT 2 What will it cost to carpet a rectangular floor measuring 18 feet by 21 feet if the carpet costs \$16 per square yard?

We can use the formula for the area of a rectangle to develop formulas for areas of other polygons. We begin with a parallelogram, a quadrilateral with opposite sides equal and parallel. The **height** of a parallelogram is the perpendicular distance between two of the parallel sides. Height is denoted by h in **Figure 10.38**. The **base**, denoted by b, is the length of either of these parallel sides.

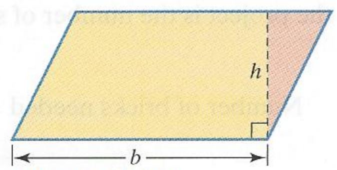


FIGURE 10.38

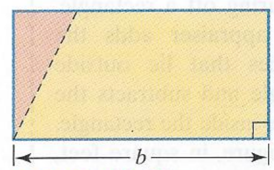


FIGURE 10.39

In **Figure 10.39**, the red triangular region has been cut off from the right of the parallelogram and attached to the left. The resulting figure is a rectangle with length b and width h. Because bh is the area of the rectangle, it also represents the area of the parallelogram.

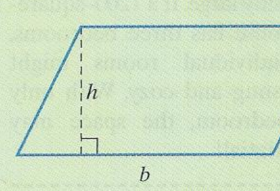
STUDY TIP

The height of a parallelogram is the perpendicular distance between two of the parallel sides. It is *not* the length of a side.

AREA OF A PARALLELOGRAM

The area, A, of a parallelogram with height h and base b is given by the formula

$$A = bh.$$



EXAMPLE 3 Using the Formula for a Parallelogram's Area

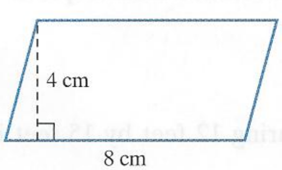


FIGURE 10.40

Find the area of the parallelogram in **Figure 10.40**.

Solution As shown in the figure, the base is 8 centimeters and the height is 4 centimeters. Thus, $b = 8$ and $h = 4$.

$$A = bh$$

$$A = 8 \text{ cm} \cdot 4 \text{ cm} = 32 \text{ cm}^2$$

The area is 32 cm^2.

CHECK POINT 3 Find the area of a parallelogram with a base of 10 inches and a height of 6 inches.

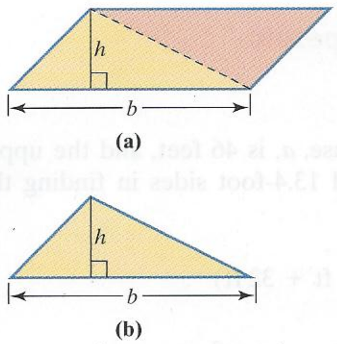


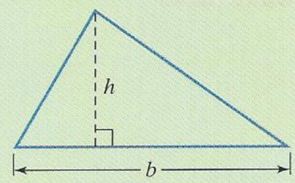
FIGURE 10.41

Figure 10.41 demonstrates how we can use the formula for the area of a parallelogram to obtain a formula for the area of a triangle. The area of the parallelogram in **Figure 10.41(a)** is given by $A = bh$. The diagonal shown in the parallelogram divides it into two triangles with the same size and shape. This means that the area of each triangle is one-half that of the parallelogram. Thus, the area of the triangle in **Figure 10.41(b)** is given by $A = \frac{1}{2}bh$.

AREA OF A TRIANGLE

The area, A, of a triangle with height h and base b is given by the formula

$$A = \frac{1}{2}bh.$$



EXAMPLE 4 Using the Formula for a Triangle's Area

Find the area of each triangle in **Figure 10.42**.

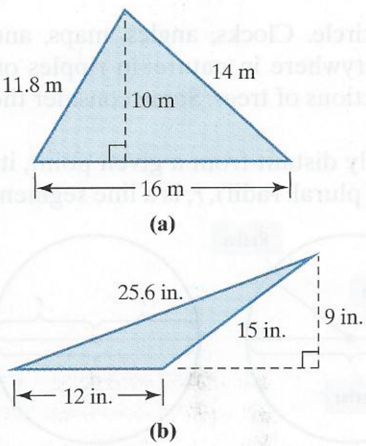


FIGURE 10.42

Solution

a. In **Figure 10.42(a)**, the base is 16 meters and the height is 10 meters, so $b = 16$ and $h = 10$. We do not need the 11.8 meters or the 14 meters to find the area. The area of the triangle is

$$A = \frac{1}{2}bh = \frac{1}{2} \cdot 16\text{ m} \cdot 10\text{ m} = 80\text{ m}^2.$$

The area is 80 square meters.

b. In **Figure 10.42(b)**, the base is 12 inches. The base line needs to be extended to draw the height. However, we still use 12 inches for b in the area formula. The height, h, is given to be 9 inches. The area of the triangle is

$$A = \frac{1}{2}bh = \frac{1}{2} \cdot 12\text{ in.} \cdot 9\text{ in.} = 54\text{ in.}^2$$

The area of the triangle is 54 square inches.

CHECK POINT 4 A sailboat has a triangular sail with a base of 12 feet and a height of 5 feet. Find the area of the sail.

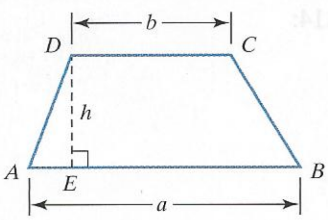


FIGURE 10.43

The formula for the area of a triangle can be used to obtain a formula for the area of a trapezoid. Consider the trapezoid shown in **Figure 10.43**. The lengths of the two parallel sides, called the **bases**, are represented by a (the lower base) and b (the upper base). The trapezoid's height, denoted by h, is the perpendicular distance between the two parallel sides.

In **Figure 10.44**, we have drawn line segment BD, dividing the trapezoid into two triangles, shown in yellow and red. The area of the trapezoid is the sum of the areas of these triangles.

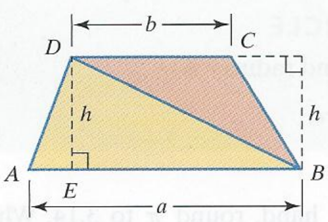


FIGURE 10.44

Area of trapezoid = Area of yellow Δ plus Area of red Δ

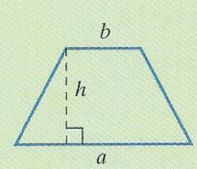
$$A = \frac{1}{2}ah + \frac{1}{2}bh$$

$$= \frac{1}{2}h(a + b) \quad \text{Factor out } \tfrac{1}{2}h.$$

AREA OF A TRAPEZOID

The area, A, of a trapezoid with parallel bases a and b and height h is given by the formula

$$A = \frac{1}{2}h(a + b).$$



EXAMPLE 5 Finding the Area of a Trapezoid

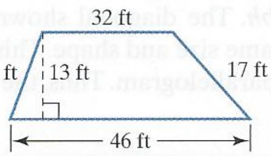


FIGURE 10.45

Find the area of the trapezoid in **Figure 10.45**.

Solution The height, h, is 13 feet. The lower base, a, is 46 feet, and the upper base, b, is 32 feet. We do not use the 17-foot and 13.4-foot sides in finding the trapezoid's area.

$$A = \frac{1}{2}h(a + b) = \frac{1}{2} \cdot 13 \text{ ft} \cdot (46 \text{ ft} + 32 \text{ ft})$$

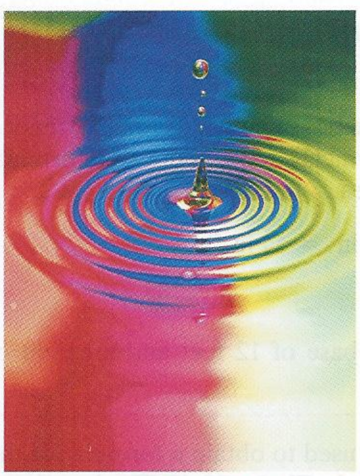
$$= \frac{1}{2} \cdot 13 \text{ ft} \cdot 78 \text{ ft} = 507 \text{ ft}^2$$

The area of the trapezoid is 507 square feet.

CHECK POINT 5 Find the area of a trapezoid with bases of length 20 feet and 10 feet and height 7 feet.

2 Use formulas for a circle's circumference and area.

It's a good idea to know your way around a circle. Clocks, angles, maps, and compasses are based on circles. Circles occur everywhere in nature: in ripples on water, patterns on a butterfly's wings, and cross sections of trees. Some consider the circle to be the most pleasing of all shapes.



The point at which a pebble hits a flat surface of water becomes the center of a number of circular ripples.

A **circle** is a set of points in the plane equally distant from a given point, its **center**. **Figure 10.46** shows two circles. The **radius** (plural: radii), r, is a line segment from the center to any point on the circle. For a given circle, all radii have the same length. The **diameter**, d, is a line segment through the center whose endpoints both lie on the circle. For a given circle, all diameters have the same length. In any circle, the **length of the diameter is twice the length of the radius.**

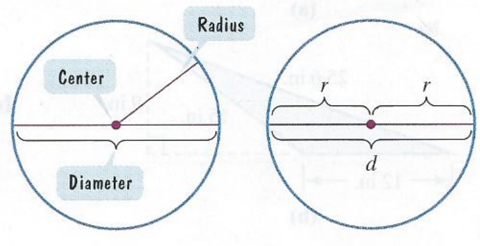


FIGURE 10.46

The words *radius* and *diameter* refer to both the line segments in **Figure 10.46** as well as to their linear measures. The distance around a circle (its perimeter) is called its **circumference**, C. For all circles, if you divide the circumference by the diameter, or by twice the radius, you will get the same number. This ratio is the irrational number π and is approximately equal to 3.14:

$$\frac{C}{d} = \pi \quad \text{or} \quad \frac{C}{2r} = \pi.$$

Thus,

$$C = \pi d \quad \text{or} \quad C = 2\pi r.$$

FINDING THE DISTANCE AROUND A CIRCLE

The circumference, C, of a circle with diameter d and radius r is

$$C = \pi d \quad \text{or} \quad C = 2\pi r.$$

When computing a circle's circumference by hand, round π to 3.14. When using a calculator, use the $\boxed{\pi}$ key, which gives the value of π rounded to approximately 11 decimal places. In either case, calculations involving π give approximate answers. These answers can vary slightly depending on how π is rounded. The symbol $\approx$ (is approximately equal to) will be written in these calculations.

EXAMPLE 6 Finding a Circle's Circumference

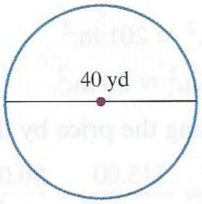


FIGURE 10.47

Find the circumference of the circle in **Figure 10.47**.

Solution The diameter is 40 yards, so we use the formula for circumference with d in it.

$$C = \pi d = \pi(40 \text{ yd}) = 40\pi \text{ yd} \approx 125.7 \text{ yd}$$

The distance around the circle is approximately 125.7 yards.

CHECK POINT 6 Find the circumference of a circle whose diameter measures 10 inches. Express the answer in terms of π and then round to the nearest tenth of an inch.

EXAMPLE 7 Using the Circumference Formula

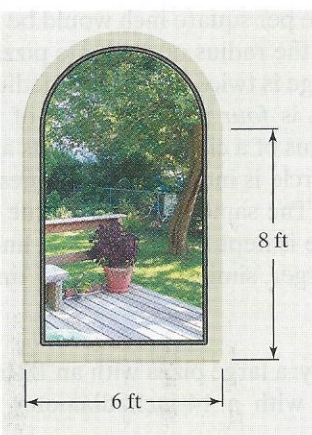


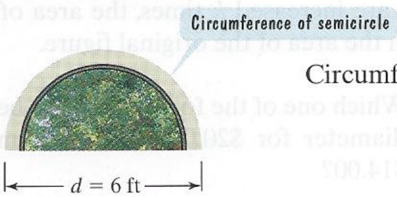
FIGURE 10.48

How much trim, to the nearest tenth of a foot, is needed to go around the window shown in **Figure 10.48**?

Solution The trim covers the 6-foot bottom of the window, the two 8-foot sides, and the half-circle (called a semicircle) on top. The length needed is

$$6 \text{ ft} + 8 \text{ ft} + 8 \text{ ft} + \text{circumference of the semicircle.}$$

The circumference of the semicircle is half the circumference of a circle whose diameter is 6 feet.



$$\text{Circumference of semicircle} = \frac{1}{2}\pi d$$

$$= \frac{1}{2}\pi(6 \text{ ft}) = 3\pi \text{ ft} \approx 9.4 \text{ ft}$$

Rounding the circumference to the nearest tenth (9.4 feet), the length of trim that is needed is approximately

$$6 \text{ ft} + 8 \text{ ft} + 8 \text{ ft} + 9.4 \text{ ft},$$

or 31.4 feet.

CHECK POINT 7 In **Figure 10.48**, suppose that the dimensions are 10 feet and 12 feet for the window's bottom and side, respectively. How much trim, to the nearest tenth of a foot, is needed to go around the window?

We also use π to find the area of a circle in square units.

FINDING THE AREA OF A CIRCLE

The area, A, of a circle with radius r is

$$A = \pi r^2.$$

EXAMPLE 8 Problem Solving Using the Formula for a Circle's Area

Which one of the following is the better buy: a large pizza with a 16-inch diameter for \$15.00 or a medium pizza with an 8-inch diameter for \$7.50?

Solution The better buy is the pizza with the lower price per square inch. The radius of the large pizza is $\frac{1}{2} \cdot 16$ inches, or 8 inches, and the radius of the medium pizza is $\frac{1}{2} \cdot 8$ inches, or 4 inches. The area of the surface of each circular pizza is determined using the formula for the area of a circle.

$$\text{Large pizza: } A = \pi r^2 = \pi\,(8\text{ in.})^2 = 64\pi\text{ in.}^2 \approx 201\text{ in.}^2$$

$$\text{Medium pizza: } A = \pi r^2 = \pi(4\text{ in.})^2 = 16\pi\text{ in.}^2 \approx 50\text{ in.}^2$$

For each pizza, the price per square inch is found by dividing the price by the area:

$$\text{Price per square inch for large pizza} = \frac{\$15.00}{64\pi\text{ in.}^2} \approx \frac{\$15.00}{201\text{ in.}^2} \approx \frac{\$0.07}{\text{in.}^2}$$

$$\text{Price per square inch for medium pizza} = \frac{\$7.50}{16\pi\text{ in.}^2} \approx \frac{\$7.50}{50\text{ in.}^2} = \frac{\$0.15}{\text{in.}^2}$$

The large pizza costs approximately \$0.07 per square inch and the medium pizza costs approximately \$0.15 per square inch. Thus, the large pizza is the better buy.

TECHNOLOGY

You can use your calculator to obtain the price per square inch for each pizza in Example 8. The price per square inch for the large pizza, $\frac{15}{64\pi}$, is approximated by one of the following sequences of keystrokes:

Many Scientific Calculators

15 [÷] [(] 64 [×] [π] [)] [=]

Many Graphing Calculators

15 [÷] [(] 64 [π] [)] [ENTER]

In Example 8, did you at first think that the price per square inch would be the same for the large and the medium pizzas? After all, the radius of the large pizza is twice that of the medium pizza, and the cost of the large is twice that of the medium. However, the large pizza's area, 64π square inches, is *four times the area* of the medium pizza's, 16π square inches. Doubling the radius of a circle increases its area by a factor of 2^2, or 4. In general, if the radius of a circle is increased by k times its original linear measure, the area is multiplied by k^2. The same principle is true for any two-dimensional figure: If the shape of the figure is kept the same while linear dimensions are increased k times, the area of the larger, similar, figure is k^2 times greater than the area of the original figure.

CHECK POINT 8 Which one of the following is the better buy: a large pizza with an 18-inch diameter for \$20.00 or a medium pizza with a 14-inch diameter for \$14.00?

Exercise Set 10.4

Practice Exercises

In Exercises 1–14, use the formulas developed in this section to find the area of each figure.

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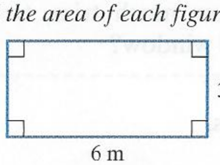

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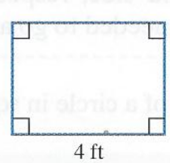

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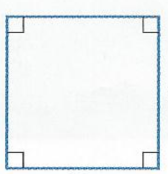

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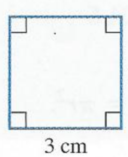

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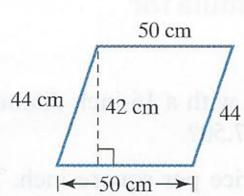

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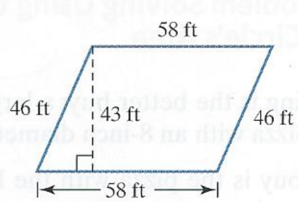

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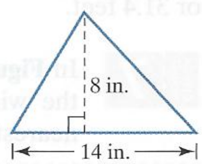

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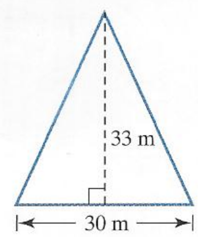

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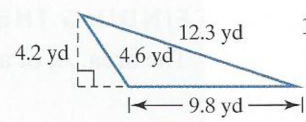
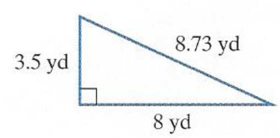


10.


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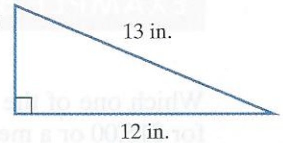

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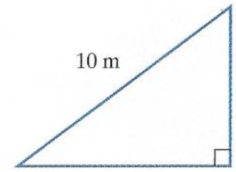

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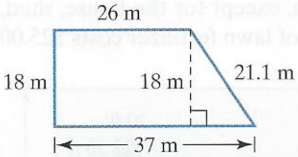

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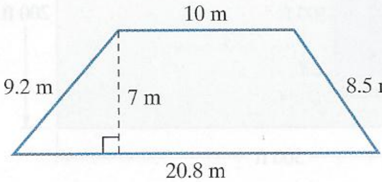
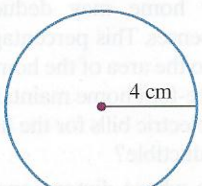


In Exercises 15–18, find the circumference and area of each circle. Express answers in terms of π and then round to the nearest tenth.

15.



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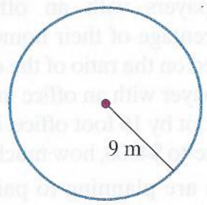

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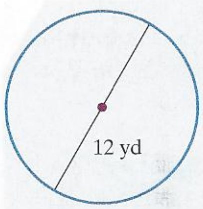

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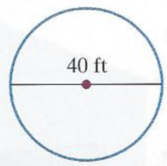
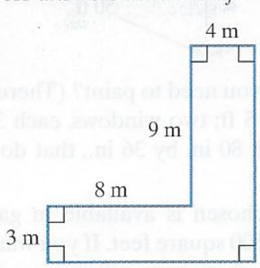


Find the area of each figure in Exercises 19–24. Where necessary, express answers in terms of π and then round to the nearest tenth.

19.



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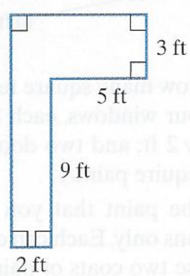

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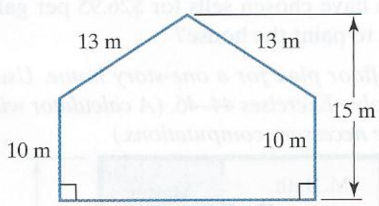

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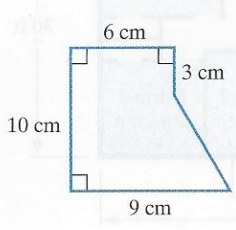

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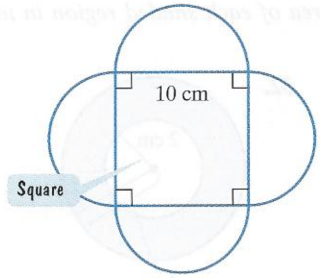

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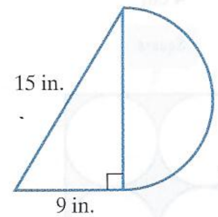
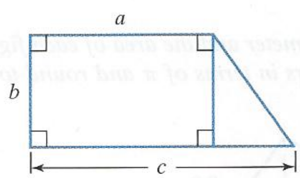


In Exercises 25–28, find a formula for the total area, A, of each figure in terms of the variable(s) shown. Where necessary, use π in the formula.

25.



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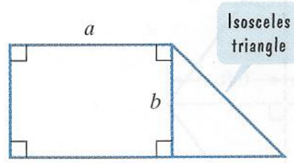
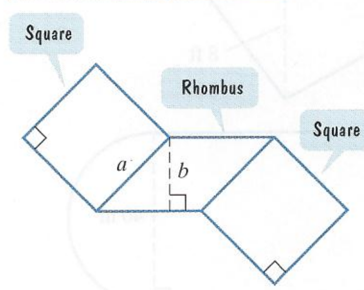
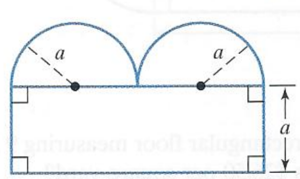


27.



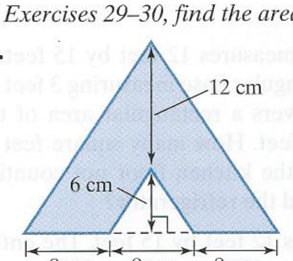
28.



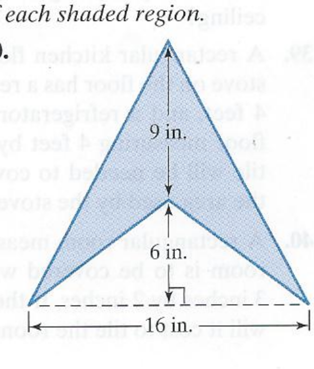
Practice Plus

In Exercises 29–30, find the area of each shaded region.

29.

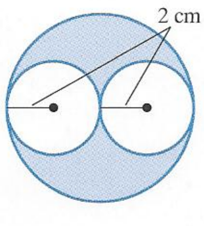


30.

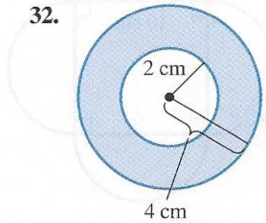


In Exercises 31–34, find the area of each shaded region in terms of π.

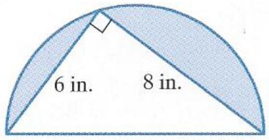
31.



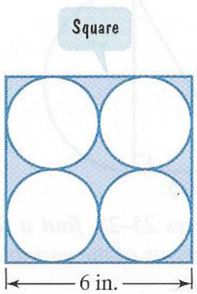
32.



33.

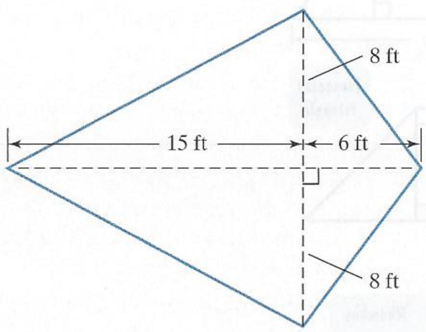


34.

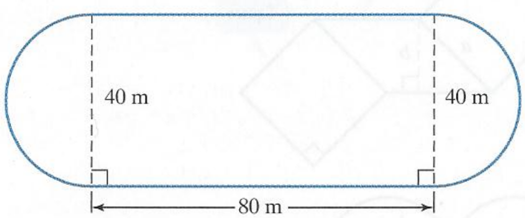


In Exercises 35–36, find the perimeter and the area of each figure. Where necessary, express answers in terms of π and round to the nearest tenth.

35.



36.



Application Exercises

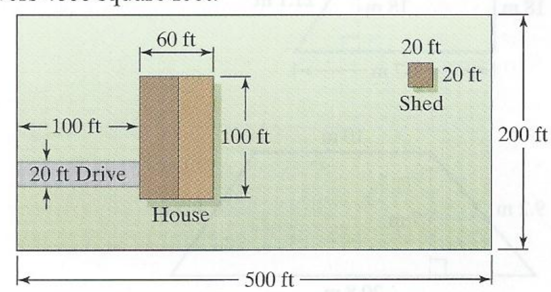
37. What will it cost to carpet a rectangular floor measuring 9 feet by 21 feet if the carpet costs $26.50 per square yard?

38. A plastering contractor charges $18 per square yard. What is the cost of plastering 60 feet of wall in a house with a 9-foot ceiling?

39. A rectangular kitchen floor measures 12 feet by 15 feet. A stove on the floor has a rectangular base measuring 3 feet by 4 feet, and a refrigerator covers a rectangular area of the floor measuring 4 feet by 5 feet. How many square feet of tile will be needed to cover the kitchen floor not counting the area used by the stove and the refrigerator?

40. A rectangular room measures 12 feet by 15 feet. The entire room is to be covered with rectangular tiles that measure 3 inches by 2 inches. If the tiles are sold at ten for 30¢, what will it cost to tile the room?

41. The lot in the figure shown, except for the house, shed, and driveway, is lawn. One bag of lawn fertilizer costs $25.00 and covers 4000 square feet.

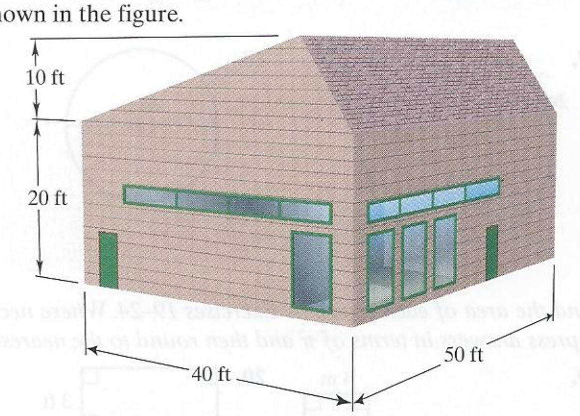


a. Determine the minimum number of bags of fertilizer needed for the lawn.

b. Find the total cost of the fertilizer.

42. Taxpayers with an office in their home may deduct a percentage of their home-related expenses. This percentage is based on the ratio of the office's area to the area of the home. A taxpayer with an office in a 2200-square-foot home maintains a 20 foot by 16 foot office. If the yearly electric bills for the home come to $4800, how much of this is deductible?

43. You are planning to paint the house whose dimensions are shown in the figure.

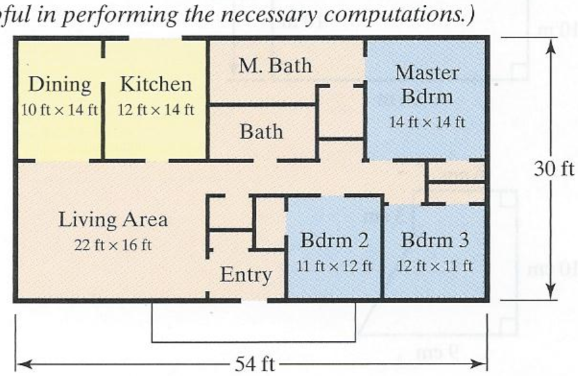


a. How many square feet will you need to paint? (There are four windows, each 8 ft by 5 ft; two windows, each 30 ft by 2 ft; and two doors, each 80 in. by 36 in., that do not require paint.)

b. The paint that you have chosen is available in gallon cans only. Each can covers 500 square feet. If you want to use two coats of paint, how many cans will you need for the project?

c. If the paint you have chosen sells for $26.95 per gallon, what will it cost to paint the house?

The diagram shows the floor plan for a one-story home. Use the given measurements to solve Exercises 44–46. (A calculator will be helpful in performing the necessary computations.)



44. If construction costs $95 per square foot, find the cost of building the home.

45. If carpet costs $17.95 per square yard and is available in whole square yards only, find the cost of carpeting the three bedroom floors.

46. If ceramic tile costs $26.95 per square yard and is available in whole square yards only, find the cost of installing ceramic tile on the kitchen and dining room floors.

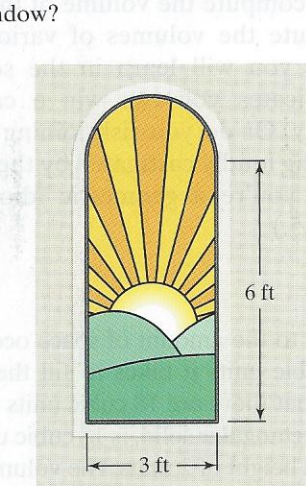
In Exercises 47–48, express the required calculation in terms of π and then round to the nearest tenth.

47. How much fencing is required to enclose a circular garden whose radius is 20 meters?

48. A circular rug is 6 feet in diameter. How many feet of fringe is required to edge this rug?

49. How many plants spaced every 6 inches are needed to surround a circular garden with a 30-foot radius?

50. A stained glass window is to be placed in a house. The window consists of a rectangle, 6 feet high by 3 feet wide, with a semicircle at the top. Approximately how many feet of stripping, to the nearest tenth of a foot, will be needed to frame the window?



51. Which one of the following is a better buy: a large pizza with a 14-inch diameter for $12.00 or a medium pizza with a 7-inch diameter for $5.00?

52. Which one of the following is a better buy: a large pizza with a 16-inch diameter for $12.00 or two small pizzas, each with a 10-inch diameter, for $12.00?

Writing in Mathematics

53. Using the formula for the area of a rectangle, explain how the formula for the area of a parallelogram ($A = bh$) is obtained.

54. Using the formula for the area of a parallelogram ($A = bh$), explain how the formula for the area of a triangle $\left(A = \frac{1}{2}bh\right)$ is obtained.

55. Using the formula for the area of a triangle, explain how the formula for the area of a trapezoid is obtained.

56. Explain why a circle is not a polygon.

57. Describe the difference between the following problems: How much fencing is needed to enclose a circular garden? How much fertilizer is needed for a circular garden?

Critical Thinking Exercises

Make Sense? *In Exercises 58–61, determine whether each statement makes sense or does not make sense, and explain your reasoning.*

58. The house is a 1500-square-foot mansion with six bedrooms.

59. Because a parallelogram can be divided into two triangles with the same size and shape, the area of a triangle is one-half that of a parallelogram.

60. I used $A = \pi r^2$ to determine the amount of fencing needed to enclose my circular garden.

61. I paid $10 for a pizza, so I would expect to pay approximately $20 for the same kind of pizza with twice the radius.

62. You need to enclose a rectangular region with 200 feet of fencing. Experiment with different lengths and widths to determine the maximum area you can enclose. Which quadrilateral encloses the most area?

63. Suppose you know the cost for building a rectangular deck measuring 8 feet by 10 feet. If you decide to increase the dimensions to 12 feet by 15 feet, by how much will the cost increase?

64. A rectangular swimming pool measures 14 feet by 30 feet. The pool is surrounded on all four sides by a path that is 3 feet wide. If the cost to resurface the path is $2 per square foot, what is the total cost of resurfacing the path?

65. A proposed oil pipeline will cross 16.8 miles of national forest. The width of the land needed for the pipeline is 200 feet. If the U.S. Forest Service charges the oil company $32 per acre, calculate the total cost. (1 mile = 5280 feet and 1 acre = 43,560 square feet.)