Introduction to Complex Analysis - excerpts

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Chapter 1

The Holomorphic Functions

We begin with the description of complex numbers and their basic algebraic properties. We will assume that the reader had some previous encounters with the complex numbers and will be fairly brief, with the emphasis on some specifics that we will need later.

1 The Complex Plane

1.1 The complex numbers

We consider the set $\mathbb{C}$ of pairs of real numbers (x, y), or equivalently of points on the plane $\mathbb{R}^2$. Two vectors $z_1 = (x_1, x_2)$ and $z_2 = (x_2, y_2)$ are *equal* if and only if $x_1 = x_2$ and $y_1 = y_2$. Two vectors $z = (x, y)$ and $\bar{z} = (x, -y)$ that are symmetric to each other with respect to the x-axis are said to be *complex conjugate* to each other. We identify the vector $(x, 0)$ with a real number x. We denote by $\mathbb{R}$ the set of all real numbers (the x-axis).

Exercise 1.1 *Show that $z = \bar{z}$ if and only if z is a real number.*

We introduce now the operations of addition and multiplication on $\mathbb{C}$ that turn it into a field. The sum of two complex numbers and multiplication by a real number $\lambda \in \mathbb{R}$ are defined in the same way as in $\mathbb{R}^2$:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2), \quad \lambda(x, y) = (\lambda x, \lambda y).$$

Then we may write each complex number $z = (x, y)$ as

$$z = x \cdot \mathbf{1} + y \cdot i = x + iy, \tag{1.1}$$

where we denoted the two unit vectors in the directions of the x and y-axes by $\mathbf{1} = (1, 0)$ and $i = (0, 1)$.

You have previously encountered two ways of defining a product of two vectors: the inner product $(z_1 \cdot z_2) = x_1 x_2 + y_1 y_2$ and the skew product $[z_1, z_2] = x_1 y_2 - x_2 y_1$. However, none of them turn $\mathbb{C}$ into a field, and, actually $\mathbb{C}$ is not even closed under these operations: both the inner product and the skew product of two vectors is a number, not a vector. This leads us to introduce yet another product on $\mathbb{C}$. Namely, we postulate that $i \cdot i = i^2 = -1$ and define $z_1 z_2$ as a vector obtained by multiplication of $x_1 + iy_1$ and $x_2 + iy_2$ using the usual rules of algebra with the additional convention that $i^2 = -1$. That is, we define

$$z_1 z_2 = x_1 x_2 - y_1 y_2 + i(x_1 y_2 + x_2 y_1). \tag{1.2}$$

More formally we may write

$$(x_1, y_1)(x_2, y_2) = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1)$$

but we will not use this somewhat cumbersome notation.

Exercise 1.2 *Show that the product of two complex numbers may be written in terms of the inner product and the skew product as* $z_1 z_2 = (\bar{z}_1 \cdot z_2) + i[\bar{z}_1, z_2]$, *where* $\bar{z}_1 = x_1 - iy_1$ *is the complex conjugate of* z_1.

Exercise 1.3 *Check that the product (1.2) turns* $\mathbb{C}$ *into a field, that is, the distributive, commutative and associative laws hold, and for any* $z \neq 0$ *there exists a number* $z^{-1} \in \mathbb{C}$ *so that* $zz^{-1} = 1$. *Hint:* $z^{-1} = \dfrac{x}{x^2 + y^2} - \dfrac{iy}{x^2 + y^2}$.

Exercise 1.4 *Show that the following operations do not turn* $\mathbb{C}$ *into a field: (a)* $z_1 z_2 = x_1 x_2 + iy_1 y_2$, *and (b)* $z_1 z_2 = x_1 x_2 + y_1 y_2 + i(x_1 y_2 + x_2 y_1)$.

The product (1.2) turns $\mathbb{C}$ into a field (see Exercise 1.3) that is called the *field of complex numbers* and its elements, vectors of the form $z = x + iy$ are called *complex numbers*. The real numbers x and y are traditionally called the real and imaginary parts of z and are denoted by

$$x = \mathrm{Re} z, \quad y = \mathrm{Im} z. \tag{1.3}$$

A number $z = (0, y)$ that has the real part equal to zero, is called *purely imaginary*.

The Cartesian way (1.1) of representing a complex number is convenient for performing the operations of addition and subtraction, but one may see from (1.2) that multiplication and division in the Cartesian form are quite tedious. These operations, as well as raising a complex number to a power are much more convenient in the *polar representation* of a complex number:

$$z = r(\cos\phi + i \sin\phi), \tag{1.4}$$

that is obtained from (1.1) passing to the polar coordinates for (x, y). The polar coordinates of a complex number z are the polar radius $r = \sqrt{x^2 + y^2}$ and the polar angle ϕ, the angle between the vector z and the positive direction of the x-axis. They are called the *modulus* and *argument* of z are denoted by

$$r = |z|, \quad \phi = \mathrm{Arg} z. \tag{1.5}$$

The modulus is determined uniquely while the argument is determined up to addition of a multiple of 2π. We will use a shorthand notation

$$\cos\phi + i\sin\phi = e^{i\phi}. \tag{1.6}$$

Note that we have not yet defined the operation of raising a number to a complex power, so the right side of (1.6) should be understood at the moment just as a shorthand for the left side. We will define this operation later and will show that (1.6) indeed holds. With this convention the polar form (1.4) takes a short form

$$z = re^{i\phi}. \tag{1.7}$$

Using the basic trigonometric identities we observe that

$$\begin{aligned} r_1e^{i\phi_1}r_2e^{i\phi_2} &= r_1(\cos\phi_1 + i\sin\phi_1)r_2(\cos\phi_2 + i\sin\phi_2) \\ &= r_1r_2(\cos\phi_1\cos\phi_2 - \sin\phi_1\sin\phi_2 + i(\cos\phi_1\sin\phi_2 + \sin\phi_1\cos\phi_2)) \\ &= r_1r_2(\cos(\phi_1+\phi_2) + i\sin(\phi_1+\phi_2)) = r_1r_2e^{i(\phi_1+\phi_2)}. \end{aligned} \tag{1.8}$$

This explains why notation (1.6) is quite natural. Relation (1.8) says that the modulus of the product is the product of the moduli, while the argument of the product is the sum of the arguments.

Exercise 1.5 *Show that if* $z = re^{i\phi}$ *then* $z^{-1} = \dfrac{1}{r}e^{-i\phi}$, *and more generally if* $z_1 = r_1e^{i\phi_1}$, $z_2 = r_2e^{i\phi_2}$ *with* $r_2 \neq 0$, *then*

$$\frac{z_1}{z_2} = \frac{r_1e^{i\phi_1}}{r_2e^{i\phi_2}} = \frac{r_1}{r_2}e^{i(\phi_1-\phi_2)}. \tag{1.9}$$

Sometimes it is convenient to consider a *compactification* of the set $\mathbb{C}$ of complex numbers. This is done by adding an ideal element that is call the point at infinity $z = \infty$. However, algebraic operations are not defined for $z = \infty$. We will call the compactified complex plane, that is, the plane $\mathbb{C}$ together with the point at infinity, the closed complex plane, denoted by $\overline{\mathbb{C}}$. Sometimes we will call $\mathbb{C}$ the open complex plane in order to stress the difference between $\mathbb{C}$ and $\overline{\mathbb{C}}$.

One can make the compactification more visual if we represent the complex numbers as points not on the plane but on a two-dimensional sphere as follows. Let ξ, η and ζ be the Cartesian coordinates in the three-dimensional space so that the ξ and η-axes coincide with the x and y-axes on the complex plane. Consider the unit sphere

$$S: \quad \xi^2 + \eta^2 + \zeta^2 = 1 \tag{1.10}$$

in this space. Then for each point $z = (x, y) \in \mathbb{C}$ we may find a corresponding point $Z = (\xi, \eta, \zeta)$ on the sphere that is the intersection of S and the segment that connects the "North pole" $N = (0, 0, 1)$ and the point $z = (x, y, 0)$ on the complex plane.

The mapping $z \to Z$ is called *the stereographic projection*. The segment Nz may be parameterized as $\xi = tx$, $\eta = ty$, $\zeta = 1 - t$, $t \in [0, 1]$. Then the intersection point $Z = (t_0 x, t_0 y, 1 - t_0)$ with t_0 being the solution of

$$t_0^2 x^2 + t_0^2 y^2 + (1 - t_0)^2 = 1$$

so that $(1 + |z|^2)t_0 = 2$. Therefore the point Z has the coordinates

$$\xi = \frac{2x}{1 + |z|^2}, \quad \eta = \frac{2y}{1 + |z|^2}, \quad \zeta = \frac{|z|^2 - 1}{1 + |z|^2}. \tag{1.11}$$

The last equation above implies that $\dfrac{2}{1 + |z|^2} = 1 - \zeta$. We find from the first two equations the explicit formulae for the inverse map $Z \to z$:

$$x = \frac{\xi}{1 - \zeta}, \quad y = \frac{\eta}{1 - \zeta}. \tag{1.12}$$

Expressions (1.11) and(1.12) show that the stereographic projection is a one-to-one map from $\mathbb{C}$ to $S \backslash N$ (clearly N does not correspond to any point z). We postulate that N corresponds to the point at infinity $z = \infty$. This makes the stereographic projection be a one-to-one map from $\bar{\mathbb{C}}$ to S. We will usually identify $\bar{\mathbb{C}}$ and the sphere S. The latter is called *the sphere of complex numbers* or *the Riemann sphere*. The open plane $\mathbb{C}$ may be identified with $S \backslash N$, the sphere with the North pole deleted.

Exercise 1.6 *Let t and u be the longitude and the latitude of a point Z. Show that the corresponding point $z = se^{it}$, where $s = \tan(\pi/4 + u/2)$.*

We may introduce two metrics (distances) on $\mathbb{C}$ according to the two geometric descriptions presented above. The first is the usual Euclidean metric with the distance between the points $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ in $\mathbb{C}$ given by

$$|z_2 - z_1| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}. \tag{1.13}$$

The second is *the spherical metric* with the distance between z_1 and z_2 defined as the Euclidean distance in the three-dimensional space between the corresponding points Z_1 and Z_2 on the sphere. A straightforward calculation shows that

$$\rho(z_1, z_2) = \frac{2|z_2 - z_1|}{\sqrt{1 + |z_1|^2}\sqrt{1 + |z_2|^2}}. \tag{1.14}$$

This formula may be extended to $\overline{\mathbb{C}}$ by setting

$$\rho(z, \infty) = \frac{2}{\sqrt{1 + |z|^2}}. \tag{1.15}$$

Note that (1.15) may be obtained from (1.14) if we let $z_1 = z$, divide the numerator and denominator by $|z_2|$ and let $|z_2| \to +\infty$.

Exercise 1.7 *Use the formula (1.11) for the stereographic projection to verify (1.14).*

Clearly we have $\rho(z_1, z_2) \le 2$ for all $z_1, z_2 \in \overline{\mathbb{C}}$. It is straightforward to verify that both of the metrics introduced above turn $\mathbb{C}$ into a metric space, that is, all the usual axioms of a metric space are satisfied. In particular, the triangle inequality for the Euclidean metric (1.13) is equivalent to the usual triangle inequality for two-dimensional plane: $|z_1 + z_2| \le |z_1| + |z_2|$.

Exercise 1.8 *Verify the triangle inequality for the metric $\rho(z_1, z_2)$ on $\overline{\mathbb{C}}$ defined by (1.14) and (1.15)*

We note that the Euclidean and spherical metrics are equivalent on bounded sets $M \subset \mathbb{C}$ that lie inside a fixed disk $\{|z| \le R\}$, $R < \infty$. Indeed, if $M \subset \{|z| \le R\}$ then (1.14) implies that for all $z_1, z_2 \in M$ we have

$$\frac{2}{1+R^2}|z_2 - z_1| \le \rho(z_1, z_2) \le 2|z_2 - z_1| \tag{1.16}$$

(this will be elaborated in the next section). Because of that the spherical metric is usually used only for unbounded sets. Typically, we will use the Euclidean metric for $\mathbb{C}$ and the spherical metric for $\overline{\mathbb{C}}$.

Now is the time for a little history. We find the first mention of the complex numbers as square rots of negative numbers in the book "Ars Magna" by Girolamo Cardano published in 1545. He thought that such numbers could be introduced in mathematics but opined that this would be useless: "Dismissing mental tortures, and multiplying $5 + \sqrt{-15}$ by $5 - \sqrt{-15}$, we obtain $25 - (-15)$. Therefore the product is 40. .... and thus far does arithmetical subtlety go, of which this, the extreme, is, as I have said, so subtle that it is useless." The baselessness of his verdict was realized fairly soon: Raphael Bombelli published his "Algebra" in 1572 where he introduced the algebraic operations over the complex numbers and explained how they may be used for solving the cubic equations. One may find in Bombelli's book the relation $(2 + \sqrt{-121})^{1/3} + (2 - \sqrt{-121})^{1/3} = 4$. Still, the complex numbers remained somewhat of a mystery for a long time. Leibnitz considered them to be "a beautiful and majestic refuge of the human spirit", but he also thought that it was impossible to factor $x^4 + 1$ into a product of two quadratic polynomials (though this is done in an elementary way with the help of complex numbers).

The active use of complex numbers in mathematics began with the works of Leonard Euler. He has also discovered the relation $e^{i\phi} = \cos\phi + i\sin\phi$. The geometric interpretation of complex numbers as planar vectors appeared first in the work of the Danish geographical surveyor Caspar Wessel in 1799 and later in the work of Jean Robert Argand in 1806. These papers were not widely known - even Cauchy who has obtained numerous fundamental results in complex analysis considered early in his career the complex numbers simply as symbols that were convenient for calculations, and equality of two complex numbers as a shorthand notation for equality of two real-valued variables.

The first systematic description of complex numbers, operations over them, and their geometric interpretation were given by Carl Friedreich Gauss in 1831 in his memoir "Theoria residuorum biquadraticorum". He has also introduced the name "complex numbers".

1.2 The topology of the complex plane

We have introduced distances on $\mathbb{C}$ and $\overline{\mathbb{C}}$ that turned them into metric spaces. We will now introduce the two topologies that correspond to these metrics.

Let $\varepsilon > 0$ then an ε-neighborhood $U(z_0, \varepsilon)$ of $z_0 \in \mathbb{C}$ in the Euclidean metric is the disk of radius ε centered at z_0, that is, the set of points $z \in \mathbb{C}$ that satisfy the inequality

$$|z - z_0| < \varepsilon. \tag{1.17}$$

An ε-neighborhood of a point $z_0 \in \overline{\mathbb{C}}$ is the set of all points $z \in \overline{\mathbb{C}}$ such that

$$\rho(z, z_0) < \varepsilon. \tag{1.18}$$

Expression (1.15) shows that the inequality $\rho(z, \infty) < \varepsilon$ is equivalent to $|z| > \sqrt{\frac{4}{\varepsilon^2} - 1}$. Therefore an ε-neighborhood of the point at infinity is the outside of a disk centered at the origin complemented by $z = \infty$.

We say that a set Ω in $\mathbb{C}$ (or $\overline{\mathbb{C}}$) is *open* if for any point $z_0 \in \Omega$ there exists a neighborhood of z_0 that is contained in Ω. It is straightforward to verify that this notion of an open set turns $\mathbb{C}$ and $\overline{\mathbb{C}}$ into *topological spaces*, that is, the usual axioms of a topological space are satisfied.

Sometimes it will be convenient to make use of the so called *punctured neighborhoods*, that is, the sets of the points $z \in \mathbb{C}$ (or $z \in \overline{\mathbb{C}}$) that satisfy

$$0 < |z - z_0| < \varepsilon, \quad 0 < \rho(z, z_0) < \varepsilon. \tag{1.19}$$

We will introduce in this Section the basic topological notions that we will constantly use in the sequel.

Definition 1.9 *A point $z_0 \in \mathbb{C}$ (resp. in $\overline{\mathbb{C}}$) is a limit point of the set $M \subset \mathbb{C}$ (resp. $\overline{\mathbb{C}}$) if there is at least one point of M in any punctured neighborhood of z_0 in the topology of $\mathbb{C}$ (resp. $\overline{\mathbb{C}}$). A set M is said to be closed if it contains all of its limit points. The union of M and all its limit points is called the closure of M and is denoted $\overline{M}$.*

Example 1.10 The set $\mathbb{Z}$ of all integers $\{0, \pm 1, \pm 2, \dots\}$ has no limit points in $\mathbb{C}$ and is therefore closed in $\mathbb{C}$. It has one limit point $z = \infty$ in $\overline{\mathbb{C}}$ that does not belong to $\mathbb{Z}$. Therefore $\mathbb{Z}$ is not closed in $\overline{\mathbb{C}}$.

Exercise 1.11 *Show that any infinite set in $\overline{\mathbb{C}}$ has at least one limit point (compactness principle).*

This principle expresses the completeness (as a metric space) of the sphere of complex numbers and may be proved using the completeness of the real numbers. We leave the proof to the reader. However, as Example 1.10 shows, this principle fails in $\mathbb{C}$. Nevertheless it holds for infinite bounded subsets of $\mathbb{C}$, that is, sets that are contained in a disk $\{|z| < R\}$, $R < \infty$.

Inequality (1.16) shows that a point $z_0 \neq \infty$ is a limit point of a set M in the topology of $\mathbb{C}$ if and only if it is a limit point of M in the topology of $\overline{\mathbb{C}}$. In other words, when we talk about finite limit points we may use either the Euclidean or the spherical metric. That is what the equivalence of these two metrics on bounded sets, that we have mentioned before, means.

Definition 1.12 *A sequence $\{a_n\}$ is a mapping from the set $\mathbb{N}$ of non-negative integers into $\mathbb{C}$ (or $\overline{\mathbb{C}}$). A point $a \in \mathbb{C}$ (or $\overline{\mathbb{C}}$) is a limit point of the sequence $\{a_n\}$ if any neighborhood of a in the topology of $\mathbb{C}$ (or $\overline{\mathbb{C}}$) contains infinitely many elements of the sequence. A sequence $\{a_n\}$ converges to a if a is its only limit point. Then we write*

$$\lim_{n\to\infty} a_n = a. \tag{1.20}$$

Remark 1.13 *The notions of the limit point of a sequence $\{a_n\}$ and of the set of values $\{a_n\}$ are different. For instance, the sequence $\{1, 1, 1, \dots\}$ has a limit point $a = 1$, while the set of values consists of only one point $z = 1$ and has no limit points.*

Exercise 1.14 *Show that 1) A sequence $\{a_n\}$ converges to a if and only if for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ so that $|a_n - a| < \varepsilon$ for all $n \geq N$ (if $a \neq \infty$), or $\rho(a_n, a) < \varepsilon$ (if $a = \infty$). 2) A point a is a limit point of a sequence $\{a_n\}$ if and only if there exists a subsequence $\{a_{n_k}\}$ that converges to a.*

The complex equation (1.20) is equivalent to two real equations. Indeed, (1.20) is equivalent to

$$\lim_{n\to\infty} |a_n - a| = 0, \tag{1.21}$$

where the limit above is understood in the usual sense of convergence of real-valued sequences. Let $a \neq \infty$, then without any loss of generality we may assume that $a_n \neq \infty$ (because if $a \neq \infty$ then there exists N so that $a_n \neq \infty$ for $n > N$ and we may restrict ourselves to $n > N$) and let $a_n = \alpha_n + i\beta_n$, $a = \alpha + i\beta$ (for $a = \infty$ the real and imaginary parts are not defined). Then we have

$$\max(|\alpha_n - \alpha|, |\beta_n - \beta|) \leq \sqrt{|\alpha_n - \alpha|^2 + |\beta_n - \beta|^2} \leq |\alpha_n - \alpha| + |\beta_n - \beta|$$

and hence (1.21) and the squeezing theorem imply that (1.20) is equivalent to a pair of equalities

$$\lim_{n\to\infty} \alpha_n = \alpha, \quad \lim_{n\to\infty} \beta_n = \beta. \tag{1.22}$$

In the case when $a \neq 0$ and $a \neq \infty$ we may assume that $a_n \neq 0$ and $a_n \neq \infty$ and write $a_n = r_n e^{i\phi_n}$, $a = re^{i\phi}$. Then

$$|a_n - a|^2 = r^2 + r_n^2 - 2rr_n \cos(\phi - \phi_n) = (r - r_n)^2 + 2rr_n(1 - \cos(\phi - \phi_n)) \tag{1.23}$$

and hence (1.20) holds if

$$\lim_{n\to\infty} r_n = r, \quad \lim_{n\to\infty} \phi_n = \phi. \tag{1.24}$$

Conversely, if (1.20) holds then (1.23) implies that the first equality in (1.24) holds and that $\lim_{n\to\infty} \cos(\phi - \phi_n) = 1$. Therefore if we choose $\phi_n \in [0, 2\pi)$ then (1.20) implies also the second equality in (1.24).

Exercise 1.15 *Show that 1) the sequence* $a_n = e^{in}$ *diverges, and 2) if a series* $\sum_{n=1}^{\infty} a_n$ *converges and* $|\arg a_n| \le \alpha < \pi/2$*, then the series converges absolutely. Here* $\arg a_n$ *is the value of Arg* a_n *that satisfies* $-\pi < \arg a_n \le \pi$*.*

We will sometimes use the notion of the distance between two sets M and N, which is equal to the least upper bound of all distances between pairs of points from M and N:

$$\rho(M,N) = \inf_{z\in M, z'\in N} \rho(z,z'). \tag{1.25}$$

One may use the Euclidean metric to define the distance between sets as well, of course.

Theorem 1.16 *Let* M *and* N *be two non-overlapping closed sets:* $M \cap N = \emptyset$*, then the distance between* M *and* N *is positive.*

Proof. Let us assume that $\rho(M,N) = 0$. Then there exist two sequences of points $z_n \in M$ and $z_n' \in N$ so that $\lim_{n\to\infty} \rho(z_n, z_n') = 0$. According to the compactness principle the sequences z_n and z_n' have limit points z and z', respectively. Moreover, since both M and N are closed, we have $z \in M$ and $z' \in N$. Then there exist a subsequence $n_k \to \infty$ so that *both* $z_{n_k} \to z$ and $z'_{n_k} \to z'$. The triangle inequality for the spherical metric implies that

$$\rho(z,z') \le \rho(z, z_{n_k}) + \rho(z_{n_k}, z'_{n_k}) + \rho(z'_{n_k}, z').$$

The right side tends to zero as $k \to \infty$ while the left side does not depend on k. Therefore, passing to the limit $k \to \infty$ we obtain $\rho(z,z') = 0$ and thus $z = z'$. However, $z \in M$ and $z' \in N$, which contradicts the assumption that $M \cap N = \emptyset$. □

1.3 Paths and curves

Definition 1.17 *A path* γ *is a continuous map of an interval* $[\alpha,\beta]$ *of the real axis into the complex plane* $\mathbb{C}$ *(or* $\overline{\mathbb{C}}$*). In other words, a path is a complex valued function* $z = \gamma(t)$ *of a real argument* t*, that is continuous at every point* $t_0 \in [\alpha,\beta]$ *in the following sense: for any* $\varepsilon > 0$ *there exists* $\delta > 0$ *so that* $|\gamma(t) - \gamma(t_0)| < \varepsilon$ *(or* $\rho(\gamma(t),\gamma(t_0)) < \varepsilon$ *if* $\gamma(t_0) = \infty$*) provided that* $|t - t_0| < \delta$*. The points* $a = \gamma(\alpha)$ *and* $b = \gamma(\beta)$ *are called the endpoints of the path* γ*. The path is closed if* $\gamma(\alpha) = \gamma(\beta)$*. We say that a path* γ *lies in a set* M *if* $\gamma(t) \in M$ *for all* $t \in [\alpha,\beta]$*.*

Sometimes it is convenient to distinguish between a path and a curve. In order to introduce the latter we say that two paths

$$\gamma_1 : [\alpha_1,\beta_1] \to \overline{\mathbb{C}} \text{ and } \gamma_2 : [\alpha_2,\beta_2] \to \overline{\mathbb{C}}$$

are *equivalent* ($\gamma_1 \sim \gamma_2$) if there exists an increasing continuous function

$$\tau : [\alpha_1,\beta_1] \to [\alpha_2,\beta_2] \tag{1.26}$$

such that $\tau(\alpha_1) = \alpha_2$, $\tau(\beta_1) = \beta_2$ and so that $\gamma_1(t) = \gamma_2(\tau(t))$ for all $t \in [\alpha_1,\beta_1]$.

Exercise 1.18 *Verify that relation $\sim$ is reflexive: $\gamma \sim \gamma$, symmetric: if $\gamma_1 \sim \gamma_2$, then $\gamma_2 \sim \gamma_1$ and transitive: if $\gamma_1 \sim \gamma_2$ and $\gamma_2 \sim \gamma_3$ then $\gamma_1 \sim \gamma_3$.*

Example 1.19 Let us consider the paths $\gamma_1(t) = t$, $t \in [0,1]$; $\gamma_2(t) = \sin t$, $t \in [0, \pi/2]$; $\gamma_3(t) = \cos t$, $t \in [0, \pi/2]$ and $\gamma_4(t) = \sin t$, $t \in [0, \pi]$. The set of values of $\gamma_j(t)$ is always the same: the interval $[0,1]$. However, we only have $\gamma_1 \sim \gamma_2$. These two paths trace $[0,1]$ from left to right once. The paths γ_3 and γ_4 are neither equivalent to these two, nor to each other: the interval $[0,1]$ is traced in a different way by those paths: γ_3 traces it from right to left, while γ_4 traces $[0,1]$ twice.

Exercise 1.20 Which of the following paths: a) $e^{2\pi it}$, $t \in [0,1]$; b) $e^{4\pi it}$, $t \in [0,1]$; c) $e^{-2\pi it}$, $t \in [0,1]$; d) $e^{4\pi i \sin t}$, $t \in [0, \pi/6]$ are equivalent to each other?

Definition 1.21 *A curve is an equivalence class of paths. Sometimes, when this will cause no confusion, we will use the word 'curve' to describe a set $\gamma \in \overline{\mathbb{C}}$ that may be represented as an image of an interval $[\alpha, \beta]$ under a continuous map $z = \gamma(t)$.*

Below we will introduce some restrictions on the curves and paths that we will consider. We say that $\gamma : [\alpha, \beta] \to \overline{\mathbb{C}}$ is a *Jordan* path if the map γ is continuous and *one-to-one*. The definition of a closed Jordan path is left to the reader as an exercise.

A path $\gamma : [\alpha, \beta] \to \mathbb{C}$ ($\gamma(t) = x(t) + iy(t)$) is *continuously differentiable* if derivative $\gamma'(t) := x'(t) + iy'(t)$ exists for all $t \in [\alpha, \beta]$. A continuously differentiable path is said to be *smooth* if $\gamma'(t) \neq 0$ for all $t \in [\alpha, \beta]$. This condition is introduced in order to avoid singularities. A path is called *piecewise smooth* if $\gamma(t)$ is continuous on $[\alpha, \beta]$, and $[\alpha, \beta]$ may be divided into a finite number of closed sub-intervals so that the restriction of $\gamma(t)$ on each of them is a smooth path.

We will also use the standard notation to describe smoothness of functions and paths: the class of continuous functions is denoted C, or C^0, the class of continuously differentiable functions is denoted C^1, etc. A function that has n continuous derivatives is said to be a C^n-function.

Example 1.22 The paths γ_1, γ_2 and γ_3 of the previous example are Jordan, while γ_4 is not Jordan. The circle $z = e^{it}$, $t \in [0, 2\pi]$ is a closed smooth Jordan path; the four-petal rose $z = e^{it} \cos 2t$, $t \in [0, 2\pi]$ is a smooth non-Jordan path; the semi-cubic parabola $z = t^2(t+i)$, $t \in [-1, 1]$ is a Jordan continuously differentiable piecewise smooth path. The path $z = t\left(1 + i \sin\left(\frac{1}{t}\right)\right)$, $t \in [-1/\pi, 1/\pi]$ is a Jordan non-piecewise smooth path.

One may introduce similar notions for curves. A *Jordan curve* is a class of paths that are equivalent to some Jordan path (observe that since the change of variables (1.26) is one-to-one, all paths equivalent to a Jordan path are also Jordan).

The definition of a smooth curve is slightly more delicate: this notion has to be invariant with respect to a replacement of a path that represents a given curve by an equivalent one. However, a continuous monotone change of variables (1.26) may map a smooth path onto a non-smooth one unless we impose some additional conditions on the functions τ allowed in (1.26).

More precisely, a smooth curve is a class of paths that may be obtained out of a smooth path by all possible re-parameterizations (1.26) with $\tau(s)$ being a continuously differentiable function with a positive derivative. One may define a piecewise smooth curve in a similar fashion: the change of variables has to be continuous everywhere, and in addition have a continuous positive derivative except possibly at a finite set of points.

Sometimes we will use a more geometric interpretation of a curve, and say that a Jordan, or smooth, or piecewise smooth curve is a set of points $\gamma \subset \mathbb{C}$ that may be represented as the image of an interval $[\alpha, \beta]$ under a map $z = \gamma(t)$ that defines a Jordan, smooth or piecewise smooth path.

1.4 Domains

We say that a set D is *pathwise-connected* if for any two points $a, b \in D$ there exists a path that lies in D and has endpoints a and b.

Definition 1.23 *A domain D is a subset of $\mathbb{C}$ (or $\overline{\mathbb{C}}$) that is both open and pathwise-connected.*

The limit points of a domain D that do not belong to D are called the *boundary points* of D. These are the points z so that any neighborhood of z contains some points in D and at least one point not in D. Indeed, if $z_0 \in \partial D$ then any neighborhood of z contains a point from D since z_0 is a limit point of D, and it also contains z_0 itself that does not lie in D. Conversely, if any neighborhood of z_0 contains some points in D and at least one point not in D then $z_0 \notin D$ since D is open, and z_0 is a limit point of D, so that $z_0 \in \partial D$. The collection of all boundary points of D is called the *boundary* of D and is denoted by ∂D. The *closure* of D is the set $\bar{D} = D \cup \partial D$. The *complement* of D is the set $D^c = \mathbb{C} \backslash \bar{D}$, the points z that lie in D^c are called the *outer points* of D.

Exercise 1.24 Show that the set D^c is open.

Theorem 1.25 *The boundary ∂D of any domain D is a closed set.*

Proof. Let ζ_0 be a limit point of ∂D. We have to show that $\zeta_0 \in \partial D$. Let U be a punctured neighborhood of ζ_0. Then U contains a point $\zeta \in \partial D$. Furthermore, there exists a neighborhood V of ζ so that $V \subset U$. However, since ζ is a boundary point of D, the set V must contain points both from D and not from D. Therefore U also contains both points from D and not in D and hence $\zeta_0 \in \partial D$. $\Box$

We will sometimes need some additional restrictions on the boundary of domains. The following definition is useful for these purposes.

Definition 1.26 *The set M is connected if it is impossible to split it as $M = M_1 \cup M_2$ so that both M_1 and M_2 are not empty while the intersections $\bar{M}_1 \cap M_2$ and $M_1 \cap \bar{M}_2$ are empty.*

Exercise 1.27 Show that a closed set is connected if and only if it cannot be represented as a union of two non-overlapping non-empty closed sets.

One may show that a pathwise connected set is connected. The converse, however, is not true.

Let M be a non-connected set. A subset $N \subset M$ is called a *connected component* if M if N is connected and is not contained in any other connected subset of M. One may show that any set is the union of its connected components (though, it may have infinitely many connected components).

A domain $D \subset \overline{\mathbb{C}}$ is *simply connected* if its boundary ∂D is a connected set.

Example 1.28 (a) The interior of figure eight is not a domain since it is not pathwise-connected. (b) The set of points between two circles tangent to each other is a simply connected domain.

Sometimes we will impose further conditions. A domain D is *Jordan* if its boundary is a union of closed Jordan curves. A domain D is *bounded* if it lies inside a bounded disk $\{|z| < R,\ R < \infty\}$. A set M is properly embedded in a domain D if its closure $\bar{M}$ in $\overline{\mathbb{C}}$ is contained in D. We will then write $M \subset\subset D$.

We will often make use of the following theorem. A neighborhood of a point z in the relative topology of a set M is the intersection of a usual neighborhood of z and M.

Theorem 1.29 *Let $M \subset \overline{\mathbb{C}}$ be a connected set and let N be its non-empty subset. If N is both open and closed in the relative topology of M then $M = N$.*

Proof. Let the set $N' = M \backslash N$ be non-empty. The closure $\bar{N}$ of N in the usual topology of $\overline{\mathbb{C}}$ is the union of its closure $(\bar{N})_M$ of N in the relative topology of M, and some other set (possibly empty) that does not intersect M. Therefore we have $\bar{N} \cap N' = (\bar{N})_M \cap N'$. However, N is closed in the relative topology of M so that $(\bar{N})_M = N$ and hence $(\bar{N})_M \cap N' = N \cap N' = \emptyset$.

Furthermore, since N is also open in the relative topology of M, its complement N' in the same topology is closed (the limit points of N' may not belong to N since the latter is open, hence they belong to N' itself). Therefore we may apply the previous argument to N' and conclude that $\bar{N}' \cap N$ is empty. This contradicts the assumption that M is connected. □

2 Functions of a complex variable

2.1 Functions

A complex valued function $f : M \to \mathbb{C}$, where $M \subset \overline{\mathbb{C}}$ is one-to-one, if for any two points $z_1 \neq z_2$ in M the images $w_1 = f(z_1)$, $w_2 = f(z_2)$ are different: $w_1 \neq w_2$. Later we will need the notion of a multi-valued function that will be introduced in Chapter 3.

Defining a function $f : M \to \mathbb{C}$ is equivalent to defining two real-valued functions

$$u = u(z), \quad v = v(z). \tag{2.1}$$

Here $u : M \to \mathbb{R}$ and $v : M \to \mathbb{R}$ are the real and imaginary parts of f: $f(x+iy) = u(x+iy) + iv(x+iy)$. Furthermore, if $f \neq 0, \neq \infty$ (this notation means that $f(z) \neq 0$ and $f(z) \neq \infty$ for all $z \in M$) we may write $f = \rho e^{i\psi}$ with

$$\rho = \rho(z), \quad \psi = \psi(z) + 2k\pi, \quad (k = 0, \pm 1, \dots). \tag{2.2}$$

At the points where $f = 0$, or $f = \infty$, the function $\rho = 0$ or $\rho = \infty$ while ψ is not defined.

We will constantly use the geometric interpretation of a complex valued function. The form (2.1) suggests representing f as two surfaces $u = u(x,y)$, $v = v(x,y)$ in the three-dimensional space. However, this is not convenient since it does not represent (u,v) as one complex number. Therefore we will represent a function $f : M \to \overline{\mathbb{C}}$ as a map of M into a sphere $\overline{\mathbb{C}}$.

We now turn to the basic notion of the limit of a function.

Definition 2.1 *Let the function f be defined in a punctured neighborhood of a point $a \in \overline{\mathbb{C}}$. We say that the number $A \in \overline{\mathbb{C}}$ is its limit as z goes to a and write*

$$\lim_{z \to a} f(z) = A, \tag{2.3}$$

if for any neighborhood U_A of A their exists a punctured neighborhood U'_a of a so that for all $z \in U'_a$ we have $f(z) \in U_A$. Equivalently, for any $\varepsilon > 0$ there exists $\delta > 0$ so that the inequality

$$0 < \rho(z,a) < \delta \tag{2.4}$$

implies

$$\rho(f(z), A) < \varepsilon. \tag{2.5}$$

If $a, A \neq \infty$ then (2.4) and (2.5) may be replaced by the inequalities $0 < |z - a| < \delta$ and $|f(z) - A| < \varepsilon$. If $a = \infty$ and $A \neq \infty$ then they may written as $\delta < |z| < \infty$, $|f(z) - A| < \varepsilon$. You may easily write them in the remaining cases $a \neq \infty$, $A = \infty$ and $a = A = \infty$.

We set $f = u + iv$. It is easy to check that for $A \neq \infty$, $A = A_1 + iA_2$, (2.3) is equivalent to two equalities

$$\lim_{z \to a} u(z) = A_1, \quad \lim_{z \to a} v(z) = A_2. \tag{2.6}$$

If we assume in addition that $A \neq 0$ and choose $\arg f$ appropriately then (2.3) may be written in polar coordinates as

$$\lim_{z \to a} |f(z)| = |A|, \quad \lim_{z \to a} \arg f(z) = \arg A. \tag{2.7}$$

The elementary theorems regarding the limits of functions in real analysis, such as on the limit of sums, products and ratios may be restated verbatim for the complex case and we do not dwell on their formulation and proof.

Sometimes we will talk about the limit of a function along a set. Let M be a set, a be its limit point and f a function defined on M. We say that f tends to A as z tends to a along M and write

$$\lim_{z \to a, z \in M} f(z) = A \tag{2.8}$$

if for any $\varepsilon > 0$ there exists $\delta > 0$ so that if $z \in M$ and $0 < \rho(z, a) < \delta$ we have $\rho(f(z), A) < \varepsilon$.

Definition 2.2 *Let f be defined in a neighborhood of $a \in \overline{\mathbb{C}}$. We say that f is continuous at a if*

$$\lim_{z \to a} f(z) = f(a). \tag{2.9}$$

For the reasons we have just discussed the elementary theorems about the sum, product and ratio of continuous functions in real analysis translate immediately to the complex case.

One may also define continuity of f at a along a set M, for which a is a limit point, if the limit in (2.9) is understood along M. A function that is continuous at every point of M (along M) is said to be continuous on M. In particular if f is continuous at every point of a domain D it is continuous in the domain.

We recall some properties of continuous functions on closed sets $K \subset \overline{\mathbb{C}}$:
1. Any function f that is continuous on K is bounded on K, that is, there exists $A \geq 0$ so that $|f(z)| \leq A$ for all $z \in K$.
2. Any function f that is continuous on K attains its maximum and minimum, that is, there exist $z_1, z_2 \in K$ so that $|f(z_1)| \leq |f(z)| \leq |f(z_2)|$ for all $z \in K$.
3. Any function f that is continuous on K is uniformly continuous, that is, for any $\varepsilon > 0$ there exists $\delta > 0$ so that $|f(z_1) - f(z_2)| < \varepsilon$ provided that $\rho(z_1, z_2) < \delta$.

The proofs of these properties are the same as in the real case and we do not present them here.

2.2 Differentiability

The notion of differentiability is intricately connected to linear approximations so we start with the discussion of linear functions of complex variables.

Definition 2.3 *A function $f : \mathbb{C} \to \mathbb{C}$ is $\mathbb{C}$-linear, or $\mathbb{R}$-linear, respectively, if*
(a) $l(z_1 + z_2) = l(z_1) + l(z_2)$ for all $z_1, z_2 \in \mathbb{C}$,
(b) $l(\lambda z) = \lambda l(z)$ for all $\lambda \in \mathbb{C}$, or, respectively, $\lambda \in \mathbb{R}$.

Thus $\mathbb{R}$-linear functions are linear over the field of real numbers while $\mathbb{C}$-linear are linear over the field of complex numbers. The latter form a subset of the former.

Let us find the general form of an $\mathbb{R}$-linear function. We let $z = x + iy$, and use properties (a) and (b) to write $l(z) = xl(1) + yl(i)$. Let us denote $\alpha = l(1)$ and $\beta = l(i)$, and replace $x = (z + \bar{z})/2$ and $y = (z - \bar{z})/(2i)$. We obtain the following theorem.

Theorem 2.4 *Any* $\mathbb{R}$*-linear function has the form*

$$l(z) = az + b\bar{z}, \tag{2.10}$$

where $a = (\alpha - i\beta)/2$ *and* $b = (\alpha + i\beta)/2$ *are complex valued constants.*

Similarly writing $z = 1 \cdot z$ we obtain

Theorem 2.5 *Any* $\mathbb{C}$*-linear function has the form*

$$l(z) = az, \tag{2.11}$$

where $a = l(1)$ *is a complex valued constant.*

Theorem 2.6 *An* $\mathbb{R}$*-linear function is* $\mathbb{C}$*-linear if and only if*

$$l(iz) = il(z). \tag{2.12}$$

Proof. The necessity of (2.12) follows immediately from the definition of a $\mathbb{C}$-linear function. Theorem 2.4 implies that $l(z) = az + b\bar{z}$, so $l(iz) = i(az - b\bar{z})$. Therefore, $l(iz) = il(z)$ if and only if

$$iaz - b\bar{z} = iaz + ib\bar{z}.$$

Therefore if $l(iz) = il(z)$ for all $z \in \mathbb{C}$ then $b = 0$ and hence l is $\mathbb{C}$-linear.

We set $a = a_1 + ia_2$, $b = b_1 + ib_2$, and also $z = x + iy$, $w = u + iv$. We may represent an $\mathbb{R}$-linear function $w = az + b\bar{z}$ as two real equations

$$u = (a_1 + b_1)x - (a_2 - b_2)y, \quad v = (a_2 + b_2)x + (a_1 - b_1)y.$$

Therefore geometrically an $\mathbb{R}$-linear function is an affine transform of a plane $\mathbf{y} = A\mathbf{x}$ with the matrix

$$A = \begin{pmatrix} a_1 + b_1 & -(a_2 - b_2) \\ a_2 + b_2 & a_1 - b_1 \end{pmatrix}. \tag{2.13}$$

Its Jacobian is

$$J = a_1^2 - b_1^2 + a_2^2 - b_2^2 = |a|^2 - |b|^2. \tag{2.14}$$

This transformation is non-singular when $|a| \neq |b|$. It transforms lines into lines, parallel lines into parallel lines and squares into parallelograms. It preserves the orientation when $|a| > |b|$ and changes it if $|a| < |b|$.

However, a $\mathbb{C}$-linear transformation $w = az$ may not change orientation since its jacobian $J = |a|^2 \geq 0$. They are not singular unless $a = 0$. Letting $a = |a|e^{i\alpha}$ and recalling the geometric interpretation of multiplication of complex numbers we find that a non-degenerate $\mathbb{C}$-linear transformation

$$w = |a|e^{i\alpha}z \tag{2.15}$$

is the composition of dilation by $|a|$ and rotation by the angle α. Such transformations preserve angles and map squares onto squares.

Exercise 2.7 Let $b = 0$ in (2.13) and decompose A as a product of two matrices, one corresponding to dilation by $|a|$, another to rotation by α.

We note that preservation of angles characterizes $\mathbb{C}$-linear transformations. Moreover, the following theorem holds.

Theorem 2.8 *If an $\mathbb{R}$-linear transformation $w = az + b\bar{z}$ preserves orientation and angles between three non-parallel vectors $e^{i\alpha_1}$, $e^{i\alpha_2}$, $e^{i\alpha_3}$, $\alpha_j \in \mathbb{R}$, $j = 1, 2, 3$, then w is $\mathbb{C}$-linear.*

Proof. Let us assume that $w(e^{i\alpha_1}) = \rho e^{i\beta_1}$ and define $w'(z) = e^{-i\beta_1} w(ze^{i\alpha_1})$. Then $w'(z) = a'z + b'\bar{z}$ with

$$a' = ae^{i(\alpha_1 - \beta_1)}a, \quad b' = be^{-i(\alpha_1+\beta_1)},$$

and, moreover $w'(1) = e^{-i\beta_1}\rho e^{i\beta_1} = \rho > 0$. Therefore we have $a' + b' > 0$. Furthermore, w' preserves the orientation and angles between vectors $v_1 = 1$, $v_2 = e^{i(\alpha_2-\alpha_1)}$ and $v_3 = e^{i(\alpha_3-\alpha_1)}$. Since both v_1 and its image lie on the positive semi-axis and the angles between v_1 and v_2 and their images are the same, we have $w'(v_2) = h_2 v_2$ with $h_2 > 0$. This means that

$$a'e^{i\beta_2} + b'e^{-i\beta_2} = h_2 e^{i\beta_2}, \quad \beta_2 = \alpha_2 - \alpha_1,$$

and similarly

$$a'e^{i\beta_3} + b'e^{-i\beta_3} = h_3 e^{i\beta_3}, \quad \beta_3 = \alpha_3 - \alpha_1,$$

with $h_3 > 0$. Hence we have

$$a' + b' > 0, \quad a' + b'e^{-2i\beta_2} > 0, \quad a' + b'e^{-2i\beta_3} > 0.$$

This means that unless $b' = 0$ there exist three different vectors that connect the vector a' to the real axis, all having the same length $|b'|$. This is impossible, and hence $b' = 0$ and w is $\mathbb{C}$-linear.

Exercise 2.9 (a) Give an example of an $\mathbb{R}$-linear transformation that is not $\mathbb{C}$-linear but preserves angles between two vectors.
(b) Show that if an $\mathbb{R}$-linear transformation preserves orientation and maps some square onto a square it is $\mathbb{C}$-linear.

Now we may turn to the notion of differentiability of complex functions. Intuitively, a function is differentiable if it is well approximated by linear functions. Two different definitions of linear functions that we have introduced lead to different notions of differentiability.

Definition 2.10 *Let $z \in \mathbb{C}$ and let U be a neighborhood of z. A function $f : U \to \mathbb{C}$ is $\mathbb{R}$-differentiable (respectively, $\mathbb{C}$-differentiable) at the point z if we have for sufficiently small $|\Delta z|$:*

$$\Delta f = f(z + \Delta z) - f(z) = l(\Delta z) + o(\Delta z), \tag{2.16}$$

where $l(\Delta z)$ (with z fixed) is an $\mathbb{R}$-linear (respectively, $\mathbb{C}$-linear) function of Δz, and $o(\Delta z)$ satisfies $o(\Delta z)/\Delta z \to 0$ as $\Delta z \to 0$. The function l is called the differential of f at z and is denoted df.

The increment of an $\mathbb{R}$-differentiable function has, therefore, the form

$$\Delta f = a\Delta z + b\overline{\Delta z} + o(\Delta z). \tag{2.17}$$

Taking the increment $\Delta z = \Delta x$ along the x-axis, so that $\overline{\Delta z} = \Delta x$ and passing to the limit $\Delta x \to 0$ we obtain

$$\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \frac{\partial f}{\partial x} = a + b.$$

Similarly, taking $\Delta z = i\Delta y$ (the increment is long the y-axis) so that $\overline{\Delta z} = -i\Delta y$ we obtain

$$\lim_{\Delta y \to 0} \frac{\Delta f}{i\Delta y} = \frac{1}{i}\frac{\partial f}{\partial y} = a - b.$$

The two relations above imply that

$$a = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right), \quad b = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

These coefficients are denoted as

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right), \quad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) \tag{2.18}$$

and are sometimes called the formal derivatives of f at the point z. They were first introduced by Riemann in 1851.

Exercise 2.11 Show that (a) $\frac{\partial z}{\partial \bar{z}} = 0$, $\frac{\partial \bar{z}}{\partial \bar{z}} = 1$; (b) $\frac{\partial}{\partial \bar{z}}(f+g) = \frac{\partial f}{\partial \bar{z}} + \frac{\partial g}{\partial \bar{z}}$, $\frac{\partial}{\partial \bar{z}}(fg) = \frac{\partial f}{\partial \bar{z}}g + f\frac{\partial g}{\partial \bar{z}}$.

Using the obvious relations $dz = \Delta z$, $d\bar{z} = \Delta\bar{z}$ we arrive at the formula for the differential of $\mathbb{R}$-differentiable functions

$$df = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z}. \tag{2.19}$$

Therefore, all the functions $f = u + iv$ such that u and v have usual differentials as functions of two real variables x and y turn out to be $\mathbb{R}$-differentiable. This notion does not bring any essential new ideas to analysis. The complex analysis really starts with the notion of $\mathbb{C}$-differentiability.

The increment of a $\mathbb{C}$-differentiable function has the form

$$\Delta f = a\Delta z + o(\Delta z) \tag{2.20}$$

and its differential is a $\mathbb{C}$-linear function of Δz (with z fixed). Expression (2.19) shows that $\mathbb{C}$-differentiable functions are distinguished from $\mathbb{R}$-differentiable ones by an additional condition

$$\frac{\partial f}{\partial \bar{z}} = 0. \tag{2.21}$$

If $f = u + iv$ then (2.18) shows that

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

so that the complex equation (2.21) may be written as a pair of real equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{2.22}$$

The notion of complex differentiability is clearly very restrictive: while it is fairly difficult to construct an example of a continuous but nowhere real differentiable function, most trivial functions turn out to be non-differentiable in the complex sense. For example, the function $f(z) = x + 2iy$ is nowhere $\mathbb{C}$-differentiable: $\dfrac{\partial u}{\partial x} = 1$, $\dfrac{\partial v}{\partial y} = 2$ and conditions (2.22) fail everywhere.

Exercise 2.12 1. Show that $\mathbb{C}$-differentiable functions of the form $u(x) + iv(y)$ are necessarily $\mathbb{C}$-linear.
2. Let $f = u + iv$ be $\mathbb{C}$-differentiable in the whole plane $\mathbb{C}$ and $u = v^2$ everywhere. Show that $f = \text{const}$.

Let us consider the notion of a derivative starting with that of the directional derivative. We fix a point $z \in \mathbb{C}$, its neighborhood U and a function $f : U \to \mathbb{C}$. Setting $\Delta z = |\Delta z| e^{i\theta}$ we obtain from (2.17) and (2.19):

$$\Delta f = \frac{\partial f}{\partial z}|\Delta z| e^{i\theta} + \frac{\partial f}{\partial \bar{z}}|\Delta z| e^{-i\theta} + o(\Delta z).$$

We divide both sides by Δz, pass to the limit $|\Delta z| \to 0$ with θ fixed and obtain the derivative of f at the point z in direction θ:

$$\frac{\partial f}{\partial z_\theta} = \lim_{|\Delta z| \to 0, \arg z = \theta} \frac{\Delta f}{\Delta z} = \frac{\partial f}{\partial z} + \frac{\partial f}{\partial \bar{z}} e^{-2i\theta}. \tag{2.23}$$

This expression shows that when z is fixed and θ changes between 0 and 2π the point $\dfrac{\partial f}{\partial z_\theta}$ traverses twice a circle centered at $\dfrac{\partial f}{\partial z}$ with the radius $\left|\dfrac{\partial f}{\partial \bar{z}}\right|$.

Hence if $\dfrac{\partial f}{\partial \bar{z}} \neq 0$ then the directional derivative depends on direction θ, and only if $\dfrac{\partial f}{\partial \bar{z}} = 0$, that is, if f is $\mathbb{C}$-differentiable, all directional derivatives at z are the same.

Clearly, the derivative of f at z exists if and only if the latter condition holds. It is defined by

$$f'(z) = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z}. \tag{2.24}$$

The limit is understood in the topology of $\mathbb{C}$. It is also clear that if $f'(z)$ exists then it is equal to $\dfrac{\partial f}{\partial z}$. This proposition is so important despite its simplicity that we formulate it as a separate theorem.

Theorem 2.13 *Complex differentiability of f at z is equivalent to the existence of the derivative $f'(z)$ at z.*

Proof. If f is $\mathbb{C}$-differentiable at z then (2.20) with $a = \dfrac{\partial f}{\partial z}$ implies that

$$\Delta f = \frac{\partial f}{\partial z}\Delta z + o(\Delta z).$$

Then, since $\lim\limits_{\Delta z \to 0} \dfrac{o(\Delta z)}{\Delta z} = 0$, we obtain that the limit $f'(z) = \lim\limits_{\Delta z \to 0} \dfrac{\Delta f}{\Delta z}$ exists and is equal to $\dfrac{\partial f}{\partial z}$. Conversely, if $f'(z)$ exists then by the definition of the limit we have

$$\frac{\Delta f}{\Delta z} = f'(z) + \alpha(\Delta z),$$

where $\alpha(\Delta z) \to 0$ as $\Delta z \to 0$. Therefore the increment $\Delta f = f'(z)\Delta z + \alpha(\Delta z)\Delta z$ may be split into two parts so that the first is linear in Δz and the second is $o(\Delta z)$, which is equivalent to $\mathbb{C}$-differentiability of f at z.□

The definition of the derivative of a function of a complex variable is exactly the same as in the real analysis, and all the arithmetic rules of dealing with derivatives translate into the complex realm without any changes. Thus the elementary theorems regarding derivatives of a sum, product, ratio, composition and inverse function apply verbatim in the complex case. We skip their formulation and proofs.

Let us mention a remark useful in computations. The derivative of a function $f = u + iv$ does not depend on direction (if it exists), so it may be computed in particular in the direction of the x-axis:

$$f'(z) = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}. \tag{2.25}$$

We should have convinced ourselves that the notion of $\mathbb{C}$-differentiability is very natural. However, as we will see later, $\mathbb{C}$-differentiability at one point is not sufficient to build an interesting theory. Therefore we will require $\mathbb{C}$-differentiability not at one point but in a whole neighborhood.

Definition 2.14 *A function f is holomorphic (or analytic) at a point $z \in \mathbb{C}$ if it is $\mathbb{C}$-differentiable in a neighborhood of z.*

Example 2.15 The function $f(z) = |z|^2 = z\bar{z}$ is clearly $\mathbb{R}$-differentiable everywhere in $\mathbb{C}$. However, $\dfrac{\partial f}{\partial \bar{z}} = 0$ only at $z = 0$, so f is only $\mathbb{C}$-differentiable at $z = 0$ but is not holomorphic at this point.

The set of functions holomorphic at a point z is denoted by $\mathcal{O}_z$. Sums and products of functions in $\mathcal{O}_z$ also belong to $\mathcal{O}_z$, so this set is a ring. We note that the ratio f/g of two functions in $\mathcal{O}_z$ might not belong to $\mathcal{O}_z$ if $g(z) = 0$.

Functions that are $\mathbb{C}$-differentiable at all points of an open set $D \subset \mathbb{C}$ are clearly also holomorphic at all points $z \in D$. We say that such functions are holomorphic in D and denote their collection by $\mathcal{O}(D)$. The set $\mathcal{O}(D)$ is also a ring. In general a function is holomorphic on a set $M \subset \mathbb{C}$ if it may extended to a function that is holomorphic on an open set D that contains M.

Finally we say that f is holomorphic at infinity if the function $g(z) = f(1/z)$ is holomorphic at $z = 0$. This definition allows to consider functions holomorphic in $\overline{\mathbb{C}}$. However, the notion of derivative at $z = \infty$ is not defined.

The notion of complex differentiability lies at the heart of complex analysis. A special role among the founders of complex analysis was played by Leonard Euler, "the teacher of all mathematicians of the second half of the XVIIIth century" according to Laplace. Let us describe briefly his life and work.

Euler was born in 1707 into a family of a Swiss pastor and obtained his Master's diploma at Basel in 1724. He studied theology for some time but then focused solely on mathematics and its applications. Nineteen-year old Euler moved to Saint Petersburg in 1727 and took the vacant position in physiology at the Russian Academy of Sciences that had been created shortly before his arrival. Nevertheless he started to work in mathematics, and with remarkable productivity on top of that: he published more than 50 papers during his first fourteen year long stay at Saint Petersburg, being also actively involved in teaching and various practical problems.

Euler moved to Berlin in 1741 where he worked until 1766 but he kept his ties to the Saint Petersburg Academy, publishing more than 100 papers and books in its publications. Then he returned to Saint Petersburg where he stayed until his death. Despite almost complete blindness Euler prepared more than 400 papers during his second seventeen year long stay in Saint Petersburg.

In his famous monographes "Introductio in analys in infinitorum" (1748), "Institutiones calculi integralis" (1755) and "Institutiones calculi integralis" (1768-70) Euler has developed mathematical analysis as a branch of mathematical science for the first time. He was the creator of calculus of variations, theory of partial differential equations and differential geometry and obtained outstanding results in number theory.

Euler was actively involved in applied problems alongside his theoretical work. For instance he took part in the creation of geographic maps of Russia and in the expert analysis of the project of a one-arc bridge over the Neva river proposed by I. Kulibin, he studied the motion of objects through the air and computed the critical stress of columns. His books include "Mechanica" (1736-37), a book on Lunar motion (1772) and a definitive book on navigation (1778). Euler died in 1783 and was buried in Saint Petersburg. His descendants stayed in Russia: two of his sons were members of the Russian Academy of Sciences and a third was a general in the Russian army.

Euler has introduced the elementary functions of a complex variable in the books mentioned above and found relations between them, such as the Euler formula $e^{i\phi} = \cos\phi + i\sin\phi$ mentioned previously and systematically used complex substitutions for computations of integrals. In his book on the basics of fluid motion (1755) Euler related the components u and v of the flow to expressions $udy - vdx$ and $udx + vdy$. Following D'Alembert who published his work three years earlier Euler formulated conditions that turn the above into exact differential forms:

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}. \tag{2.26}$$

He found the general form of a solution of such system:

$$\begin{aligned} u - iv &= \frac{1}{2}\phi(x+iy) - \frac{i}{2}\psi(x+iy) \\ u + iv &= \frac{1}{2}\phi(x-iy) + \frac{i}{2}\psi(x-iy), \end{aligned}$$

where ϕ and ψ are arbitrary (according to Euler) functions. Relations (2.26) are simply the conditions for complex differentiability of the function $f = u - iv$ and have a simple physical interpretation (see the next section). Euler has also written down the usual conditions of differentiability (2.22) that differ from (2.26) by a sign. In 1776 the 69 year old Euler wrote a paper where he pointed out that these conditions imply that the expression $(u+iv)(dx+idy)$ is an exact differential form, and in 1777 he pointed out their application to cartography. Euler was the first mathematician to study systematically the functions of complex variables and their applications in analysis, hydrodynamics and cartography.

However, Euler did not have the total understanding of the full implications of complex differentiability. The main progress in this direction was started by the work of Cauchy 70 years later and then by Riemann 30 years after Cauchy. The two conditions of $\mathbb{C}$-differentiability,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

are called the Cauchy-Riemann equations, though historically they should probably be called D'Alembert-Euler equations.

2.3 Geometric and Hydrodynamic Interpretations

The differentials of an $\mathbb{R}$-differentiable and, respectively, a $\mathbb{C}$-differentiable function at a point z have form

$$df = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z}, \quad df = f'(z)dz. \tag{2.27}$$

The Jacobians of such maps are given by (see (2.14))

$$J_f(z) = \left|\frac{\partial f}{\partial z}\right|^2 - \left|\frac{\partial f}{\partial \bar{z}}\right|^2, \quad J_f(z) = |f'(z)|^2. \tag{2.28}$$

Let us assume that f is $\mathbb{R}$-differentiable at z and z is not a critical point of f, that is, $J_f(z) \neq 0$. The implicit function theorem implies that locally f is a homeomorphism, that is, there exists a neighborhood U of z so that f maps U continuously and one-to-one onto a neighborhood of $f(z)$. Expressions (2.28) show that in general J_f may have an arbitrary sign if f is just $\mathbb{R}$-differentiable. However, the critical points of a $\mathbb{C}$-differentiable map coincide with the points where derivative vanishes, while such maps preserve orientation at non-critical points: $J_f(z) = |f'(z)'|^2 > 0$.

Furthermore, an $\mathbb{R}$-differentiable map is said to be conformal at $z \in \mathbb{C}$ if its differential df at z is a non-degenerate transformation that is a composition of dilation and rotation. Since the latter property characterizes $\mathbb{C}$-linear maps we obtain the following geometric interpretation of $\mathbb{C}$-differentiability:

Complex differentiability of f at a point z together with the condition $f'(z) \neq 0$ is equivalent to f being a conformal map at z.

A map $f : D \to \mathbb{C}$ conformal at every point $z \in D$ is said to be conformal in D. It is realized by a holomorphic function in z with no critical points ($f'(z) \neq 0$ in D). Its differential at every point of the domain is a composition of a dilation and a rotation, in particular it conserves angles. Such mappings were first considered by Euler in 1777 in relation to his participation in the project of producing geographic maps of Russia. The name "conformal mapping" was introduced by F. Schubert in 1789.

So far we have studied differentials of maps. Let us look now at how the properties of the map itself depend on it being conformal. Assume that f is conformal in a neighborhood U of a point z and that f' is continuous in U[1]. Consider a smooth path $\gamma : I = [0, 1] \to U$ that starts at z, that is, $\gamma'(t) \neq 0$ for all $t \in I$ and $\gamma(0) = z$. Its image $\gamma_* = f \circ \gamma$ is also a smooth path since

$$\gamma_*'(t) = f'[\gamma(t)]\gamma'(t), \quad t \in I, \tag{2.29}$$

and f' is continuous and different from zero everywhere in U by assumption.

Geometrically $\gamma'(t) = \dot{x}(t) + i\dot{y}(t)$ is the vector tangent to γ at the point $\gamma(t)$, and $|\gamma'(t)|dt = \sqrt{\dot{x}^2 + \dot{y}^2}dt = ds$ is the differential of the arc length of γ at the same point. Similarly, $|\gamma_*(t)|dt = ds_*$ is the differential of the arc length of γ_* at the point $\gamma_*(t)$. We conclude from (2.29) at $t = 0$ that

$$|f'(z)| = \frac{|\gamma_*'(0|}{|\gamma'(0)|} = \frac{ds_*}{ds}. \tag{2.30}$$

Thus the modulus of $f'(z)$ is equal to the dilation coefficient at z under the mapping f.

The left side does not depend on the curve γ as long as $\gamma(0) = z$. Therefore under our assumptions all arcs are dilated by the same factor. Therefore a conformal map f has a circle property: it maps small circles centered at z into curves that differ from circles centered at $f(z)$ only by terms of the higher order.

Going back to (2.29) we see that

$$\arg f'(z) = \arg \gamma_*'(0) - \arg \gamma'(0), \tag{2.31}$$

so that $\arg f'(z)$ is the rotation angle of the tangent lines at z under f.

The left side also does not depend on the choice of γ as long as $\gamma(0) = z$, so that all such arcs are rotated by the same angle. Thus a conformal map f preserves angles: the angle between any two curves at z is equal to the angle between their images at $f(z)$.

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[1] We will later see that existence of f' implies its continuity and, moreover, existence of derivatives of all orders.

If f is holomorphic at z but z is a critical point then the circle property holds in a degenerate form: the dilation coefficient of all curves at z is equal to 0. Angle preservation does not hold at all, for instance under the mapping $z \to z^2$ the angle between the lines $\arg z = \alpha_1$ and $\arg z = \alpha_2$ doubles! Moreover, smoothness of curves may be violated at a critical point. For instance a smooth curve $\gamma(t) = t+it^2$, $t \in [-1,1]$ is mapped under the same map $z \to z^2$ into the curve $\gamma_*(t) = t^2(1-t^2) + 2it^3$ with a cusp at $\gamma_*(0) = 0$.

Exercise 2.16 Let $u(x,y)$ and $v(x,y)$ be real valued $\mathbb{R}$-differentiable functions and let $\nabla u = \dfrac{\partial u}{\partial x} + i\dfrac{\partial u}{\partial y}$, $\nabla v = \dfrac{\partial v}{\partial x} + i\dfrac{\partial v}{\partial y}$. Find the geometric meaning of the conditions $(\nabla u, \nabla v) = 0$ and $|\nabla u| = |\nabla v|$, and their relation to the $\mathbb{C}$-differentiability of $f = u+iv$ and the conformity of f.

Let us now find the hydrodynamic meaning of complex differentiability and derivative. We consider a steady two-dimensional flow. That means that the flow vector field $v = (v_1, v_2)$ does not depend on time. The flow is described by

$$v = v_1(x,y) + iv_2(x,y). \tag{2.32}$$

Let us assume that in a neighborhood U of the point z the functions v_1 and v_2 have continuous partial derivatives. We will also assume that the flow v is irrotational in U, that is,

$$\mathrm{curl} v = \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} = 0 \tag{2.33}$$

and incompressible:

$$\mathrm{div} v = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} = 0 \tag{2.34}$$

at all $z \in U$.

Condition (2.33) implies the existence of a potential function ϕ such that $v = \nabla\phi$, that is,

$$v_1 = \frac{\partial \phi}{\partial x}, \quad v_2 = \frac{\partial \phi}{\partial y}. \tag{2.35}$$

The incompressibility condition (2.34) implies that there exists a stream function ψ so that

$$v_2 = -\frac{\partial \psi}{\partial x}, \;\; v_1 = \frac{\partial \psi}{\partial y}. \tag{2.36}$$

We have $d\psi = -v_2 dx + v_1 dy = 0$ along the level set of ψ and thus $\dfrac{dy}{dx} = \dfrac{v_2}{v_1}$. This shows that the level set is an integral curve of v.

Consider now a complex function

$$f = \phi + i\psi, \tag{2.37}$$

that is called the complex potential of v. Relations (2.35) and (2.36) imply that ϕ and ψ satisfy

$$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y}, \quad \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x}. \tag{2.38}$$

The above conditions coincide with (2.22) and show that the complex potential f is holomorphic at $z \in U$.

Conversely let the function $f = \phi + i\psi$ be holomorphic in a neighborhood U of a point z, and let the functions ϕ and ψ be twice continuously differentiable. Define the vector field $v = \nabla\phi = \frac{\partial\phi}{\partial x} + i\frac{\partial\phi}{\partial y}$. It is irrotational in U since curl$v = \frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x} = 0$. It is also incompressible since div$v = \frac{\partial^2\phi}{\partial^2 x} + \frac{\partial^2\phi}{\partial^2 y} = \frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x} = 0$. The complex potential of the vector field v is clearly the function f.

Therefore the function f is holomorphic if and only if it is the complex potential of a steady fluid flow that is both irrotational and incompressible.

It is easy to establish the hydrodynamic meaning of the derivative:

$$f' = \frac{\partial\phi}{\partial x} + i\frac{\partial\psi}{\partial x} = v_1 - iv_2, \tag{2.39}$$

so that the derivative of the complex potential is the vector that is the complex conjugate of the flow vector. The critical points of f are the points where the flow vanishes.

Example 2.17 Let us find the complex potential of an infinitely deep flow over a flat bottom with a line obstacle of height h perpendicular to the bottom. This is a flow in the upper half-plane that goes around an interval of length h that we may consider lying on the imaginary axis.

The boundary of the domain consists, therefore, of the real axis and the interval $[0, ih]$ on the imaginary axis. The boundary must be the stream line of the flow. We set it to be the level set $\psi = 0$ and will assume that $\psi > 0$ everywhere in D. In order to find the complex potential f it suffices to find a conformal mapping of D onto the upper half-plane $\psi > 0$. One function that provides such a mapping may be obtained as follows. The mapping $z_1 = z^2$ maps D onto the plane without the half-line Re$z_1 \geq -h^2$, Im$z_1 = 0$. The map $z_2 = z_1 + h^2$ maps this half-line onto the positive semi-axis Re$z_2 \geq 0$, Im$z_2 = 0$. Now the mapping $w_2 = \sqrt{z_2} = \sqrt{|z_2|}e^{i(\arg z_2)/2}$ with $0 < \arg z_2 < 2\pi$ maps the complex plane without the positive semi-axis onto the upper half-plane. It remains to write explicitly the resulting map

$$w = \sqrt{z_2} = \sqrt{z_1 + h^2} = \sqrt{z^2 + h^2} \tag{2.40}$$

that provides the desired mapping of D onto the upper half-plane. We may obtain the equation for the stream-lines of the flow by writing $(\phi + i\psi)^2 = (x + iy)^2 + h^2$. The streamline $\psi = \psi_0$ is obtained by solving

$$\phi^2 - \psi_0^2 = h^2 + x^2 - y^2, \quad 2\phi\psi_0 = 2xy.$$

This leads to $\phi = xy/\psi_0$ and

$$y = \psi_0 \sqrt{1 + \frac{h^2}{x^2 + \psi_0^2}}. \tag{2.41}$$

The magnitude of the flow is $|v| = \left|\frac{dw}{dz}\right| = \frac{|z|}{\sqrt{|z|^2 + h^2}}$ and is equal to one at infinity. The point $z = 0$ is the critical point of the flow. One may show that the general form of the solution is

$$f(z) = v_\infty \sqrt{z^2 + h^2}, \tag{2.42}$$

where $v_\infty > 0$ is the flow speed at infinity.

3 Properties of Fractional Linear Transformations

We will now study some simplest classes of functions of a complex variable.

3.1 Fractional Linear Transformations

Fractional linear transformations are functions of the form

$$w = \frac{az + b}{cz + d}, \quad ad - bc \neq 0, \tag{3.1}$$

where a, b, c, d are fixed complex numbers, and z is the complex variable. The condition $ad - bc \neq 0$ is imposed to exclude the degenerate case when $w = \text{const}$ (if $ad - bc = 0$ then the numerator is proportional to the denominator for all z). When $c = 0$ one must have $d \neq 0$, then the function (3.1) takes the form

$$w = \frac{a}{d} z + \frac{b}{d} = Az + B \tag{3.2}$$

and becomes an *entire linear function*. Such function is either constant if $A = 0$, or a composition of a shift $z \to z' = z + B/A$ and dilation and rotation $z' \to w = Az'$, as can be seen from the factorization $w = A(z + B/A)$ if $A \neq 0$.

The function (3.1) is defined for all $z \neq -d/c, \infty$ if $c \neq 0$, and for all finite z if $c = 0$. We define it at the exceptional points setting $w = \infty$ at $z = -d/c$ and $w = a/c$ at $z = \infty$ (it suffices to set $w = \infty$ at $z = \infty$ if $c = 0$). The following theorem holds.

Theorem 3.1 *A fractional linear transformation (3.1) is a homeomorphism (that is, a continuous and one-to-one map) of* $\overline{\mathbb{C}}$ *onto* $\overline{\mathbb{C}}$.

Proof. We assume that $c \neq 0$ - the simplifications in the case $c = 0$ are obvious. The function $w(z)$ is defined everywhere in $\overline{\mathbb{C}}$. We may solve (3.1) for z to obtain

$$z = \frac{dw - b}{a - cw} \tag{3.3}$$

and find that each $w \neq a/c, \infty$ has exactly one pre-image. Moreover, the extension of $w(z)$ to $\overline{\mathbb{C}}$ defined above shows that $\infty = w(-d/c)$ and $a/c = w(\infty)$. Therefore the function (3.1) is bijection of $\overline{\mathbb{C}}$ onto itself. It remains to show that (3.1) is continuous. However, its continuity is obvious at $z \neq -d/c, \infty$. The continuity of (3.1) at those points follows from the fact that

$$\lim_{z \to -d/c} \frac{az+b}{cz+d} = \infty, \qquad \lim_{z \to \infty} \frac{az+b}{cz+d} = \frac{a}{c}. \square$$

We would like to show now that the map (3.1) preserves angles everywhere in $\overline{\mathbb{C}}$. This follows from the existence of the derivative

$$\frac{dw}{dz} = \frac{ad-bc}{(cz+d)^2} \neq 0$$

for $z \neq -d/c, \infty$. In order to establish this property for the two exceptional points (both are related to infinity: one is infinity and the other is mapped to infinity) we have to define the notion of the angle at infinity.

Definition 3.2 *Let γ_1 and γ_2 be two paths that pass through the point $z = \infty$ and have tangents at the North Pole in the stereographic projection. The angle between γ_1 and γ_2 at $z = \infty$ is the angle between their images Γ_1 and Γ_2 under the map*

$$z \to 1/z = Z \tag{3.4}$$

at the point $Z = 0$.

Exercise 3.3 The readers who are not satisfied with this formal definition should look at the following problems:
(a) Show that the stereographic projection $\mathbb{C} \to S$ preserves angles, that is, it maps a pair of intersecting lines in $\mathbb{C}$ onto a pair of circles on S that intersect at the same angle.
(b) Show that the mapping $z \to 1/z$ of the plane $\mathbb{C}$ corresponds under the stereographic projection to a rotation of the sphere S around its diameter passing through the points $z = \pm 1$. (Hint: use expressions (1.14).)

Theorem 3.4 *Fractional linear transformations (3.1) are conformal[2] everywhere in $\overline{\mathbb{C}}$.*

Proof. The theorem has already been proved for non-exceptional points. Let γ_1 and γ_2 be two smooth (having tangents) paths intersecting at $z = -d/c$ at an angle α. The angle between their images γ_1^* and γ_2^* by definition is equal to the angle between the images Γ_1^* and Γ_2^* of γ_1^* and γ_2^* under the map $W = 1/w$ at the point $W = 0$. However, we have

$$W(z) = \frac{cz+d}{az+b},$$

[2]A map is conformal at $z = \infty$ if it preserves angles at this point.

so that Γ_1^* and Γ_2^* are the images of γ_1 and γ_2 under this map. The derivative

$$\frac{dW}{dz} = \frac{bc - ad}{(az + b)^2}$$

exists at $z = -d/c$ and is different from zero. Therefore the angle between Γ_1^* and Γ_2^* at $W = 0$ is equal to α, and the theorem is proved for $z = -d/c$. It suffices to apply the same consideration to the inverse function of (3.1) that is given by (3.3) in order to prove the theorem at $z = \infty$. □

We would like now to show that fractional linear transformations form a group. Let us denote the collection of all such functions by Λ. Let L_1 and L_2 be two fractional linear transformations:

$$L_1 : \; z \longrightarrow \frac{a_1 z + b_1}{c_1 z + d_1}, \quad a_1 d_1 - b_1 c_1 \neq 0$$

$$L_2 : \; z \longrightarrow \frac{a_2 z + b_2}{c_2 z + d_2}, \quad a_2 d_2 - b_2 c_2 \neq 0.$$

Their *product* is the composition of L_1 and L_2:

$$L : \; z \longrightarrow L_1 \circ L_2(z).$$

The map L is clearly a fractional linear transformation (this may be checked immediately by a direct substitution)

$$L : \; w = \frac{az + b}{cz + d},$$

and, moreover, $ad - bc \neq 0$ since L maps $\overline{\mathbb{C}}$ onto $\overline{\mathbb{C}}$ and does not degenerate into a constant.

We check that the group axioms hold.

(a) *Associativity*: for any three maps $L_1, L_2, L_3 \in \Lambda$ we have

$$L_1 \circ (L_2 \circ L_3) = (L_1 \circ L_2) \circ L_3. \tag{3.5}$$

Indeed, both sides of (3.5) represent the fractional linear transformation $L_1(L_2(L_3(z)))$.

(b) *Existence of unity*: the unity is clearly the identity transformation

$$E : \; z \longrightarrow z. \tag{3.6}$$

(c) *Existence of the inverse*: for any $L \in \Lambda$ there exists an inverse map $L^{-1} \in \Lambda$ so that

$$L^{-1} \circ L = L \circ L^{-1} = E. \tag{3.7}$$

Indeed, the inverse to (3.1) is given by the map (3.3).

Therefore we have proved the following theorem.

Theorem 3.5 *Fractional linear transformations form a group with respect to composition.*

The group Λ is not commutative. For instance, if $L_1(z) = z + 1$, $L_2(z) = 1/z$, then $L_1 \circ L_2(z) = \dfrac{1}{z} + 1$ while $L_2 \circ L_1(z) = \dfrac{1}{z+1}$.

The entire linear transformations (3.2) with $A \neq 0$ form a subgroup $\Lambda_0 \subset \Lambda$ of mappings from Λ that have $z = \infty$ as a fixed point.

3.2 Geometric properties

Let us present two elementary properties of fractional linear transformations. In order to formulate the first one we introduce the convention that a circle in $\overline{\mathbb{C}}$ is either a circle or a straight line on the complex plane $\mathbb{C}$ (both are mapped onto circles under the stereographic projection).

Theorem 3.6 *Fractional linear transformations map a circle in* $\overline{\mathbb{C}}$ *onto a circle in* $\overline{\mathbb{C}}$.

Proof. The statement is trivial if $c = 0$ since entire linear transformations are a composition of a shift, rotation and dilation that all have the property stated in the theorem. If $c \neq 0$ then the mapping may be written as

$$L(z) = \frac{az+b}{cz+d} = \frac{a}{c} + \frac{bc-ad}{c(cz+d)} = A + \frac{B}{z+C}. \tag{3.8}$$

Therefore L is a composition $L = L_1 \circ L_2 \circ L_3$ of three maps:

$$L_1(z) = A + Bz, \quad L_2(z) = \frac{1}{z}, \quad L_3(z) = z + C.$$

It is clear that L_1 (dilation with rotation followed by a shift) and L_3 (a shift) map circles in $\overline{\mathbb{C}}$ onto circles in $\overline{\mathbb{C}}$. It remains to prove this property for the map

$$L_2(z) = \frac{1}{z}. \tag{3.9}$$

Observe that any circle in $\overline{\mathbb{C}}$ may be represented as

$$E(x^2+y^2) + F_1 x + F_2 y + G = 0, \tag{3.10}$$

where E may vanish (then this is a straight line). Conversely, any such equation represents a circle in $\overline{\mathbb{C}}$ that might degenerate into a point or an empty set (we rule out the case $E = F_1 = F_2 = G = 0$). Using the complex variables $z = x + iy$ and $\bar{z} = x - iy$, that is, $x = (z+\bar{z})/2$, $y = \frac{1}{2i}(z - \bar{z})$ we may rewrite (3.10) as

$$Ez\bar{z} + Fz + \bar{F}\bar{z} + G = 0, \tag{3.11}$$

with $F = (F_1 - iF_2)/2$, $\bar{F} = (F_1 + iF_2)/2$.

In order to obtain the equation for the image of the circle (3.11) under the map (3.9) it suffices to set $z = 1/w$ in (3.11) to get

$$E + F\bar{w} + \bar{W}w + Gw\bar{w} = 0. \tag{3.12}$$

This is an equation of the same form as (3.11). The cases when such an equation degenerates to a point or defines an empty set are ruled out by the fact that (3.9) is a bijection. Therefore the image of the circle defined by (3.10) is indeed a circle in $\overline{\mathbb{C}}$.

We have seen above that a holomorphic function f at a non-critical point z_0 maps infinitesimally small circles centered at z_0 onto curves that are close to circles centered at $f(z_0)$ up to higher order corrections. Theorem 3.6 shows that fractional linear transformations map *all* circles in $\overline{\mathbb{C}}$ onto circles exactly. It is easy to see, however, that the center of a circle is not mapped onto the center of the image.

In order to formulate the second geometric property of the fractional linear transformations we introduce the following definition.

Definition 3.7 *Two points z and z^* are said to be conjugate with respect to a circle $\Gamma = \{|z - z_0| = R\}$ in $\mathbb{C}$ if*
(a) they lie on the same half-line originating at z_0 ($\arg(z - z_0) = \arg(z^* - z_0)$) and $|z - z_0||z^* - z_0| = R^2$, or, equivalently,
(b) any circle γ in $\overline{\mathbb{C}}$ that passes through z and z^* is orthogonal to Γ.

The equivalence of the two definitions is shown as follows. Let z and z^* satisfy part (a) and γ be any circle that passes through z and z^*. Elementary geometry implies that if ζ is the point where the tangent line to γ that passes though z_0 touches γ, then $|\zeta - z_0|^2 = |z - z_0||z^* - z_0| = R^2$ and hence $\zeta \in \Gamma$ so that the circles γ and Γ intersect orthogonally. Conversely, if any circle that passes through z and z^* is orthogonal to Γ then in particular so is the straight line that passes through z and z^*. Hence z_0, z and z^* lie on one straight line. It easy to see that z and z^* must lie on the same side of z_0. Then the same elementary geometry calculation implies that $|z - z_0||z^* - z_0| = R^2$.

The advantage of the geometric definition (b) is that it may be extended to circles in $\overline{\mathbb{C}}$: if Γ is a straight line it leads to the usual symmetry. Definition (a) leads to a simple formula that relates the conjugate points: the conditions

$$\arg(z - z_0) = \arg(z^* - z_0), \quad |z - z_0||z^* - z_0| = R^2,$$

may be written as

$$z^* - z = \frac{R^2}{\overline{z - z_0}}. \tag{3.13}$$

The mapping $z \to z^*$ that maps each point $z \in \overline{\mathbb{C}}$ into the point z^* conjugate to z with respect to a fixed circle Γ is called inversion with respect to Γ.

Expression (3.13) shows that inversion is a function that is complex conjugate of a fractional linear transformation. Therefore inversion is an anticonformal transformation in $\overline{\mathbb{C}}$: it preserves "absolute value of angles" but changes orientation.

We may now formulate the desired geometric property of fractional linear transformations and prove it in a simple way.

Theorem 3.8 *A fractional linear transformation L maps points z and z^* that are conjugate with respect to a circle Γ onto points w and w^* that are conjugate with respect to the image $L(\Gamma)$.*

Proof. Consider the family $\{\gamma\}$ of all circles in $\overline{\mathbb{C}}$ that pass through z and z^*. All such circles are orthogonal to Γ. Let γ' be a circle that passes through w and w^*.

According to Theorem 3.6 the pre-image $\gamma = L^{-1}(\gamma')$ is a circle that passes through z and z^*. Therefore the circle γ is orthogonal to Γ. Moreover, since L is a conformal map, $\gamma' = L(\gamma)$ is orthogonal to $L(\Gamma)$, and hence the points w and w^* are conjugate with respect to $L(\Gamma)$. □

3.3 Fractional linear isomorphisms and automorphisms

The definition of a fractional linear transformation

$$L(z) = \frac{az+b}{cz+d} \tag{3.14}$$

involves four complex parameters a, b, c and d. However, the mapping really depends only on *three* parameters since one may divide the numerator and denominator by one of the coefficients that is not zero. Therefore it is natural to expect that three given points may be mapped onto three other given points by a unique fractional linear transformation.

Theorem 3.9 *Given any two triplets of different points $z_1, z_2, z_3 \in \overline{\mathbb{C}}$ and $w_1, w_2, w_3 \in \overline{\mathbb{C}}$ there exists a unique fractional linear transformation L so that $L(z_k) = w_k$, $k = 1, 2, 3$.*

Proof. First we assume that none of z_k and w_k is infinity. The existence of L is easy to establish. We first define fractional linear transformations L_1 and L_2 that map z_1, z_2, z_3 and w_1, w_2, w_3, respectively, into the points 0, 1 and ∞:

$$L_1(z) = \frac{z-z_1}{z-z_2} \cdot \frac{z_3-z_2}{z_3-z_1}, \quad L_2(w) = \frac{w-w_1}{w-w_2} \cdot \frac{w_3-w_2}{w_3-w_1}. \tag{3.15}$$

Then the mapping

$$w = L(z) = L_2^{-1} \circ L_1(z), \tag{3.16}$$

that is determined by solving $L_2(w) = L_1(z)$ for $w(z)$:

$$\frac{z-z_1}{z-z_2} \cdot \frac{z_3-z_2}{z_3-z_1} = \frac{w-w_1}{w-w_2} \cdot \frac{w_3-w_2}{w_3-w_1}, \tag{3.17}$$

satisfies $L(z_k) = w_k$, $k = 1, 2, 3$ by construction.

We show next uniqueness of such L. Let $\lambda(z)$ be a fractional linear transformation that satisfies $\lambda(z_k) = w_k$, $k = 1, 2, 3$. Let us define $\mu(z) = L_2 \circ \lambda \circ L_1^{-1}(z)$ with L_1 and L_2 defined by (3.15). Then we have $\mu(0) = 0$, $\mu(1) = 1$, $\mu(\infty) = \infty$. The last condition implies that μ is an entire linear transformation: $\mu(z) = \alpha z + \beta$. Then $\mu(0) = 0$ implies $\beta = 0$ and finally $\mu(1) = 1$ implies that $\alpha = 1$ so that $\mu(z) = z$. Therefore we have $L_2 \circ \lambda \circ L_1^{-1} = E$ is the identity transformation and hence $\lambda = L_2^{-1} \circ L_1 = L$.

Let us consider now the case when one of z_k or w_k may be infinity. Then expression (3.17) still makes sense provided that the numerator and denominator of the fraction where such z_k or w_k appears are replaced by one. This is possible since each z_k and w_k

appears exactly once in the numerator and once in the denominator. For instance, if $z_1 = w_3 = \infty$ expression (3.17) takes the form

$$\frac{1}{z - z_2} \cdot \frac{z_3 - z_2}{1} = \frac{w - w_1}{w - w_2} \cdot \frac{1}{1}.$$

Therefore Theorem 3.9 holds for $\overline{\mathbb{C}}$. □

Theorems 3.9 and 3.6 imply that any circle Γ in $\overline{\mathbb{C}}$ may be mapped onto any other circle Γ^* in $\overline{\mathbb{C}}$: it suffices to map three points on Γ onto three points on Γ^* using Theorem 3.9 and use Theorem 3.6. It is clear from the topological considerations that the disk B bounded by Γ is mapped onto one of the two disks bounded by Γ^* (it suffices to find out to which one some point $z_0 \in B$ is mapped). It is easy to conclude from this that any disk $B \subset \overline{\mathbb{C}}$ may be mapped onto any other disk $B^* \subset \overline{\mathbb{C}}$.

A fractional linear transformation of a domain D on D^* is called a fractional linear isomorphism. The domains D and D^* for which such an isomorphism exists are called FL-isomorphic. We have just proved that

Theorem 3.10 *Any two disks in $\overline{\mathbb{C}}$ are FL-isomorphic.*

Let us find for instance all such isomorphisms of the upper half plane $H = \{\text{Im}z > 0\}$ onto the unit disk $D = \{|z| < 1\}$. Theorem 3.9 would produce an ugly expression so we take a different approach. We fix a point $a \in H$ that is mapped into the center of the disk $w = 0$. According to Theorem 3.9 the point $\bar{a}$ that is conjugate to a with respect to the real axis should be mapped onto the point $w = \infty$ that is conjugate to $w = 0$ with respect to the unit circle $\{|w| = 1\}$. However, a fractional linear transformation is determined by the points that are mapped to zero and infinity, up to a constant factor. Therefore the map should be of the form $w = k\dfrac{z - a}{z - \bar{a}}$.

We have $|z - a| = |z - \bar{a}|$ when $z = x$ is real. Therefore in order for the real axis to be mapped onto the unit circle by such $w(z)$ we should have $|k| = 1$, that is, $k = e^{i\theta}$. Thus, all FL-isomorphisms of the upper half plane $H = \{\text{Im}z > 0\}$ onto the unit disk $D = \{|z| < 1\}$ have the form

$$w = e^{i\theta}\frac{z - a}{z - \bar{a}}, \tag{3.18}$$

where a is an arbitrary point in the upper half plane ($\text{Im}a > 0$) and $\theta \in \mathbb{R}$ is an arbitrary real number. The map (3.18) depends on three real parameters: θ and two coordinates of the point a that is mapped onto the center of the disk. The geometric meaning of θ is clear from the observation that $z = \infty$ is mapped onto $w = e^{i\theta}$ - the change of θ leads to rotation of the disk.

An FL-isomorphism of a domain on itself is called an FL-automorphism. Clearly the collection of all FL-isomorphisms of a domain is a group that is a subgroup of the group Λ of all fractional linear transformations.

The set of all FL-automorphisms $\overline{\mathbb{C}} \to \overline{\mathbb{C}}$ coincides, obviously, with the group Λ. It is also clear that the collection of all FL-automorphisms $\mathbb{C} \to \mathbb{C}$ coincides with the set Λ_0 of all entire linear transformations of the form $z \to az + b$, $a \neq 0$. We compute the group of FL-automorphisms of the unit disk before we conclude.

We fix a point a, $|a| < 1$ that is mapped onto the center $w = 0$. The point $a^* = 1/\bar{a}$ that is conjugate to a with respect to the unit circle $\{|z| = 1\}$ should be mapped to $z = \infty$. Therefore any such map should have the form

$$w = k\frac{z-a}{z-1/\bar{a}} = k_1\frac{z-a}{1-\bar{a}z},$$

where k and k_1 are some constants. The point $z = 1$ is mapped onto a point on the unit circle and thus $|k_1|\left|\frac{1-a}{1-\bar{a}}\right| = |k_1| = 1$. hence we have $k_1 = e^{i\theta}$ with $\theta \in \mathbb{R}$. Therefore such maps have the form

$$w = e^{i\theta}\frac{z-a}{1-\bar{a}z}. \tag{3.19}$$

Conversely, any function of the form (3.19) maps the unit disk onto the unit disk. Indeed, it maps the points a and $1/\bar{a}$ that are conjugate with respect to the unit circle to $w = 0$ and $w = \infty$, respectively. Therefore $w = 0$ must be the center of the image $w(\Gamma)$ of the unit circle Γ (since it is conjugate to infinity with respect to the image circle). However, $|w(1)| = \left|\frac{1-a}{1-\bar{a}}\right| = 1$ and hence $w(\Gamma)$ is the unit circle. Moreover, $w(0) = -e^{i\theta}a$ lies inside the unit disk so the unit disk is mapped onto the unit disk.

3.4 Some elementary functions

The function

$$w = z^n, \tag{3.20}$$

where n is a positive integer, is holomorphic in the whole plane $\mathbb{C}$. Its derivative $\frac{dw}{dz} = nz^{n-1}$ when $n > 1$ is different from zero for $z \neq 0$, hence (3.20) is conformal at all $z \in \mathbb{C}\backslash\{0\}$. Writing the function (3.20) in the polar coordinates as $z = re^{i\phi}$, $w = \rho e^{i\psi}$ we obtain

$$\rho = r^n, \quad \psi = n\phi. \tag{3.21}$$

We see that this mapping increases angles by the factor of n at $z = 0$ and hence the mapping is not conformal at this point.

Expressions (3.21) also show that two points z_1 and z_2 that have the same absolute value and arguments that differ by a multiple of $2\pi/n$:

$$|z_1| = |z_2|, \quad \arg z_1 = \arg z_2 + k\frac{2\pi}{n} \tag{3.22}$$

are mapped onto the same point w. Therefore, when $n > 1$ this is not a one-to-one map in $\mathbb{C}$. In order for it to be an injection $D \to \mathbb{C}$ the domain D should not contain any points z_1 and z_2 related as in (3.22).

An example of a domain D so that (3.20) is an injection from D into $\mathbb{C}$ is the sector $D = \{0 < \arg z < 2\pi/n\}$. This sector is mapped one-to-one onto the domain $D^* = \{0 < \arg z < 2\pi\}$, that is, the complex plane without the positive semi-axis.

The rational function

$$w = \frac{1}{2}\left(z + \frac{1}{z}\right) \tag{3.23}$$

is called the Joukovsky function. It is holomorphic in $\mathbb{C}\backslash\{0\}$. Its derivative

$$\frac{dw}{dz} = \frac{1}{2}\left(1 - \frac{1}{z^2}\right)$$

is different from zero everywhere except $z = \pm 1$. Thus (3.23) is conformal at all finite points $z \neq 0, \pm 1$. The point $z = 0$ is mapped onto $w = \infty$. The fact that $w(z)$ is conformal at $z = 0$ follows from the existence and non-vanishing of the derivative

$$\frac{d}{dz}\left(\frac{1}{w}\right) = 2\frac{1 - z^2}{(1 + z^2)^2}$$

at $z = 0$. According to our definition the conformality of $w = f(z)$ at $z = \infty$ is equivalent to the conformality of $\tilde{w} = f(1/z)$ at $z = 0$. However, we have $\tilde{w}(z) = w(z)$ for the Joukovsky function and we have just proved that $w(z)$ is conformal at $z = 0$. Therefore it is also conformal at $z = \infty$.

The function (3.23) maps two different points z_1 and z_2 onto the same point w if

$$z_1 + \frac{1}{z_1} - z_2 - \frac{1}{z_2} = (z_1 - z_2)\left(1 - \frac{1}{z_1 z_2}\right) = 0,$$

that is, if

$$z_1 z_2 = 1. \tag{3.24}$$

An example of a domain where $w(z)$ is one-to-one is the outside of the unit disk: $D = \left\{z \in \overline{\mathbb{C}}:\ |z| > 1\right\}$. In order to visualize the mapping (3.23) we let $z = re^{i\phi}$, $w = u + iv$ and rewrite (3.23) as

$$u = \frac{1}{2}\left(r + \frac{1}{r}\right)\cos\phi, \quad v = \frac{1}{2}\left(r - \frac{1}{r}\right)\sin\phi. \tag{3.25}$$

We see that the Joukovsky function transforms the circles $\{|z| = r_0\}$, $r_0 > 1$ into ellipses with semi-axes $a_{r_0} = \dfrac{1}{2}\left(r + \dfrac{1}{r}\right)$ and $b_{r_0} = \frac{1}{2}\left(r - \frac{1}{r}\right)$ and focal points at $w = \pm 1$ (since $a_{r_0}^2 - b_{r_0}^2 = 1$ for all r_0). Note that as $r \to 1$ the ellipses tend to the interval $[-1, 1] \subset \mathbb{R}$, while for large r the ellipses are close to the circle $\{|z| = r\}$. The rays $\{\phi = \phi_0,\ 1 < r < \infty\}$ are mapped onto parts of hyperbolas

$$\frac{u^2}{\cos^2\phi_0} - \frac{v^2}{\sin^2\phi_0} = 1$$

with the same focal points $w = \pm 1$. Conformality of (3.23) implies that these hyperbolas are orthogonal to the family of ellipses described above.

The above implies that the Joukovsky function maps one-to-one and conformally the outside of the unit disk onto the complex plane without the interval $[-1, 1]$.

The mapping is not conformal at $z = \pm 1$. It is best seen from the representation

$$\frac{w-1}{w+1} = \left(\frac{z-1}{z+1}\right)^2. \tag{3.26}$$

This shows that (3.23) is the composition of three mappings

$$\zeta = \frac{z+1}{z-1}, \quad \omega = \zeta^2, \quad w = \frac{1+\omega}{1-\omega} \tag{3.27}$$

(the last mapping is the inverse of $\omega = \frac{w-1}{w+1}$). The first and the last maps in (3.27) are fractional linear transformations and so are conformal everywhere in $\overline{\mathbb{C}}$. The mapping $\omega = \zeta^2$ doubles the angles at $\zeta = 0$ and $\zeta = \infty$ that correspond to $z = \pm 1$. Therefore the Joukovsky function doubles the angles at these points.

Exercise 3.11 Use the decomposition (3.27) to show that the Joukovsky function maps the outside of a circle γ that passes through $z = \pm 1$ and forms an angle α with the real axis onto the complex plane without an arc that connects $z = \pm 1$ and forms angle 2α with the real axis. One may also show that circles that are tangent to γ at $z = 1$ or $z = -1$ are mapped onto curves that look like an airplane wing. This observation allowed Joukovsky (1847-1921) to create the first method of computing the aerodynamics of the airplane wings.

3.5 The exponential function

We define the function e^z in the same way as in real analysis:

$$e^z = \lim_{n\to+\infty} \left(1 + \frac{z}{n}\right)^n. \tag{3.28}$$

Let us show the existence of this limit for any $z \in \mathbb{C}$. We set $z = x + iy$ and observe that

$$\left|\left(1 + \frac{z}{n}\right)^n\right| = \left(1 + \frac{2x}{n} + \frac{x^2+y^2}{n^2}\right)^{n/2}$$

and

$$\arg\left(1 + \frac{z}{n}\right)^n = n \arctan \frac{y/n}{1 + x/n}.$$

This shows that the limits

$$\lim_{n\to\infty} \left|\left(1 + \frac{z}{n}\right)^n\right| = e^x, \quad \lim_{n\to\infty} \arg\left(1 + \frac{z}{n}\right)^n = y$$

exist. Therefore the limit (3.28) also exists and may be written as

$$e^{x+iy} = e^x(\cos y + i \sin y). \tag{3.29}$$

Therefore

$$|e^z| = e^{Rez}, \quad \arg e^z = \text{Im}z. \tag{3.30}$$

We let $x = 0$ in (3.29) and obtain the Euler formula

$$e^{iy} = \cos y + i \sin y, \tag{3.31}$$

that we have used many times. However, so far we have used the symbol e^{iy} as a short-hand notation of the right side, while now we may understand it as a complex power of the number e.

Let us list some basic properties of the exponential function.

1. The function e^z is holomorphic in the whole plane $\mathbb{C}$. Indeed, letting $e^z = u+iv$ we find that $u = e^x \cos y$, $v = e^y \sin y$. The functions u and v are everywhere differentiable in the real sense and the Cauchy-Riemann equations hold everywhere:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = e^x \cos y, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = e^x \sin y.$$

Therefore the function (3.29) defines an extension of the real exponential function to the whole complex plane and the extended function is holomorphic. We will later see that such extension is unique.

2. The usual formula for the derivative of e^z holds. Indeed, we may compute the derivative along the direction x since we know that it exists. Therefore

$$(e^z)' = \frac{\partial}{\partial x}\left(e^x \cos y + ie^x \sin y\right) = e^z. \tag{3.32}$$

The exponential function never vanishes since $|e^z| = e^x > 0$ and hence $(e^z)' \neq 0$ so that the mapping $w = e^z$ is conformal everywhere in $\mathbb{C}$.

3. The usual product formula holds

$$e^{z_1+z_2} = e^{z_1}e^{z_2}. \tag{3.33}$$

Indeed, setting $z_k = x_k + iy_k$, $k = 1, 2$ and using the expressions for sine and cosine of a sum we may write

$$e^{x_1}(\cos y_1 + i \sin y_1)e^{x_2}(\cos y_2 + i \sin y_2) = e^{x-1+x-2}(\cos(y_1 + y_2) + i \sin(y_1 + y_2)).$$

Thus addition of complex numbers z_1 and z_2 corresponds to multiplication of the images e^{z_1} and e^{z_2}. In other words the function e^z transforms the additive group of the field of complex numbers into its multiplicative group: under the map $z \to e^z$:

$$z_1 + z_2 \to e^{z_1} \cdot e^{z_2}. \tag{3.34}$$

4. The function e^z is periodic with an imaginary period $2\pi i$. Indeed, using the Euler formula we obtain $e^{2\pi i} = \cos(2\pi) + i \sin(2\pi) = 1$ and hence we have for all $z \in \mathbb{C}$:

$$e^{z+2\pi i} = e^z \cdot e^{2\pi i} = e^z.$$

On the other hand, assume that $e^{z+T} = e^z$. Multiplying both sides by e^{-z} we get $e^T = 1$, which implies $e^{T_1}(\cos T_2 + i \sin T_2) = 1$, with $T = T_1 + iT_2$. Evaluating the absolute value of both sides we see that $e^{T_1} = 1$ so that $T_1 = 0$. Then the real part of the above implies that $\cos T_2 = 1$, and the imaginary part shows that $\sin T_2 = 0$. We conclude that $T = 2\pi ni$ and $2\pi i$ is indeed the basic period of e^z.

The above mentioned considerations also show that for the map $e^z : D \to \mathbb{C}$ to be one-to-one the domain D should contain no points that are related by

$$z_1 - z_2 = 2\pi in, \quad n = \pm 1, \pm 2, \ldots \tag{3.35}$$

An example of such a domain is the strip $\{0 < \text{Im} z < 2\pi$. Setting $z = x + iy$ and $w = \rho e^{i\psi}$ we may write $w = e^z$ as

$$\rho = e^x, \quad \psi = y. \tag{3.36}$$

This shows that this map transforms the lines $y = y_0$ into the rays $\psi = y_0$ and the intervals $\{x = x_0,\ 0 < y < 2\pi\}$ into circles without a point $\{\rho = e^{x_0},\ 0 < \psi < 2\pi\}$. The strip $\{0 < y < 2\pi\}$ is therefore transformed into the whole plane without the positive semi-axis. The twice narrower strip $\{0 < y < \pi\}$ is mapped onto the upper half-plane $\text{Im} w > 0$.

3.6 The trigonometric functions

The Euler formula shows that we have $e^{ix} = \cos x + i \sin x$, $e^{-ix} = \cos x - i \sin x$ for all real $x \in \mathbb{R}$ so that

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}.$$

These expressions may be used to continue cosine and sine as holomorphic functions in the whole complex plane setting

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i} \tag{3.37}$$

for all $z \in \mathbb{C}$. It is clear that the right side in (3.38) is holomorphic.

All properties of these functions follow from the corresponding properties of the exponential function. They are both periodic with the period 2π: the exponential function has the period $2\pi i$ but expressions in (3.37) have the factor of i in front of z. Cosine is an even function, sine is odd. The usual formulas for derivatives hold:

$$(\cos z)' = i\frac{e^{iz} - e^{-iz}}{2} = -\sin z$$

and similarly $(\sin z)' = \cos z$. The usual trigonometric formulas hold, such as

$$\sin^2 z + \cos^2 z = 1, \quad \cos z = \sin\left(z + \frac{\pi}{2}\right),$$

etc. The reader will have no difficulty deriving these expressions from (3.37).

The trigonometric functions of a complex variable are closely related to the hyperbolic ones defined by the usual expressions

$$\cosh z = \frac{e^z + e^{-z}}{2}, \quad \sinh z = \frac{e^z - e^{-z}}{2}. \tag{3.38}$$

They are related to sine and cosine by

$$\begin{aligned} \cosh z &= \cos iz, \quad \sinh z = -\sin iz \\ \cos z &= \cosh iz, \quad \sin z = -i \sinh iz \end{aligned} \tag{3.39}$$

as may be seen by comparing (3.37) and (3.38).

Using the formulas for cosine of a sum and relations (3.39) we obtain

$$\cos(x + iy) = \cos x \cosh y - i \sin x \sinh y,$$

so that

$$\begin{aligned} |\cos z| &= \sqrt{\cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y} \\ &= \sqrt{\cos^2 x(1 + \sinh^2 y) + (1 - \cos^2 x)\sinh^2 y} = \sqrt{\cos^2 x + \sinh^2 y}. \end{aligned} \tag{3.40}$$

We see that $|\cos z|$ goes to infinity as $y \to \infty$.

Let us consider for example the map of half-strip $D = \{-\pi/2 < x < \pi/2, \quad y > 0\}$ by the function $w = \sin z$. We represent this map as a composition of the familiar maps

$$z_1 = iz, \quad z_2 = e^{z_1}, \quad z_3 = \frac{z_2}{i}, \quad w = \frac{1}{2}\left(z_3 + \frac{1}{z_3}\right).$$

This shows that $w = \sin z$ maps conformally and one-to-one the half-strip D onto the upper half-plane. Indeed, z_1 maps D onto the half-strip $D_1 = \{x_1 < 0, \quad -\pi/2 < y_1 < \pi/2\}$; z_2 maps D_1 onto the semi-circle $D_2 = \{|z| < 1, \quad -\pi/2 < \arg z < \pi/2\}$; z_3 maps D_2 onto the semi-circle $D_3 = \{|z| < 1, \ \pi < \arg z < 2\pi\}$. Finally, the Joukovksy function w maps D_3 onto the upper half-plane. The latter is best seen from (3.25): the interval $[0, 1]$ is mapped onto the half-line $[1, +\infty)$, the interval $[-1, 0)$ is mapped onto the half-line $(-\infty, 1]$, and the arc $\{|z| = 1, \ \pi < \arg z < 2\pi\}$ is mapped onto the interval $(-1, 1)$ of the x-axis. This shows that the boundary of D_3 is mapped onto the real axis. Furthermore, (3.25) shows that for $z_3 = \rho e^{i\phi}$ we have $\text{Im } w = \frac{1}{2}\left(\rho - \frac{1}{\rho}\right)\sin\phi > 0$ so that the interior of D_3 is mapped onto the upper half plane (and not onto the lower one).

Tangent and cotangent of a complex variable are defined by

$$\tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{\cos z}{\sin z} \tag{3.41}$$

and are rational functions of the complex exponential:

$$\tan z = -i\frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}}, \quad \cot z = i\frac{e^{iz} + e^{-iz}}{e^{iz} - e^{-iz}}. \tag{3.42}$$

These functions are holomorphic everywhere in $\mathbb{C}$ except for the points where the denominators in (3.42) vanish (the numerators do not vanish at these points). Let us find such points for $\cot z$. We have $\sin z = 0$ there, or $e^{iz} = e^{-iz}$ so that $z = n\pi$, $n = \pm 1, \pm 2, \dots$ - we see that the singularities are all on the real line.

Tangent and cotangent remain periodic in the complex plane with the real period π, and all the usual trigonometric formulas involving these functions still hold. Expression (3.40) and the corresponding formula for sine shows that

$$|\tan z| = \sqrt{\frac{\sin^2 x + \sinh^2 y}{\cos^2 x + \sinh^2 y}}. \tag{3.43}$$

The mappings realized by the functions $w = \tan z$ and $w = \cot z$ are a composition of known maps. For instance, $w = \tan z$ can be reduced to the following:

$$z_1 = 2iz, \quad z_2 = e^{z_1}, \quad w = -i\frac{z_2 - 1}{z_2 + 1}.$$

This function maps conformally and one-to-one the strip $D = \{-\pi/4 < x < \pi/4\}$ onto the interior of the unit disk: z_1 maps D onto the strip $D_1 = \{-\pi/2 < y_1 < \pi/2\}$; z_2 maps D_1 onto the half plane $D_2 = \{x_2 > 0\}$; z_3 maps the imaginary axis onto the unit circle: $\left| -i\frac{iy-1}{iy+1} \right| = \frac{|1-iy|}{|1+iy|} = 1$, and the interior point $z_2 = 1$ of D_2 is mapped onto $w = 0$, an interior point of the unit disk.

4 Exercises for Chapter 1

1. Let us define multiplication for two vectors $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ in $\mathbb{R}^2$ by

$$z_1 \star z_2 = (x_1 x_2 + y_1 y_2, x_1 y_2 + x_2 y_1).$$

This corresponds to the "$i^2 = 1$" rule.
(a) Show that this set is not a field and find divisors of zero.
(b) Let $\bar{z} = (x_1, -y_1)$ and define the absolute value as $\|z\| = \sqrt{|z \star \bar{z}|}$. Find the set of points such that $\|z\| = 0$. Show that absolute value of a product is the product of absolute values. Show that $\|z\| = 0$ is a necessary and sufficient condition for z to be a divisor of zero.
(c) Given z_2 so that $\|z_\| \neq 0$ define the ratio as

$$z_1 \star\star z_2 = \frac{z_1 \star \bar{z}_2}{z_2 \star \bar{z}_2}$$

with the denominator on the right side being a real number. Show that $(z_1 \star z_2) \star\star z_2 = z_1$.
(d) Let us define a derivative of a function $w = f(z) = u + iv$ as

$$f'(z) = \lim_{\Delta z \to 0, \|\Delta z\| \neq 0} \Delta w \star\star \Delta z$$

if the limit exists. Show that in order for such a derivative to exist if f is continuously differentiable in the real sense it is necessary and sufficient that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}.$$

(e) Find the geometric properties of the maps $w = z \star z$ and $w = 1 \star \star z$.
(f) Define $e_*^z = e^x(\cosh y, \sinh y)$ and $\sin_* z = (\sin x \cos y, \cos x \sin y)$. Find the similarities and differences of these functions from the usual exponential and trigonometric functions and describe their geometric properties.

2. Prove that
(a) if the points $z_1, \ldots, z_n$ lie on the same side of a line passing through $z = 0$ then $\sum_{k=1}^{n} z_k \neq 0$.
(b) if $\sum_{k=1}^{n} z_k^{-1} = 0$ then the points $\{z_k\}$ may not lie on the same side of a line passing through $z = 0$.

3. Show that for any polynomial $P(z) = \prod_{k=1}^{n}(z - a_k)$ the zeros of the derivative $P'(z) = \sum_{k=1}^{n} \prod_{j \neq k}(z - a_j)$ belong to the convex hull of the set of zeros $\{a_k\}$ of the polynomial $P(z)$ itself.

4. Show that the set of limit points of the sequence $a_n = \prod_{k=1}^{n}\left(1 + \frac{i}{k}\right)$, $n = 1, 2, \ldots$ is a circle. (Hint: show that first that $|a_n|$ is an increasing and bounded sequence and then analyze the behavior of $\arg a_n$).

5. let $f = u + iv$ have continuous partial derivatives in a neighborhood of $z_0 \in \mathbb{C}$. Show that the Cauchy-Riemann conditions for its $\mathbb{C}$-differentiability may be written in a more general form: there exist two directions s and n such that n is the rotation of s counterclockwise by 90 degrees, and the directional derivatives of u and v are related by

$$\frac{\partial u}{\partial s} = \frac{\partial v}{\partial n}, \quad \frac{\partial u}{\partial n} = -\frac{\partial v}{\partial s}.$$

In particular the conditions of $\mathbb{C}$-differentiability in the polar coordinates have the form

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}.$$

6. Let the point z move on the complex plane according to $z = re^{it}$, where r is constant and t is time. Find the velocity of the point $w = f(z)$, where f is a holomorphic function on the circle $\{|z| = r\}$. (Answer: $izf'(z)$.)

7. Let f be holomorphic in the disk $\{|z| \leq r\}$ and $f'(z) \neq 0$ on $\gamma = \{|z| = r\}$. Show that the image $f(\gamma)$ is a convex curve if and only if $\mathrm{Re}\left(\frac{zf''(z)}{f'(z)}\right) + 1 \geq 0$. (Hint: first

show that convexity is equivalent to $\frac{\partial}{\partial \phi}\left(\frac{\pi}{2} + \phi + \arg f'(r^{e^{i\phi}})\right) \geq 0$.)

8. Find the general form of a fractional linear transformation that corresponds to the rotation of the Riemann sphere in the stereographic projection around two points lying on the same diameter. (Answer: $\frac{w-a}{1+\bar{a}w} = e^{i\theta}\frac{z-a}{1+\bar{a}z}$.)

9. Show that a map $w = \frac{az+b}{cz+d}$, $ad - bc = 1$ preserves the distances on the Riemann sphere if and only if $c = -\bar{b}$ and $d = \bar{a}$.

Chapter 2

Properties of Holomorphic Functions

We will consider in this chapter some of the most important methods in the study of holomorphic functions. They are based on the representation of such functions as special integrals (the Cauchy integral) and as sums of power series (the Taylor and the Laurent series). We begin with the notion of the integral of a function of a complex variable.

1 The Integral

1.1 Definition of the integral

Definition 1.1 *Let $\gamma : I \to \mathbb{C}$ be a piecewise smooth path, where $I = [\alpha, \beta]$ is an interval on the real axis. Let a complex-valued function f be defined on $\gamma(I)$ so that the function $f \circ \gamma$ is a continuous function on I. The integral of f along the path γ is*

$$\int_\gamma f dz = \int_\alpha^\beta f(\gamma(t))\gamma'(t)dt. \qquad (1.1)$$

The integral in the right side of (1.1) is understood to be $\int_\alpha^\beta g_1(t)dt + i\int_\alpha^\beta g_2(t)dt$, where g_1 and g_2 are the real and imaginary parts of the function $f(\gamma(t))\gamma'(t) = g_1(t) + ig_2(t)$.

Note that the functions g_1 and g_2 may have only finitely many discontinuities on I so that the integral (1.1) exists in the usual Riemann integral sense. If we set $f = u + iv$ and $dz = \gamma'(t)dt = dx + idy$ then (1.1) may be rewritten as

$$\int_\gamma f dz = \int_\gamma udx - vdy + i\int_\gamma vdx + udy. \qquad (1.2)$$

One could also define the integral (1.1) as the limit of partial sums: divide the curve $\gamma(I)$ into finally many pieces $z_0 = \gamma(\alpha)$, $z_1 = \gamma(t_1)$, $\dots$, $z_n = \gamma(\beta)$, $\alpha < t_1 < \cdots < \beta$, choose arbitrary points $\zeta_k = \gamma(\tau_k)$, $\tau_k \in [t_k, t_{k+1}]$ and define

$$\int_\gamma f dz = \lim_{\delta \to 0} \sum_{k=0}^{n-1} f(\zeta_k)\Delta z_k, \tag{1.3}$$

where $\Delta z_k = z_{k+1} - z_k$, $k = 0, 1, \ldots, n-1$ and $\delta = \max |\Delta z_k|$. Nevertheless we will use only the first definition and will not prove its equivalence to the other two.

If the path γ is just a rectifiable curve, then the Riemann integral is not defined even for continuous functions f because of the factor $\gamma'(t)$ in the right side of (1.1). One would have to use the Lebesgue integral in that case and assume that the function $f(\gamma(t))$ is Lebesgue integrable on I.

Example 1.2 Let γ be a circle $\gamma(t) = a + re^{it}$, $t \in [0, 2\pi]$, and $f(z) = (z-a)^n$, where $n = 0, \pm 1, \ldots$ is an integer. Then we have $\gamma'(t) = re^{it}$, $f(\gamma(t)) = r^n e^{int}$ so that

$$\int_\gamma (z-a)^n dz = r^{n+1} i \int_0^{2\pi} e^{i(n+1)t} dt.$$

We have to consider two cases: when $n \neq 1$ we have

$$\int_\gamma (z-a)^n dz = r^{n+1} \frac{e^{2\pi i(n+1)} - 1}{n+1} = 0,$$

because of the periodicity of the exponential function, while when $n = -1$

$$\int_\gamma \frac{dz}{z-a} = i \int_0^{2\pi} dt = 2\pi i.$$

Therefore the integer powers of $z - a$ have the "orthogonality" property

$$\int_\gamma (z-a)^n = \begin{cases} 0, & \text{if } n \neq -1 \\ 2\pi i, & \text{if } n = -1 \end{cases} \tag{1.4}$$

that we will use frequently.

Example 1.3 Let $\gamma: I \to \mathbb{C}$ be an arbitrary piecewise smooth path and $n \neq 1$. We also assume that the path $\gamma(t)$ does not pass through the point $z = 0$ in the case $n < 0$. The chain rule implies that $\frac{d}{dt}\gamma^{n+1}(t) = (n+1)\gamma^n(t)\gamma'(t)$ so that

$$\int_\gamma z^n dz = \int_\alpha^\beta \gamma^n(t)\gamma'(t) dt = \frac{1}{n+1}[\gamma^{n+1}(\beta) - \gamma^{n+1}(\alpha)]. \tag{1.5}$$

We observe that the integrals of z^n, $n \neq -1$ depend not on the path but only on its endpoints. Their integrals over a closed path vanish.

We summarize the basic properties of the integral of a complex-valued function.

1. *Linearity.* If f and g are continuous on a piecewise smooth path γ then for any complex numbers α and β we have

$$\int_\gamma (\alpha f + \beta g) dz = \alpha \int_\gamma f dz + \beta \int_\gamma g dz. \tag{1.6}$$

This follows immediately from the definition.

2. *Additivity.* Let $\gamma_1 : [\alpha_1, \beta_1] \to \mathbb{C}$ and $\gamma_2 : [\beta_1, \beta_2] \to \mathbb{C}$ be two piecewise smooth paths so that $\gamma_1(\beta_1) = \gamma_2(\beta_1)$. The union $\gamma = \gamma_1 \cup \gamma_2$ is a path $\gamma : [\alpha_1, \beta_2] \to \mathbb{C}$ so that

$$\gamma(t) = \begin{cases} \gamma_1(t), & \text{if } t \in [\alpha_1, \beta_1] \\ \gamma_2(t), & \text{if } t \in [\beta_1, \beta_2] \end{cases}.$$

We have then for any function f that is continuous on $\gamma_1 \cup \gamma_2$:

$$\int\limits_{\gamma_1 \cup \gamma_2} f dz = \int\limits_{\gamma_1} f dz + \int\limits_{\gamma_2} f dz. \tag{1.7}$$

One may drop the condition $\gamma_1(\beta_1) = \gamma_2(\beta_2)$ in the definition of the union $\gamma_1 \cup \gamma_2$. Then $\gamma_1 \cup \gamma_2$ will no longer be a continuous path but property (1.7) would still hold.

3. *Invariance.* Integral is invariant under a re-parameterization of the path.

Theorem 1.4 *Let a path $\gamma_1 : [\alpha_1, \beta_1] \to \mathbb{C}$ be obtained from a piecewise smooth path $\gamma : [\alpha, \beta] \to \mathbb{C}$ by a legitimate re-parameterization, that is $\gamma = \gamma_1 \circ \tau$ where τ is an increasing piecewise smooth map $\tau : [\alpha, \beta] \to [\alpha_1, \beta_1]$. Then we have for any function f that is continuous on γ (and hence on γ_1):*

$$\int_{\gamma_1} f dz = \int_\gamma f dz. \tag{1.8}$$

Proof. The definition of the integral implies that

$$\int_{\gamma_1} f dz = \int_{\alpha_1}^{\beta_1} f(\gamma_1(s)) \gamma_1'(s) ds.$$

Introducing the new variable t so that $\tau(t) = s$ and using the usual rules for the change of real variables in an integral we obtain

$$\int_{\gamma_1} f dz = \int_{\alpha_1}^{\beta_1} f(\gamma_1(s)) \gamma_1'(s) ds = \int_\alpha^\beta f(\gamma_1(\tau(t))) \gamma_1'(\tau(t)) \tau'(t) dt$$
$$= \int_\alpha^\beta f(\gamma(t)) \gamma'(t) dt = \int_\gamma f dz. \quad \square$$

This theorem has an important corollary: the integral that we defined for a path makes sense also for a *curve* that is an equivalence class of paths. More precisely, the value of the integral along any path that defines a given curve is independent of the choice of path in the equivalence class of the curve.

As we have previously mentioned we will often identify the curve and the set of points on the complex plane that is the image of a path that defines this curve. Then we will talk about integral over this set understanding it as the integral along the corresponding set. For instance, expressions (1.4) may be written as

$$\int\limits_{\{|z-a|=r\}} \frac{dz}{z-a} = 2\pi i, \qquad \int\limits_{\{|z-a|=r\}} (z-a)^n dz = 0, \quad n \in \mathbb{Z}\backslash\{-1\}.$$

4. *Orientation.* Let γ^- be the path that is obtained out of a piecewise smooth path $\gamma : [\alpha, \beta] \to \mathbb{C}$ by a change of variables $t \to \alpha + \beta - t$, that is, $\gamma^-(t) = \gamma(\alpha + \beta - t)$, and let f be a function continuous on γ. Then we have

$$\int_{\gamma^-} f dz = -\int_\gamma f dz. \tag{1.9}$$

This statement is proved exactly as Theorem 1.4.

We say that the path γ^- is obtained from γ by a change of orientation.

5. *A bound for the integral.*

Theorem 1.5 *Let f be a continuous function defined on a piecewise smooth path $\gamma : [\alpha, \beta] \to \mathbb{C}$. Then the following inequality holds:*

$$\left| \int_\gamma f dz \right| \le \int_\gamma |f||d\gamma|, \tag{1.10}$$

where $|d\gamma| = |\gamma'(t)|dt$ is the differential of the arc length of γ and the integral on the right side is the real integral along a curve.

Proof. Let us denote $J = \int_\gamma f dz$ and let $J = |J|e^{i\theta}$, then we have

$$|J| = \int_\gamma e^{-i\theta} f dz = \int_\alpha^\beta e^{-i\theta} f(\gamma(t))\gamma'(t)dt.$$

The integral on the right side is a real number and hence

$$|J| = \int_\alpha^\beta \mathrm{Re}\left[e^{-i\theta} f(\gamma(t))\gamma'(t)\right] dt \le \int_\alpha^\beta |f(\gamma(t))||\gamma'(t)|dt = \int_\gamma |f||d\gamma|. \ \square$$

Corollary 1.6 *Let assumptions of the previous theorem hold and assume that $|f(z)| \le M$ for a constant M, then*

$$\left| \int_\gamma f dz \right| \le M|\gamma|, \tag{1.11}$$

where $|\gamma|$ is the length of the path γ.

Inequality (1.11) is obtained from (1.10) if we estimate the integral on the right side of (1.10) and note that $\int_\gamma |d\gamma| = |\gamma|$.

Exercise 1.7 *Show that if a function f is $\mathbb{R}$-differentiable in a neighborhood of a point $a \in \mathbb{C}$ then*

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2} \int\limits_{\{|z-a|=\varepsilon\}} f(z)dz = 2\pi i \frac{\partial f}{\partial \bar{z}}(a).$$

Hint: use the formula

$$f(z) = f(a) + \frac{\partial f}{\partial z}(a)(z-a) + \frac{\partial f}{\partial \bar{z}}(a)(\bar{z}-\bar{a}) + o(|z-a|)$$

and Example 1.3.

1.2 The anti-derivative

Definition 1.8 *An anti-derivative of a function f in a domain D is a holomorphic function F such that at every point $z \in D$ we have*

$$F'(z) = f(z). \tag{1.12}$$

If F is an anti-derivative of f in a domain D then any function of the form $F(z)+C$ where C is an arbitrary constant is also an anti-derivative of f in D. Conversely, let F_1 and F_2 be two anti-derivatives of f in D and let $\Phi = F_1 - F_2$. The function Φ is holomorphic in D and thus $\dfrac{\partial \Phi}{\partial \bar{z}} = 0$ in D. Moreover, $\dfrac{\partial \Phi}{\partial z} = F_1' - F_2' = 0$ in D and therefore $\dfrac{\partial \Phi}{\partial x} = \dfrac{\partial \Phi}{\partial y} = 0$ in D. The familiar result of the real analysis applied to the real-valued functions $\mathrm{Re}\Phi$ and $\mathrm{Im}\Phi$ implies that $\Phi = C$ is a constant in D. We have proved the following theorem.

Theorem 1.9 *If F is an anti-derivative of f in D then the collection of all anti-derivatives of f in D is described by*

$$F(z) + C, \tag{1.13}$$

where C is an arbitrary constant.

Therefore an anti-derivative of f in D if it exists is defined up to an arbitrary constant.

Let us now address the existence of anti-derivative. First we will look at the question of existence of a local anti-derivative that exists in a neighborhood of a point. We begin with a theorem that expresses in the simplest form the Cauchy theorem that lies at the core of the theory of integration of holomorphic functions.

Theorem 1.10 *(Cauchy) Let $f \in \mathcal{O}(D)$, that is, f is holomorphic in D. Then the integral of f along the oriented boundary*[1] *of any triangle Δ that is properly contained*[2] *in D is equal to zero:*

$$\int_{\partial\Delta} f dz = 0. \tag{1.14}$$

Proof. Let us assume that this is false, that is, there exists a triangle Δ properly contained in D so that

$$\left|\int_{\partial\Delta} f dz\right| = M > 0. \tag{1.15}$$

Let us divide Δ into four sub-triangles by connecting the midpoints of all sides and assume that the boundaries both of Δ and these triangles are oriented counter-clockwise. Then clearly the integral of f over $\partial\Delta$ is equal to the sum of the integrals over the boundaries of the small triangles since each side of a small triangle that is not part of the boundary $\partial\Delta$ belongs to two small triangles with two different orientations so that they do not contribute to the sum. Therefore there exists at least one small triangle that we denote Δ_1 so that

$$\left|\int_{\partial\Delta_1} f dz\right| \geq \frac{M}{4}.$$

We divide Δ_1 into four smaller sub-triangles and using the same considerations we find one of them denoted Δ_2 so that $\left|\int_{\partial\Delta_2} f dz\right| \geq \frac{M}{4^2}$.

Continuing this procedure we construct a sequence of nested triangles Δ_n so that

$$\left|\int_{\partial\Delta_n} f dz\right| \geq \frac{M}{4^n}. \tag{1.16}$$

The closed triangles Δ_n have a common point $z_0 \in \Delta \subset D$. The function f is holomorphic at z_0 and hence for any $\varepsilon > 0$ there exists $\delta > 0$ so that we may decompose

$$f(z) - f(z_0) = f'(z_0)(z - z_0) + \alpha(z)(z - z_0) \tag{1.17}$$

with $|\alpha(z)| < \varepsilon$ for all $z \in U = \{|z - z_0| < \delta\}$.

We may find a triangle Δ_n that is contained in U. Then (1.17) implies that

$$\int_{\partial\Delta_n} f dz = \int_{\partial\Delta_n} f(z_0)dz + \int_{\partial\Delta_n} f'(z_0)(z - z_0)dz + \int_{\partial\Delta_n} \alpha(z)(z - z_0)dz.$$

However, the first two integrals on the right side vanish since the factors $f(z_0)$ and $f'(z_0)$ may be pulled out of the integrals and the integrals of 1 and $z - z_0$ over a closed path $\partial\Delta_n$ are equal to zero (see Example 1.3). Therefore, we have $\int_{\partial\Delta_n} f dz = \int_{\partial\Delta_n} \alpha(z)(z - z_0)dz$,

[1] We assume that the boundary $\partial\Delta$ (that we treat as a piecewise smooth curve) is oriented in such a way that the triangle Δ remains on one side of $\partial\Delta$ when one traces $\partial\Delta$.

[2] A set S is properly contained in a domain S' if S is contained in a compact subset of S'.

where $|\alpha(z)| < \varepsilon$ for all $z \in \partial\Delta_n$. Furthermore, we have $|z - z_0| \le |\partial\Delta_n|$ for all $z \in \partial\Delta_n$ and hence we obtain using Theorem 1.5

$$\left|\int_{\partial\Delta_n} f dz\right| = \left|\int_{\partial\Delta_n} \alpha(z)(z - z_0)dz\right| < \varepsilon|\partial\Delta_n|^2.$$

However, by construction we have $|\partial\Delta_n| = |\partial\Delta|/2^n$, where $|\partial\Delta|$ is the perimeter of Δ, so that

$$\left|\int_{\partial\Delta_n} f dz\right| < \varepsilon|\partial\Delta|^2/4^n.$$

This together with (1.16) implies that $M < \varepsilon|\partial\Delta|^2$ which in turn implies $M = 0$ since ε is an arbitrary positive number. This contradicts assumption (1.15) and the conclusion of Theorem 1.10 follows. □

We will consider the Cauchy theorem in its full generality in the next section. At the moment we will deduce the local existence of anti-derivative from the above Theorem.

Theorem 1.11 *Let $f \in \mathcal{O}(D)$ then it has an anti-derivative in any disk $U = \{|z-a| < r\} \subset D$:*

$$F(z) = \int\limits_{[a,z]} f(\zeta)d\zeta, \tag{1.18}$$

where the integral is taken along the straight segment $[a, z] \subset U$.

Proof. We fix an arbitrary point $z \in U$ and assume that $|\Delta z|$ is so small that the point $z + \Delta z \in U$. Then the triangle Δ with vertices a, z and $z + \Delta z$ is properly contained in D so that Theorem 1.10 implies that

$$\int\limits_{[a,z]} f(\zeta)d\zeta + \int\limits_{[z,z+\Delta z]} f(\zeta)d\zeta + \int\limits_{[z+\Delta z,a]} f(\zeta)d\zeta = 0.$$

The first term above is equal to $F(z)$ and the third to $-F(z + \Delta z)$ so that

$$F(z + \Delta z) - F(z) = \int\limits_{[z,z+\Delta z]} f(\zeta)d\zeta. \tag{1.19}$$

On the other hand we have

$$f(z) = \frac{1}{\Delta z}\int\limits_{[z,z+\Delta z]} f(z)d\zeta$$

(we have pulled the constant factor $f(z)$ out of the integral sign above), which allows us to write

$$\frac{F(z + \Delta z) - F(z)}{\Delta z} - f(z) = \frac{1}{\Delta z}\int\limits_{[z,z+\Delta z]} [f(\zeta) - f(z)]d\zeta. \tag{1.20}$$

We use now continuity of the function f: for any $\varepsilon > 0$ we may find $\delta > 0$ so that if $|\Delta z| < \delta$ then we have $|f(\zeta) - f(z)| < \varepsilon$ for all $\zeta \in [z, z + \Delta z]$. We conclude from (1.20) that

$$\left|\frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z)\right| < \frac{1}{|\Delta z|}\varepsilon|\Delta z| = \varepsilon$$

provided that $|\Delta z| < \delta$. The above implies that $F'(z)$ exists and is equal to $f(z)$. □

Remark 1.12 We have used only two properties of the function f in the proof of Theorem 1.11: f is continuous and its integral over any triangle Δ that is contained properly in D vanishes. Therefore we may claim that the function F defined by (1.18) is a local anti-derivative of any function f that has these two properties.

The problem of existence of a global anti-derivative in the whole domain D is somewhat more complicated. We will address it only in the next section, and now will just show how an anti-derivative that acts along a given path may be glued together out of local anti-derivatives.

Definition 1.13 *Let a function f be defined in a domain D and let $\gamma : I = [\alpha, \beta] \to D$ be an arbitrary continuous path. A function $\Phi : I \to \mathbb{C}$ is* an anti-derivative of f along the path γ *if (i) Φ is continuous on I, and (ii) for any $t_0 \in I$ there exists a neighborhood $U \subset D$ of the point $z_0 = \gamma(t_0)$ so that f has an anti-derivative F_U in U such that*

$$F_U(\gamma(t)) = \Phi(t) \tag{1.21}$$

for all t in a neighborhood $u_{t_0} \subset I$.

We note that if f has an anti-derivative F in the whole domain D then the function $F(\gamma(t))$ is an anti-derivative along the path γ. However, the above definition does not require the existence of a global anti-derivative in all of D – it is sufficient for it to exist *locally*, in a neighborhood of each point $z_0 \in \gamma$. Moreover, if $\gamma(t') = \gamma(t'')$ with $t' \neq t''$ then the two anti-derivatives of f that correspond to the neighborhoods $u_{t'}$ and $u_{t''}$ need not coincide: they may differ by a constant (observe that they are anti-derivatives of f in a neighborhood of the same point z' and hence Theorem 1.9 implies that their difference is a constant). Therefore anti-derivative along a path being a function of the parameter t might not be a function of the point z.

Theorem 1.14 *Let $f \in \mathcal{O}(D)$ and let $\gamma : I \to D$ be a continuous path. Then anti-derivative of f along γ exists and is defined up to a constant.*

Proof. Let us divide the interval $I = [\alpha, \beta]$ into n sub-intervals $I_k = [t_k, t'_k]$ so that each pair of adjacent sub-intervals overlap on an interval ($t_k < t'_{k-1} < t_{k+1} < t'_k$, $t_1 = \alpha$, $t'_n = \beta$). Using uniform continuity of the function $\gamma(t)$ we may choose I_k so small that the image $\gamma(I_k)$ is contained in a disk $U_k \subset D$. Theorem 1.10 implies that f has an anti-derivative F in each disk U_k. Let us choose arbitrarily an anti-derivative of f in U_1 and denote it F_1. Consider an anti-derivative of f defined in U_2. It may differ only by a constant from F_1 in the intersection $U_1 \cap U_2$. Therefore we may choose the anti-derivative F_2 of f in U_2 that coincides with F_1 in $U_1 \cap U_2$.

We may continue in this fashion choosing the anti-derivative F_k in each U_k so that $F_k = F_{k-1}$ in the intersection $U_{k-1} \cap U_k$, $k = 1, 2, \ldots, n$. The function

$$\Phi(t) = F_k \circ \gamma(t), \quad t \in I_k, \quad k = 1, 2, \ldots, n,$$

is an anti-derivative of f along γ. Indeed it is clearly continuous on γ and for each $t_0 \in I$ one may find a neighborhood u_{t_0} where $\Phi(t) = F_U \circ \gamma(t)$ where F_U is an anti-derivative of f in a neighborhood of the point $\gamma(t_0)$.

It remains to prove the second part of the theorem. Let Φ_1 and Φ_2 be two anti-derivatives of f along γ. We have $\Phi_1 = F^{(1)} \circ \gamma(t)$, $\Phi_2 = F^{(2)} \circ \gamma(t)$ in a neighborhood u_{t_0} of each point $t_0 \in I$. Here $F^{(1)}$ and $F^{(2)}$ are two anti-derivatives of f defined in a neighborhood of the point $\gamma(t_0)$. They may differ only by a constant so that $\phi(t) = \Phi_1(t) - \Phi_2(t)$ is constant in a neighborhood u_{t_0} of t_0. However, a locally constant function defined on a connected set is constant on the whole set [3]. Therefore $\Phi_1(t) - \Phi_2(t) = \text{const}$ for all $t \in I$. □

If the anti-derivative of f along a path γ is known then the integral of f over γ is computed using the usual Newton-Leibnitz formula.

Theorem 1.15 *Let $\gamma : [\alpha, \beta] \to \mathbb{C}$ be a piecewise smooth path and let f be continuous on γ and have an anti-derivative $\Phi(t)$ along γ, then*

$$\int_\gamma f dz = \Phi(\beta) - \Phi(\alpha). \tag{1.22}$$

Proof. Let us assume first that γ is a smooth path and its image is contained in a domain D where f has an anti-derivative F. Then the function $F \circ \gamma$ is an anti-derivative of f along γ and hence differs from Φ only by a constant so that $\Phi(t) = F \circ \gamma(t) + C$. Since γ is a smooth path and $F'(z) = f(z)$ the derivative $\Phi'(t) = f(\gamma(t))\gamma'(t)$ exists and is continuous at all $t \in [\alpha, \beta]$. However, using the definition of the integral we have

$$\int_\gamma f dz = \int_\alpha^\beta f(\gamma(t))\gamma'(t) dt = \int_\alpha^\beta \Phi'(t) dt = \Phi(\beta) - \Phi(\alpha)$$

and the theorem is proved in this particular case.

In the general case we may divide γ into a finite number of paths $\gamma_\nu : [\alpha_\nu, \alpha_{n+1}] \to \mathbb{C}$ $(\alpha_0 = \alpha < \alpha_1 < \alpha_2 < \cdots < \alpha_n = \beta)$ so that each of them is smooth and is contained in a domain where f has an anti-derivative. As we have just shown,

$$\int_{\gamma_\nu} f dz = \Phi(\alpha_{\nu+1}) - \Phi(\alpha_\nu),$$

and summing over ν we obtain (1.22). □

[3] Indeed, let $E = \{t \in I : \phi(t) = \phi(t_0)\}$. This set is not empty since it contains t_0. It is open since ϕ is locally constant so that if $t \in E$ and $\phi(t) = \phi(t_0)$ then $\phi(t') = \phi(t) = \phi(t_0)$ for all t' in a neighborhood u_t and thus $u_t \subset E$. However, it is also closed since ϕ is a continuous function (because it is locally constant) so that $\phi(t_n) = \phi(t_0)$ and $t_n \to t''$ implies $\phi(t'') = \phi(t_0)$. Therefore $E = I$.

Remark 1.16 We may extend our definition of the integral to continuous paths (from piecewise smooth) by defining the integral of f over an arbitrary continuous path γ as the increment of its anti-derivative along the this path over the interval $[\alpha, \beta]$ of the parameter change. Clearly the right side of (1.22) does not change under a re-parameterization of the path. Therefore one may consider integrals of holomorphic functions over arbitrary continuous curves.

Remark 1.17 Theorem 1.15 allows us to verify that a holomorphic function might have no global anti-derivative in a domain that is not simply connected. Let $D = \{0 < |z| < 2\}$ be a punctured disk and consider the function $f(z) = 1/z$ that is holomorphic in D. This function may not have an anti-derivative in D. Indeed, were the anti-derivative F of f to exist in D, the function $F(\gamma(t))$ would be an anti-derivative along any path γ contained in D. Theorem 1.15 would imply that

$$\int_\gamma f dz = F(b) - F(a),$$

where $a = \gamma(\alpha)$, $b = \gamma(\beta)$ are the end-points of γ. In particular the integral of f along any closed path γ would vanish. However, we know that the integral of f over the unit circle is

$$\int\limits_{|z|=1} f dz = 2\pi i.$$

1.3 The Cauchy Theorem

We will prove now the Cauchy theorem in its general form - the basic theorem of the theory of integration of holomorphic functions (we have proved it in its simplest form in the last section). This theorem claims that the integral of a function holomorphic in some domain does not change if the path of integration is changed continuously inside the domain provided that its end-points remain fixed or a closed path remains closed. We have to define first what we mean by a continuous deformation of a path. We assume for simplicity that all our paths are parameterized so that $t \in I = [0, 1]$. This assumption may be made without any loss of generality since any path may be re-parameterized in this way without changing the equivalence class of the path and hence the value of the integral.

Definition 1.18 *Two paths $\gamma_0 : I \to D$ and $\gamma_1 : I \to D$ with common ends $\gamma_0(0) = \gamma_1(0) = a$, $\gamma_0(1) = \gamma_1(1) = b$ are* homotopic *to each other in a domain D if there exists a continuous map $\gamma(s, t) : I \times I \to D$ so that*

$$\begin{aligned} \gamma(0, t) = \gamma_0(t), \quad \gamma(1, t) = \gamma_1(t), \quad t \in I \\ \gamma(s, 0) = a, \qquad \gamma(s, 1) = b, \qquad s \in I. \end{aligned} \tag{1.23}$$

The function $\gamma(s_0, t) : I \to D$ defines a path inside in the domain D for each fixed $s_0 \in I$. These paths vary continuously as s_0 varies and their family "connects" the paths γ_0 and γ_1 in D. Therefore the homotopy of two paths in D means that one path may be deformed continuously into the other inside D.

Similarly two closed paths $\gamma_0 : I \to D$ and $\gamma_1 : I \to D$ are homotopic in a domain D if there exists a continuous map $\gamma(s,t) : I \times I \to D$ so that

$$\begin{aligned} \gamma(0,t) = \gamma_0(t), \quad \gamma(1,t) = \gamma_1(t), \quad t \in I \\ \gamma(s,0) = \gamma(s,1), \qquad s \in I. \end{aligned} \tag{1.24}$$

Homotopy is usually denoted by the symbol $\sim$, we will write $\gamma_0 \sim \gamma_1$ if γ_0 is homotopic to γ_1.

It is quite clear that homotopy defines an equivalence relation. Therefore all paths with common end-points and all closed paths may be separated into equivalence classes. Each class contains all paths that are homotopic to each other.

A special homotopy class is that of paths homotopic to zero. We say that a closed path γ is homotopic to zero in a domain D if there exists a continuous mapping $\gamma(s,t) : I \times I \to D$ that satisfies conditions (1.24) and such that $\gamma_1(t) = \text{const.}$ That means that γ may be contracted to a point by a continuous transformation.

Any closed path is homotopic to zero in a simply connected domain, and thus any two paths with common ends are homotopic to each other. Therefore the homotopy classes in a simply connected domains are trivial.

Exercise 1.19 Show that the following two statements are equivalent: (i) any closed path in D is homotopic to zero, and (ii) any two paths in D that have common ends are homotopic to each other.

The notion of homotopy may be easily extended from paths to curves since homotopy is preserved under re-parameterizations of paths. Two curves (either with common ends or closed) are homotopic in D if the paths γ_1 and γ_2 that represent those curves are homotopic to each other.

We have introduced the notion of the integral first for a path and then verified that the value of the integral is determined not by a path but by a curve, that is, by an equivalence class of paths. The general Cauchy theorem claims that integral of a holomorphic function is determined not even by a curve but by the homotopy class of the curve. In other words, the following theorem holds.

Theorem 1.20 *(Cauchy) Let $f \in \mathcal{O}(D)$ and γ_0 and γ_1 be two paths homotopic to each other in D either as paths with common ends or as closed paths, then*

$$\int_{\gamma_0} f dz = \int_{\gamma_1} f dz. \tag{1.25}$$

Proof. Let $\gamma : I \times I \to D$ be a function that defines the homotopy of the paths γ_0 and γ_1. We construct a system of squares K_{mn}, $m, n = 1, \ldots, N$ that covers the square

$K = I \times I$ so that each K_{mn} overlaps each neighboring square. Uniform continuity of the function γ implies that the squares K_{mn} may be chosen so small that each K_{mn} is contained in a disk $U_{mn} \subset D$. The function f has an anti-derivative F_{mn} in each of those disks (we use the fact that a holomorphic function has an anti-derivative in any disk). We fix the subscript m and proceed as in the proof of theorem 1.14. We choose arbitrarily the anti-derivative F_{m1} defined in U_{m1} and pick the anti-derivative F_{m2} defined in U_{m2} so that $F_{m1} = F_{m2}$ in the intersection $U_{m1} \cap U_{m2}$. Similarly we may choose $F_{m3}, \dots, F_{mN}$ so that $F_{m,n+1} = F_{mn}$ in the intersection $U_{m,n+1} \cap U_{mn}$ and define the function

$$\Phi_m(s,t) = F_{mn} \circ \gamma(s,t) \text{ for } (s,t) \in K_{mn},\ n = 1, \dots, N. \tag{1.26}$$

The function Φ_{mn} is clearly continuous in the rectangle $K_m = \cup_{n=1}^{N} K_{mn}$ and is defined up to an arbitrary constant. We choose arbitrarily Φ_1 and pick Φ_2 so that $\Phi_1 = \Phi_2$ in the intersection $K_1 \cap K_2$ [4]. The functions $\Phi_3, \dots, \Phi_N$ are chosen in exactly the same fashion so that $\Phi_{m+1} = \Phi_m$ in $K_{m+1} \cap K_m$. This allows us to define the function

$$\Phi(s,t) = \Phi_m(s,t) \text{ for } (s,t) \in K_m,\ m = 1, \dots, N. \tag{1.27}$$

the function $\Phi(s,t)$ is clearly an anti-derivative along the path $\gamma_s(t) = \gamma(s,t) : \ I \to D$ for each fixed s. Therefore the Newton-Leibniz formula implies that

$$\int_{\gamma_s} f dz = \Phi(s,1) - \Phi(s,0). \tag{1.28}$$

We consider now two cases separately.

(a) *The paths γ_0 and γ_1 have common ends.* Then according to the definition of homotopy we have $\gamma(s,0) = a$ and $\gamma(s,1) = b$ for all $s \in I$. Therefore the functions $\Phi(s,0)$ and $\Phi(s,1)$ are locally constant as functions of $s \in I$ at all s and hence they are constant on I. Therefore $\Phi(0,0) = \Phi(1,0)$ and $\Phi(1,0) = \Phi(1,1)$ so that (1.28) implies 1.25. □

(b) The paths γ_0 and γ_1 are closed. In this case we have $\gamma(s,0) = \gamma(s,1)$ so that the function $\Phi(s,0) - \Phi(s,1)$ is locally constant on I, and hence this function is a constant on I. Therefore once again (1.28) implies (1.25).

Exercise 1.21 Show that if f is holomorphic in an annulus $V = \{r < |z-a| < R\}$ then the integral $\displaystyle\int_{|z-a|=\rho} f dz$ has the same value for all ρ, $r < \rho < R$.

1.4 Some special cases

We consider in this section some special cases of the Cauchy theorem that are especially important and deserve to be stated separately.

[4] This is possible since the function $\Phi_1 - \Phi_2$ is locally constant on a connected set $K_1 \cap K_2$ and is therefore constant on this set

Theorem 1.22 *Let $f \in \mathcal{O}(D)$ then its integral along any path that is contained in D and is homotopic to zero vanishes:*

$$\int_\gamma f dz = 0 \text{ if } \gamma \sim 0. \tag{1.29}$$

Proof. Since $\gamma \sim 0$ this path may be continuously deformed into a point $a \in D$ and thus into a circle $\gamma_\varepsilon = \{|z-a| = \varepsilon\}$ of an arbitrarily small radius $\varepsilon > 0$. The general Cauchy theorem implies that

$$\int_\gamma f dz = \int_{\gamma_\varepsilon} f dz.$$

The integral on the right side vanishes in the limit $\varepsilon \to 0$ since the function f is bounded in a neighborhood of the point a. However, the left side is independent of ε and thus it must be equal to zero. □

Any closed path is homotopic to zero in a simply connected domain and thus the Cauchy theorem has a particularly simple form for such domains - this is its classical statement:

Theorem 1.23 *If a function f is holomorphic in a simply connected domain $D \subset \mathbb{C}$ then its integral over any closed path $\gamma: I \to D$ vanishes.*

Due to the importance of this theorem we also present its elementary proof under two additional assumptions: (1) the derivative f' is continuous[5], and (2) γ is a piecewise smooth Jordan path.

The second assumption implies that γ is the boundary of a domain G contained in D since the latter is simply connected. The first assumption allows to apply the Green's formula

$$\int_{\partial G} P dx + Q dy = \iint_G \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy. \tag{1.30}$$

Its derivation assumes the continuity of the partial derivatives of P and Q in $\bar{G}$ (here ∂G is the boundary of G traced counter-clockwise). Applying this formula to the real and imaginary parts of the integral

$$\int_{\partial G} f dz = \int_{\partial G} u dx - v dy + i \int_{\partial G} v dx + u dy,$$

we obtain

$$\int_{\partial G} f dz = \iint_G \left\{ -\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + i \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) \right\} dx dy.$$

The last equation may be re-written as

$$\int_{\partial G} f dz = 2i \iint_G \frac{\partial f}{\partial \bar{z}} dx dy, \tag{1.31}$$

which may be considered as the complex form of the Green's formula.

It is easy to deduce from the Cauchy theorem the global theorem of existence of an anti-derivative in a *simply connected* domain.

[5]We will soon see that this assumption holds automatically.

Theorem 1.24 *Any function f holomorphic in a simply connected domain D has an anti-derivative in this domain.*

Proof. We first show that the integral of f along a path in D is independent of the choice of the path and is completely determined by the end-points of the path. Indeed, let γ_1 and γ_2 be two paths that connect in D two points a and b. Without any loss of generality we may assume that the path γ_1 is parameterized on an interval $[\alpha, \beta_1]$ and γ_2 is parameterized on an interval $[\beta_1, \beta]$, $\alpha < \beta_1 < \beta$. Let us denote by γ the union of the paths γ_1 and γ_2^-, this is a closed path contained in γ, and, moreover,

$$\int_\gamma f dz = \int_{\gamma_1} f dz - \int_{\gamma_2} f dz.$$

However, Theorem 1.23 integral of f over any closed path vanishes and this implies our claim[6].

We fix now a point $a \in D$ and let z be a point in D. Integral of f over any path $\gamma = \widetilde{az}$ that connects a and z depends only on z and not on the choice of γ:

$$F(z) = \int_{\widetilde{az}} f(\zeta) d\zeta. \tag{1.32}$$

Repeating verbatim the arguments in the proof of theorem 1.11 we verify that $F(z)$ is holomorphic in D and $F'(z) = f(z)$ for all $z \in D$ so that F is an anti-derivative of f in D. $\Box$

The example of the function $f = 1/z$ in an annulus $\{0 < |z| < 2\}$ (see Remark 1.17) shows that the assumption that D is simply connected is essential: the global existence theorem of anti-derivative does not hold in general for multiply connected domains.

The same example shows that the integral of a holomorphic function over a closed path in a multiply connected domain might not vanish, so that the Cauchy theorem in its classical form (Theorem 1.23) may not be extended to non-simply connected domains. However, one may present a reformulation of this theorem that allows such a generalization.

The boundary ∂D of a nice simply connected domain D is a closed curve that is homotopic to zero in the closer $\bar{D}$. One may not apply Theorem 1.22 to ∂D because f is defined only in D and it may be impossible to extend it to ∂D. If we require that $f \in \mathcal{O}(\bar{D})$, that is, that f may be extended into a domain G that contains D, then Theorem 1.29 may be applied. We obtain the following re-statement of the Cauchy theorem.

Theorem 1.25 *Let f be holomorphic in the closure $\bar{D}$ of a simply connected domain D that is bounded by a continuous curve, then the integral of f over the boundary of this domain vanishes.*

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[6] One may also obtain this result directly from the general Cauchy theorem using the fact that any two paths with common ends are homotopic to each other in a simply connected domain.

Exercise 1.26 Sometimes the assumptions of Theorem 1.25 may be weakened requiring only that f may be extended continuously to $\bar{D}$. For instance, let D be a star-shaped domain with respect to $z = 0$, that is, its boundary ∂D may be represented in polar coordinates as $r = r(\phi)$, $0 \leq \phi \leq 2\pi$ with $r(\phi)$ a single-valued function. Assume in addition that $r(\phi)$ is a piecewise smooth function. Show that the statement of theorem 1.25 holds for functions f that are holomorphic in D and continuous in $\bar{D}$.

Theorem 1.25 may be extended to multiply connected domains with the help of the following definition.

Definition 1.27 *Let the boundary of a compact domain D*[7] *consist of a finite number of closed curves γ_ν, $\nu = 0, \dots, n$. We assume that the outer boundary γ_0, that is, the curve that separates D from infinity, is oriented counterclockwise while the other boundary curves γ_ν, $\nu = 1, \dots, n$ are oriented clockwise. In other words, all the boundary curves are oriented in such a way that D remains on the left side as they are traced. The boundary of D with this orientation is called* the oriented boundary *and denote by ∂D.*

We may now state the Cauchy theorem for multiply connected domains as follows.

Theorem 1.28 *Let a compact domain D be bounded by a finite number of continuous curves and let f be holomorphic in its closure $\bar{D}$. Then the integral of f over its oriented boundary ∂D is equal to zero:*

$$\int_{\partial D} f dz = \int_{\gamma_0} f dz + \sum_{\nu=1}^{n} \int_{\gamma_\nu} f dz = 0. \tag{1.33}$$

Proof. Let us introduce a finite number of cuts $\lambda_\nu^\pm$ that connect the components of the boundary of this domain. It is clear that the closed curve Γ that consists of the oriented boundary ∂D and the unions $\Lambda^+ = \cup \lambda_\nu^+$ and $\Lambda^- = \cup \lambda_\nu^-$ is homotopic to zero in the domain G that contains $\bar{D}$, and such that f is holomorphic in D. Theorem 1.22 implies that the integral of f along Γ vanishes so that

$$\int_\Gamma f dz = \int_{\partial D} f dz + \int_{\Lambda^+} f dz + \int_{\Lambda^-} f dz = \int_{\partial D} f dz$$

since the integrals of f along Λ^+ and Λ^- cancel each other.□

Example 1.29 Let $D = \{r < |z - a| < R\}$ be an annulus and $f \in \mathcal{O}(\bar{D})$ is a function holomorphic in a slightly larger annulus that contains $\bar{D}$. The oriented boundary of D consists of the circle $\gamma_0 = \{|z - a| = R\}$ oriented counterclockwise and the circle $\gamma_1^- = \{|z - a| = R\}$ oriented clockwise. According to Theorem 1.28

$$\int_{\partial D} f dz = \int_{\gamma_0} f dz + \int_{\gamma_1^-} f dz = 0,$$

or

$$\int_{\gamma_0} f dz = \int_{\gamma_1} f dz.$$

The last relation also follows from the Cauchy theorem for homotopic paths.

[7] Recall that a domain D is compact if its closure does not contain the point at infinity.

1.5 The Cauchy Integral Formula

We will obtain here a representation of functions holomorphic in a compact domain with the help of the integral over the boundary of the domain. This representation finds numerous applications both in theoretical and practical problems.

Theorem 1.30 *Let the function f be holomorphic in the closure of a compact domain D that is bounded by a finite number of continuous curves. Then the function f at any point $z \in D$ may be represented as*

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} d\zeta, \tag{1.34}$$

where ∂D is the oriented boundary of D.

The right side of (1.34) is called the Cauchy integral.

Proof. Let $\rho > 0$ be such that the disk $U_\rho = \{z' : \ |z - z'| < \rho\}$ is properly contained in D and let $D_\rho = \bar{D} \backslash \bar{U}_\rho$. The function $g(\zeta) = \dfrac{f(\zeta)}{\zeta - z}$ is holomorphic in $\bar{D}_\rho$ as a ratio of two holomorphic functions with the numerator different from zero. The oriented boundary of D_ρ consists of the union of ∂D and the circle $\partial U_\rho = \{\zeta : \ |\zeta - z| = \rho\}$ oriented clockwise. Therefore we have

$$\frac{1}{2\pi i} \int_{\partial D_\rho} g(\zeta) d\zeta = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i} \int_{\partial U_\rho} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

However, the function g is holomorphic in $\bar{D}_\rho$ (its singular point $\zeta = z$ lies outside this set) and hence the Cauchy theorem for multiply connected domains may be applied. We conclude that the integral of g over ∂D_ρ vanishes.

Therefore,

$$\frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \int_{\partial U_\rho} \frac{f(\zeta)}{\zeta - z} d\zeta, \tag{1.35}$$

where ρ may be taken arbitrarily small. Since the function f is continuous at the point z, for any $\varepsilon > 0$ we may choose $\delta > 0$ so that

$$|f(\zeta) - f(z)| < \varepsilon \text{ for all } \zeta \in \partial U_\rho$$

for all $\rho < \delta$. Therefore the difference

$$f(z) - \frac{1}{2\pi i} \int_{\partial U_\rho} \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \int_{\partial U_\rho} \frac{f(z) - f(\zeta)}{\zeta - z} d\zeta \tag{1.36}$$

does not exceed $\dfrac{1}{2\pi} \varepsilon \cdot 2\pi = \varepsilon$ and thus goes to zero as $\rho \to 0$. However, (1.35) shows that the left side in (1.36) is independent of ρ and hence is equal to zero for all sufficiently small ρ, so that

$$f(z) = \frac{1}{2\pi i} \int_{\partial U_\rho} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

This together with (1.35) implies (1.34). □

Remark 1.31 If the point z lies outside $\bar{D}$ and conditions of Theorem 1.30 hold then

$$\frac{1}{2\pi i}\int_{\partial D}\frac{f(\zeta)}{\zeta - z}d\zeta = 0. \tag{1.37}$$

This follows immediately from the Cauchy theorem since now the function $g(\zeta) = \dfrac{f(\zeta)}{\zeta - z}$ is holomorphic in $\bar{D}$.

The integral Cauchy theorem expresses a very interesting fact: the values of a function f holomorphic in a domain $\bar{G}$ are completely determined by its values on the boundary ∂G. Indeed, if the values of f on ∂G are given then the right side of (1.34) is known and thus the value of f at any point $z \in D$ is also determined. This property is the main difference between holomorphic functions and differentiable functions in the real analysis sense.

Exercise 1.32 Let the function f be holomorphic in the closure of a domain D that contains the point at infinity and the boundary ∂D is oriented so that D remains on the left as the boundary is traced. Show that then

$$f(z) = \frac{1}{2\pi i}\int_{\partial D}\frac{f(\zeta)}{\zeta - z}d\zeta + f(\infty).$$

An easy corollary of Theorem 1.30 is

Theorem 1.33 *The value of the function $f \in \mathcal{O}(D)$ at each point $z \in D$ is equal to the average of its values on any sufficiently small circle centered at z:*

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(z + \rho e^{it})dt. \tag{1.38}$$

Proof. Consider the disk $U_\rho = \{z' : \ |z - z'| < \rho\}$ so that U_ρ is properly contained in D. The Cauchy integral formula implies that

$$f(z) = \frac{1}{2\pi i}\int_{\partial U_\rho}\frac{f(\zeta)}{\zeta - z}d\zeta. \tag{1.39}$$

Introducing the parameterization $\zeta = z + \rho e^{it}$, $t \in [0, 2\pi]$ of U_ρ and replacing $d\zeta = \rho i e^{it}dt$ we obtain (1.38) from (1.39). $\Box$

The mean value theorem shows that holomorphic functions are built in a very regular fashion, so to speak, and their values are intricately related to the values at other points. This explains why these functions have specific properties that the real differentiable functions lack. We will consider many other such properties later.

Before we conclude we present an integral representation of $\mathbb{R}$-differentiable functions that generalizes the Cauchy integral formula.

Theorem 1.34 *Let $f \in C^1(\bar{D})$ be a continuously differentiable function in the real sense in the closure of a compact domain D bounded by a finite number of piecewise smooth curves. Then we have*

$$f(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{\pi}\iint_D \frac{\partial f}{\partial \bar{\zeta}} \frac{d\xi d\eta}{\zeta - z} \tag{1.40}$$

for all $z \in D$ (here $\zeta = \xi + i\eta$ inside the integral).

Proof. Let us delete a small disk $\bar{U}_\rho = \{\zeta : \ |\zeta - z| \leq \rho\}$ out of D and apply the Green's formula in its complex form (1.31) to the function $g(\zeta) = \dfrac{f(\zeta)}{\zeta - z}$ that is continuously differentiable in the domain $D_\rho = D \backslash \bar{U}_\rho$

$$\int_{\partial D} \frac{f(\zeta)}{\zeta - z} d\zeta - \int_{\partial U_\rho} \frac{f(\zeta)}{\zeta - z} d\zeta = 2i \iint_{D_\rho} \frac{\partial f}{\partial \bar{\zeta}} \frac{d\xi d\eta}{\zeta - z}{}^{8}. \tag{1.41}$$

The function f is continuous at z so that $f(\zeta) = f(z) + O(\rho)$ for $\zeta \in U_\rho$, where $O(\rho) \to 0$ as $\rho \to 0$, and thus

$$\int_{\partial U_\rho} \frac{f(\zeta)}{\zeta - z} d\zeta = f(z)\int_{\partial U_\rho} \frac{1}{\zeta - z} d\zeta + \int_{\partial U_\rho} \frac{O(\rho)}{\zeta - z} d\zeta = 2\pi i f(z) + O(\rho).$$

Passing to the limit in (1.41) and using the fact that the double integrals in (1.40) and (1.41) are convergent[9] we obtain (1.40). □

Having described the basic facts of the theory of complex integration let us describe briefly its history. The main role in its development was played by the outstanding French mathematician A. Cauchy.

Augustin-Louis Cauchy was born in 1789 into an aristocratic family. He graduated from Ecole Polytechnique in Paris in 1807. This school was created in the time of the French revolution in order to prepare highly qualified engineers. Its graduates received fundamental training in mathematics, mechanics and technical drawing for two years and were afterward sent for two more years of engineering training to on one of the four specialized institutes. Cauchy was trained at Ecole des Ponts et Chaussées from which he graduated in 1810. At that time he started his work at Cherbourg on port facilities for Napoleon's English invasion fleet.

The work of Cauchy was quite diverse - he was occupied with elasticity theory, optics, celestial mechanics, geometry, algebra and number theory. But the basis of his interests was mathematical analysis, a branch of mathematics that underwent a serious transformation started by his work. Cauchy became a member of the Academy of Sciences in 1816 and a

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[8]We have $\dfrac{\partial g}{\partial \bar{\zeta}} = \dfrac{1}{\zeta - z}\dfrac{\partial f}{\partial \bar{\zeta}}$ since the function $1/(\zeta - z)$ is holomorphic in ζ so that its derivative with respect to $\bar{\zeta}$ vanishes.

[9]Our argument shows that the limit $\lim\limits_{\rho \to 0} \iint_{D_\rho} \dfrac{\partial f}{\partial \bar{\zeta}} \dfrac{d\xi d\eta}{\zeta - z}$ exists. Moreover, since $f \in C^1(D)$ the double integral in (1.40) exists as can be easily seen by passing to the polar coordinates and thus this limit coincides with it.

professor at College de France and Ecole Polytechnique in 1817. He presented there his famous course in analysis that were published in three volumes as Cours d'analyse (1821-1828).

Baron Cauchy was a devoted royalist. He followed the royal family and emigrated to Italy after the July revolution of 1830. His failure to return to Paris caused him to lose all his positions there. He returned to Paris in 1838 and regained his position at the Academy but not his teaching positions because he had refused to take an oath of allegiance. He taught at a Jesuit college and became a professor at Sorbonne when Louis Philippe was overthrown in 1848.

The first results on complex integration by Cauchy were presented in his memoir on the theory of definite integrals presented to the Academy in 1814 and published only in 1825. Similarly to Euler Cauchy came to these problems from hydrodynamics. He starts with the relation

$$\int_{x_0}^{X}\int_{y_0}^{Y} f(x,y)dxdy = \int_{y_0}^{Y} dy \int_{x_0}^{X} f(x,y)dx \tag{1.42}$$

and considers two real valued functions S and V put together in one complex function $F = S + iV$. Inserting $f = \dfrac{\partial V}{\partial y} = \dfrac{\partial S}{\partial x}$ into (1.42) Cauchy obtains the formula that relates the integrals of these functions:

$$\int_{x_0}^{X} [V(x,Y) - V(x,y_0)]dx = \int_{y_0}^{Y} [S(X,y) - S(x_0,y)]dy.$$

He obtained a similar formula using $f = \dfrac{\partial V}{\partial x} = -\dfrac{\partial S}{\partial y}$ but only in 1822 he arrived at the idea of putting together in the complex form that he put as a footnote in his memoir of 1825. This is the Cauchy theorem for a rectangular contour though the geometric meaning of that identity is missing here.

We note that his work differs little from the work of Euler presented in 1777 at the Saint Petersburg Academy of Sciences that contains the formula

$$\int (u+iv)(dx+idy) = \int udx - vdy + i\int vdx + udy$$

and describes some of its applications. However, in the same year 1825 Cauchy published separately his memoir on definite integrals with imaginary limits, where he considered the complex integral as the limit of partial sums and observed that to make its meaning precise one should define the continuous monotone functions $x = \phi(t)$, $y = \chi(t)$ on an interval $t_0 \le t \le T$ such that $\phi(t_0) = x_0$, $\chi(t_0) = y_0$, $\phi(T) = X$, $\chi(T) = Y$. It seems that Cauchy was not yet aware of the geometric interpretation of the integral as a path in the complex plane as well as of the geometric interpretation of complex numbers in general at that time.

He has formulated his main theorem as follows: "if $F(x + y\sqrt{-1})$ is finite and continuous for $x_0 \le x \le X$ and $y_0 \le y \le Y$ then the value of the integral does not depend on the nature of the functions $\phi(t)$ and $\chi(t)$." He proves it varying the functions ϕ and χ and verifying that the variation of the integral is equal to zero. We should note that the clear notion of the integral of a function of a complex variable as integral along a path in the complex plane and the formulation of the independence of the integral from the path appeared first in the letter by Gauss to Bessel in 1831.

The Cauchy integral formula was first proved by him in 1831 in a memoir on celestial mechanics. Cauchy proved it for a disk which is quite sufficient for developing functions in power series (see the next section). We will describe other results by Cauchy as they are presented in the course.

2 The Taylor series

We will obtain the representation of holomorphic functions as sums of power series (the Taylor series) in this section.

Let us recall the simplest results regarding series familiar from the real analysis. A series (of complex numbers) $\sum_{n=0}^{\infty} a_n$ is convergent if the sequence of its partial sums $s_k = \sum_{n=0}^{k} a_n$ has a finite limit s. This limit is called the sum of the series.

A functional series $\sum_{n=0}^{\infty} f_n(z)$ with the functions f_n defined on a set $M \subset \bar{\mathbb{C}}$ converges uniformly on M if it converges at all $z \in M$, and, moreover, for any $\varepsilon > 0$ there exists $N = N(\varepsilon)$ such that for all $n \geq N$ the remainder of the series satisfies $|\sum\limits_{k=n+1}^{\infty} f_k(z)| < \varepsilon$ for all $z \in M$.

The series $\sum_{n=0}^{\infty} f_n(z)$ converges uniformly on M if the series $\sum_{n=0}^{\infty} \|f_n\|$ converges. Here $\|f_n\| = \sup\limits_{z \in M} |f_n(z)|$, and the proof is identical to that in the real analysis. This condition implies that the functional series is majorized by a convergent series of numbers. We also recall that the sum of a uniformly convergent series of continuous functions $f_n(z)$ on M is also continuous on M, and that one may integrate term-wise a uniformly convergent series along a smooth curve. The proofs are once again identical to those in the real analysis.

2.1 The Taylor series

One of the main theorems of the theory of functions of a complex variable is

Theorem 2.1 *Let $f \in \mathcal{O}(D)$ and let $z_0 \in D$ be an arbitrary point in D. Then the function f may be represented as a sum of a convergent power series*

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n \tag{2.1}$$

inside any disk $U = \{|z - z_0| < R\} \subset D$.

Proof. Let $z \in U$ be an arbitrary point. Choose $r > 0$ so that $|z - z_0| < r < R$ and denote by $\gamma_r = \{\zeta : \ |\zeta - z_0| = r\}$ The integral Cauchy formula implies that

$$f(z) = \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

In order to represent f as a power series let us represent the kernel of this integral as the sum of a geometric series:

$$\frac{1}{\zeta - z} = \left[(\zeta - z_0)\left(1 - \frac{z - z_0}{\zeta - z_0}\right)\right]^{-1} = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}}. \tag{2.2}$$

We multiply both sides by $\frac{1}{2\pi i} f(\zeta)$ and integrate the series term-wise along γ_r. The series (2.2) converges uniformly on γ_r since

$$\left|\frac{z - z_0}{\zeta - z_0}\right| = \frac{|z - z_0|}{r} = q < 1$$

for all $\zeta \in \gamma_r$. Uniform convergence is preserved under multiplication by a continuous and hence bounded function $\frac{1}{2\pi i} f(\zeta)$. Therefore our term-wise integration is legitimate and we obtain

$$f(z) = \frac{1}{2\pi i} \int_{\gamma_r} \sum_{n=0}^{\infty} \frac{f(\zeta) d\zeta}{(\zeta - z_0)^{n+1}} (z - z_0)^n = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

where[10]

$$c_n = \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(\zeta) d\zeta}{(\zeta - z_0)^{n+1}}, \quad n = 0, 1, \ldots . \square \tag{2.3}$$

Definition 2.2 *The power series (2.1) with coefficients given by (2.3) is* the Taylor series *of the function f at the point z_0 (or centered at z_0).*

The Cauchy theorem 1.20 implies that the coefficients c_n of the Taylor series defined by (2.3) do not depend on the radius r of the circle γ_r, $0 < r < R$.

Exercise 2.3 Find the radius of the largest disk where the function $z/\sin z$ may be represented by a Taylor series centered at $z_0 = 0$.

Exercise 2.4 Let f be holomorphic in $\mathbb{C}$. Show that (a) f is even if and only if its Taylor series at $z = 0$ contains only even powers; (b) f is real on the real axis if and only if $f(\bar{z}) = \overline{f(z)}$ for all $z \in \mathbb{C}$.

We present some simple corollaries of Theorem 2.1.

The Cauchy inequalities. *Let the function f be holomorphic in a closed disk $\bar{U} = \{|z - z_0| \le r\}$ and let its absolute value on the circle $\gamma_r = \partial U$ be bounded by a constant M. Then the coefficients of the Taylor series of f at z_0 satisfy the inequalities*

$$|c_n| \le M/r^n, \quad (n = 0, 1, \ldots). \tag{2.4}$$

[10] This theorem was presented by Cauchy in 1831 in Turin. Its proof was first published in Italy, and it appeared in France in 1841. However, Cauchy did not justify the term-wise integration of the series. This caused a remark by Chebyshev in his paper from 1844 that such integration is possible only in some "particular cases".

Proof. We deduce from (2.3) using the fact that $|f(\zeta)| \leq M$ for all $\zeta \in \gamma_r$:

$$|c_n| \leq \frac{1}{2\pi} \frac{M}{r^{n+1}} 2\pi r = \frac{M}{r^n}. \square$$

Exercise 2.5 Let $P(z)$ be a polynomial in z of degree n. Show that if $|P(z)| \leq M$ for $|z| = 1$ then $|P(z)| \leq M|z|^n$ for all $|z| \geq 1$.

The Cauchy inequalities imply the interesting

Theorem 2.6 *(Liouville[11]) If the function f is holomorphic in the whole complex plane and bounded then it is equal identically to a constant.*

Proof. According to Theorem 2.1 the function f may be represented by a Taylor series

$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$

in any closed disk $\bar{U} = \{|z| \leq R\}$, $R < \infty$ with the coefficients that do not depend on R. Since f is bounded in $\mathbb{C}$, say, $|f(z)| \leq M$ then the Cauchy inequalities imply that for any $n = 0, 1, \ldots$ we have $|c_n| \leq M/R^n$. We may take R to be arbitrarily large and hence the right side tends to zero as $R \to +\infty$ while the left side is independent of R. Therefore $c_n = 0$ for $n \geq 1$ and hence $f(z) = c_0$ for all $z \in \mathbb{C}$. $\square$

Therefore the two properties of a function – to be holomorphic and bounded are realized simultaneously only for the trivial functions that are equal identically to a constant.

Exercise 2.7 Prove the following properties of functions f holomorphic in the whole plane $\mathbb{C}$:

(1) Let $M(r) = \sup_{|z|=r} |f(z)|$, then if $M(r) = Ar^N + B$ where r is an arbitrary positive real number and A, B and N are constants, then f is a polynomial of degree not higher than N.

(2) If all values of f belong to the right half-plane then $f = \text{const}$.

(3) If $\lim_{z\to\infty} f(z) = \infty$ then the set $\{z \in \mathbb{C} : \ f(z) = 0\}$ is not empty.

The Liouville theorem may be reformulated:

Theorem 2.8 *If a function f is holomorphic in the closed complex plane $\overline{\mathbb{C}}$ then it is equal identically to a constant.*

Proof. if the function f is holomorphic at infinity the limit $\lim_{z\to\infty} f(z)$ exists and is finite. Therefore f is bounded in a neighborhood $U = \{|z| > R\}$ of this point. However, f is also bounded in the complement $\bar{U}^c = \{|z| \leq R\}$ since it is continuous there and the set $\bar{U}^c$ is compact. Therefore f is holomorphic and bounded in $\mathbb{C}$ and thus Theorem 2.6 implies that is equal to a constant. $\square$

[11] Actually this theorem was proved by Cauchy in 1844 while Liouville has proved only a partial result in the same year. The wrong attribution was started by a student of Liouville who has learned the theorem at one of his lectures.

Exercise 2.9 Show that a function $f(z)$ that is holomorphic at $z = 0$ and satisfies $f(z) = f(2z)$, is equal identically to a constant.

Theorem 2.1 claims that any function holomorphic in a disk may be represented as a sum of a convergent power series inside this disk. We would like to show now that, conversely, the sum of a convergent power series is a holomorphic function. Let us first recall some properties of power series that are familiar from the real analysis.

Lemma 2.10 *If the terms of a power series*

$$\sum_{n=0}^{\infty} c_n (z-a)^n \tag{2.5}$$

are bounded at some point $z_0 \in \mathbb{C}$, that is,

$$|c_n(z_0-a)^n| \le M, \quad (n = 0, 1, \dots), \tag{2.6}$$

then the series converges in the disk $U = \{z : \ |z-a| < |z_0 - a|\}$. Moreover, it converges absolutely and uniformly on any set K that is properly contained in U.

Proof. We may assume that $z_0 \neq a$, so that $|z_0 - a| = \rho > 0$, otherwise the set U is empty. Let K be properly contained in U, then there exists $q < 1$ so that $|z-a|/\rho \le q < 1$ for all $z \in K$. Therefore for any $z \in K$ and any $n \in \mathbb{N}$ we have $|c_n(z-a)^n| \le |c_n|\rho^n q^n$. However, assumption (2.6) implies that $|c_n|\rho^n \le M$ so that the series (2.5) is majorized by a convergent series $M \sum_{n=0}^{\infty} q^n$ for all $z \in K$. Therefore the series (2.5) converges uniformly and absolutely on K. This proves the second statement of this lemma. The first one follows from the second since any point $z \in U$ belongs to a disk $\{|z-a| < \rho'$, with $\rho' < \rho$, that is properly contained in U. $\square$

Theorem 2.11 *(Abel[12]) Let the power series (2.5) converge at a point $z_0 \in \mathbb{C}$. Then this series converges in the disk $U = \{z : \ |z-a| < |z_0 - a|\}$ and, moreover, it converges uniformly and absolutely on every compact subset of U.*

Proof. Since the series (2.5) converges at the point z_0 the terms $c_n(z_0-a)^n$ converge to zero as $n \to \infty$. However, every converging sequence is bounded, and hence the assumptions of the previous lemma are satisfied and both claims of the present theorem follow from this lemma. $\square$

The Cauchy-Hadamard formula. *Let the coefficients of the power series (2.5) satisfy*

$$\limsup_{n\to\infty} |c_n|^{1/n} = \frac{1}{R}, \tag{2.7}$$

with $0 \le R \le \infty$ (we set $1/0 = \infty$ and $1/\infty = 0$). Then the series (2.5) converges at all z such that $|z-a| < R$ and diverges at all z such that $|z-a| > R$.

Proof. Recall that $A = \limsup_{n\to\infty} \alpha_n$ if (1) there exists a subsequence $\alpha_{n_k} \to A$ as $k \to \infty$, and (2) for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ so that $\alpha_n < A + \varepsilon$ for all $n \ge N$. This

[12] This theorem was published in 1826 by a Norwegian mathematician Niels Abel (1802-1829).

includes the cases $A = \pm\infty$. However, if $A = +\infty$ then condition (2) is not necessary, and if $A = -\infty$ then the number $A + \varepsilon$ in condition (2) is replaced by an arbitrary number (in the latter case condition (1) holds automatically and $\lim_{n\to\infty} \alpha_n = -\infty$). It is shown in real analysis that $\limsup \alpha_n$ exists for any sequence $\alpha_n \in \mathbb{R}$ (either finite or infinite).

Let $0 < R < \infty$, then for any $\varepsilon > 0$ we may find N such that for all $n \geq N$ we have $|c_n|^{1/n} \leq \dfrac{1}{R} + \varepsilon$. Therefore, we have

$$|c_n(z-a)^n| < \left\{ \left(\frac{1}{R} + \varepsilon \right) |z-a| \right\}^n . \tag{2.8}$$

Furthermore, given $z \in \mathbb{C}$ such that $|z - a| < R$ we may choose ε so small that we have $\left(\dfrac{1}{R} + \varepsilon \right) |z-a| = q < 1$. Then (2.8) shows that the terms of the series (2.5) are majorized by a convergent geometric series q^n for $n \geq N$, and hence the series (2.5) converges when $|z - a| < R$.

Condition (1) in the definition of lim sup implies that for any $\varepsilon > 0$ one may find a subsequence c_{n_k} so that $|c_{n_k}|^{1/n_k} > \dfrac{1}{R} - \varepsilon$ and hence

$$|c_{n_k}(z-a)^{n_k}| > \left\{ \left(\frac{1}{R} - \varepsilon \right) |z-a| \right\}^{n_k} . \tag{2.9}$$

Then, given $z \in \mathbb{C}$ such that $|z - a| > R$ we may choose ε so small that we have $\left(\dfrac{1}{R} - \varepsilon \right) |z-a| > 1$. then (2.9) implies that $|c_{n_k}(z-a)^{n_k}| > 1$ for all k and hence the n-th term of the power series (2.5) does not vanish as $n \to \infty$ so that the series diverges if $|z - a| > R$.

We leave the proof in the special case $R = 0$ and $R = \infty$ as an exercise for the reader. □

Definition 2.12 *The domain of convergence of a power series (2.5) is the interior of the set E of the points $z \in \mathbb{C}$ where the series converges.*

Theorem 2.13 *The domain of convergence of the power series (2.5) is the open disk $\{|z - a| < R\}$, where R is the number determined by the Cauchy-Hadamard formula.*

Proof. The previous proposition shows that the set E where the series (2.5) converges consists of the disk $U = \{|z - a| < R\}$ and possibly some other set of points on the boundary $\{|z - a| = R\}$ of U. Therefore the interior of E is the open disk $\{|z - a| < R\}$. □

The open disk in Theorem 2.13 is called the disk of convergence of the power series (2.5), and the number R is its radius of convergence.

Example 2.14 1. The series

$$(a)\ \sum_{n=1}^{\infty}(z/n)^n, \quad (b)\sum_{n=1}^{\infty} z^n, \quad (c)\sum_{n=1}^{\infty}(nz)^n \tag{2.10}$$

have the radii of convergence $R = \infty$, $R = 1$ and $R = 0$, respectively. Therefore the domain of convergence of the first is $\mathbb{C}$, of the second – the unit disk $\{|z| < 1\}$ and of the third – an empty set.

2. The same formula shows that the domain of convergence of all three series

$$(a)\ \sum_{n=1}^{\infty} z^n, \quad (b)\sum_{n=1}^{\infty} z^n/n, \quad (c)\sum_{n=1}^{\infty} z^n/n^2 \tag{2.11}$$

is the unit disk $\{|z| < 1\}$. However, the sets where the three series converge are different. The series (a) diverges at all points on the circle $\{|z| = 1\}$ since its n-th term does not vanish as $n \to +\infty$. The series (b) converges at some points of the circle $\{|z| = 1\}$ (for example, at $z = -1$) and diverges at others (for example, at $z = 1$). The series (c) converges at all points on this circle since it is majorized by the converging series $sum_{n=1}^{\infty} 1/n^2$ at all z such that $|z| = 1$.

We pass now to the proof that the sum of a power series is holomorphic.

Theorem 2.15 *The sum of a power series*

$$f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n \tag{2.12}$$

is holomorphic in its domain of convergence.

Proof. We assume that the radius of convergence $R > 0$, otherwise there is nothing to prove. Let us define the formal series of derivatives

$$\sum_{n=1}^{\infty} nc_n(z-a)^{n-1} = \phi(z). \tag{2.13}$$

Its convergence is equivalent to that of the series $\sum_{n=1}^{\infty} nc_n(z-a)^n$. However, since $\limsup_{n\to\infty} |nc_n|^{1/n} = \limsup_{n\to\infty} |c_n|^{1/n}$ the radius of convergence of the series (2.13) is also equal to R. Therefore this series converges uniformly on compact subsets of the disk $U = \{|z-a| < R\}$ and hence the function $\phi(z)$ is continuous in this disk.

Moreover, for the same reason the series (2.13) may be integrated term-wise along the boundary of any triangle Δ that is properly contained in U:

$$\int_{\partial\Delta} \phi dz = \sum_{n=1}^{\infty} nc_n \int_{\partial\Delta} (z-a)^{n-1} dz = 0.$$

The integrals on the right side vanish by the Cauchy theorem. Therefore we may apply Theorem 1.11 and Remark 1.12 which imply that the function

$$\int_{[a,z]} \phi(\zeta)d\zeta = \sum_{n=1}^{\infty} nc_n \int_{[a,z]} (\zeta - a)^{n-1} d\zeta = \sum_{n=1}^{\infty} c_n (z-a)^n$$

has a derivative at all $z \in U$ that is equal to $\phi(z)$. Once again we used uniform convergence to justify the term-wise integration above. However, then the function

$$f(z) = c_0 + \int_{[a,z]} \phi(\zeta)d\zeta$$

has a derivative at all $z \in U$ that is also equal to $\phi(z)$. □

2.2 Properties of holomorphic functions

We discuss some corollaries of Theorem 2.15.

Theorem 2.16 *Derivative of a function $f \in \mathcal{O}(D)$ is holomorphic in the domain D.*

Proof. Given a point $z_0 \in D$ we construct a disk $U = \{|z - z_0| < R\}$ that is contained in D. Theorem 2.1 implies that f may be represented as a sum of a converging power series in this disk. Theorem 2.15 implies that its derivative $f' = \phi$ may also be represented as a sum of a power series converging in the same disk. Therefore one may apply Theorem 2.15 also to the function ϕ and hence ϕ is holomorphic in the disk U. □

This theorem also implies directly the necessary condition for the existence of anti-derivative that we have mentioned in Section 1.2:

Corollary 2.17 *If a continuous function f has an anti-derivative F in a domain D then f is holomorphic in D.*

Using Theorem 2.16 once again we obtain

Theorem 2.18 *Any function $f \in \mathcal{O}(D)$ has derivatives of all orders in D that are also holomorphic in D.*

The next theorem establishes uniqueness of the power series representation of a function relative to a given point.

Theorem 2.19 *Let a function f have a representation*

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n \tag{2.14}$$

in a disk $\{|z - z_0| < R\}$. Then the coefficients c_n are determined uniquely as

$$c_n = \frac{f^{(n)}(z_0)}{n!}, \quad n = 0, 1, \dots \tag{2.15}$$

Proof. Inserting $z = z_0$ in (2.14) we find $c_0 = f(z_0)$. Differentiating (2.14) termwise we obtain

$$f'(z) = c_1 + 2c_2(z - z_0) + 3c_3(z - z_0)^2 + \dots$$

Inserting $z = z_0$ above we obtain $c_1 = f'(z_0)$. Differentiating (2.14) n times we obtain (we do not write out the formulas for $\tilde{c}_j$ below)

$$f^{(n)}(z) = n!c_n + \tilde{c}_1(z - z_0) + \tilde{c}_1(z - z_0)^2 + \dots$$

and once again using $z = z_0$ we obtain $c_n = f^{(n)}(z_0)/n!$. □

Sometimes Theorem 2.19 is formulated as follows: "Every converging power series is the Taylor series for its sum."

Exercise 2.20 Show that a differential equation $dw/dz = P(w, z)$ where P is a polynomial both in z and w has no more than solution $w(z)$ holomorphic near a given point $z = a$ such that $w(a) = b$ with a given $b \in \mathbb{C}$.

Expression (2.14) allows to calculate the Taylor series of elementary functions. For example, we have

$$e^z = 1 + z + \frac{z^2}{2!} + \dots + \frac{z^n}{n!} + \dots \tag{2.16}$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots, \quad \sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots \tag{2.17}$$

with all three expansions valid at all $z \in \mathbb{C}$ (they have infinite radius of convergence $R = \infty$).

Comparing expressions (2.15) for the coefficients c_n with their values given by (2.3) we obtain the formulas for the derivatives of holomorphic functions:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{\gamma_r} \frac{f(\zeta)d\zeta}{(\zeta - z_0)^{n+1}}, \quad n = 1, 2 \dots \tag{2.18}$$

If the function f is holomorphic in a domain D and G is a sub-domain of D that is bounded by finitely many continuous curves and such that $z_0 \in G$ then we may replace the contour γ_r in (2.18) by the oriented boundary ∂G, using the invariance of the integral under homotopy of paths. Then we obtain *the Cauchy integral formula for derivatives* of holomorphic functions:

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial G} \frac{f(\zeta)d\zeta}{(\zeta - z)^{n+1}}, \quad n = 1, 2 \dots \tag{2.19}$$

These formulas may be also obtained from the Cauchy integral formula

$$f(z)\frac{1}{2\pi i} \int_{\partial G} \frac{f(\zeta)d\zeta}{(\zeta - z)},$$

by differentiating with respect to the parameter z inside the integral. Our indirect argument allowed us to bypass the justification of this operation.

Theorem 2.21 *(Morera[13]) If a function f is continuous in a domain D and its integral over the boundary $\partial\Delta$ of any triangle Δ vanishes then f is holomorphic in D.*

Proof. Given $a \in D$ we construct a disk $U = \{|z-a| < r\} \subset D$. The function $F(z) = \int_{[a,z]} f(\zeta)d\zeta$ is holomorphic in U (see remark after Theorem 1.11). Theorem 2.16 implies then that f is also holomorphic in D. This proves that f is holomorphic at all $a \in D$. $\square$

Remark 2.22 The Morera Theorem states the converse to the Cauchy theorem as formulated in Theorem 1.10, that is, that integral of a holomorphic function over the boundary of any triangle vanishes. However, the Morera theorem also requires that f is continuous in D. This assumption is essential: for instance, the function f that is equal to zero everywhere in $\mathbb{C}$ except at $z = 0$, where it is equal to one, is not even continuous at $z = 0$ but its integral over any triangle vanishes.

However, the Morera theorem does not require any differentiability of f: from the modern point of view we may say that a function satisfying the assumptions of this theorem is a generalized solution of the Cauchy-Riemann equations. The theorem asserts that any generalized solution is a classical solution, that is, it has partial derivatives that satisfy the Cauchy-Riemann equations.

Exercise 2.23 Let f be continuous in a disk $U = \{|z| < 1\}$ and holomorphic everywhere in U except possibly on the diameter $[-1, 1]$. Show that f is holomorphic in all of U.

Finally, we present the list of equivalent definitions of a holomorphic function.

Theorem 2.24 *The following are equivalent:*
(R) The function f is $\mathbb{C}$-differentiable in a neighborhood U of the point a.
(C) The function f is continuous in a neighborhood U of the point a and its integral over the boundary of any triangle in $\Delta \subset U$ vanishes.
(W) the function f may be represented as the sum of a converging power series in a neighborhood U of the point a.

These three statements reflect three concepts in the development of the theory of functions of a complex variable. Usually a function f that satisfies (R) is called holomorphic in the sense of Riemann, those that satisfy (C) - holomorphic in the sense of Cauchy, and (W) - holomorphic in the sense of Weierstrass[14] The implication (R)→(C) was proved in the Cauchy theorem 1.11, (C)→(W) in the Taylor theorem 2.1, and (W)→(R) in Theorem 2.15.

Remark 2.25 We have seen that the representation as a power series in a disk $\{|z-a| < R\}$ is a necessary an sufficient condition for f to be holomorphic in this disk. However, convergence of the power series on the boundary of the disk is not related to it being

[13]The theorem was proved by an Italian mathematician Giacinto Morera in 1889.
[14]These names approximately correspond to the true order of the events.

holomorphic at those points. This may be sen on simplest examples. Indeed, the geometric series

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \tag{2.20}$$

converges in the open disk $\{|z| < 1\}$. The series (2.20) diverges at all points on $\{|z| = 1\}$ since its n-th term does not vanish in the limit $n \to \infty$. On the other hand, the series

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n^2} \tag{2.21}$$

converges at all points of $\{|z| = 1\}$ since it is majorized by the convergent number series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. However, its sum may not be holomorphic at $z = 1$ since its derivative $f'(z) = \sum_{n=1}^{\infty} \frac{z^{n-1}}{n}$ is unbounded as z tends to one along the real axis.

2.3 The Uniqueness theorem

Definition 2.26 *A zero of the function f is a point $a \in \overline{\mathbb{C}}$ where f vanishes, that is, solution of $f(z) = 0$.*

Zeroes of differentiable functions in the real analysis may have limit points where the function f remains differentiable, for example, $f(x) = x^2 \sin(1/x)$ behaves in this manner at $x = 0$. The situation is different in the complex analysis: zeroes of a holomorphic function must be isolated, they may have limit points only on the boundary of the domain where the function is holomorphic.

Theorem 2.27 *Let the point $a \in \mathbb{C}$ be a zero of the function f that is holomorphic at this point, and f is not equal identically to zero in a neighborhood of a. Then there exists a number $n \in \mathbb{N}$ so that*

$$f(z) = (z-a)^n \phi(z), \tag{2.22}$$

where the function ϕ is holomorphic at a and is different from zero in a neighborhood of a.

Proof. Indeed, f may be represented by a power series in a neighborhood of a: $f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$. The first coefficient $c_0 = 0$ but not all c_n are zero, otherwise f would vanish identically in a neighborhood of a. Therefore there exists the smallest n so that $c_n \neq 0$ and the power series has the form

$$f(z) = c_n (z-a)^n + c_{n+1}(z-a)^{n+1} + \ldots, \quad c_n \neq 0. \tag{2.23}$$

Let us denote by

$$\phi(z) = c_n + c_{n+1}(z-a) + \ldots \tag{2.24}$$

so that $f(z) = (z-a)^n \phi(z)$. The series (2.24) converges in a neighborhood of a (it has the same radius of convergence as f) and thus ϕ is holomorphic in this neighborhood. Moreover, since $\phi(a) = c_n \neq 0$ and ϕ is continuous at a, $\phi(z) \neq 0$ in a neighborhood of a. □

Theorem 2.28 *(Uniqueness) Let $f_1, f_2 \in \mathcal{O}(D)$, then if $f_1 = f_2$ on a set E that has a limit point in D then $f_1(z) = f_2(z)$ for all $z \in D$.*

Proof. The function $f = f_1 - f_2$ is holomorphic in D. We should prove that $f \equiv 0$ in D, that is, that the set $F = \{z \in D : \; f(z) = 0\}$, that contains in particular the set E, coincides with D. The limit point a of E belongs to E (and hence to F) since f is continuous. Theorem 2.23 implies that $f \equiv 0$ in a neighborhood of a, otherwise it would be impossible for a to e a limit point of the set of zeroes of f.

Therefore the interior F^o of F is not empty - it contains a. Moreover, F^o is an open set as the interior of a set. However, it is also closed in the relative topology of D. Indeed, let $b \in D$ be a limit point of F^o, then the same Theorem 2.27 implies that $f \equiv 0$ in a neighborhood of b so that $b \in F^o$. Finally, the set D being a domain is connected, and hence $F^o = D$ by Theorem 1.29 of Chapter 1. □

This theorem shows another important difference of a holomorphic function from a real differentiable function in the sense of real analysis. Indeed, even two infinitely differentiable functions may coincide on an open set without being identically equal to each other everywhere else. However, according to the previous theorem tow holomorphic functions that coincide on a set that has a limit point in the domain where they are holomorphic (for instance on a small disk, or an arc inside the domain) have to be equal identically in the whole domain.

Exercise 2.29 Show that if f is holomorphic at $z = 0$ then there exists $n \in \mathbb{N}$ so that $f(1/n) \neq (-1)^n/n^3$.

We note that one may simplify the formulation of Theorem 2.27 using the Uniqueness theorem. That is, the assumption that f is not equal identically to zero in any neighborhood of the point a may be replaced by the assumption that f is not equal identically to zero everywhere (these two assumptions coincide by the Uniqueness theorem).

Theorem 2.27 shows that holomorphic functions vanish as an integer power of $(z-a)$.

Definition 2.30 *The order, or multiplicity, of a zero $a \in \mathbb{C}$ of a function f holomorphic at this point, is the order of the first non-zero derivative $f^{(k)}(a)$. in other words, a point a is a zero of f of order n if*

$$f(a) = \cdots = f^{(k-1)}(a) = 0, \quad f^{(n)}(a) \neq 0, \quad n \geq 1. \tag{2.25}$$

Expressions $c_k = f^{(k)}(a)/k!$ for the coefficients of the Taylor series show that the order of zero is the index of the first non-zero Taylor coefficient of the function f at the point a, or, alternatively, the number n in Theorem 2.27. The Uniqueness theorem shows that holomorphic functions that are not equal identically to zero may not have zeroes of infinite order.

Similar to what is done for polynomials, one may define the order of zeroes using division.

Theorem 2.31 *The order of zero $a \in \mathbb{C}$ of a holomorphic function f coincides with the order of the highest degree $(z-a)^k$ that is a divisor of f in the sense that the ratio $\frac{f(z)}{(z-a)^k}$ (extended by continuity to $z = a$) is a holomorphic function at a.*

Proof. Let us denote by n the order of zero a and by N the highest degree of $(z-a)$ that is a divisor of f. Expression (2.22) shows that f is divisible by any power $k \le n$:

$$\frac{f(z)}{(z-a)^k} = (z-a)^{n-k}\phi(z),$$

and thus $N \ge n$. Let f be divisible by $(z-a)^N$ so that the ratio

$$\psi(z) = \frac{f(z)}{(z-a)^N}$$

is a holomorphic function at a. Developing ψ as a power series in $(z-a)$ we find that the Taylor expansion of f at a starts with a power not smaller than N. Therefore $n \ge N$ and since we have already shown that $n \le N$ we conclude that $n = N$. $\square$

Example 2.32 The function $f(z) = \sin z - z$ has a third order zero at $z = 0$. Indeed, we have $f(0) = f'(0) = f''(0)$ but $f'''(0) \ne 0$. This may also be seen from the representation

$$f(z) = -\frac{z^3}{3!} + \frac{z^5}{5!} + \dots$$

Remark 2.33 Let f be holomorphic at infinity and equal to zero there. It is natural to define the order of zero at this point as the order of zero the order of zero at $z = 0$ of the function $\phi(z) = f(1/z)$. The theorem we just proved remains true also for $a = \infty$ if instead of dividing by $(z-a)^k$ we consider multiplication by z^k. For example, the function $f(z) = \frac{1}{z^3} + \frac{1}{z^2}$ has order 3 at infinity.

2.4 The Weierstrass theorem

Recall that termwise differentiation of a series in real analysis requires uniform convergence of the series in a neighborhood of a point as well as uniform convergence of the series of derivatives. The situation is simplified in the complex analysis. The following theorem holds.

Theorem 2.34 *(Weierstrass) If the series*

$$f(z) = \sum_{n=0}^{\infty} f_n(z) \tag{2.26}$$

of functions holomorphic in a domain D converges uniformly on any compact subset of this domain then
(i) the sum of this series is holomorphic in D;
(ii) the series may be differentiated termwise arbitrarily many times at any point in D.

Proof. Let a be arbitrary point in D and consider the disk $U = \{|z-a| < r\}$ that is properly contained in D. The series (2.26) converges uniformly in U by assumption and thus its sum is continuous in U. Let $\Delta \subset U$ be a triangle contained in U and let $\gamma = \partial\Delta$. Since the series (2.26) converges uniformly in U we may integrate it termwise along γ:

$$\int_\gamma f(z)dz = \sum_{n=0}^{\infty} \int_\gamma f_n(z)dz.$$

However, the Cauchy theorem implies that all integrals on the right side vanish since the functions f_n are holomorphic. Hence the Morera theorem implies that the function f is holomorphic and part (i) is proved.

In order to prove part (ii) we once again take an arbitrary point $a \in D$, consider the same disk U as in the proof of part (i) and denote by $\gamma_r = \partial U = \{|z-a| = r\}$. The Cauchy formulas for derivatives imply that

$$f^{(k)}(a) = \frac{k!}{2\pi i} \int_{\gamma_r} \frac{f(\zeta)}{(\zeta-a)^{k+1}} d\zeta. \tag{2.27}$$

The series

$$\frac{f(\zeta)}{(\zeta-a)^{k+1}} = \sum_{n=0}^{\infty} \frac{f_n(\zeta)}{(\zeta-a)^{k+1}} \tag{2.28}$$

differs from (2.26) by a factor that has constant absolute value $\frac{1}{r^{k+1}}$ for all $\zeta \in \gamma_r$. Therefore it converges uniformly on γ_r and may be integrated termwise in (2.27). Using expressions (2.27) in (2.28) we obtain

$$f^{(k)}(a) = \frac{k!}{2\pi i} \sum_{n=0}^{\infty} \int_{\gamma_r} \frac{f_n(\zeta)}{(\zeta-a)^{k+1}} d\zeta = \sum_{n=0}^{\infty} f_n^{(k)}(a),$$

and part (ii) is proved. □

Exercise 2.35 Explain why the series $\sum_{n=1}^{\infty} \frac{\sin(n^3 z)}{n^2}$ may not be differentiated termwise.

3 The Laurent series and singular points

The Taylor series are well suited to represent holomorphic functions in a disk. We will consider here more general power series with both positive and negative powers of $(z-a)$. Such series represent functions holomorphic in annuli

$$V = \{z \in \mathbb{C} : \ r < |z-a| < R\}, \quad r \geq 0, \ R \leq \infty.$$

Such representations are especially important when the inner radius is zero, that is, in punctured neighborhoods. They allow to study functions near the singular points where they are not holomorphic.

3.1 The Laurent series

Theorem 3.1 *(Laurent[15]) Any function f holomorphic in an annulus*

$$V = \{z \in \mathbb{C} : \ r < |z-a| < R\}$$

may be represented in this annulus as a sum of a converging power series

$$f(z) = \sum_{-\infty}^{\infty} c_n (z-a)^n. \tag{3.1}$$

Its coefficients are determined by the formulas

$$c_n = \frac{1}{2\pi i} \int\limits_{|z-a|=\rho} \frac{f(\zeta)d\zeta}{(\zeta-a)^{n+1}}, \quad n = 0, \pm 1, \pm 2, \ldots, \tag{3.2}$$

where $r < \rho < R$.

Proof. We fix an arbitrary point $z \in V$ and consider the annulus $V' = \{\zeta : \ r' < |z-a| < R'\}$ such that $z \in V' \subset V$. The Cauchy integral formula implies that

$$f(z) = \frac{1}{2\pi i} \int_{\partial V'} \frac{f(\zeta)d\zeta}{\zeta - z} = \frac{1}{2\pi i} \int_{\Gamma'} \frac{f(\zeta)d\zeta}{\zeta - z} - \frac{1}{2\pi i} \int_{\gamma'} \frac{f(\zeta)d\zeta}{\zeta - z}. \tag{3.3}$$

The circles $\Gamma' = \{|z-a| = R'\}$ and $\gamma' = \{|z-a| = r'\}$ are both oriented counterclockwise. We have $\left|\dfrac{z-a}{\zeta-a}\right| = q < 1$ for all $\zeta \in \Gamma'$. Therefore the geometric series

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta-a)\left(1 - \frac{z-a}{\zeta-a}\right)} = \sum_{n=0}^{\infty} \frac{(z-a)^n}{(\zeta-a)^{n+1}}$$

converges uniformly and absolutely for $\zeta \in \Gamma'$. We multiply this series by a bounded function $\dfrac{1}{2\pi i} f(\zeta)$ (this does not violate uniform convergence) and integrating termwise along Γ' we obtain

$$\frac{1}{2\pi i} \int_{\Gamma'} \frac{f(\zeta)d\zeta}{\zeta - z} = \sum_{0}^{\infty} c_n (z-a)^n \tag{3.4}$$

with

$$c_n = \frac{1}{2\pi i} \int\limits_{\Gamma'} \frac{f(\zeta)d\zeta}{(\zeta-a)^{n+1}}, \quad n = 0, 1, 2, \ldots \tag{3.5}$$

[15]This theorem was proved by Weierstrass in his Münster notebooks in 1841, but they were not published until 1894. A French engineer and mathematician Pierre Alphonse Laurent has proved this theorem in his memoir submitted in 1842 for the Grand Prize after the deadline has passed. It was not approved for the award.

The second integral in (3.3) has to be treated differently. We have $\left|\frac{\zeta - a}{z - a}\right| = q_1 < 1$ for all $\zeta \in \gamma'$. Therefore we obtain an absolutely and uniformly converging on γ' geometric series as

$$-\frac{1}{\zeta - z} = \frac{1}{(z-a)\left(1 - \frac{\zeta - a}{z-a}\right)} = \sum_{n=1}^{\infty} \frac{(\zeta - a)^{n-1}}{(z-a)^n}.$$

Once again multiplying this series by $\frac{1}{2\pi i} f(\zeta)$ and integrating termwise along γ' we get

$$-\frac{1}{2\pi i}\int_{\gamma'} \frac{f(\zeta)d\zeta}{\zeta - z} = \sum_{1}^{\infty} \frac{d_n}{(z-a)^n} \tag{3.6}$$

with

$$d_n = \frac{1}{2\pi i}\int\limits_{\Gamma'} f(\zeta)(\zeta - a)^{n-1} d\zeta, \quad n = 1, 2, \dots \tag{3.7}$$

We replace now the index n in (3.6) and (3.7) that takes values $1, 2, \dots$ by index $-n$ that takes values $-1, -2, \dots$ (this does not change anything) and denote[16]

$$c_n = -d_n = \frac{1}{2\pi i}\int\limits_{\Gamma'} f(\zeta)(\zeta - a)^{-n-1} d\zeta. \quad n = 1, 2, \dots \tag{3.8}$$

Now decomposition (3.6) takes the form

$$-\frac{1}{2\pi i}\int_{\gamma'} \frac{f(\zeta)d\zeta}{\zeta - z} = \sum_{n=-1}^{\infty} c_n (z-a)^n. \tag{3.9}$$

We now insert (3.4) and (3.9) into (3.1) and obtain the decomposition (3.1): $f(z) = \sum\limits_{n=-\infty}^{\infty} c_n(z-a)^n$, where the infinite series is understood as the sum of the series (3.4) and (3.9). It remains to observe that the Cauchy theorem 1.20 implies that the circles γ' and Γ' in (3.5) and (3.8) may be replaced by any circle $\{|\zeta - a| = \rho\}$ with any $r < \rho < R$. Then these expressions becomes (3.2). □

Definition 3.2 *The series (3.1) with the coefficients determined by (3.2) is called the Laurent series of the function f in the annulus V. The terms with non-negative powers constitute its regular part, while the terms with the negative powers constitute the principal part (we will see in the next section that these names are natural).*

Let us consider the basic properties of the power series in integer powers of $(z - a)$. As before we define such a series

$$\sum_{n=-\infty}^{\infty} c_n (z-a)^n \tag{3.10}$$

[16]Note that we have so far we used only c_n with positive indices so we do not interfere with previously defined c_n's.

as the sum of two series

$$(\Sigma_1): \sum_{n=0}^{\infty} c_n(z-a)^n \text{ and } (\Sigma_2): \sum_{n=-1}^{-\infty} c_n(z-a)^n. \tag{3.11}$$

The series (Σ_1) is a usual power series, its domain of converges is the disk $\{|z-a| < R\}$ where the radius R is determined by the Cauchy-Hadamard formula

$$\frac{1}{R} = \limsup_{n\to\infty} |c_n|^{1/n}. \tag{3.12}$$

The series (Σ_2) is a power series in the variable $Z = 1/(z-a)$:

$$(\Sigma_2): \sum_{n=1}^{\infty} c_{-n} Z^n. \tag{3.13}$$

Therefore its domain of convergence is the outside of the disk $\{|z-a| > r\}$ where

$$r = \limsup_{n\to\infty} |c_{-n}|^{1/n} \tag{3.14}$$

as follows from the Cauchy-Hadamard formula applied to the series (3.13). The number R is not necessarily larger than r therefore the domain of convergence of the series (3.10) may be empty. However, if $r < R$ then the domain of convergence of the series (3.10) is the annulus $V = \{r < |z-a| < R\}$. We note that the set of points where (3.10) converges may differ from V by a subset of the boundary ∂V.

The series (3.10) converges uniformly on any compact subset of V according to the Abel theorem. Therefore the Weierstrass theorem implies that its sum is holomorphic in V.

These remarks imply immediately the uniqueness of the representation of a function as a power series in both negative and positive powers in a given annulus.

Theorem 3.3 *If a function f may be represented by a series of type (3.1) in an annulus $V = \{r < |z-a| < R\}$ then the coefficients of this series are determined by formulas (3.2).*

Proof. Consider a circle $\gamma = \{|z-a| = \rho\}$, $r < \rho < R$. The series

$$\sum_{k=-\infty}^{\infty} c_k(z-a)^k = f(z)$$

converges uniformly on γ. This is still true if we multiply both sides by an arbitrary power $(z-a)^{-n-1}$, $n = 0, \pm 1, \pm 2, \ldots$:

$$\sum_{n=-\infty}^{\infty} c_k(z-a)^{k-n-1} = \frac{f(z)}{(z-a)^{n+1}}.$$

Integrating this series term-wise along γ we obtain

$$\sum_{n=-\infty}^{\infty} c_k \int_\gamma (z-a)^{k-n-1} dz = \int_\gamma \frac{f(z)dz}{(z-a)^{n+1}}.$$

The orthogonality (1.4) implies that all integrals on the left side vanish except the one with $k = n$ that is equal to $2\pi i$. We get

$$2\pi c_n = \int_\gamma \frac{f(z)dz}{(z-a)^{n+1}}$$

which is nothing but (3.2). □

Theorem 3.3 may be reformulated as follows: any converging series in negative and positive powers is the Laurent series of its sum.

Expression (3.2) for the coefficients of the Laurent series are rarely used in practice since they require computation of integrals. The uniqueness theorem that we have just proved implies that any legitimate way of getting the Laurent series may be used: they all lead to the same result.

Example 3.4 The function $f(z) = \dfrac{1}{(z-1)(z-2)}$ is holomorphic in the annuli $V_1 = \{0 < |z| < 1\}$, $V_2 = \{1 < |z| < 2\}$, $V_3 = \{2 < |z| < \infty\}$. In order to obtain its Laurent series we represent f as $f = \dfrac{1}{z-2} - \dfrac{1}{z-1}$. The two terms may be represented by the following geometric series in the annulus V_1:

$$\frac{1}{z-2} = -\frac{1}{2}\frac{1}{1-\frac{z}{2}} = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n \quad \text{(converges for } |z| < 2) \tag{3.15}$$

$$-\frac{1}{z-1} = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad \text{(converges for } |z| < 1).$$

Therefore the function f is given in V_1 by the series

$$f(z) = \sum_{n=0}^{\infty}\left(1 - \frac{1}{2^{n+1}}\right) z^n,$$

that contains only positive powers (the Taylor series). The first series in (3.15) still converges in V_2 but the second ones needs to be replaced by the decomposition

$$-\frac{1}{z-1} = -\frac{1}{z}\frac{1}{1-\frac{1}{z}} = -\sum_{n=-1}^{-\infty} z^n \quad \text{(converges for } |z| > 1). \tag{3.16}$$

The function f is represented by the Laurent series in this annulus:

$$f(z) = -\sum_{n=-1}^{-\infty} z^n - \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n.$$

Finally, the series (3.16) converges in V_3 while the first expansion in (3.15) should be replaced by

$$\frac{1}{z-2} = \frac{1}{z}\frac{1}{1-\dfrac{2}{z}} = \frac{1}{2}\sum_{n=-1}^{-\infty}\left(\frac{z}{2}\right)^n \quad \text{(converges for } |z|>2\text{)}.$$

Therefore we have in V_3:

$$f(z) = \sum_{n=-1}^{-\infty}\left(\frac{1}{2^{n+1}} - 1\right) z^n.$$

We observe that the coefficients of the Laurent series are determined by formulas (3.2) that coincide with the integral formulas for the coefficients of the Taylor series[17] Repeating the arguments in the derivation of the Cauchy inequalities for the coefficients of the Taylor series we obtain

Theorem 3.5 *The Cauchy inequalities (for the coefficients of the Laurent series). Let the function f be holomorphic in the annulus $V = \{r < |z-a| < R\}$ and let its absolute value be bounded by M on a circle $\gamma_\rho = \{|z-a| = \rho\}$ then the coefficients of the Laurent series of the function f in V satisfy the inequalities*

$$|c_n| \le M/\rho^n, \quad n = 0, \pm1, \pm2 \ldots \tag{3.17}$$

We now comment on the relation between the Laurent and Fourier series. The Fourier series of a function ϕ that is integrable on $[0, 2\pi]$ is the series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt, \tag{3.18}$$

where

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{2\pi} \phi(t)\cos nt dt, \\ b_n &= \frac{1}{\pi}\int_0^{2\pi} \phi(t)\sin nt dt, \quad n = 0, 1, 2, \ldots \end{aligned} \tag{3.19}$$

with $b_0 = 0$. Such a series may be re-written in the complex form using the Euler formulas $\cos nt = \dfrac{e^{int} + e^{-int}}{2}$, $\sin nt = \dfrac{e^{int} - e^{-int}}{2i}$:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt = \sum_{n=-\infty}^{\infty} c_n e^{int},$$

[17]However, the coefficients of the Laurent series may not be written as $c_n = f^{(n)}(a)/n!$ – for the simple reason that f might be not defined for $z = a$.

where we set

$$c_n = \frac{a_n - ib_n}{2} = \frac{1}{2\pi}\int_0^{2\pi} \phi(t)e^{-int}dt, \quad n = 0, 1, \ldots,$$

and

$$c_n = \frac{a_{-n} + ib_{-n}}{2} = \frac{1}{2\pi}\int_0^{2\pi} \phi(t)e^{-int}dt, \quad n = -1, -2, \ldots$$

The series

$$\sum_{n=-\infty}^{\infty} c_n e^{int} \tag{3.20}$$

with the coefficients

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} \phi(t)e^{-int}dt \tag{3.21}$$

is the Fourier series of the function ϕ written in the complex form.

Let us now set $e^{it} = z$ and $\phi(t) = f(e^{it}) = f(z)$, then the series (3.20) takes the form

$$\sum_{n=-\infty}^{\infty} c_n z^n \tag{3.22}$$

and its coefficients are

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})e^{-int}dt = \frac{1}{2\pi i}\int_{|z|=1} f(z)\frac{dz}{z^{n+1}}. \tag{3.23}$$

Therefore the Fourier series of a function $\phi(t)$, $t \in [0, 2\pi]$ written in the complex form is the Laurent series of the function $f(z) = \phi(t)$ with $z = e^{it}$, on the unit circle $|z| = 1$.

Clearly, conversely, the Laurent series of a function $f(z)$ on the unit circle is the Fourier series of the function $f(e^{it}) = \phi(t)$ on the interval $[0, 2\pi]$.

We note that in general even if the Fourier series converges to the function ϕ at all points $[0, 2\pi]$ the corresponding Laurent series may have $R = r = 1$ so that its domain of convergence is empty. Domain of convergence is not empty only under fairly restrictive assumptions on the function ϕ.

Example 3.6 Let $\phi(t) = \dfrac{a\sin t}{a^2 - 2a\cos t + 1}$, then we set $z = e^{it}$ and find

$$f(z) = \frac{a(z - \frac{1}{z})}{2i\left\{a^2 - a(z + \frac{1}{z}) + 1\right\}} = \frac{1}{2i}\cdot\frac{1 - z^2}{z^2 - (a + \frac{1}{a})z + 1} = \frac{1}{2i}\left(\frac{1}{1 - az} - \frac{1}{1 - \frac{a}{z}}\right)$$

. This function is holomorphic in the annulus $\{|a| < |z| < 1/|a|\}$. As in the previous example we obtain its Laurent series in this annulus:

$$f(z) = \frac{1}{2i}\sum_{n=1}^{\infty} a^n\left(z^n - \frac{1}{z^n}\right).$$

Replacing again $z = e^{it}$ we obtain the Fourier series of the function ϕ:

$$\phi(t) = \sum_{n=1}^{\infty} a^n \sin nt.$$

3.2 Isolated singular points

We begin to study the points where analyticity of a function is violated. We first consider the simplest type of such points.

Definition 3.7 *A point $a \in \bar{\mathbb{C}}$ is an isolated singular point of a function f if there exists a punctured neighborhood of this point (that is, a set of the form $0 < |z-a| < r$ if $a \neq \infty$, or of the form $R < |z| < \infty$ if $a = \infty$), where f is holomorphic.*

We distinguish three types of singular points depending on the behavior of f near such point.

Definition 3.8 *An isolated singular point a of a function f is said to be*
(I) removable if the limit $\lim_{z\to a} f(z)$ exists and is finite;
(II) a pole if the limit $\lim_{z\to a} f(z)$ exists and is equal to ∞.
(III) an essential singularity if f has neither a finite nor infinite limit as $z \to a$.

Example 3.9 1. All three types of singular points may be realized. For example, the function $z/\sin z$ has a removable singularity at $z = 0$ as may be seen from the Taylor expansion

$$\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{4!} - \dots$$

that implies that the limit $\lim_{z=0} \frac{\sin z}{z} = 1$ exists and thus so does $\lim_{z=0} \frac{z}{\sin z} = 1$. The functions $1/z^n$, where n is a positive integer have a pole at $z = 0$. The function e^z has an essential singularity at $z = 0$, since, for instance, its limits as $z = x$ tends to zero from the left and right are different (the limit on the left is equal to zero, and the limit on the right is infinite), while it has no limit as z goes to zero along the imaginary axis: $e^{iy} = \cos(1/y) + i\sin(1/y)$ has no limit as $y \to 0$.

Non-isolated singular points may exist as well. For instance, the function $\dfrac{1}{\sin(\pi z)}$ has poles at the points $z = 1/n$, $n \in \mathbb{Z}$ and hence $z = 0$ is non-isolated singular point of this function - a limit point of poles.

2. A more complicated set of singular points is exhibited by the function

$$f(z) = \sum_{n=0}^{\infty} z^{2^n} = 1 + z^2 + z^4 + z^8 + \dots \qquad (3.24)$$

According to the Cauchy-Hadamard formula the series (3.24) converges in the open disk $\{|z|, 1\}$ and hence f is holomorphic in this disk. Furthermore, $f(z)$ tends to infinity as $z \to 1$ along the real axis and hence $z = 1$ is a singular point of this function. However, we have

$$f(z^2) = 1 + z^4 + z^8 + \dots = f(z) - z^2$$

and hence $f(z)$ tends to infinity also when $z \to -1$ along the radial direction. Similarly $f(z) = z^2 + z^4 + f(z^4)$ and hence $f \to \infty$ as $z \to \pm i$ along the radius of the disk. In general,

$$f(z) = z^2 + \dots + z^{2^n} + f(z^{2^n})$$

for any $n \in \mathbb{N}$. Therefore $f \to \infty$ as z tends to any "dyadic" point $z = e^{ik\cdot 2\pi/2^n}$, $k = 0, 1, \ldots, 2^n - 1$ on the circle along the radial direction. Since the set of "dyadic" points is dense on the unit circle each point on this circle is a singular point of f. Therefore f is singular along a whole curve that consists of non-isolated singular points.

The type of an isolated singular point $z = a$ is closely related to the Laurent expansion of f in a punctured neighborhood of a. This relation is expressed by the following three theorems for finite singular points.

Theorem 3.10 *An isolated singular point $a \in \mathbb{C}$ of a function f is a removable singularity if and only if its Laurent expansion around a contains no principal part:*

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n. \tag{3.25}$$

Proof. Let a be a removable singularity of f, then the limit $\lim_{z\to a} f(z) = A$ exists and is finite. Therefore f is bounded in a punctured neighborhood $\{0 < |z-a| < R\}$ of f, say, $|f| \le M$. Let ρ be such that $0 < \rho < R$ and use the Cauchy inequalities:

$$|c_n| \le M/\rho^n, \quad n = 0, \pm 1, \pm 2, \ldots$$

If $n < 0$ then the right side vanishes in the limit $\rho \to 0$ while the left side is independent of ρ. Therefore $c_n = 0$ when $n < 0$ and the Laurent series has no principal part.

Conversely, let $f(z)$ has a Laurent expansion around a that has no principal part. This is a Taylor expansion and hence the limit

$$\lim_{z\to a} f(z) = c_0$$

exists and is finite. Therefore a is a removable singularity of f. □

Remark 3.11 The same argument proves the following.

Theorem 3.12 *An isolated singular point a of a function f is removable if and only if f is bounded in a neighborhood of the point a.*

Extending f to a removable singular point a by continuity we set $f(a) = \lim_{z\to a} f(z)$ and obtain a function holomorphic at this point – this removes the singularity. That explains the name "removable singularity". In the future we will consider such points as regular and not singular points.

Exercise 3.13 *Show that if f is holomorphic in a punctured neighborhood of a point a and we have $Re f > 0$ at all points in this neighborhood, then a is a removable singularity of f.*

Theorem 3.14 *An isolated singular point $a \in \mathbb{C}$ is a pole if and only if the principal part of the Laurent expansion near a contains only finite (and positive) number of non-zero terms:*

$$f(z) = \sum_{n=-N}^{\infty} c_n (z-a)^n, \quad N > 0. \tag{3.26}$$

Proof. Let a be a pole of f. There exists a punctured neighborhood of a where f is holomorphic and different from zero since $\lim_{z \to a} f(z) = \infty$. The function $\phi(z) = 1/f(z)$ is holomorphic in this neighborhood and the limit $\lim_{z \to a} \phi(z) = 0$ exists. Therefore a is a removable singularity of ϕ (and its zero) and the Taylor expansion holds:

$$\phi(z) = b_N (z-a)^N + b_{N+1}(z-a)^{N+1} + \dots, \quad b_N \neq 0.$$

Therefore we have in the same neighborhood

$$f(z) = \frac{1}{\phi(z)} = \frac{1}{(z-a)^N} \cdot \frac{1}{b_N + b_{N+1}(z-a) + \dots}. \tag{3.27}$$

The second factor above is a holomorphic function at a and thus admits the Taylor expansion

$$\frac{1}{b_N + b_{N+1}(z-a) + \dots} = c_{-N} + c_{-N+1}(z-a) + \dots, \quad c_{-N} = \frac{1}{b_N} \neq 0.$$

Using this expansion in (3.27) we find

$$f(z) = \frac{c_{-N}}{(z-a)^N} + \frac{c_{-N+1}}{(z-a)^{N-1}} + \dots + \sum_{n=0}^{\infty} c_n (z-a)^n.$$

This is the Laurent expansion of f near a and we see that its principal part contains finitely many terms.

Let f be represented by a Laurent expansion (3.26) in a punctured neighborhood of a with the principal part that contains finitely many terms, and $c_{-N} \neq 0$. Then both f and $\phi(z) = (z-a)^N f(z)$ are holomorphic in this neighborhood. The latter has the expansion

$$\phi(z) = c_{-N} + c_{-N+1}(z-a) + \dots$$

that shows that a is a removable singularity of ϕ and the limit $\lim_{z \to a} \phi(z) = c_{-N}$ exists. Then the function $f(z) = \phi(z)/(z-a)^N$ tends to infinity as $z \to a$ and hence a is a pole of f. □

We note another simple fact that relates poles and zeros.

Theorem 3.15 *A point a is a pole of the function f if and only if the function $\phi = 1/f$ is holomorphic in a neighborhood of a and $\phi(a) = 0$.*

Proof. The necessity of this condition has been proved in the course of the proof of Theorem 3.14. Let us show it is also sufficient. If ϕ is holomorphic at a and $\phi(a) = 0$ but ϕ is not equal identically to a constant then the uniqueness theorem implies that there exists a punctured neighborhood of this point where $\phi \neq 0$. the function $f = 1/\phi$ is holomorphic in this neighborhood and hence a is an isolated singular point of f. However, $\lim_{z\to a} f(z) = \infty$ and thus a is a pole of f. □

This relation allows to introduce the following definition.

Definition 3.16 *The order of the pole a of a function f is the order of this point as a zero of $\phi = 1/f$.*

The proof of Theorem 3.14 shows that the order of a pole coincides with the index N of the leading term in the Laurent expansion of the function around the pole.

Theorem 3.17 *An isolated singular point of a is an essential singularity if and only if the principal part of the Laurent expansion of f near a contains infinitely many non-zero terms.*

Proof. This theorem is essentially contained in Theorems 3.10 and 3.14 (if the principal part contains infinitely many terms then a may be neither removable singularity nor a pole; if a is an essential singularity then the principal part may neither be absent nor contain finitely many terms). □

Exercise 3.18 Show that if a is an essential singularity of a function f then

$$\rho^k \sup_{|z-a|=\rho} |f(z)| \to \infty$$

as $\rho \to 0$ for any natural k.

Behavior of a function near an essential singularity is characterized by the following interesting

Theorem 3.19 *If a is an essential singularity of a function f then for any $A \in \overline{\mathbb{C}}$ we may find a sequence $z_n \to a$ so that*

$$\lim_{n\to\infty} f(z) = A. \tag{3.28}$$

Proof. Let $A = \infty$. Since f may not be bounded in a punctured neighborhood $\{0 < |z-a| < r\}$ there exists a point z_1 so that $|f(z_1)| > 1$. Similarly there exists a point z_2 in $\{0 < |z-a| < |z_1 - a|/2\}$ such that $|f(z_2)| > 2$ etc.: there exists a point z_n in the neighborhood $\{0 < |z-a| < |z_{n-1} - a|/2\}$ so that $|f(z_n)| > n$. Clearly we have both $z_n \to a$ and $f(z_n) \to \infty$.

Let us consider now the case $A \neq \infty$. Then either there exists a sequence of points $\zeta_k \to a$ so that $f(\zeta_k) = A$ or there exists a neighborhood $\{0 < |z-a| < r\}$ so that $f(z) \neq A$ in this neighborhood. The function $\phi(z) = 1/(f(z) - A)$ is holomorphic in this neighborhood. Moreover, a is an essential singularity of ϕ (otherwise $f(z) = A + \dfrac{1}{\phi(z)}$

would have a limit as $z \to a$). The first part of this proof implies that there exists a sequence $z_k \to a$ so that $\phi(z_k) \to \infty$ which in turn implies that

$$\lim_{n\to\infty} f(z_n) = A + \lim_{n\to\infty} \frac{1}{\phi(z)} = A. \square$$

The collection of all possible limits of $f(z_k)$ for all sequences $z_k \to a$ is called the indeterminacy set of f at the point a. If a is a removable singularity or a pole of f the indeterminacy set of f at a consists of one point (either finite or infinite). Theorem 3.19 claims that the other extreme is realized at an essential singularity: the indeterminacy set fills the whole closed complex plane $\overline{\mathbb{C}}$.

Exercise 3.20 (i) Show that the conclusion of Theorem 3.19 holds also for a singular point that is a limit point of poles.
(ii) Let a be an essential singularity of f: which type of singularity may the function $1/f$ have at a? (Hint: it is either an essential singularity or a limit point of poles.)

We briefly comment now on the isolated singularities at infinity. The classification and Theorems 3.12, 3.15 and 3.19 are applicable in this case without any modifications. However, Theorems 3.10, 3.14 and 3.17 related to the Laurent expansion require changes. The reason is that the type of singularity at a finite singular point is determined by the principal part of the Laurent expansion that contains the negative powers of $(z-a)$ that are singular at those points. However, the negative powers are regular at infinity and the type of singularity is determined by the positive powers of z. Therefore it is natural to define the principal part of the Laurent expansion at infinity as the collection of the positive powers of z of this expansion. Theorems 3.10, 3.14 and 3.17 hold after that modification also for $a = \infty$.

This result may be obtained immediately with the change of variables $z = 1/w$: if we denote $f(z) = f(1/w) = \phi(w)$ then clearly

$$\lim_{z\to\infty} f(z) = \lim_{w\to 0} \phi(w)$$

and hence ϕ has the same type of singularity at $w = 0$ as f at the point $z = \infty$. For example, in the case of a pole ϕ has an expansion in $\{0 < |w| < r\}$

$$\phi(w) = \frac{b_{-N}}{w^N} + \cdots + \frac{b_{-1}}{w} + \sum_{n=0}^{\infty} b_n w^n, \quad b_{-N} \neq 0.$$

Replacing w by $1/z$ we get the expansion for f in the annulus $\{R < |z| < \infty\}$ with $R = 1/r$:

$$f(z) = \sum_{n=-1}^{-\infty} c_n z^n + c_0 + c_1 z + \cdots + c_N z^N$$

with $c_n = b_{-n}$. Its principal part contains finitely many terms. We may consider the case of a removable or an essential singularity in a similar fashion.

We describe now the classification of the simplest holomorphic functions according to their singular points. According to the Liouville theorem the functions that have no singularities in $\overline{\mathbb{C}}$ are constants. The next level of complexity is exhibited by the entire functions.

Definition 3.21 *A function $f(z)$ is called entire if it is holomorphic in $\mathbb{C}$, that is, if it has no finite singular points.*

The point $a = \infty$ is therefore an isolated singularity of an entire function f. If it is a removable singularity then $f = \text{const}$. If it is a pole then the principal part of the Laurent expansion at infinity is a polynomial $g(z) = c_1 z + \cdots + c_N z^N$. Subtracting the principal part from f we observe that the function $f - g$ is entire and has a removable singularity at infinity. Therefore it is a constant and hence f is a polynomial. Therefore an entire function with a pole at infinity must be a polynomial.

Entire functions with an essential singularity at infinity are called entire transcendental functions, such as e^z, $\sin z$ or $\cos z$.

Exercise 3.22 (i) Show that an entire function such that $|f(z)| \geq |z|^N$ for sufficiently large $|z|$ is a polynomial.
(ii) Deduce Theorem 3.19 for entire functions and $a = \infty$ from the Liouville theorem.

Definition 3.23 *A function f is meromorphic if it has no singularities in $\mathbb{C}$ except poles.*

Entire functions form a sub-class of meromorphic functions that have no singularities in $\mathbb{C}$. Since each pole is an isolated singular point a meromorphic function may have no more than countably many poles in $\mathbb{C}$. Indeed, every disk $\{|z| < n\}$ contains finitely many poles (otherwise the set of poles would have a limit point that would be a non-isolated singular point and not a pole) and hence all poles may be enumerated. Examples of meromorphic functions with infinitely many poles are given by functions $\tan z$ and cotan z.

Theorem 3.24 *If a meromorphic function f has a pole or a removable singularity at infinity (that is, if all its singularities in $\overline{\mathbb{C}}$ are poles) then f is a rational function.*

Proof. The number of poles of f is finite - otherwise a limit point of poles would exist in $\overline{\mathbb{C}}$ since the latter is compact, and it would be a non-isolated singular point and not a pole. Let us denote by a_ν, $\nu = 1, \ldots, n$ the finite poles of f and let

$$g_\nu(z) = \frac{c_{-N_\nu}^{(\nu)}}{(z - a_\nu)^{N_\nu}} + \cdots + \frac{c_{-1}^{(\nu)}}{z - a_\nu} \tag{3.29}$$

be the principal part of f near the pole a_ν. We also let

$$g(z) = c_1 z + \cdots + c_N z^N \tag{3.30}$$

be the principal part of f at infinity. If $a = \infty$ is a removable singularity of f we set $g = 0$.

Consider the function

$$\phi(z) = f(z) - g(z) - \sum_{\nu=1}^{n} g_\nu(z).$$

It has no singularities in $\overline{\mathbb{C}}$ and hence $\phi(z) = c_0$. Therefore

$$f(z) = c_0 + g(z) + \sum_{\nu=1}^{n} g_\nu(z) \tag{3.31}$$

is a rational function. □

Remark 3.25 Expression (3.31) is the decomposition of f into an entire part and simple fractions. Our argument gives a simple existence proof for such a decomposition.

Sometimes we will use the term "meromorphic function" in a more general sense. We say that f is meromorphic in a domain D if it has no singularities in D other than poles. Such function may also have no more than countably many poles. Indeed we may construct a sequence of compact sets $K_1 \subset K_2 \cdots \subset K_n \subset \dots$ so that $D = \cup_{n=1}^{\infty} K_n$: it suffices to take $K_n = \{z : \ |z| \le n, \ \mathrm{dist}(z, \partial D) \ge 1/n\}$. Then f may have only finitely many poles in each K_n and hence it has no more than countably many poles in all of D. If the set of poles of f in D is infinite then the limit points of this set belong to the boundary ∂D.

Theorem 3.24 may now be formulated as follows: any function meromorphic in the closed complex plane $\overline{\mathbb{C}}$ is rational.

3.3 The Residues

Somewhat paradoxically the most interesting points in the study of holomorphic functions are those where functions cease being holomorphic – the singular points. We will encounter many observations in the sequel that demonstrate that the singular points and the Laurent expansions around them contain the basic information about the holomorphic functions.

We illustrate this point on the problem of computing integrals of holomorphic functions. Let f be holomorphic in a domain D everywhere except possibly at a countable set of isolated singular points. Let G be properly contained in D, and let the boundary ∂G consist of finitely many continuous curves and not contain any singular points of f. There is a finite number of singular points contained inside G that we denote by $a_1, a_2, \dots, a_n$. Let us consider the circles $\gamma_\nu = \{|z - a_\nu| = t\}$ oriented counterclockwise, and of so small a radius that the disks $\bar{U}_\nu$ bounded by them do not overlap and are all contained in G. Let us also denote the domain $G_r = G \backslash (\cup_{\nu=1}^{n} \bar{U}_\nu)$. The function f is holomorphic in $\bar{G}_r$ and hence the Cauchy theorem implies that

$$\int_{\partial G_r} f dz = 0. \tag{3.32}$$

However, the oriented boundary ∂G_r consists of ∂G and the circles γ_ν^- oriented clockwise so that

$$\int_{\partial G} f dz = \sum_{\nu=1}^{n} \int_{\gamma_\nu} f dz. \tag{3.33}$$

Therefore the computation of the integral of a function along the boundary of a domain is reduced to the computation of the integrals over arbitrarily small circles around its singular points.

Definition 3.26 *The integral of a function f over a sufficiently small circle centered at an isolated singular point $a \in \mathbb{C}$ of this function, divided by $2\pi i$ is called the residue of f at a and is denoted by*

$$res_a f = \frac{1}{2\pi i} \int_{\gamma_r} f dz. \tag{3.34}$$

The Cauchy theorem on invariance of the integral under homotopic variations of the contour implies that the residue does not depend on the choice of r provided that r is sufficiently small and is completely determined by the local behavior of f near a.

Relation (3.33) above expresses the Cauchy theorem on residues[18]:

Theorem 3.27 *Let the function f be holomorphic everywhere in a domain D except at an isolated set of singular points. Let the domain G be properly contained in D and let its boundary ∂G contain no singular points of f. Then we have*

$$\int_{\partial G} f dz = 2\pi i \sum_{(G)} res_{a_\nu} f, \tag{3.35}$$

where summation is over all singular points of f contained in G.

This theorem is of paramount importance as it allows to reduce the computation of a global quantity such as integral over a curve to a computation of local quantities – residues of the function at its singular points.

As we will now see the residues of a function at its singular points are determined completely by the principal part of its Laurent expansion near the singular points. This will show that it suffices to have the information about the singular points of a function and the principal parts of the corresponding Laurent expansions in order to compute its integrals.

Theorem 3.28 *The residue of a function f at an isolated singular point $a \in \mathbb{C}$ is equal to the coefficient in front of the term $(z-a)^{-1}$ in its Laurent expansion around a:*

$$res_a f = c_{-1}. \tag{3.36}$$

[18]Cauchy first considered residues in his memoirs of 1814 and 1825 where he studied the difference of integrals with common ends that contain a pole of the function between them. This explains the term "residue" that first appeared in a Cauchy memoir of 1826. Following this work Cauchy has published numerous papers on the applications of residues to calculations of integrals, series expansions, solution of differential equations etc.

Proof. The function f has the Laurent expansion around a:

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n.$$

The series converges uniformly on a sufficiently small circle $\gamma_r = \{|z-a| = r\}$. Integrating the series termwise over γ_r and using (1.4) we find $\int_{\gamma_r} f dz = 2\pi i c_{-1}$. The definition (3.34) of the residue implies (3.36). □

Corollary 3.29 *The residue at a removable singularity $a \in \mathbb{C}$ vanishes.*

We present now some formulas for the computation of the residue at a pole. First we let a be a pole of order one. The Laurent expansion of the function near a has the form

$$f(z) = \frac{c_{-1}}{z-a} + \sum_{n=0}^{\infty} c_n (z-a)^n.$$

This immediately leads to the formula for the residue at a pole of order one:

$$c_{-1} = \lim_{z \to a} (z-a) f(z). \tag{3.37}$$

A simple modification of this formula is especially convenient. Let

$$f(z) = \frac{\phi(z)}{\psi(z)}$$

with the functions ϕ and ψ holomorphic at a so that $\psi(a) = 0$, $\psi'(a) \neq 0$, and $\phi(a) \neq 0$. This implies that a is a pole of order one of the function f. Then (3.37) implies that

$$c_{-1} = \lim_{z \to a} \frac{(z-a)\phi(z)}{\psi(z)} = \lim_{z \to a} \frac{\phi(z)}{\frac{\psi(z)-\psi(a)}{z-a}}$$

so that

$$c_{-1} = \frac{\phi(a)}{\psi'(a)}. \tag{3.38}$$

Let f now have a pole of order n at a, then its Laurent expansion near this point has the form

$$f(z) = \frac{c_{-n}}{(z-a)^n} + \cdots + \frac{c_{-1}}{z-a} + \sum_{n=0}^{\infty} c_n (z-a)^n.$$

We multiply both sides by $(z-a)^n$ in order to get rid of the negative powers in the Laurent expansion and then differentiate $n-1$ times in order to single out c_{-1} and pass to the limit $z \to a$. We obtain the expression for the residue at a pole of order n:

$$c_{-1} = \frac{1}{(n-1)!} \lim_{z \to a} \frac{d^{n-1}}{dz^{n-1}} \left[(z-a)^n f(z) \right]. \tag{3.39}$$

There no analogous formulas for the calculation of residues at an essential singularity: one has to compute the principal part of the Laurent expansion.

A couple of remarks on residue at infinity.

Definition 3.30 *Let infinity be an isolated singularity of the function f. The residue of f at infinity is*

$$res_{\infty} f = \frac{1}{2\pi i} \int_{\gamma_R^-} f dz, \tag{3.40}$$

where γ_R^- is the circle $\{|z| = R\}$ of a sufficiently large radius R oriented clockwise.

The orientation of γ_R^- is chosen so that the neighborhood $\{R < |z| < \infty\}$ remains on the left as the circle is traversed. The Laurent expansion of f at infinity has the form

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n.$$

Integrating the series termwise along γ_R^- and using (1.4) we obtain

$$\mathrm{res}_{\infty} f = c_{-1}. \tag{3.41}$$

The terms with the negative powers constitute the regular part of the Laurent expansion at infinity. Therefore unlike at finite singular points the residue at infinity may be non-zero even if $z = \infty$ is not a singular point of the function.

We present a simple theorem on the total sum of residues.

Theorem 3.31 *Let the function f be holomorphic everywhere in the complex plane $\mathbb{C}$ except at a finite number of points a_ν, $\nu = 1, \dots, n$. Then the sum of its residues at all of its finite singular points and the residue at infinity vanishes:*

$$\sum_{\nu=1}^{n} res_{a_\nu} f + res_{\infty} f = 0. \tag{3.42}$$

Proof. We consider the circle $\gamma_R = \{|z| = R\}$ of such a large radius that it contains all finite singular points a_ν of f. Let γ_R be oriented counterclockwise. The Cauchy theorem on residues implies that

$$\frac{1}{2\pi} \int_{\gamma_R} f dz = \sum_{\nu=1}^{n} \mathrm{res}_{a_\nu} f,$$

while the Cauchy theorem 1.20 implies that the left side does not change if R is increased further. Therefore it is equal to the negative of the residue of f at infinity. Thus the last equality is equivalent to (3.42).

Example 3.32 One needs not compute the residues at all the eight poles of the second order in order to compute the integral $I = \int\limits_{|z|=2} \frac{dz}{(z^8+1)^2}$. It suffices to apply the theorem on the sum of residues that implies that

$$\sum_{\nu=1}^{n} \mathrm{res}_{a_\nu} \frac{1}{(z^8+1)^2} + \mathrm{res}_{\infty} \frac{1}{(z^8+1)^2} = 0.$$

However, the function f has a zero of order sixteen at infinity. Thus its Laurent expansion at infinity has negative powers starting at z^{-16}. Hence its residue at infinity is equal to zero, and hence the sum of residues at finite singular points vanishes so that $I = 0$.

We present several examples of the application of the Cauchy theorem on residues to the computation of definite integrals of functions of a real variable. Let us compute the integral along the real axis

$$\phi(t) = \int_{-\infty}^{\infty} \frac{e^{itx}}{1+x^2} dx, \tag{3.43}$$

where t is a real number. The integral converges absolutely since it is majorized by the converging integral of $1/(1+x^2)$.

The residues are used as follows. We extend the integrand to the whole complex plane

$$f(z) = \frac{e^{itz}}{1+z^2}$$

and choose a closed contour so that it contains the interval $[-R, R]$ of the real axis and an arc that connects the end-points of this segment. The Cauchy theorem on residues is applied to this closed contour and then the limit $R \to \infty$ is taken. If the limit of the integral along the arc may be found then the problem is solved.

Let $z = x + iy$, given that $|e^{izt}| = e^{-yt}$ we consider separately two cases: $t \geq 0$ and $t < 0$. In the former case we close the contour by using the upper semi-circle $\gamma'_R = \{|z| = R, \ \mathrm{Im} z > 0\}$ that is traversed counterclockwise. When $R > 1$ the resulting contour contains on pole $z = i$ of f of the first order. The residue at this point is easily found using (3.38):

$$\mathrm{res}_i \frac{e^{izt}}{1+z^2} = \frac{e^{-t}}{2i}.$$

The Cauchy theorem on residues implies then that

$$\int_{-R}^{R} f(x)dx + \int_{\gamma'_R} f dz = \pi e^{-t}. \tag{3.44}$$

The integral over γ'_R is bounded as follows. We have $|e^{itz}| = e^{-ty} \leq 1$, $|1+z^2| \geq R^2 - 1$ when $t \geq 0$ and $z \in \gamma'_R$. Therefore we have an upper bound

$$\left| \int_{\gamma'_R} \frac{e^{itz}}{1+z^2} dz \right| \leq \frac{\pi R}{R^2 - 1} \tag{3.45}$$

that shows that this integral vanishes in the limit $R \to \infty$. Therefore passing to the limit $R \to \infty$ in (3.44) we obtain for $t \geq 0$:

$$\int_{-\infty}^{\infty} f(x)dx = \pi e^{-t}. \tag{3.46}$$

The estimate (3.45) fails when $t < 0$ since $|e^{izt}| = e^{-yt}$ grows as $y \to +\infty$. Therefore we replace the semi-circle γ_R' by the lower semi-circle $\gamma_R'' = \{|z| = R,\ \text{Im} z < 0\}$ that is traversed clockwise. Then the Cauchy theorem on residues implies for $R > 1$:

$$\int_{-R}^{R} f(x)dx + \int_{\gamma_R''} f dz = -2\pi \text{res}_{-i} f = \pi e^t. \tag{3.47}$$

We have $|e^{itz}| = e^{ty} \le 1$, $|1+z^2| \ge R^2 - 1$ when $t < 0$ and $z \in \gamma_R''$. Therefore the integral over γ_R'' also vanishes in the limit $R \to \infty$ and (3.47) becomes in the limit $R \to \infty$

$$\int_{-\infty}^{\infty} f(x)dx = \pi e^t. \tag{3.48}$$

Putting (3.46) and (3.48) together we obtain the final answer

$$\phi(t) = \int_{-\infty}^{\infty} \frac{e^{itx}}{1+x^2} dx = \pi e^{-|t|}. \tag{3.49}$$

We will often use residues to compute various integrals. We present a lemma useful in such calculations.

Lemma 3.33 *(Jordan[19]) Let the function f be holomorphic everywhere in $\{Imz \ge 0\}$ except possibly at an isolated set of singular points and $M(R) = \sup_{\gamma_R} |f(z)|$ over the semi-circle $\gamma_R = \{|z| = R,\ Imz \ge 0\}$ tends to zero as $R \to \infty$ (or along a sequence $R_n \to \infty$ such that γ_{R_n} do not contain singular points of f). Then the integral*

$$\int_{\gamma_R} f(z) e^{i\lambda z} dz \tag{3.50}$$

tends to zero as $R \to \infty$ (or along the corresponding sequence $R_n \to \infty$) for all $\lambda > 0$.

The main point of this lemma is that $M(R)$ may tend to zero arbitrary slowly so that the integral of f over γ_R needs not vanish as $R \to \infty$. Multiplication by the exponential $e^{i\lambda z}$ with $\lambda > 0$ improves convergence to zero.

Proof. Let us denote by $\gamma_R' = \{z = Re^{i\phi},\ 0 \le \phi \le \pi/2\}$ the right half of γ_R. We have $\sin\phi \ge \frac{2}{\pi}\phi$ for $\phi \in [0, \pi/2]$ because $\sin\phi$ is a concave function on the interval. Therefore the bound $\left|e^{i\lambda z}\right| = e^{-\lambda R \sin\phi} \le e^{-2\lambda R\phi/\pi}$ holds and thus

$$\left| \int_{\gamma_R} f(z) e^{i\lambda z} dz \right| \le M(R) \int_0^{\pi/2} e^{-2\lambda R\phi/\pi} R d\phi = M(R) \frac{\pi}{2\lambda}(1 - e^{-\lambda R}) \to 0$$

as $R \to \infty$. The bound for $\gamma_R'' = \gamma_R \backslash \gamma_R'$ is obtained similarly. $\square$

As the proof of this lemma shows the assumption that f is holomorphic is not essential in this lemma.

[19]This lemma appeared first in 1894 in the textbook on analysis written by Camille Jordan (1838-1922).

4 Exercises for Chapter 2

1. An integral of the Cauchy type is an integral of the form

$$F(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)d\zeta}{\zeta - z}$$

where γ is a smooth curve in $\mathbb{C}$ and f is a continuous function on γ. Show that F is a holomorphic function in $\overline{\mathbb{C}}\backslash\gamma$ that vanishes at infinity.

2. Let γ be a smooth closed Jordan curve that bounds a domain D: $\gamma = \partial D$, and let $f \in C^1(\gamma)$. Show that the value of the integral of the Cauchy type jumps by the value of f at the crossing point when we cross γ. More precisely, if $\zeta_0 \in \gamma$ and $z \to \zeta_0$ from one side of γ then F has two limiting values $F^+(\zeta_0)$ and $F^-(\zeta_0)$ so that

$$F^+(\zeta_0) - F^-(\zeta_0) = f(\zeta_0).$$

Here $+$ corresponds to inside of D and $-$ to the outside. Hint: write F as

$$F(z) = \frac{1}{2\pi i}\int_\gamma \frac{(f(\zeta) - f(\zeta_0))d\zeta}{\zeta - z} + \frac{f(\zeta_0)}{2\pi i}\int_\gamma \frac{d\zeta}{\zeta - z}.$$

3. Under the assumptions of the previous problem show that each of the following conditions is necessary and sufficient for the integral of the Cauchy type to be the Cauchy integral:

$$(a) \int_\gamma \frac{f(\zeta)}{\zeta - z} = 0 \ \text{ for all } z \in \overline{\mathbb{C}}\backslash\bar{D}$$

and

$$(b) \int_\gamma \zeta^n f(\zeta)d\zeta = 0 \ \text{ for all } n = 0, 1, 2\ldots$$

4. Let f be holomorphic in the disk $\{|z| < R\}$, $R > 1$. Show that the average of the square of its absolute value on the unit circle $\{|z| = 1\}$ is equal to $\sum_{n=0}^\infty |c_n|^2$, where c_n are the Taylor coefficients of f at $z = 0$.

5. The series $\displaystyle\sum_{n=0}^{\infty} \frac{x^2}{n^2x^2+1}$ converges for all real x but its sum may not be expanded in the Taylor series at $z = 0$. Explain.

6. Show that any entire function that satisfies the conditions $f(z + i) = f(z)$ and $f(z + 1) = f(z)$ is equal to a constant.

7. Show that the function $f(z) = \displaystyle\int_0^1 \frac{\sin tz}{t}dt$ is entire.

8. Let $f(z) = \sum_{n=0}^\infty a_n z^n$ be holomorphic in a closed disk $\bar{U} = \{|z| \le R\}$ and $a_0 \neq 0$. Show that f is different from zero in the disk $\left\{|z| < \dfrac{|a_0|R}{|a_0| + M}\right\}$ where $M = \sup_{z\partial U} |f(z)|$.

9. Show that a power series may not converge absolutely at any boundary point of the disk of convergence if the boundary contains at least one pole of the function.

10. Show that a function holomorphic outside two non-intersecting compact sets may be represented as a sum of two functions, one of which is holomorphic outside of one compact set and the other outside the other compact set.

Chapter 3

The Basics of the Geometric Theory

This chapter introduces the reader to the basics of the geometric theory of functions of a complex variable. We will consider here the main problems of the theory of conformal mappings as well as the geometric principles that concern the most general properties of holomorphic functions.

1 The Geometric Principles

1.1 The Argument Principle

Let the function f be holomorphic in a punctured neighborhood $\{0 < |z-a| < r\}$ of a point $a \in \mathbb{C}$. We assume also that f does not vanish in this neighborhood. The logarithmic residue of the function f at the point a is the residue of the logarithmic derivative

$$\frac{f'(a)}{f(a)} = \frac{d}{dz}\mathrm{Ln} z \tag{1.1}$$

of this function at the point a.

Apart from isolated singular points the function f may have a non-zero logarithmic residue at its zeros. Let $a \in \mathbb{C}$ be a zero of order n of a function f holomorphic at a. Then we have $f(z) = (z-a)^n \phi(z)$ in a neighborhood U_a of a with the function ϕ holomorphic and different from zero in U_a. Therefore we have in U_a

$$\frac{f'(z)}{f(z)} = \frac{n(z-a)^{n-1}\phi(z) + (z-a)^n \phi'(z)}{(z-a)^n \phi(z)} = \frac{1}{z-a} \cdot \frac{n\phi(z) + (z-a)\phi'(z)}{\phi(z)}$$

with the second factor holomorphic in U_a. Hence it may be expanded into the Taylor series with the zero order term equal to n. Therefore we have in U_a

$$\frac{f'(z)}{f(z)} = \frac{1}{z-a}\left\{n + c_1(z-a) + c_2(z-a)^2 + \ldots\right\} = \frac{n}{z-a} + c_1 + c_2(z-a) + \ldots \tag{1.2}$$

This shows that the logarithmic derivative has a pole of order one with residue equal to n at the zero of order n of f: *the logarithmic residue at a zero of a function is equal to the order of this zero.*

If a is a pole of f of the order p then $1/f$ has a zero of order p at this point. Observing that

$$\frac{f'(z)}{f(z)} = -\frac{d}{dz}\mathrm{Ln}\frac{1}{f(z)},$$

and using (1.2) we conclude that the logarithmic derivative has residue equal to $-p$ at a pole of order p: *the logarithmic residue at a pole is equal to the order of this pole with the minus sign.*

Those observations allow to compute the number of zeros and poles of meromorphic functions. We adopt the convention that a pole and a zero are counted as many times as their order is.

Theorem 1.1 *Let the function f be meromorphic in a domain $D \subset \mathbb{C}$ and let G be a domain properly contained in D with the boundary ∂G that is a continuous curve. Let us assume that ∂G contains neither poles nor zeros of f and let N and P be the total number of zeros and poles of f in the domain G, then*

$$N - P = \frac{1}{2\pi i}\int_{\partial G}\frac{f'(z)}{f(z)}dz. \tag{1.3}$$

Proof. The function f has only finitely many poles $a_1, \dots, a_l$ and zeros $b_1, \dots, b_m$ in G since G is properly contained in D. The function $g = f'/f$ is holomorphic in a neighborhood of ∂G since the boundary of G does not contain poles or zeros. Applying the Cauchy theorem on residues to g we find

$$\frac{1}{2\pi i}\int_{\partial G}\frac{f'}{f}dz = \sum_{\nu=1}^{l}\mathrm{res}_{a_\nu} g + \sum_{\nu=1}^{m}\mathrm{res}_{b_\nu} g. \tag{1.4}$$

However, according to our previous remark,

$$\mathrm{res}_{a_\nu} g = n_\nu, \quad \mathrm{res}_{b_\nu} g = p_\nu.$$

Here n_ν and p_ν are the order of zero a_ν and pole b_ν, respectively. Using this in (1.4) and counting the multiplicities of zeros and poles we obtain (1.3) since $N = \sum n_\nu$ and $P = \sum p_\nu$. $\square$

Exercise 1.2 Let the function f satisfy assumptions of Theorem 1.1 and let g be holomorphic in $\bar{G}$. Show that then

$$\frac{1}{2\pi i}\int_{\partial G} g(z)\frac{f'(z)}{f(z)}dz = \sum_{k=1}^{l} g(a_k) - \sum_{k=1}^{m} g(b_m), \tag{1.5}$$

where the first some includes all the zeros and the second all the poles of f in G. This generalizes Theorem 1.1 that follows from (1.5) when $g \equiv 1$.

The theorem that we have just proved has a geometric interpretation. Let us parameterize ∂G as $z = z(t)$, $\alpha \le t \le \beta$ and denote by $\Phi(t)$ the anti-derivative of $\frac{f'}{f}$ along this path. The Newton-Leibnitz formula implies that

$$\int_{\partial G} \frac{f'(z)}{f(z)} dz = \Phi(\beta) - \Phi(\alpha). \tag{1.6}$$

However, clearly, $\Phi(t) = \ln[f(z(t))]$, where ln denotes any branch of the logarithm that varies continuously along the path ∂G. It suffices to choose a branch of $\arg f$ that varies continuously along ∂G since $\mathrm{Ln} f = \ln|f| + i\mathrm{Arg} f$ and the function $\ln|f|$ is single-valued. The increment of $\ln|f|$ along a closed path ∂G is equal to zero and thus

$$\Phi(\beta) - \Phi(\alpha) = i\{\arg f(z(\beta)) - \arg f(z(\alpha))\}.$$

We denote the increment of the argument of f in the right side by $\Delta_{\partial G} f$ and re-write (1.6) as

$$\int_{\partial G} \frac{f'}{f} dz = i\Delta_{\partial G} \arg f.$$

Theorem 1.1 may now be expressed as

Theorem 1.3 *(The argument principle) Under the assumptions of Theorem 1.1 the difference between the number of zeros N and the number of poles P of a function f in a domain G is equal to the increment of the argument of this function along the oriented boundary of G divided by 2π:*

$$N - P = \frac{1}{2\pi} \Delta_{\partial G} \arg f. \tag{1.7}$$

Geometrically the right side of (1.7) is the total number of turns the vector $w = f(z)$ makes around $w = 0$ as z varies along ∂G. Let us denote by ∂G^* the image of ∂G under the map f, that is, the path $w = f(z(t))$, $\alpha \le t \le \beta$. Then this number is equal to the total number of times the vector w rotates around $w = 0$ as it varies along ∂G^*. This number is called the winding number of ∂G^* around $w = 0$, we will denote it by $\mathrm{ind}_0 \partial G^*$. The argument principle states that

$$N - P = \frac{1}{2\pi} \Delta_{\partial G} \arg f = \mathrm{ind}_0 \partial G^*. \tag{1.8}$$

Remark 1.4 *We may consider the a-points of f, solutions of $f(z) = a$ and not only its zeros: it suffices to replace f by $f(z) - a$ in our arguments. If ∂G contains neither poles nor a-points of f then*

$$N_a - P = \frac{1}{2\pi i} \int_{\partial G} \frac{f'(z)}{f(z) - a} dz = \frac{1}{2\pi} \Delta_{\partial G} \arg\{f(z) - a\}, \tag{1.9}$$

where N_a is the number of a-points of f in the domain D. Passing to the plane $w = f(z)$ and introducing the index of the path ∂G^* around the point a we may re-write (1.9) as

$$N_a - P = \frac{1}{2\pi} \Delta_{\partial G} \arg\{f(z) - a\} = ind_a \partial G^*. \tag{1.10}$$

The next theorem is an example of the application of the argument principle.

Theorem 1.5 *(Rouche[1]) Let the functions f and g be holomorphic in a closed domain $\bar{G}$ with a continuous boundary ∂G and let*

$$|f(z)| > |g(z)| \quad \textit{for all } z \in \partial G. \tag{1.11}$$

Then the functions f and $f+g$ have the same number of zeros in G.

Proof. Assumption (1.11) shows that neither f nor $f+g$ vanish on ∂G and thus the argument principle might be applied to both of these functions. Moreover, since $f \neq 0$ on ∂G, we have $f + g = f\left(1 + \dfrac{g}{f}\right)$ and thus we have with the appropriate choice of a branch of the argument:

$$\Delta_{\partial G} \arg(f+g) = \Delta_{\partial G} \arg f + \Delta_{\partial G} \arg\left(1 + \frac{g}{f}\right). \tag{1.12}$$

However, since $\left|\dfrac{g}{f}\right| < 1$ on ∂G, the point $\omega = \dfrac{g}{f}$ lies in $\{|\omega| < 1\}$ for all $z \in \partial G$. Therefore the vector $w = 1 + \omega$ may not turn around zero and hence the second term in the right side of (1.12) vanishes. Therefore, $\Delta_{\partial G} \arg(f+g) = \Delta_{\partial G} \arg f$ and the argument principle implies the statement of the theorem. $\square$

The Rouche theorem is useful in counting the zeros of holomorphic functions. In particular it implies the main theorem of algebra in a very simple way.

Theorem 1.6 *Any polynomial P_n of degree n has exactly n roots in $\mathbb{C}$.*

Proof. All zeros of P_n must lie in a disk $\{|z| < R\}$ since P_n has a pole at infinity. Let $P_n = f + g$ where $f = a_0 z^n$, $a_0 \neq 0$ and $g = a_1 z^{n-1} + \cdots + a_n$, then, possibly after increasing R, we may assume that $|f| > |g|$ on $\{|z| = R\}$ since $|f| = |a_0|R^n$ while g is a polynomial of degree less than n. The Rouche theorem implies that P_n has as many roots in $\{|z| < R\}$ as $f = a_0 z^n$, that is, exactly n of them. $\square$

Exercise 1.7 1. Find the number of roots of the polynomial $z^4 + 10z + 1$ in the annulus $\{1 < |z| < 2\}$.
2. Show that any polynomial with real coefficients may be decomposed as a product of linear and quadratic factors with real coefficients.

1.2 The Open Mapping Theorem

This is the name of the following basic

Theorem 1.8 [2] *If a function f holomorphic in a domain D is not equal identically to a constant then the image $D^* = f(D)$ is also a domain.*

[1] Eugene Rouche (1832-1910) was a French mathematician.
[2] This theorem was proved by Riemann in 1851.

Proof. We have to show that D^* is connected and open. Let w_1 and w_2 be two arbitrary points in D^* and let z_1 and z_2 be some pre-images of w_1 and w_2, respectively. Since the domain D is path-wise connected there exists a path $\gamma : [\alpha, \beta] \to D$ that connects z_1 and z_2. Its image $\gamma^* = f \circ \gamma$ connects w_1 and w_2 and is a path since the function f is continuous. Moreover, it is clearly contained in D^* and hence the set D^* is path-wise connected.

Let w_0 be an arbitrary point in D^* and let z_0 be a pre-image of w_0. There exists a disk $\{|z - z_0| < r\}$ centered at z_0 that is properly contained in D since D is open. After decreasing r we may assume that $\{|z - z_0| \leq r\}$ contains no other w_0-points of f except z_0: since $f \neq$ const its w_0 points are isolated in D. We denote by $\gamma = \{|z - z_0| = r\}$ the boundary of this disk and let

$$\mu = \min_{z \in \gamma} |f(z) - w_0|. \tag{1.13}$$

Clearly $\mu > 0$ since the continuous function $|f(z) - w_0|$ attains its minimum on γ, so that if $\mu = 0$ then there would exist a w_0-point of f on γ contrary to our construction of the disk.

Let us now show that the set $\{|w - w_0| < \mu\}$ is contained in D^*. Indeed, let w_1 be an arbitrary point in this disk, that is, $|w_1 - w_0| < \mu$. Then we have

$$f(z) - w_1 = f(z) - w_0 + (w_0 - w_1), \tag{1.14}$$

and, moreover, $|f(z) - w_0| \geq \mu$ on γ. Then, since $|w_0 - w_1| < \mu$, the Rouche theorem implies that the function $f(z) - w_1$ has as many roots inside γ as $f(z) - w_0$. Hence it has at least one zero (the point z_0 may be a zero of order higher than one of $f(z) - w_0$). Thus the function f takes the value w_1 and hence $w_1 \in D^*$. However, w_1 is an arbitrary function in the disk $\{|w - w_0| < \mu\}$ and hence this whole disk is contained in D^* so that D^* is open. □

Exercise 1.9 Let f be holomorphic in $\{\text{Im} z \geq 0\}$, real on the real axis and bounded. Show that $f \equiv$ const.

A similar but more detailed analysis leads to the solution of the problem of local inversion of holomorphic functions. This problem is formulated as follows.

A holomorphic function $w = f(z)$ is defined at z_0, find a function $z = g(w)$ analytic at $w_0 = f(z_0)$ so that $g(w_0) = z_0$ and $f(g(w)) = w$ in a neighborhood of w_0.

We should distinguish two cases in the solution of this problem:

I. The point z_0 is not a critical point: $f'(z_0) \neq 0$. As in the proof of the open mapping theorem we choose a disk $\{|z - z_0| \leq r\}$ that contains no w_0-points except z_0, and define μ according to (1.13). Let w_1 be an arbitrary point in the disk $\{|w - w_0| < \mu\}$. Then the same argument (using (1.14) and the Rouche theorem) shows that the function f takes the value w_1 as many times as w_0. However, the value w_0 is taken only once and, moreover, z_0 is a simple zero of $f(z) - w_0$ since $f'(z_0) \neq 0$.

Therefore the function f takes all values in the disk $\{|w - w_0| < \mu\}$ once in the disk $\{|z - z_0| < r\}$. In other words, the function f is a local bijection at z_0.

Then the function $z = g(w)$ is defined in the disk $\{|w - w_0| < r\}$ so that $g(w_0) = z_0$ and $f \circ g(w) = w$. Furthermore, derivative $g'(w)$ exists at every point of the disk $\{|w - w_0| < r\}$:

$$g'(w) = \frac{1}{f'(z)} \tag{1.15}$$

and thus g is holomorphic in this disk[3].

II. The point z_0 is a critical point: $f'(z_0) = \cdots = f^{(p-1)}(z_0) = 0$, $f^{(p)} \neq 0$, $p \geq 2$. Repeating the same argument as before choosing a disk $\{|z - z_0| < r\}$ that contains neither w_0-points of f nor zeros of the derivative f' (we use the uniqueness theorem once again). As before, we choose $\mu > 0$, take an arbitrary point w_1 in the disk $\{|w-w_0| < \mu\}$ and find that f takes the value w_1 as many times as w_0. However, in the present case the w_0-point z_0 has multiplicity p: z_0 is a zero of order p of $f(z) - w_0$. Furthermore, since $f'(z) \neq 0$ for $0 < |z - z_0| < r$ the value w_1 has to be taken at p different points. Therefore, the function f takes each value p times in $\{|z - z_0| < r\}$.

The above analysis implies the following

Theorem 1.10 *Condition $f'(z_0) \neq 0$ is necessary and sufficient for the local invertibility of a holomorphic function f at the point z_0.*

Remark 1.11 The general inverse function theorem of the real analysis implies that the assumption $f'(z_0) \neq 0$ is sufficient for the local invertibility since the Jacobian $J_f(z) = |f'(z)|^2$ of the map $(x, y) \to (u, v)$ is non-zero at this point. However, for an arbitrary differentiable map to be locally invertible one needs not $J_f(z) \neq 0$ to hold. This may be seen on the example of the map $f = x^3 + iy$ that has Jacobian equal to zero at $z = 0$ but that is nevertheless one-to-one.

Remark 1.12 The local invertibility condition $f'(z) \neq 0$ for all $z \in D$ is not sufficient for the global invertibility of the function in the whole domain D. This may be seen on the example of $f(z) = e^z$ that is locally invertible at every point in $\mathbb{C}$ but is not one-to-one in any domain that contains two points that differ by $2k\pi i$ where $k \neq 0$ is an integer.

We have described above a qualitative solution of the problem of local invertibility. Methods of the theory of analytic functions also allow to develop an effective quantitative solution of this problem. Let us consider for simplicity the case $f'(z_0) \neq 0$.

Let us construct as before the disks $\{|z - z_0| \leq r\}$ and $\{|w - w_0| < \mu\}$. Given a fixed w in the latter we consider the function $h(\zeta) = \dfrac{\zeta f'(\zeta)}{f(\zeta) - w}$. It is holomorphic everywhere in the former disk except possibly at the point $z = g(w)$ where g is the inverse of the function f. The residue of h at this point (a pole of multiplicity one) is equal to z. Therefore, according to the Cauchy theorem on residues we have

$$z = \frac{1}{2\pi i} \int_\gamma \frac{\zeta f'(\zeta)}{f(\zeta) - w} d\zeta, \tag{1.16}$$

[3]Expression (1.15) shows that in order for derivative to exist we need $f' \neq 0$. Using continuity of f' we may conclude that $f' \neq 0$ in the disk $\{|z - z_0| < r\}$, possibly decreasing r if needed.

where $\gamma = \{|\zeta - z_0| = r\}$.

The integral in the right side depends on w so that we have obtained the integral representation of the inverse function $g(w)$. We may use it in order to obtain the Taylor expansion of the function g in the same way as we used the Cauchy integral formula in order to obtain the Taylor expansion of a holomorphic function. We have

$$\frac{1}{f(\zeta) - w} = \frac{1}{f(\zeta) - w_0} \cdot \frac{1}{1 - \dfrac{w - w_0}{f(\zeta) - w_0}} = \sum_{n=0}^{\infty} \frac{(w - w_0)^n}{(f(\zeta) - w_0)^{n+1}}.$$

This series converges uniformly in ζ on the circle γ (we have $|f(\zeta) - w_0| \geq \mu$ while $|w - w_0| < \mu$). Multiplying this expansion by $\dfrac{\zeta f'(\zeta)}{2\pi i}$ and integrating term-wise along γ we obtain

$$z = g(w) = \sum_{n=0}^{\infty} d_n (w - w_0)^n, \tag{1.17}$$

where

$$d_n = \frac{1}{2\pi i} \int_\gamma \frac{\zeta f'(\zeta) d\zeta}{(f(\zeta) - w_0)^{n+1}}, \quad n = 0, 1, \ldots$$

We clearly have $d_0 = z_0$, while we may integrate by parts in the above integral when $n \geq 1$ to get

$$d_n = \frac{1}{2\pi i n} \int_\gamma \frac{d\zeta}{(f(\zeta) - w_0)^n}.$$

The integrand has pole of order n at the point z_0. We may find the residue at this point to obtain the final expression for the coefficients:

$$d_0 = z_0, \quad d_n = \frac{1}{n!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} \left(\frac{z - z_0}{f(z) - w_0} \right)^n, \quad n = 1, 2, \ldots \tag{1.18}$$

The series (1.17) may be effectively used to invert holomorphic functions.

Example 1.13 Let us find the inverse function of $f(z) = ze^{-az}$ at the point $w_0 = 0$ that corresponds to $z_0 = 0$. Using expression (1.18) we obtain

$$d_n = \frac{1}{n!} \lim_{z \to 0} \frac{d^{n-1}}{dz^{n-1}} \left(\frac{z}{f(z)} \right)^n = \frac{(an)^{n-1}}{n!}.$$

The inverse function has the representation

$$g(w) = \sum_{n=1}^{\infty} \frac{(an)^{n-1}}{n!} w^n.$$

1.3 The maximum modulus principle and the Schwartz lemma

The maximum modulus principle is expressed by the following theorem.

Theorem 1.14 *If the function f is holomorphic in a domain D and its modulus $|f|$ achieves its (local) maximum at a point $z_0 \in D$ then f is constant.*

Proof. We use the open mapping theorem. If $f \neq \text{const}$ then it maps z_0 into a point w_0 of the domain D^*. There exists a disk $\{|w - w_0| < \mu\}$ centered at w_0 that is contained in D^*. There must be a point w_1 in this disk so that $|w_1| > |w_0|$. The value w_1 is taken by the function f in a neighborhood of the point z_0 which contradicts the fact that $|f|$ achieves its maximum at this point. □

Taking into account the properties of continuous functions on a closed set the maximum modulus principle may be reformulated as

Theorem 1.15 *If a function f is holomorphic in a domain D and continuous in $\bar{D}$ then $|f|$ achieves its maximum on the boundary ∂D.*

Proof. If $f = \text{const}$ in D (and hence in $\bar{D}$ by continuity) the statement is trivial. Otherwise if $f \neq \text{const}$ then $|f|$ may not attain its maximum at the points of D. However, since this maximum is attained in $\bar{D}$ it must be achieved on ∂D. □

Exercise 1.16 1. Let $P(z)$ be a polynomial of degree n in z and let $M(r) = \max\limits_{|z|=r} |P(z)|$. Show that $M(r)/r^n$ is a decreasing function.

2. Formulate and prove the maximum principle for the real part of a holomorphic function.

A similar statement for the minimum of modulus is false in general. This may be seen on the example of the function $f(z) = z$ in the disk $\{|z| < 1\}$ (the minimum of $|f|$ is attained at $z = 0$). However, the following theorem holds.

Theorem 1.17 *Let a function f be holomorphic in a domain D and not vanish anywhere in D. Then $|f|$ may attain its (local) minimum in D only if $f = \text{const}$.*

For the proof of this theorem it suffices to apply Theorem 1.14 to the function $g = 1/f$ that is holomorphic since $f \neq 0$.

Exercise 1.18 1. Let a function f be holomorphic in $U = \{|z| < 1\}$ and continuous in $\bar{U}$. Assume also that $f \neq 0$ anywhere in U and, moreover, that $|f| = 1$ on $\{|z| = 1\}$. Show that then $f = \text{const}$.

2. Let the functions f and g be holomorphic in U and continuous in $\bar{U}$. Show that $|f(z)| + |g(z)|$ attains its maximum on $\{|z| = 1\}$. Hint: consider the function $h = e^{i\alpha} f + e^{i\beta} g$ with suitably chosen constants α and β.

A simple corollary of the maximum modulus principle is

Lemma 1.19 *(The Schwartz lemma[4]) Let a function f be holomorphic in the unit disk $\{|z| \leq 1\}$, satisfy $|f(z)| \leq 1$ for all $z \in U$ and $f(0) = 0$. Then we have*

$$|f(z)| \leq |z| \tag{1.19}$$

for all $z \in U$. Moreover, if the equality in (1.19) holds for at least one $z \neq 0$ then it holds everywhere in U and in this case $f(z) = e^{i\alpha} z$, where α is a real constant.

Proof. Consider the function $\phi(z) = f(z)/z$, it is holomorphic in U since $f(0) = 0$. Let $U_r = \{|z| < r\}$, $r < 1$ be an arbitrary disk centered at zero. The function $\phi(z)$ attains its maximum in U_r on its boundary $\gamma_r = \{|z| = r\}$ according to Theorem 1.15. However, we have $|\phi| \leq 1/r$ on γ_r since $|f| \leq 1$ by assumption. Therefore we have

$$|\phi(z)| \leq 1/r \tag{1.20}$$

everywhere in U_r. We fix $z \in U$ and observe that $z \in U_r$ for $r > |z|$. Therefore (1.20) holds for any given z with all $r > |z|$. We let $r \to 1$, and passing to the limit $r \to 1$ we obtain $|\phi(z)| \leq 1$ or $|f(z)| \leq |z|$. This proves the inequality (1.19).

Let us assume that equality in (1.19) holds for some $z \in U$, then $|\phi|$ attains its maximum equal to 1 at this point. Then ϕ is equal to a constant so that $\phi(z) = e^{i\alpha}$ and $f(z) = e^{i\alpha} z$. □

The Schwartz lemma implies that a holomorphic map f that maps the disk $\{|z| < 1\}$ into the disk $\{|w| < 1\}$ and that takes the center to the center, maps any circle $\{|z| = r\}$ inside the disk $\{|w| < r\}$. The image of $\{|z| = r\}$ may intersect $\{|w| = r\}$ if and only f is a rotation around $z = 0$.

Exercise 1.20 *1. Show that under the assumptions of the Schwartz lemma we have $|f'(0)| \leq 1$ and equality is attained if and only if $f(z) = e^{i\alpha} z$.*
2. Let $f \in \mathcal{O}(D)$, $f : U \to U$ and $f(0) = \cdots = f^{(k-1)}(0) = 0$. Show that then $|f(z)| \leq |z|^k$ for all $z \in U$.

2 The Riemann Theorem

Any holomorphic one-to-one function defined in a domain D defines a conformal map of this domain since the above assumptions imply that f has no critical points in D. We have encountered such maps many times before. Here we consider a more difficult and important for practical purposes problem:

Given two domains D_1 and D_2 find a one-to-one conformal map $f : D_1 \to D_2$ of one of these domains onto the other.

[4]Hermann Schwartz (1843-1921) was a German mathematician, a student of Weierstrass. This important lemma has appeared in his papers of 1869-70.

2.1 Conformal isomorphisms and automorphisms

Definition 2.1 *A conformal one-to-one map of a domain D_1 onto D_2 is said to be a (conformal) isomorphism, while the domains D_1 and D_2 that admit such a map are isomorphic (or conformally equivalent). Isomorphism of a domain onto itself is called a (conformal) automorphism.*

It is easy to see that the set of all automorphisms $\phi : \ D \to D$ of a domain D forms a group that is denoted Aut$D$. The group operation is the composition $\phi_1 \circ \phi_2$, the unity is the identity map and the inverse is the inverse map $z = \phi^{-1}(w)$.

The richness of the group of automorphisms of a domain allows to understand the richness of the family of the conformal maps onto it of a different domain, as may be seen from the next

Theorem 2.2 *Let $f_0 : \ D_1 \to D_2$ be a fixed isomorphism. Then any other isomorphism of D_1 onto D_2 has the form*

$$f = \phi \circ f_0 \tag{2.1}$$

where ϕ is an automorphism of D_2.

Proof. First, it is clear that all maps of the form of the right side of (2.1) are isomorphisms from D_1 onto D_2. Furthermore, if $f : \ D_1 \to D_2$ is an arbitrary isomorphism then $\phi = f \circ f_0^{-1}$ is a conformal map of D_2 onto itself, that is, an automorphism of D_2. Then (2.1) follows. □

In the sequel we will only consider simply connected domains D. We will distinguish three special domains that we will call canonical: the closed plane $\overline{\mathbb{C}}$, the open plane $\mathbb{C}$ and the unit disk $\{|z| < 1\}$. We have previously found the group of all fractional-linear automorphisms of those domains. However, the following theorem holds.

Theorem 2.3 *Any conformal automorphism of a canonical domain is a fractional-linear transformation.*

Proof. Let ϕ be automorphism of $\overline{\mathbb{C}}$. There exists a unique point z_0 that is mapped to infinity. Therefore ϕ is holomorphic everywhere in $\mathbb{C}$ except at z_0 where it has a pole. This pole has multiplicity one since in a neighborhood of a pole of higher order the function ϕ could not be one-to-one. Therefore since the only singularities of ϕ are poles ϕ is a rational function. Since it has only one simple pole, ϕ should be of the form $\phi(z) = \dfrac{A}{z - z_0} + B$ if $z_0 \neq \infty$ and $\phi(z) = Az + B$ if $z_0 = \infty$. The case of the open complex plane $\mathbb{C}$ is similar.

Let ϕ be an arbitrary automorphism of the unit disk U. Let us denote $w_0 = \phi(0)$ and consider a fractional linear transformation

$$\lambda : \ w \to \frac{w - w_0}{1 - \bar{w}_0 w}$$

of the disk U that maps w_0 into 0. The composition $f = \lambda \circ \phi$ is also an automorphism of U so that $f(0) = 0$. Moreover, $|f(z)| < 1$ for all $z \in U$. Therefore the Schwartz lemma implies that $|f(z)| \le |z|$ for all $z \in U$. However, the inverse map $z = f^{-1}(w)$ also satisfies the assumptions of the Schwartz lemma and hence $|f^{-1}(w)| \le |w|$ for all $w \in U$ that in turn implies that $|z| \le |f(z)|$ for all $z \in U$. Thus $|f(z)| = |z|$ for all $z \in U$ so that the Schwartz lemma implies that $f(z) = e^{i\alpha} z$. Then $\phi = \lambda^{-1} \circ f = \lambda^{-1}(e^{i\alpha} z)$ is also a fractional-linear transformation. □

Taking into account our results from Chapter 1 we obtain the complete description of all conformal automorphisms of the canonical domains.

(I) The closed complex plane:

$$\mathrm{Aut}\overline{\mathbb{C}} = \left\{ z \to \frac{az+b}{cz+d},\ ad - bc \neq 0 \right\}. \tag{2.2}$$

(II) The open plane:

$$\mathrm{Aut}\mathbb{C} = \{ z \to az + b,\ a \neq 0 \}. \tag{2.3}$$

(III) The unit disk:

$$\mathrm{Aut}U = \left\{ z \to e^{i\alpha} \frac{z-a}{1-\bar{a}z},\ |a| < 1, \alpha \in \mathbb{R} \right\}. \tag{2.4}$$

It is easy to see that different canonical domains are not isomorphic to each other. Indeed, the closed complex plane $\overline{\mathbb{C}}$ is not even homeomorphic to $\mathbb{C}$ and U and hence it may not be mapped conformally onto these domains. The domains $\mathbb{C}$ and U are homeomorphic but there is no conformal map of $\mathbb{C}$ onto U since such a map would have to be realized by an entire function such that $|f(z)| < 1$ which has then to be equal to a constant by the Liouville theorem.

A domain that has no boundary (boundary is an empty set) coincides with $\overline{\mathbb{C}}$. Domains with boundary that consists of one point are the plane $\overline{\mathbb{C}}$ without a point which are clearly conformally equivalent to $\mathbb{C}$ (even by a fractional linear transformation). The main result of this section is the Riemann theorem that asserts that any simply connected domain D with a boundary that contains more than one point (and hence infinitely many points since boundary of a simply connected domain is connected) is conformally equivalent to the unit disk U.

This theorem will be presented later while at the moment we prove the uniqueness theorem for conformal maps.

Theorem 2.4 *If a domain D is conformally equivalent to the unit disk U then the set of all conformal maps of D onto U depends on three real parameters. In particular there exists a unique conformal map f of D onto U normalized by*

$$f(z_0) = 0, \quad \arg f'(z_0) = \theta, \tag{2.5}$$

where z_0 is an arbitrary point of D and θ is an arbitrary real number.

Proof. The first statement follows from Theorem 2.2 since the group Aut$U$ depends on three real parameters: two coordinates of the point a and the number α in (2.4).

In order to prove the second statement let us assume that there exist two maps f_1 and f_2 of the domain D onto U normalized as in (2.5). Then $\phi = f_1 \circ f_2^{-1}$ is an automorphism of U such that $\phi(0) = 0$ and $\arg f'(0) = 0$. Expression (2.4) implies that then $a = 0$ and $\alpha = 0$, that is $\phi(z) = z$ and $f_1 = f_2$.

Exercise 2.5 Show that there exists no more than one conformal map of a domain D onto the unit disk U that is continuous in $\bar{D}$ and is normalized by one of the following two conditions: (i) the images of one internal and one boundary point in D are prescribed, and (ii) the images of three boundary points of D are prescribed.

In order to prove the Riemann theorem we need to develop some methods that are useful in other areas of the complex analysis.

2.2 The compactness principle

Definition 2.6 *A family $\{f\}$ of functions defined in a domain D is locally uniformly bounded if for any domain K properly contained in D there exists a constant $M = M(K)$ such that*

$$|f(z)| \leq M \text{ for all } z \in K \text{ and all } f \in \{f\}. \tag{2.6}$$

A family $\{f\}$ is locally equicontinuous if for any $\varepsilon > 0$ and any domain K properly contained in D there exists $\delta = \delta(\varepsilon, K)$ so that

$$|f(z') - f(z'')| < \varepsilon \tag{2.7}$$

for all $z', z'' \in K$ so that $|z' - z''| < \delta$ and all $f \in \{f\}$.

Theorem 2.7 *If a family $\{f\}$ of holomorphic functions in a domain D is locally uniformly bounded then it is locally equicontinuous.*

Proof. Let K be a domain properly contained in D. Let us denote by 2ρ the distance between the closed sets $\bar{K}$ and ∂D[5] and let

$$K^{(\rho)} = \cup_{z_0 \in K}\{z : \ |z - z_0| < \rho\}$$

be a ρ-enlargement of K. The set $K^{(\rho)}$ is properly contained in D and thus there exists a constant M so that $|f(z)| \leq M$ for all $z \in K^{(\rho)}$ and $f \in \{f\}$. Let z' and z'' be arbitrary points in K so that $|z' - z''| < \rho$. The disk $U_\rho = \{z : \ |z - z'| < \rho\}$ is contained in $K^{(\rho)}$ and hence $|f(z) - f(z')| < 2M$ for all $z \in U_\rho$. The mapping $\zeta = \dfrac{1}{\rho}(z - z')$ maps U_ρ onto the disk $|\zeta| < 1$ and the function $g(\zeta) = \frac{1}{2M}\{f(z' + \zeta\rho) - f(z')\}$ satisfies the assumptions of the Schwartz lemma.

This lemma implies that $g(\zeta)| \leq |\zeta|$ for all ζ, $|\zeta| < 1$, which means

$$|f(z) - f(z')| \leq \frac{2M}{\rho}|z - z'| \text{ for all } z \in U_\rho. \tag{2.8}$$

[5]Note that ρ is positive except when $D = \mathbb{C}$ or $\overline{\mathbb{C}}$ when the statement of the theorem is trivial.

Given $\varepsilon > 0$ we choose $\delta = \min\left(\rho, \dfrac{\varepsilon\rho}{2M}\right)$ and obtain from (2.8) that $|f(z') - f(z'')| < \varepsilon$ for all $f \in \{f\}$ provided that $|z' - z''| < \delta$. □

Definition 2.8 *A family of functions $\{f\}$ defined in a domain D is compact in D if any sequence f_n of functions of this family has a subsequence f_{n_k} that converges uniformly on any domain K properly contained in D.*

Theorem 2.9 *(Montel[6]) If a family of functions $\{f\}$ holomorphic in a domain D is locally uniformly bounded then it is compact in D.*

Proof. (a) We first show that if a sequence $f_n \subset \{f\}$ converges at every point of an everywhere dense set $E \subset D$ then it converges uniformly on every compact subset K of D. We fix $\varepsilon > 0$ and the set K. Using equicontinuity of the family $\{f\}$ we may choose a partition of D into squares with sides parallel to the coordinate axes and so small that that for any two points $z', z'' \in K$ that belong to the same square and any $f \in \{f\}$ we have

$$|f(z') - f(z'')| < \frac{\varepsilon}{3}. \tag{2.9}$$

The set K is covered by a finite number of such squares q_p, $p = 1, \ldots, P$. Each q_p contains a point $z_p \in E$ since the set E is dense in D. Moreover, since the sequence$\{f_n\}$ converges on E there exists N so that

$$|f_m(z_p) - f_n(z_p)| < \frac{\varepsilon}{3} \tag{2.10}$$

for all $m, n > N$ and all z_p, $p = 1, \ldots, P$.

Let now z be an arbitrary point in K. Then there exists a point z_p that belongs to the same square as z. We have for all $m, n > N$:

$$|f_m(z) - f_n(z)| \leq |f_m(z) - f_m(z_p)| + |f_m(z_p) - f_n(z_p)| + |f_n(z_p) - f_n(z)| < \varepsilon$$

due to (2.9) and (2.10). The Cauchy criterion implies that the sequence $\{z_n\}$ converges for all $z \in K$ and convergence is uniform on K.

(b) Let us show now that any sequence $\{f_n\}$ has a subsequence that converges at every point of a dense subset E of D. We choose E as the set $z = x + iy \in D$ with both coordinates $x$and y rational numbers. This set is clearly countable and dense in D, let $E = \{z_\nu\}_{\nu=1}^{\infty}$.

The sequence $f_n(z_1)$ is bounded and hence it has a converging subsequence $f_{k1} = f_{n_k}(z_1)$, $k = 1, 2, \ldots$. The sequence $f_{n1}(z_2)$ is also bounded so we may extract its subsequence $f_{k2} = f_{n_k1}$, $k = 1, 2, \ldots$. The sequence f_{n2} converges at least at the points z_1 and z_2. Then we extract a subsequence $f_{k3} = f_{n_k2}$ of the sequence $f_{n2}(z_3)$ so that f_{n3} converges at least at z_1, z_2 and z_3. We may continue this procedure indefinitely. It remains to choose the diagonal sequence

$$f_{11}, f_{22}, \ldots, f_{nn}, \ldots$$

[6]Paul Montel (1876-1937) was a French mathematician.

This sequence converges at any point $z_p \in E$ since by construction all its entries after index p belong to the subsequence f_{np} that converges at z_p.

Parts (a) and (b) together imply the statement of the theorem. □

The Montel theorem is often called the compactness principle.

Exercise 2.10 Show that any sequence $\{f_n\}$ of functions holomorphic in a domain D with $\text{Re} f_n \geq 0$ everywhere in D has a subsequence that converges locally uniformly either to a holomorphic function or to infinity.

Definition 2.11 *A functional J of a family $\{f\}$ of functions defined in a domain D is a mapping $J : \{f\} \to \mathbb{C}$, that is, $J(f)$ is a complex number. A functional J is continuous if given any sequence of functions $f_n \in \{f\}$ that converges uniformly to a function $f_0 \in \{f\}$ on any compact set $K \subset D$ we have*

$$\lim_{n\to\infty} J(f_n) = J(f_0).$$

Example 2.12 Let $\mathcal{O}(D)$ be the family of all functions f holomorphic in D and let a be an arbitrary point in D. Consider the p-th coefficient of the Taylor series in a:

$$c_p(f) = \frac{f^{(p)}(a)}{p!}.$$

This is a functional on the family $\mathcal{O}(D)$. Let us show that it is continuous. if $f_n \to f_0$ uniformly on every compact set $K \subset D$ we may let K be the circle $\gamma = \{|z-a| = r\} \subset D$. Then given any $\varepsilon > 0$ we may find N so that $|f_n(z) - f_0(z)| < \varepsilon$ for all $n > N$ and all $z \in \gamma$. The Cauchy formula for c_p

$$c_p = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)^{n+1}} dz$$

implies that

$$|c_p(f_n) - c_p(f_0)| \leq \frac{\varepsilon}{r^n}$$

for all $n > N$ which in turn implies the continuity of the functional $c_p(f)$.

Definition 2.13 *A compact family of functions $\{f\}$ is sequentially compact if the limit of any sequence f_n that converges uniformly on every compact subset $K \subset D$ belongs to the family $\{f\}$.*

Theorem 2.14 *Any functional J that is continuous on a sequentially compact family $\{f\}$ is bounded and attains its lowest upper bound. That is, there exists a function $f_0 \in \{f\}$ so that we have*

$$|J(f_0)| \geq |J(f)|$$

for all $f \in \{f\}$.

Proof. We let $A = \sup_{f\in\{f\}} |J(f)|$ - this is a number that might be equal to infinity. By definition of the supremum there exists a sequence $f_n \in \{f\}$ so that $|J(f_n)| \to A$. Since $\{f\}$ is a sequentially compact family there exists a subsequence f_{n_k} that converges to a function $f_0 \in \{f\}$. Continuity of the functional J implies that

$$|J(f_0)| = \lim_{k\to\infty} |J(f_{n_k})| = A.$$

This means that first $A < \infty$ and second, $|J(f_0)| \geq |J(f)|$ for all $f \in \{f\}$. □

We will consider below families of univalent functions in a domain D. The following theorem is useful to establish sequential compactness of such families.

Theorem 2.15 *(Hurwitz[7]) Let a sequence of functions f_n holomorphic in a domain D converge uniformly on any compact subset K of D to a function $f \neq$ const. Then if $f(z_0) = 0$ then given any disk $U_r = \{|z - z_0| < r\}$ there exists N so that all functions f_n vanish at some point in U_r when $n > N$.*

Proof. The Weierstrass theorem implies that f is holomorphic in D. The uniqueness theorem implies that there exists a punctured disk $\{0 < |z - z_0| \leq \rho\} \subset D$ where $f \neq 0$ (we may assume that $\rho < r$). We denote $\gamma = \{|z - z_0| = \rho\}$ and $\mu = \min_{z\in\gamma} |f(z)|$, and observe that $\mu > 0$. However, f_n converges uniformly to f on γ and hence there exists N so that

$$|f_n(z) - f(z)| < \mu$$

for all $z \in \gamma$ and all $n > N$. The Rouche theorem implies that for such n the function $f_n = f + (f_n - f)$ has as many zeros (with multiplicities) as f inside γ, that is, f_n has at least one zero inside U_ρ. □

Corollary 2.16 *If a sequence of holomorphic and univalent functions f_n in a domain D converges uniformly on every compact subset K of D then the limit function f is either a constant or univalent.*

Proof. Assume that $f(z_1) = f(z_2)$ but $z_1 \neq z - 2$, $z_{1,2} \in D$ and] $f \neq$ const. Consider a sequence of functions $g_n(z) = f_n(z) - f_n(z_2)$ and a disk $\{|z - z_1| < r\}$ with $r < |z_1 - z_2|$. The limit function $g(z)$ vanishes at the point z_1. Hence according to the Hurwitz theorem all functions f_n starting with some N vanish in this disk. This, however, contradicts the assumption that $f_n(z)$ are univalent. □

2.3 The Riemann theorem

Theorem 2.17 *Any simply connected domain D with a boundary that contains more than one point is conformally equivalent to the unit disk U.*

[7] Adolf Hurwitz (1859-1919) was a German mathematician, a student of Weierstrass.

Proof. The idea of the proof is as follows. Consider the family S of holomorphic and univalent functions f in D bounded by one in absolute value, that is, those that map D into the unit disk U. We fix a point $a \in D$ and look for a function f that maximizes the dilation coefficient $|f'(a)|$ at the point a. Restricting ourselves to a sequentially compact subset S_1 of S and using continuity of the functional $J(f) = |f'(a)|$ we may find a function f_0 with the maximal dilation at the point a. Finally we check that f_0 maps D onto U and not just into U as other functions in S.

Such a variational method when one looks for a function that realizes the extremum of a functional is often used in analysis.

(i) Let us show that there exists a holomorphic univalent function in D that is bounded by one in absolute value. By assumption the boundary ∂D contains at least two points α and β. The square root $\sqrt{\dfrac{z-\alpha}{z-\beta}}$ admits two branches ϕ_1 and ϕ_2 that differ by a sign. Each one of them is univalent in D[8] since the equality $\phi_\nu(z_1) = \phi_\nu(z_2)$ ($\nu = 1$ or 2) implies

$$\frac{z_1-\alpha}{z_1-\beta} = \frac{z_2-\alpha}{z_2-\beta} \tag{2.11}$$

which implies $z_1 = z_2$ since fractional linear transformations are univalent. The two branches ϕ_1 and ϕ_2 map D onto domains $D_1^* = \phi_1(D)$ and $D_2^* = \phi_2(D)$ that have no overlap. Otherwise there would exist two points $z_{1,2} \in D$ so that $\phi_1(z_1) = \phi_2(z_2)$ which would in turn imply (2.11) so that $z_1 = z_2$ and then $\phi_1(z_1) = -\phi_2(z_2)$. This is a contradiction since $\phi_\nu 9z) \neq 0$ in D.

The domain D_2^* contains a disk $\{|w - w_0| < \rho\}$. Hence ϕ_1 does not take values in this disk. Therefore the function

$$f_1(z) = \frac{\rho}{\phi_1(z) - w_0} \tag{2.12}$$

is clearly holomorphic and univalent in D and takes values inside the unit disk: we have $|f_1(z)| \leq 1$ for all $z \in D$.

(ii) Let us denote by S the family of functions that are holomorphic and univalent in D, and are bounded by one in absolute value. This family is not empty since it contains the function f_1. It is compact by the Montel theorem. The subset S_1 of the family S that consists of all functions $f \in S$ such that

$$|f'(a)| \geq |f_1'(a)| > 0 \tag{2.13}$$

at some fixed point $a \in D$ is sequentially compact. Indeed Corollary 2.16 implies that the limit of any sequence of functions $f_n \in S_1$ that converges on any compact subset K of D may be only a univalent function (and hence belong to S_1) or be a constant but the latter case is ruled out by (2.13).

Consider the functional $J(f) = |f'(a)|$ defined on S_1. It is a continuous functional as was shown in Example 2.12. Therefore there exists a function $f_0 \in S$ that attains its

[8] In general we may define a univalent branch of $\sqrt{\dfrac{z-a}{z-b}}$ in a domain D if neither a nor b are in D.

maximum, that is, such that

$$|f'(a)| \le |f_0'(a)| \tag{2.14}$$

for all $f \in S$.

(iii) The function $f_0 \in S_1$ maps D conformally into the unit disk U. Let us show that $f_0(a) = 0$. Otherwise, the function

$$g(z) = \frac{f_0(z) - f_0(a)}{1 - \overline{f_0(a)} f_0(z)}$$

would belong to S_1 and have $|g'(a)| = \frac{1}{1-|f_0(a)|^2}|f_0'(a)| > |f_0'(a)|$, contrary to the extremum property (2.14) of the function f.

Finally, let us show that f_0 maps D onto U. Indeed, let f_0 omit some value $b \in U$. Then $b \ne 0$ since $f_0(a) = 0$. However, the value $b^* = 1/b$ is also not taken by f_0 in D since $|b^*| > 1$. Therefore one may define in D a single valued branch of the square root

$$\psi(z) = \sqrt{\frac{f_0(z) - b}{1 - \bar{b} f_0(z)}} \tag{2.15}$$

that also belongs to S: it is univalent for the same reason as in the square root in part (i), and $|\psi(z)| \le 1$. However, then the function

$$h(z) = \frac{\psi(z) - \psi(a)}{1 - \overline{\psi(a)} \psi(z)}$$

also belongs to S. We have $|h'(a)| = \dfrac{1+|b|}{2\sqrt{|b|}}|f_0'(a)|$. However, $1 + |b| > 2\sqrt{|b|}$ since $|b| < 1$ and thus $h \in S_1$ and $|h'(a)| > |f_0'(a)|$ contrary to the extremal property of f_0. □

The Riemann theorem implies that any two simply connected domains D_1 and D_2 with boundaries that contain more than one point are conformally equivalent. Indeed, as we have shown there exist conformal isomorphisms $f_j : D_j \to U$ of these domains onto the unit disk. Then $f = f_2^{-1} \circ f_1$ is a conformal isomorphism between D_1 and D_2. Theorem 2.4 implies that an isomorphism $f : D_1 \to D_2$ is uniquely determined by a normalization

$$f(z_0) = w_0, \quad \arg f'(z_0) = \theta, \tag{2.16}$$

where $z_0 \in D_1$, $w_0 \in D_2$ and θ is a real number.