Appendix A

Practical implementation using Mathematica

The purpose of this appendix is to illustrate some of the set partitions and diagram formulae we have encountered. We do this through a practical implementation of these formulae by using the software program *Mathematica*[1 2]. The source code and more detailed examples are contained in a companion *Mathematica* notebook. This notebook can be downloaded by using the link:

http://extras.springer.com, Password: 978-88-470-1678-1

To view the notebook you may use the free *Mathematica Reader* which you can download from the Internet or you can use the *Mathematica* program itself. The *Mathematica* program (version 7.0 or later) must be installed on your computer or remotely through your system if you want to execute the source code.

A.1 General introduction

The filename of a *Mathematica* notebook ends with a ".nb". Place the notebook and the style file *StyleFile978-88-470-1678-1.nb* in the same folder. This style file formats the notebook nicely but it is not absolutely necessary to have it.

In Windows, double-clicking on the notebook will open the notebook and start *Mathematica* (or the *Mathematica Reader*). It is best to begin by executing the entire notebook because some functions depend on previously defined functions. To execute the entire notebook (on Windows), press Control-A (to select all the cells in the notebook), followed by Shift-Enter. (On a Mac, press Command-A followed by Shift-Enter.) These actions will execute all of the function definitions and examples in this notebook. This may take a little time. You will see "Running" on the top left of the

[1] *Mathematica* is a computational software program developed by Wolfram Research.
[2] For further references on the use of *Mathematica* in the context of multiple stochastic integrals, see for example, [54], [155].

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notebook. After the execution has completed, you will be able to apply the functions created in this notebook to new set partitions or to create additional examples of your own. You may type, paste, or outline a command and then press Shift-Enter to execute it. You may do this in the present notebook or, preferably, in a new notebook that you would open.

Please note that the execution of any of these functions requires the use of the *Combinatorica* package in *Mathematica* (which can be activated by executing the line `Needs["Combinatorica"]`. We automatically execute this package in this notebook.

To type a command, simply type the appropriate function and the desired argument and press Shift-Enter to execute the cell in which the command is located. *Mathematica* will label the command as a numbered input and the output will be numbered correspondingly. One can recall previous outputs with commands like %5, which would recover the fifth output in the current *Mathematica* session. Unexecuted cells have no input label.

Warning. Some of the functions in this notebook use a brute-force methodology. The number of set partitions associated to a positive integer n grows very rapidly, and so any brute-force methodology may suffer from combinatorial explosion. Unless you have lots of time and computational resources at your disposal, it may be better to investigate examples for which n is small.

Comment about In[ ] and Out[ ]. *Mathematica*, after execution of a command, automatically numbers it, for example, "In[1]:=". The output lines are also numbered, for example, the corresponding output line will start with: "Out[1]=". Here, we symbolically start a command input line with "*In:=*" and the corresponding output starts with "*Out*=" . Thus, in the sequel do not type, for example, "In:=" nor "In[269]:=", as this is automatically generated by *Mathematica* after you execute the command by pressing Shift-Enter.

Comments about the time it takes to evaluate a function. To evaluate the time it takes to simulate 1,000,000 realizations of a standard normal random variable, type:

In := RandomReal[NormalDistribution[0, 1], {1 000 000}]; // AbsoluteTiming
Out= 0.2187612, Null

The semicolon suppresses the display of the output (except for the timing result), which is generated by AbsoluteTiming. The first part of the output is the number of seconds it took *Mathematica* to generate 1,000,000 realizations of a standard normal random variable. The output was suppressed, which is why the symbol Null appears in the second position.

Comments about the function names. We typically give the functions long, informative names. In practical use, one may sometimes want to use a shorter name. For example, to have the function Chi do the same thing as the function CumulantVectorToMoments, set:

In := Chi = CumulantVectorToMoments

A few similar functions may have different names because they are used in different contexts. This is the case, for example, of the functions MeetSolve, PiStarMeet Solve and MZeroSets.

Comments about typing Greek symbols. To type symbols, such as μ and κ, either copy and paste them with the mouse from somewhere else in the notebook or type respectively \[Mu] and \[Kappa]. *Mathematica* converts these immediately to greek.

Comments about *Mathematica* notation. In *Mathematica*, a set partition is denoted in a particular way, for example, $\{\{1,2,4\},\{3\},\{5\}\}$ is a set partition of the set $\{1,2,3,4,5\}$. The trivial, or maximal, partition consisting of the entire set is denoted $\{\{1,2,3,4,5\}\}$ and the set partition consisting entirely of singletons is denoted $\{\{1\},\{2\},\{3\},\{4\},\{5\}\}$.

Comments about hard returns. *Mathematica* ignores hard returns. Therefore, if the input line is long, one can press "Enter" in order to subdivide it.

Comments about stopping the execution of a function. Hold down the "Alt" key and then press the "." key, or go to the menu at the top, choose "Evaluation" and then choose "Abort Evaluation."

Comments about the examples below. The examples are particularly simple. Their only purpose is to illustrate the *Mathematica* commands. The *Mathematica* output will sometimes not be listed below, because it is either too long or would require complicated formatting.

A.2 When Starting

Type first the following two commands:

In := Needs ["Combinatorica"] ; Off[First::first]

The first command loads the package *Combinatorica*. The second command shuts off an error message that does not impact any of the function outputs. These commands will be executed in sequence.

Note: These two commands are listed at the beginning of the notebook. They will be automatically executed if you execute the whole notebook.

A.3 Integer Partitions

A.3.1 IntegerPartitions

Find all nonnegative integers that sum up to an integer n, ignoring order; this is a built-in *Mathematica* function.

► Please refer to Section 2.1.

Example. Find all integer partitions of the number 5. The output $\{3, 1, 1\}$ should be read as the decomposition $5 = 3 + 1 + 1$.

In := IntegerPartitions[5]
Out= $\{\{5\}, \{4, 1\}, \{3, 2\}, \{3, 1, 1\}, \{2, 2, 1\}, \{2, 1, 1, 1\}, \{1, 1, 1, 1, 1\}\}$

Example. Count the number of integer partitions of the number 5.

In := Length[IntegerPartitions[5]]
Out= 7

A.3.2 IntegerPartitionsExponentRepresentation

This function represents the output of IntegerPartitions in exponent notation; namely, $1^{r_1}2^{r_2}\cdots n^{r_n}$, where r_i is the number of times that integer i appears in the decomposition, $i = 1, ..., n$.

Example. Find all integer partitions of the number 5, and express them in exponent notation. Thus $5 = 3 + 1 + 1$ is expressed as $\{1^2 2^0 3^1 4^0 5^0\}$ since there are 2 ones and 1 three.

In := IntegerPartitionsExponentRepresentation[5]
Out= $\{\{1^0 2^0 3^0 4^0 5^1\}, \{1^1 2^0 3^0 4^1 5^0\}, \{1^0 2^1 3^1 4^0 5^0\}, \{1^2 2^0 3^1 4^0 5^0\},$
$\{1^1 2^2 3^0 4^0 5^0\}, \{1^3 2^1 3^0 4^0 5^0\}, \{1^5 2^0 3^0 4^0 5^0\}\}$

A.4 Basic Set Partitions

► Please refer to Section 2.2.

A.4.1 SingletonPartition

Generates the singleton set partition of order n, namely $\hat{0} = \{\{1\}, \{2\}, \cdots, \{n\}\}$. This is a partition made up of singletons.

Example. Generate the singleton set partition of order 12.

In := SingletonPartition[12]
Out= $\{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}\}$

Example. Same example with grid display.

In := Grid[SetPartitions[3], Frame → All]

Out=

$\{1,2,3\}$		
$\{1\}$	$\{2,3\}$	
$\{1,2\}$	$\{3\}$	
$\{1,3\}$	$\{2\}$	
$\{1\}$	$\{2\}$	$\{3\}$

A.4.2 MaximalPartition

Generates the maximal set partition of order n, namely $\hat{1} = \{\{1, 2, ..., n\}\}$.

Example. Generate the maximal set partition of order 12.

In := MaximalPartition[12]
Out= $\{\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}\}$

A.4.3 SetPartitions

Enumerates all of the set partitions associated with a particular integer n, namely all set partitions of $\{1, 2, \cdots, n\}$; this is a built-in *Mathematica* function.

► Please refer to Section 2.4.

Example. Enumerate all set partitions of $\{1, 2, 3\}$.

In := SetPartitions[3]
Out= $\{\{\{1,2,3\}\},\ \{\{1\},\{2,3\}\},\ \{\{1,2\},\{3\}\},\ \{\{1,3\},\{2\}\},$
$\{\{1\},\{2\},\{3\}\}\}$

A.4.4 PiStar

Generates a set partition of $\{1, 2, \cdots, n_1 + \cdots + n_k\}$, with k blocks, the first of size n_1 and the last of size n_k, that is,

$$\pi^* = \{\{1, \cdots, n_1\}, \{n_1 + 1, \cdots, n_1 + n_2\}, \cdots, \{n_1 + \cdots + n_{k-1} + 1, \cdots, n\}\}.$$

► Please refer to Formula (6.1.1).

Example. Create a set partition π^* with blocks of size 1, 2, 4, 7, and 9.

In := PiStar[{1, 2, 4, 7, 9}]
Out= $\{\{1\}, \{2, 3\}, \{4, 5, 6, 7\}, \{8, 9, 10, 11, 12, 13, 14\}, \{15, 16, 17, 18, 19, 20, 21, 22, 23\}\}$

A.5 Operations with Set Partitions

A.5.1 PartitionIntersection

Finds the meet $\sigma \wedge \pi$ between two set partitions σ and π; each block of $\sigma \wedge \pi$ is a non-empty intersection between one block of σ and one block of π.

Example. Find the meet between the set partitions $\{\{1, 4, 5\}, \{2\}, \{3\}\}$ and $\{\{1, 2, 5\}, \{3\}, \{4\}\}$.

In := PartitionIntersection [$\{\{1, 4, 5\}, \{2\}, \{3\}\}, \{\{1, 2, 5\}, \{3\}, \{4\}\}$]
Out= $\{\{1, 5\}, \{2\}, \{3\}, \{4\}\}$

Example. If the partitions σ and π are not of the same order, then PartitionIntersection returns a message.

In := PartitionIntersection[SingletonPartition[8], {{1, 2, 5, 6, 7}, {3}, {4}}]
Out= Error

A.5.2 PartitionUnion

Finds the join $\sigma \vee \pi$ between two set partitions σ and π; that is, the union of blocks of σ and π that have at least one element in common.

Example. Find the join between the set partitions $\{\{1, 4, 5\}, \{2\}, \{3\}\}$ and $\{\{1, 2, 5\}, \{3\}, \{4\}\}$.

In := PartitionUnion [{{1,4,5},{2},{3}} , {{1,2,5},{3},{4}}]
Out= {{1, 2, 4, 5}, {3}}

A.5.3 MeetSolve

For a fixed set partition π , find all set partitions σ such that the meet of π and σ is the singleton partition $\hat{0}$, that is, $\sigma \wedge \pi = \hat{0}$.

► Please refer to Section 4.2.

Example. Find all set partitions σ such that the meet of σ and $\{\{1,4,5\},\{2\},\{3\}\}$ is the singleton partition $\hat{0}$.

In := MeetSolve$[\{\{1,4,5\},\{2\},\{3\}\}]$
Out= $\{\{\{1\},\{2,3,4\},\{5\}\},\ \{\{1\},\{2,5\},\{3,4\}\},\ \{\{1\},\{2,3,5\},\{4\}\},$
$\{\{1\},\ \{2,4\},\{3,5\}\},\{\{1,2\},\{3,4\},\{5\}\},\ \{\{1,2\},\{3,5\},\{4\}\},$
$\{\{1,2,3\},\{4\},\{5\}\},\ \{\{1,3\},\{2,4\},\{5\}\},\ \{\{1,3\},\{2,5\},\{4\}\},$
$\{\{1\},\{2\},\{3,4\},\{5\}\},\ \{\{1\},\{2\},\{3,5\},\{4\}\},\ \{\{1\},\{2,3\},\{4\},\{5\}\},$
$\{\{1\},\{2,4\},\{3\},\{5\}\},\ \{\{1\},\{2,5\},\{3\},\{4\}\},\ \{\{1,2\},\{3\},\{4\},\{5\}\},$
$\{\{1,3\},\{2\},\{4\},\{5\}\},\ \{\{1\},\{2\},\{3\},\{4\},\{5\}\}\}$

The output also displays the number of solutions (namely 17).

A.5.4 JoinSolve

For a fixed set partition π, find all set partitions σ such that the join of π and σ is the maximal partition $\hat{1}$, that is, $\sigma \vee \pi = \hat{1}$.

Example. Find all set partitions σ such that the join of σ and $\{\{1,4,5\},\{2\},\{3\}\}$ is the maximal partition $\hat{1}$.

In := JoinSolve$[\{\{1,4,5\},\{2\},\{3\}\}]$
Out= $\{\{\{1,2,3,4,5\}\},\ \{\{1\},\{2,3,4,5\}\},\ \{\{1,2\},\{3,4,5\}\},$
$\{\{1,2,3\},\{4,5\}\},\ \{\{1,3\},\{2,4,5\}\},\ \{\{1,2,3,4\},\{5\}\},$
$\{\{1,5\},\{2,3,4\}\},\ \{\{1,2,5\},\{3,4\}\},\{\{1,3,4\},\{2,5\}\},$
$\{\{1,2,3,5\},\{4\}\},\ \{\{1,4\},\{2,3,5\}\},\ \{\{1,2,4\},\{3,5\}\},$
$\{\{1,3,5\},\{2,4\}\},\ \{\{1\},\{2,3,4\},\{5\}\},\ \{\{1\},\{2,5\},\{3,4\}\},$
$\{\{1\},\{2,3,5\},\{4\}\},\ \{\{1\},\{2,4\},\{3,5\}\},\ \{\{1,2\},\{3,4\},\{5\}\},$
$\{\{1,2\},\{3,5\},\{4\}\},\ \{\{1,2,3\},\{4\},\{5\}\},$
$\{\{1,3\},\{2,4\},\{5\}\},\ \{\{1,3\},\{2,5\},\{4\}\}\}$

A.5.5 JoinSolveGrid

Display the output of JoinSolve in a grid.

Example. Find all set partitions σ such that the join of σ and $\{\{1,2\},\{3,4\}\}$ is the maximal partition $\hat{1}$.

In := JoinSolveGrid$[\{\{1,2\},\{3,4\}\}]$

Out=

$\{1,2,3,4\}$		
$\{1\}$	$\{2,3,4\}$	
$\{1,3,4\}$	$\{2\}$	
$\{1,2,3\}$	$\{4\}$	
$\{1,4\}$	$\{2,3\}$	
$\{1,2,4\}$	$\{3\}$	
$\{1,3\}$	$\{2,4\}$	
$\{1\}$	$\{2,3\}$	$\{4\}$
$\{1\}$	$\{2,4\}$	$\{3\}$
$\{1,3\}$	$\{2\}$	$\{4\}$
$\{1,4\}$	$\{2\}$	$\{3\}$

A.5.6 MeetAndJoinSolve

For a fixed set partition π, find all set partitions σ such that the join of π and σ is the maximal partition $\hat{1}$ AND the meet of π and σ is the singleton partition $\hat{0}$. that is, $\sigma \wedge \pi = \hat{0}$ and $\sigma \vee \pi = \hat{1}$.

Example. Find all set partitions σ such that the join of σ and $\{\{1,4,5\},\{2\},\{3\}\}$ is the maximal partition $\hat{1}$ and such that the meet of σ and $\{\{1,4,5\},\{2\},\{3\}\}$ is the singleton partition $\hat{0}$. Include the timing of the computation.

In := MeetAndJoinSolve $[\{\{1,4,5\},\{2\},\{3\}\}]$ // AbsoluteTiming
Out= $\{0.2031250, \{\{\{1\},\{2,3,4\},\{5\}\}, \{\{1\},\{2,5\},\{3,4\}\},$
$\{\{1\},\{2,3,5\},\{4\}\}, \{\{1\},\{2,4\},\{3,5\}\}, \{\{1,2\},\{3,4\},\{5\}\},$
$\{\{1,2\},\{3,5\},\{4\}\}, \{\{1,2,3\},\{4\},\{5\}\},$
$\{\{1,3\},\{2,4\},\{5\}\}, \{\{1,3\},\{2,5\},\{4\}\}\}\}$

The output in the first position is the number of seconds it took to generate the rest of the output. The number of solutions will also be specified (namely 9).

A.5.7 CoarserSetPartitionQ

This is a built-in *Mathematica* function; CoarserSetPartitionQ[σ,π] yields "True" if the set partition π is coarser than the set partition σ; that is, if $\sigma \leq \pi$, i.e., if every block in σ is contained in a block of π.

Example. Is the set partition $\pi = \{1,2\},\{3,4\},\{5\}$ coarser than the set partition $\sigma = \{\{1,2,3,4\},\{5\}\}$?

In := CoarserSetPartitionQ[$\{\{1,2,3,4\},\{5\}\},\{\{1,2\},\{3,4\},\{5\}\}$]
Out= False

A.5.8 CoarserThan

For a fixed set partition π, find all set partitions σ such that $\sigma \geq \pi$, i.e., all set partitions such that every block of π is fully contained in some block of σ.

Example. Find all of the set partitions σ such that $\sigma \geq \{\{1,2\},\{3,4\},\{5\}\}$.

In := CoarserThan[$\{\{1,2\},\{3,4\},\{5\}\}$]
Out= $\{\{\{1,2,3,4,5\}\},\{\{1,2\},\{3,4,5\}\},\{\{1,2,3,4\},\{5\}\},$
$\{\{1,2,5\},\{3,4\}\},\{\{1,2\},\{3,4\},\{5\}\}\}$

A.6 Partition Segments

A.6.1 PartitionSegment

For set partitions σ and π, find the partition segment $[\sigma,\pi]$, that is, all set partitions ρ such that $\sigma \leq \rho \leq \pi$.

► Please refer to Definition 2.2.4.

Example. Find all of the set partitions ρ such that

$$\{\{1\},\{2\},\{3\},\{4,5\}\} = \sigma \leq \rho \leq \pi = \{\{1,2,3\},\{4,5\}\}.$$

In := PartitionSegment[$\{\{1\},\{2\},\{3\},\{4,5\}\},\{\{1,2,3\},\{4,5\}\}$]
Out= $\{\{\{1,2,3\},\{4,5\}\},\ \{\{1\},\{2,3\},\{4,5\}\},\ \{\{1,2\},\{3\},\{4,5\}\},$
$\{\{1,3\},\{2\},\{4,5\}\},\ \{\{1\},\{2\},\{3\},\{4,5\}\}\}$

The output also specifies the number of partitions in the segment (namely 5).

A.6.2 ClassSegment

For set partitions σ and π, with $\sigma \leq \pi$, obtain the "class" of the segment $[\sigma, \pi]$, namely

$$\lambda(\sigma, \pi) = (1^{r_1} 2^{r_2} \cdots |\sigma|^{r_{|\sigma|}}),$$

where r_i is the number of blocks of π that contain exactly i blocks of σ (terms with exponent zero are ignored in the output).

► Please refer to Definition 2.3.1.

Example. Compute the class segment when $\sigma = \{\{1,2\},\{3\},\{4,5\}\}$ and $\pi = \{\{1,2,3\},\{4,5\}\}$.

In := ClassSegment$[\{\{1,2\},\{3\},\{4,5\}\},\{\{1,2,3\},\{4,5\}\}]$
Out= $(1^1 2^1)$

This is because only one block of π (namely, {4,5}) contains one block of σ and one block of π (namely, {1,2,3}) contains two blocks of σ. The answer could be more fully written as $(1^1 2^1 3^0)$.

A.6.3 ClassSegmentSolve

Given a partition σ, find all coarser partitions π with a specific class segment $\lambda(\sigma, \pi) = (1^{r_1} 2^{r_2} \cdots |\sigma|^{r_{|\sigma|}})$ written as $\{\{1, r_1\}, \{2, r_2\}, \cdots, \{|\sigma|, r_{|\sigma|}\}\}$ with $r_i \geq 0$ or merely $r_i > 0$. We want r_i blocks of π to contain exactly i blocks of σ, for $i = 1, ..., n$.

Input format. ClassSegmentSolve$[\sigma, \lambda(\sigma, \pi)]$

Example. Find the set of all partitions π such that the class segment of $\sigma = \{\{1,2\},\{3\},\{4,5\}\}$ and π is equal to $\lambda(\sigma, \pi) = (1^1 2^1)$. We want $r_1 = 1$ block of π to contain one block of σ and $r_2 = 1$ block of π to contain two blocks of σ.

In := ClassSegmentSolve$[\{\{1,2\},\{3\},\{4,5\}\},\{\{1,1\},\{2,1\}\}]$
Out= $\{\{\{1,2\},\{3,4,5\}\}, \quad \{\{1,2,3\},\{4,5\}\}, \quad \{\{1,2,4,5\},\{3\}\}\}$

If $\pi = \{\{1,2\},\{3,4,5\}\}$, then {1,2} contains the block {1,2} in $\sigma = \{\{1,2\},\{3\},\{4,5\}\}$ and {3,4,5} contains the blocks {3} and {4,5} of σ.

A.6.4 SquareChoose

For a list of non-negative integers $\{r_1, r_2, ..., r_n\}$, with $1r_1 + 2r_2 + \cdots + nr_n = n$, compute

$$\begin{bmatrix} n \\ \lambda \end{bmatrix} = \begin{bmatrix} n \\ r_1, ..., r_n \end{bmatrix} = \frac{n!}{(1!)^{r_1}\, r_1!\,(2!)^{r_2}\, r_2! \cdots (n!)^{r_n}\, r_n!}.$$

If one relabels the integers and denote by $\{r_1, r_2, \cdots, r_k\}$, with $k \leq n$ and k equal to the greatest integer in $[n]$ such that $r_k > 0$, then $1r_1 + 2r_2 + \cdots + kr_k = n$, and

$$\begin{bmatrix} n \\ \lambda \end{bmatrix} = \begin{bmatrix} n \\ r_1, ..., r_k \end{bmatrix} = \frac{n!}{(1!)^{r_1}\, r_1!\,(2!)^{r_2}\, r_2! \cdots (k!)^{r_k}\, r_k!}.$$

► Please refer to Formula (2.3.8).

Example. For the list $\{1, 3, 0, 0, 0, 0, 0\}$ or the list $\{1, 3\}$ compute

$$\frac{7!}{[(1!)^1(1!)][(2!)^3(3!)]}.$$

In := SquareChoose[{1, 3, 0, 0, 0, 0, 0}]
In := SquareChoose[$\{1, 3\}$]

The output is 105 in both cases.

A.7 Bell and Touchard polynomials

► Please refer to Section 2.4.

A.7.1 StirlingS2

Obtain the Stirling numbers $S(n, k)$ of the second kind, where $1 \leq k \leq n$. This is a built-in mathematica function. Recall that

$$S(n, k) = \sum_{r_1, \ldots, r_{n-k+1}} \frac{n!}{(1!)^{r_1}\, r_1!\,(2!)^{r_2}\, r_2! \cdots (n-k+1)!^{r_{n-k+1}} r_{n-k+1}!},$$

where the sum runs over all vectors on nonnegative integers $(r_1, ..., r_{n-k+1})$ satisfying $r_1 + \cdots + r_{n-k+1} = k$ and $1r_1 + 2r_2 \cdots + (n-k+1)\, r_{n-k+1} = n$.

► Please refer to Proposition 2.3.4.

Example. Get $S(3,2)$.

In := StirlingS2[3, 2]
Out= = 3

Example. Get all the Stirling numbers of the second kind for $n = 10$.

In := Table[StirlingS2[10, k], k, 10]
Out= 1, 511, 9330, 34105, 42525, 22827, 5880, 750, 45, 1

Example. Make a table of the Stirling numbers of the second kind for $n = 1, ..., 10$ and $k = 1, ..., n$.

In := Table[If[n >= k, StirlingS2[n, k]], {n, 1, 10}, {k, 1, 10}] // MatrixForm

Out=

1	Null	Null	Null	Null	Null	Null	Null	Null	Null
1	1	Null	Null	Null	Null	Null	Null	Null	Null
1	3	1	Null	Null	Null	Null	Null	Null	Null
1	7	6	1	Null	Null	Null	Null	Null	Null
1	15	25	10	1	Null	Null	Null	Null	Null
1	31	90	65	15	1	Null	Null	Null	Null
1	63	301	350	140	21	1	Null	Null	Null
1	127	966	1701	1050	266	28	1	Null	Null
1	255	3025	7770	6951	2646	462	36	1	Null
1	511	9330	34105	42525	22827	5880	750	45	1

The output has "Null" in the upper quadrant since the $S(n,k)$ are either not defined or are set equal to 0 when $k > n$.

A.7.2 BellPolynomialPartial

Generate the partial Bell polynomials $B_{n,k}(x_1, ..., x_{n-k+1})$, defined for $1 \leq k \leq n$. Recall that

$$B_{n,k}(x_1, ..., x_{n-k+1}) = \sum_{r_1,\ldots,r_{n-k+1}} \frac{n!}{r_1!r_2!\cdots r_{n-k+1}!} \left(\frac{x_1}{1!}\right)^{r_1} \cdots \left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{r_{n-k+1}}$$

where the sum runs over all vectors on nonnegative integers $(r_1, ..., r_{n-k+1})$ satisfying $r_1 + \cdots + r_{n-k+1} = k$ and $1r_1 + 2r_2 \cdots + (n-k+1)\, r_{n-k+1} = n$.

► Please refer to Definition 2.4.1.

Input format. BellPolynomialPartial$[n, k, x]$ where $1 \leq k \leq n$ and x is a vector of length at least $n - k + 1$ (the extra components are ignored). The index k can also be "All". This outputs $B_{n,1}, \cdots, B_{n,n}$ in a list.

Example. Generate the partial Bell polynomial $n = 4, k = 2$ and 3 variables $x1, x2, x3, x4$. These form a vector and must be inserted with curly brackets.

In := BellPolynomialPartial$[4, 2, \{x1, x2, x3\}]$
Out= $3x2^2 + 4x1x3$

The output should be read as $3x_2^2 + 4x_1x_3$.

Example. Numerical computation with $n = 70, k = 50$, and the variables taking consecutive numerical values.

In := BellPolynomialPartial$[40, 33, \{1, 2, 3, 4, 5, 6, 7, 8\}]$
Out= 794559498748278120

Example. Numerical computation with $n = 70, k = 50$, and the variables taking consecutive numerical values, defined through a table.

In := BellPolynomialPartial[70, 50, Table[i, {i, 1, 70 - 50 + 1}]]
Out= 15438518873468196487426757812500000000000000000000

A.7.3 BellPolynomialPartialAuto

Automatically generates symbolic form of BellPolynomialPartial$[n, k]$ without having to input a vector. Moreover k can be 'All', in which case the output is $B_{n,1}, \cdots, B_{n,n}$ in a list.

Example. Generate the partial Bell polynomial $n = 4, k = 2$ but with unspecified variables.

In := BellPolynomialPartialAuto[4, 2]
Out= $3x_2^2 + 4x_1x_3$

Example. Generate all the partial Bell polynomials with $n = 4$.

In := BellPolynomialPartialAuto[4,"All"]
Out= $\{x_4,\ 3x_2^2 + 4x_1x_2,\ 6x_1^2x_3,\ x_1^4\}$

A.7.4 BellPolynomialComplete

Generate the complete Bell polynomial of order n, that is,

$$B_n(x_1, x_2, \cdots, x_n) = \sum \begin{bmatrix} n \\ r_1, \ldots, r_n \end{bmatrix} x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n}$$
$$= \sum \frac{n!}{(1!)^{r_1} r_1! (2!)^{r_2} r_2! \cdots (n!)^{r_n} r_n!} x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n},$$

where the sum is over all non-negative $r_1, r_2, \cdots, r_n$ satisfying $1r_1 + 2r_2 + \cdots + nr_n = n$.

► Please refer to Definition 2.4.1.

Input format. BellPolynomialComplete$[x]$ where x is a vector. The length of the vector determines the value of n.

Example. Generate the complete Bell polynomial with 4 variables $x1, x2, x3, x4$. These form a vector and must be inserted with curly brackets.

In := BellPolynomialComplete[$\{x1, x2, x3, x4\}$]
Out= $x1^4 + 6x1^2x2 + 3x2^2 + 4x1x3 + x4$

The output should be read as $x_1^4 + 6x_1^2x_2 + 3x_2^2 + 4x_1x_3 + x_4$.

Example. Generate the complete Bell polynomial with 4 identical variables.

In := BellPolynomialComplete[$\{x, x, x, x\}$]
Out= $x + 7x^2 + 6x^3 + x^4$

A.7.5 BellPolynomialCompleteAuto

Automatically generates a symbolic form of complete Bell polynomial of order n. There is no need to input a vector.

Input format. BellPolynomialCompleteAuto$[n]$.

Example. Obtain the complete Bell polynomial of order 4.

In := BellPolynomialCompleteAuto[4]
Out= $x_1^4 + 6x_1^2x_2 + 3x_2^2 + 4x_1x_3 + x_4$

Example. Make a table of Complete Bell polynomials of order 1 to 6.

In := TableForm[Table[BellPolynomialCompleteAuto[n], n, 1, 6], TableHeadings → "n=1", "n=2", "n=3", "n=4", "n=5", "n=6", None]

Out=

n=1 x_1
n=2 $x_1^2 + x_2$
n=3 $x_1^3 + 3x_1x_2 + x_3$
n=4 $x_1^4 + 6x_1^2x_2 + 3x_2^2 + 4x_1x_3 + x_4$
n=5 $x_1^5 + 10x_1^3x_2 + 15x_1x_2^2 + 10x_1^2x_3 + 10x_2x_3 + 5x_1x_4 + x_5$
n=6 $x_1^6 + 15x_1^4x_2 + 45x_1^2x_2^2 + 15x_2^3 + 20x_1^3x_3 + 60x_1x_2x_3 + 10x_3^2$
$+15x_1^2x_4 + 15x_2x_4 + 6x_1x_5 + x_6$

A.7.6 BellB

Gives the complete Bell polynomial when all the arguments are identical and computes the Bell number of order n. These are buit-in mathematica functions.

Input format. BellB$[n, x]$ where n is a non-negative integer and x is a scalar. This is equivalent to BellPolynomialComplete$[\{x, x, .., x, x\}]$ with the scalar x appearing n times.

Example. Get BellPolynomialComplete$[\{x, x, x, x\}]$ using BellB$[4, x]$.

In := BellB$[4, x]$
Out= $x + 7x^2 + 6x^3 + x^4$

Input format. BellB$[n]$.

This is the Bell number of order n. It is equivalent to BellPolynomialComplete$[\{1, ..., 1\}]$ with 1 appearing n times and also equivalent to $T_n(1)$, where T_n is the Touchard polynomial.

Example. Get BellPolynomialComplete$[\{1, 1, 1, 1\}]$ using BellB$[4]$.

In := BellB[4]
Out= 15

Example. Get the Bell numbers up to 20.

In := Table[BellB[k], {k, 20}]
Out= {1, 2, 5, 15, 52, 203, 877, 4140, 21147, 115975, 678570, 4213597, 27644437, 190899322, 1382958545, 10480142147, 82864869804, 682076806159, 5832742205057, 51724158235372 }

Example. Compute the sum of *SquareChoose* of over all values of its non-negative arguments, that is:

$$\sum \left[{n \atop r_1, \ldots, r_n} \right] = \sum \frac{n!}{(1!)^{r_1}\, r_1!\, (2!)^{r_2}\, r_2! \cdots (n!)^{r_n}\, r_n!}$$

This amounts to evaluate the corresponding complete Bell polynomial with the variables x all equal to 1 or BellB[n]. Do it for $n = 5$.

In := BellB[5]
Out= 52

A.7.7 TouchardPolynomial

Obtain the Touchard polynomial $T_n(x)$ of order n, where x is a scalar. It is, in fact, identical to BellB$[n, x]$. The Touchard polynomial of order n is equal to the moment of order n of a Poisson with parameter x.

► Please refer to Definition 2.4.1.

Input format. TouchardPolynomial$[n, x]$ where n is a non-negative integer and x is a scalar.

Example. Get the Touchard polynomial of order $n = 4$.

In := TouchardPolynomial[4, x]
Out= $x + 7x^2 + 6x^3 + x^4$

Input format. TouchardPolynomial$[n, x,$ "All"$]$. This generates a full list of Touchard Polynomials of order 0 up to order n.

Example. Get the Touchard polynomial of order 0 up to order 5.

In := TouchardPolynomial[5, x, "All"]
Out= $= \{1, x, x+x^2, x+3x^2+x^3, x+7x^2+6x^3+x^4, x+15x^2+25x^3+10x^4+x^5\}$

A.7.8 TouchardPolynomialTable

Generates a table of Touchard polynomials from order 0 to n.

Example. Get a table of the Touchard polynomials of order 0 up to order 5.

In := TouchardPolynomialTable[5]

Out=

n	T_n
0	1
1	x
2	$x + x^2$
3	$x + 3x^2 + x^3$
4	$x + 7x^2 + 6x^3 + x^4$
5	$x + 15x^2 + 25x^3 + 10x^4 + x^5$

A.7.9 TouchardPolynomialCentered

Computes the centered Touchard polynomial of order n. It is equal to the moment of order n of a centered Poisson with parameter x.

► Please refer to Proposition 3.3.4.

Input format. TouchardPolynomialCentered$[n, x]$ where n is a non-negative integer and x is a scalar.

Example. Get the Touchard polynomial of order $n = 5$.

In := TouchardPolynomialCentered$[5, x]$
Out= $x + 10x^2$

Input format. TouchardPolynomialCentered$[n, x,$ "All"$]$. This generates a full list of centered Touchard polynomials of order 0 up to order n.

Example. Get the centered Touchard polynomial of order 0 up to order 5.

In := TouchardPolynomialCentered$[5, x,$ "All"$]$
Out= $\{1, 0, x, x, x + 3x^2, x + 10x^2\}$

A.7.10 TouchardPolynomialCenteredTable

Generates a table of centered Touchard polynomials from order 0 to n.

Example. Get a table of the centered Touchard polynomials of order 0 up to order 5.

In := TouchardPolynomialCenteredTable[5]

Out=

n	$\widetilde{T}_n$
0	1
1	0
2	x
3	x
4	$x + 3x^2$
5	$x + 10x^2$

A.8 Mobius Formula

► Please refer to Section 2.5.

A.8.1 MobiusFunction

For set partitions $\sigma \leq \pi$, compute the Mobius function

$$\begin{aligned}\mu(\sigma,\pi) &= (-1)^{n-r}(2!)^{r_3}(3!)^{r_4}\cdots((n-1)!)^{r_n} \\ &= (-1)^{n-r}\prod_{j=2}^{n-1}(j!)^{r_{j+1}} = (-1)^{n-r}\prod_{j=0}^{n-1}(j!)^{r_{j+1}},\end{aligned}$$

where $n = |\sigma|$, $r = |\pi|$, and $\lambda(\sigma,\pi) = (1^{r_1}2^{r_2}\cdots n^{r_n})$ (that is, there are exactly r_i blocks of π containing exactly i blocks of σ).

Input format. MobiusFunction[σ,π].

Example. Evaluation of the Mobius function with $\sigma = \{\{1\},\{2\},\{3,4\},\{5,6\},\{7\},\{8,9\}\}$ and $\pi = \{\{1,2\},\{3,4,5,6,7,8,9\}\}$.

In := MobiusFunction[$\{\{1\},\{2\},\{3,4\},\{5,6\},\{7\},\{8,9\}\},\{\{1,2\},$
$\{3,4,5,6,7,8,9\}\}$]

Out= 6

Let's check this result with ClassSegment.

In := $[\{\{1\},\{2\},\{3,4\},\{5,6\},\{7\},\{8,9\}\},\{\{1,2\},\{3,4,5,6,7,8,9\}\}]$
Out= $(2^1 4^1)$

Thus $r_2 = 1, r_4 = 1, n = 6, r = 2$ which explains why $\mu(\sigma,\pi) = (-1)^{6-2}$ $(3!)^1 = 6$.

A.8.2 MobiusRecursionCheck

Verify that the Mobius function recursion

$$\mu(\sigma,\pi) = -\sum_{\sigma\le\rho<\pi} \mu(\sigma,\rho),$$

holds; in the output, the number in the first coordinate is $\mu(\sigma,\pi)$ and the number in the second coordinate is $-\sum_{\sigma\le\rho<\pi}\mu(\sigma,\rho)$; they should be equal.

Example. Check the Mobius function recursion when $\sigma = \{\{1\},\{2\},\{3\},\{4\}\}$ and $\pi = \{\{1,2,3,4\}\}$.

In := MobiusRecursionCheck $[\{\{1\},\{2\},\{3\},\{4\}\},\{\{1,2,3,4\}\}]$

The output is $\{-6,-6\}$, confirming equality.

A.8.3 MobiusInversionFormulaZero

Compute the Mobius inversion formula given by

$$F(\pi) = \sum_{\hat{0}\le\sigma\le\pi} \mu(\sigma,\pi)\, G(\sigma)$$

where F and G are functions taking set partitions as their arguments.

Input format. MobiusInversionFormulaZero[π].

Example. Let $\pi = \{\{1,2,3\},\{4,5\}\}$. For all $\hat{0} \le \sigma \le \pi$, obtain the Mobius function coefficient $\mu(\sigma,\pi)$ and print $G(\sigma)$.

In := MobiusInversionFormulaZero$[\{\{1,2,3\},\{4,5\}\}]$

Out= Number of partitions in the segment = 10

σ	$\mu(\sigma,\pi)$	G(σ)
$\{\{1,2,3\},\{4,5\}\}$	1	$G[\{\{1,2,3\},\{4,5\}\}]$
$\{\{1\},\{2,3\},\{4,5\}\}$	-1	$G[\{\{1\},\{2,3\},\{4,5\}\}]$
$\{\{1,2\},\{3\},\{4,5\}\}$	-1	$G[\{\{1,2\},\{3\},\{4,5\}\}]$
$\{\{1,3\},\{2\},\{4,5\}\}$	-1	$G[\{\{1,3\},\{2\},\{4,5\}\}]$
$\{\{1,2,3\},\{4\},\{5\}\}$	-1	$G[\{\{1,2,3\},\{4\},\{5\}\}]$
$\{\{1\},\{2\},\{3\},\{4,5\}\}$	2	$G[\{\{1\},\{2\},\{3\},\{4,5\}\}]$
$\{\{1\},\{2,3\},\{4\},\{5\}\}$	1	$G[\{\{1\},\{2,3\},\{4\},\{5\}\}]$
$\{\{1,2\},\{3\},\{4\},\{5\}\}$	1	$G[\{\{1,2\},\{3\},\{4\},\{5\}\}]$
$\{\{1,3\},\{2\},\{4\},\{5\}\}$	1	$G[\{\{1,3\},\{2\},\{4\},\{5\}\}]$
$\{\{1\},\{2\},\{3\},\{4\},\{5\}\}$	-2	$G[\{\{1\},\{2\},\{3\},\{4\},\{5\}\}]$

Here G is not specified so the output merely lists the various terms in the sum. For example, in the output, the term corresponding to $\sigma = \{\{1\},\{2,3\},\{4,5\}\}$ is $(-1)G(\{\{1\},\{2,3\},\{4,5\}\})$.

The function MobiusInversionFormulaZero is also useful for computing

$$G(\pi) = \sum_{\hat{0}\leq\sigma\leq\pi} F(\sigma).$$

Merely ignore $\mu(\sigma,\pi)$ and replace $G(\sigma)$ by $F(\sigma)$.

A.8.4 MobiusInversionFormulaOne

Compute the Mobius inversion formula given by

$$F(\pi) = \sum_{\pi\leq\sigma\leq\hat{1}} \mu(\pi,\sigma)\, G(\sigma)$$

where F and G are functions taking set partitions as their arguments.

Input format. MobiusInversionFormulaOne[π].

Example. Let $\pi = \{\{1,2,3,\},\{4,5\}\}$. For all $\pi \leq \sigma \leq \hat{1}$, obtain the Mobius function coefficient $\mu(\pi,\sigma)$ and print $G(\sigma)$.

In := MobiusInversionFormulaOne$[\{\{1,2,3\},\{4,5\}\}]$

Out= Number of partitions in the segment = 2

σ	$\mu(\sigma,\pi)$	G(σ)
$\{\{1,2,3,4,5\}\}$	-1	$G[\{\{1,2,3,4,5\}\}]$
$\{\{1,2,3\},\{4,5\}\}$	1	$G[\{\{1,2,3\},\{4,5\}\}]$

The output has 2 terms only in this case:

$$(-1)G[\{\{1,2,3,4,5\}\}] + (1)G[\{\{1,2,3\},\{4,5\}\}].$$

The function MobiusInversionFormulaOne is also useful for computing

$$G(\pi) = \sum_{\pi \le \sigma \le \hat{1}} F(\sigma).$$

Merely ignore $\mu(\sigma,\pi)$ and replace $G(\sigma)$ by $F(\sigma)$.

A.9 Moments and Cumulants

► Please refer to Chapter 3. Recall the notation:

For every subset $b = \{j_1, ..., j_k\} \subseteq [n] = \{1, ..., n\}$, we write

$$\boxed{\mathbf{X}_b = (X_{j_1}, ..., X_{j_k})} \quad \text{and} \quad \boxed{\mathbf{X}^b = X_{j_1} \times \cdots \times X_{j_k},}$$

where $\times$ denotes the usual product. For instance, $\forall m \le n$,

$$\mathbf{X}_{[m]} = (X_1, .., X_m) \quad \text{and} \quad \mathbf{X}^{[m]} = X_1 \times \cdots \times X_m.$$

The function names below acquire their meaning when read from left to right. Thus, the function *MomentToCumulants* expresses a given moment in terms of cumulants. In practice, this function is used to transform cumulants into moments.

A.9.1 MomentToCumulants

Express the moment of a random variable in terms of the cumulants.

Input format. To express $\mathbb{E}[X^m]$ in terms of cumulants, use MomentToCumulants[m].

► Please refer to Formula (3.2.16).

Example. Express the third moment $\mathbb{E}[X^3]$ of a random variable X in terms of cumulants.

In := MomentToCumulants[3]
Out= $\chi_1[X]^3 + 3\chi_1[X]\chi_2[X] + \chi_3[X]$

Notation: $\chi_1[X]^3$ denotes the third power of the first cumulant of X.

A.9.2 MomentToCumulantsBell

Use complete Bell polynomials to express a moment in terms of cumulants. One can use either the function *MomentToCumulantsBell* or directly *BellPolynomialComplete*.

► Please refer to Proposition 3.3.1.

Input format. MomentToCumulantsBell$[x]$ where x is a vector (of cumulants).

Example. Express the sixth moment of a random variable X in terms of cumulants (using Bell polynomials). Let $k1, \cdots, k6$ denote the cumulants.

In := MomentToCumulantsBell[$\{k1, k2, k3, k4, k5, k6\}$]
Out= $k1^6 + 15k1^4k2 + 45k1^2k2^2 + 15k2^3 + 20k1^3k3 + 60k1k2k3 + 10k3^2 + 15k1^2k4 + 15k2k4 + 6k1k5 + k6$

In := BellPolynomialComplete[$\{k1, k2, k3, k4, k5, k6\}$]
Out= $k1^6 + 15k1^4k2 + 45k1^2k2^2 + 15k2^3 + 20k1^3k3 + 60k1k2k3 + 10k3^2 + 15k1^2k4 + 15k2k4 + 6k1k5 + k6$

Example. Express the second moment of a random variable X in terms of cumulants (using complete Bell polynomials). Let $\chi_1[X], \chi_2[X]$ denote the cumulants. This example demonstrates that the arguments can be any expression.

In := BellPolynomialComplete[$\{\chi_1[X], \chi_2[X]\}$]
Out= $\chi_1[X]^2 + \chi_2[X]$

Example. Express the sixth central moment of a random variable X in terms of cumulants (using complete Bell polynomials). Let $k1, \cdots, k6$ denote the cumulants. This amounts to supposing that the mean, that is, the first cumulant $k1 = 0$.

In := BellPolynomialComplete[$\{0, k2, k3, k4, k5, k6\}$]
Out= $15k2^3 + 10k3^2 + 15k2k4 + k6$

Example. In the previous example, suppose that the random variable X is normalized to have mean 0 and variance 1. This amounts to setting $k1 = 0$ and $k2 = 1$.

In := BellPolynomialComplete[$\{0, 1, k3, k4, k5, k6\}$]
Out= $15 + 10k3^2 + 15k4 + k6$

Example. Obtain the sixth moment of a standard normal. This amounts to setting in addition all cumulants higher than 2 equal to 0.

In := BellPolynomialComplete[$\{0, 1, 0, 0, 0, 0\}$]
Out= 15

Remark. From now on, we shall sometimes adopt the notation $\chi_j[X] = \kappa_j$, $j = 1, 2, \ldots$.

Example. Display the results in a table (up to order 5). For a more involved table use *MomentToCumulantsBellTable*.

In := Column[Table[BellPolynomialComplete[Table[Subscript[\[Kappa], i], i, 1, j]], j, 1, 5]]

The output is a table:

$$\kappa_1$$
$$\kappa_1^2 + \kappa_2$$
$$\kappa_1^3 + 3\kappa_1\kappa_2 + \kappa_3$$
$$\kappa_1^4 + 6\kappa_1^2\kappa_2 + 3\kappa_2^2 + 4\kappa_1\kappa_3 + \kappa_4$$
$$\kappa_1^5 + 10\kappa_1^3\kappa_2 + 15\kappa_1\kappa_2^2 + 10\kappa_1^2\kappa_3 + 10\kappa_2\kappa_3 + 5\kappa_1\kappa_4 + \kappa_5.$$

A.9.3 MomentToCumulantsBellTable

Express moments in terms of cumulants and display the result in a table.

Input format. MomentToCumulantsBellTable[κ] , where κ is a vector of cumulants.

Proceed interactively. How many rows do you want to see?
In := NumRows = 5
Out= 5

Sometimes you may need to clear the previous definitions of μ and κ. To type these symbols copy and paste them with the mouse from somewhere else in the notebook or type respectively as \[Mu] and \[Kappa].

In := ClearAll[μ, κ]

Input is a list of cumulants, for example:
In := κ = Table[Subscript[κ, i], i, 1, NumRows]
Out= $\{\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5\}$

Then, you make a table by plugging this into *MomentToCumulantsBellTable*:

In := MomentToCumulantsBellTable[κ]

Out=

n	μ_n
1	κ_1
2	$\kappa_1^2 + \kappa_2$
3	$\kappa_1^3 + 3\kappa_1\kappa_2 + \kappa_3$
4	$\kappa_1^4 + 6\kappa_1^2\kappa_2 + 3\kappa_2^2 + 4\kappa_1\kappa_3 + \kappa_4$
5	$\kappa_1^5 + 10\kappa_1^3\kappa_2 + 15\kappa_1\kappa_2^2 + 10\kappa_1^2\kappa_3 + 10\kappa_2\kappa_3 + 5\kappa_1\kappa_4 + \kappa_5$

To obtain centered moments, make the first element of your input equal to 0:

In := \[Kappa][[1]] = 0

Out= 0

Check the value of the vector κ and generate the table:

In := κ

Out= $\{0, \kappa_2, \kappa_3, \kappa_4, \kappa_5\}$

In := MomentToCumulantsBellTable[κ]

Out=

n	μ_n
1	0
2	κ_2
3	κ_3
4	$3\kappa_2^2 + \kappa_4$
5	$10\kappa_2\kappa_3 + \kappa_5$

A.9.4 CumulantToMoments

Express the cumulant of a random variable in terms of moments.

► Please refer to Proposition 3.3.1.

Input format. To express $\chi_m[X]$ in terms of moments, use CumulantToMoments[m].

► Please refer to Formula (3.2.19).

Example. Express the third cumulant $\chi_3[X]$ of a random variable X in terms of moments.

In := CumulantToMoments[3]

Out= $2\mathbb{E}[X]^3 - 3\mathbb{E}[X]\mathbb{E}[X^2] + \mathbb{E}[X^3]$

Notation: $\mathbb{E}[X]^3$ denotes the third power of the first moment of X.

A.9.5 CumulantToMomentsBell

Use partial Bell polynomials to express a cumulant in terms of moments.

Input format. CumulantToMomentsBell$[x]$ where x is a vector (of moments).

Example. Express the moments in terms of cumulants (using partial Bell polynomials).

In := CumulantToMomentsBell$[\{m1, m2\}]$
Out= $-m1^2 + m2$

Example. Display the conversion results in a table (up to order 5). For a more involved table use *CumulantToMomentsBellTable*.

In := Column[Table[CumulantToMomentsBell[Table[Subscript[\[Mu], i], i, 1, j]], j, 1, 5]]
The output is a table:

$$\mu_1$$
$$-\mu_1^2 + \mu_2$$
$$2\mu_1^3 - 3\mu_1\mu_2 + \mu_3$$
$$-6\mu_1^4 + 12\mu_1^2\mu_2 - 3\mu_2^2 - 4\mu_1\mu_3 + \mu_4$$
$$24\mu_1^5 - 60\mu_1^3\mu_2 + 30\mu_1\mu_2^2 + 20\mu_1^2\mu_3 - 10\mu_2\mu_3 - 5\mu_1\mu_4 + \mu_5.$$

A.9.6 CumulantToMomentsBellTable

Express cumulants in terms of moments and display the result in a table.

Input format. CumulantToMomentsBellTable$[\mu]$, where μ is a vector of moments.

Proceed interactively. How many rows do you want to see?
In := NumRows = 5
Out= 5

Sometimes you may need to clear the previous definitions of μ and κ. To type these symbols copy and paste them with the mouse from somewhere else in the notebook or type respectively as \[Mu] and \[Kappa].

In := ClearAll$[\mu, \kappa]$

Input is a list of moments, for example:

In := μ = Table[Subscript[μ, i], i, 1, NumRows]

Out= $\{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5\}$

Then, you make a table by plugging this into *CumulantToMomentsBellTable*:

In := CumulantToMomentsBellTable[μ]

Out=

n	κ_n
1	μ_1
2	$-\mu_1^2 + \mu_2$
3	$2\mu_1^3 - 3\mu_1\mu_2 + \mu_3$
4	$-6\mu_1^4 + 12\mu_1^2\mu_2 - 3\mu_2^2 - 4\mu_1\mu_3 + \mu_4$
5	$24\mu_1^5 - 60\mu_1^3\mu_2 + 30\mu_1\mu_2^2 + 20\mu_1^2\mu_3 - 10\mu_2\mu_3 - 5\mu_1\mu_4 + \mu_5$

To obtain a table involving centered moments and cumulants, make the first element of your input equal to 0:

In := \[Mu][[1]] = 0

Out= 0

Check the value of the vector μ and generate the table:

In := μ

Out= $\{0, \mu_2, \mu_3, \mu_4, \mu_5\}$

In := CumulantToMomentsBellTable[μ]

Out=

n	κ_n
1	0
2	μ_2
3	μ_3
4	$-3\mu_2^2 + \mu_4$
5	$-10\mu_2\mu_3 + \mu_5$

A.9.7 MomentProductToCumulants

Express the expected value of the product of random variables $\mathbf{X}^b = X_{j_1} \cdots X_{j_k}$, where $b = \{j_1, ..., j_k\} \subseteq [n] = \{1, ..., n\}$, in terms of cumulants.

► Please refer to Formula (3.2.7).

Example. Express the expected value $\mathbb{E}\mathbf{X}^{\{1,2\}} = \mathbb{E}X_1X_2$ in terms of cumulants.

In := MomentProductToCumulants $[X, \{\{1,2\}\}]$
Out= $\chi[X_1]\chi[X_2] + \chi[X_1, X_2]$

A.9.8 CumulantVectorToMoments

Express the cumulant of a random vector $\mathbf{X}_b = (X_{j_1}, ..., X_{j_k})$, where $b = \{j_1, ..., j_k\} \subseteq [n] = \{1, ..., n\}$, in terms of moments.

► Please refer to Formula (3.2.6).

Example. Express the cumulant of $X_{\{1,2\}} = (X_1, X_2)$ which is the covariance of X_1 and X_2, in terms of moments.

In := CumulantVectorToMoments$[X, \{\{1,2\}\}]$
Out= $-\mathbb{E}[X_1]\mathbb{E}[X_2] + \mathbb{E}[X_1X_2]$

A.9.9 CumulantProductVectorToCumulantVectors

Express the joint cumulant of a random vector $(\mathbf{X}_{b_1}, \mathbf{X}_{b_2}, \cdots, \mathbf{X}_{b_k})$ where $\{b_1, b_2, \cdots, b_k\} \subseteq [n]$, in terms of cumulants of random vectors whose components are not products (Malyshev's formula).

► Please refer to Formula (3.2.8).

Example. Express the cumulants $\chi[X_1X_2, X_3]$ in terms of cumulants of random vectors whose components are not products.

In := CumulantProductVectorToCumulantVectors$[X, \{\{1,2\}, \{3\}\}]$
Out= $\chi(X_2)\chi(X_1, X_3) + \chi(X_1)\chi(X_2, X_3) + \chi(X_1, X_2, X_3)$

A.9.10 GridCumulantProductVectorToCumulantVectors

Display the output of CumulantProductVectorToCumulantVectors in a grid.

Example. Display in a grid CumulantProductVectorToCumulantVectors of $\chi[X_1X_2X_3, X_4]$.

In := GridCumulantProductVectorToCumulantVectors $[X, \{\{1,2,3\}, \{4\}\}]$

Out= Number of terms in summation = 10

$$\begin{array}{lr}
\chi\left[X_2\right]\chi\left[X_3\right]\chi\left[X_1,X_4\right] & + \\
\chi\left[X_1,X_4\right]\chi\left[X_2,X_3\right] & + \\
\chi\left[X_1\right]\chi\left[X_3\right]\chi\left[X_2,X_4\right] & + \\
\chi\left[X_1,X_3\right]\chi\left[X_2,X_4\right] & + \\
\chi\left[X_1\right]\chi\left[X_2\right]\chi\left[X_3,X_4\right] & + \\
\chi\left[X_1,X_2\right]\chi\left[X_3,X_4\right] & + \\
\chi\left[X_3\right]\chi\left[X_1,X_2,X_4\right] & + \\
\chi\left[X_2\right]\chi\left[X_1,X_3,X_4\right] & + \\
\chi\left[X_1\right]\chi\left[X_2,X_3,X_4\right] & + \\
\chi\left[X_1,X_2,X_3,X_4\right] & +
\end{array}$$

A.9.11 CumulantProductVectorToMoments

Express the joint cumulant of a random vector $(\mathbf{X}^{b_1},\mathbf{X}^{b_2},\cdots,\mathbf{X}^{b_k})$ where $\{b_1,b_2,\cdots,b_k\}\subseteq[n]$, in terms of moments.

Example. Express the cumulants $\chi[X_1X_2,X_3]$ in terms of moments.

In := CumulantProductVectorToMoments$[X,\{\{1,2\},\{3\}\}]$
Out= $-\mathbb{E}[X_1X_2]\mathbb{E}[X_3]+\mathbb{E}[X_1X_2X_3]$

A.10 Gaussian Multiple Integrals

A.10.1 GaussianIntegral

This function can be used to compute multiple Gaussian integrals

$$\int_\sigma f(x_1,...,x_n)G(dx_1)\cdots G(dx_n)$$

over a set partition σ and

$$\int_{\geq\sigma} f(x_1,...,x_n)G(dx_1)\cdots G(dx_n)$$

over all partitions at least as coarse as σ, where $f(x_1,...,x_n)$ is a symmetric function, for example, $g(x_1)\cdots g(x_n)$, and where G is a Gaussian measure with control measure ν. This function generates the set partition data necessary to compute multiple Gaussian integrals. This data is listed under two lists: “control measure ν” and “Gaussian measure G”. The output is “The integral is zero” when this is the case.

► Please refer to Theorem 5.11.1.

Example. When computing the multiple Gaussian integral over a set partition with a block size greater than or equal to three, the integral is zero.

In := GaussianIntegral$[\{\{1,2,3\},\{4,5\},\{6,7\},\{8\}\}]$
Out= The integral is zero.

Rules for interpreting the data listed in output:

Rule 1. To obtain a Gaussian multiple integral over σ, equate the variables listed under “control measure ν” and integrate them with respect to ν; the variables listed under “Gaussian measure G” are integrated with respect to G over the off-diagonals.

Rule 2. To obtain a Gaussian multiple integral over “$\geq \sigma$”, equate the variables listed under “control measure ν” and integrate.

Example. Obtain the block data necessary to decompose the multiple Gaussian integral over a particular set partition.

In := GaussianIntegral$[\{\{1,3\},\{4,5\},\{6\},\{7\},\{2\}\}]$

Out=

Control measure ν
$\{1,3\}$
$\{4,5\}$

Gaussian measure G
$\{6\}$
$\{7\}$
$\{2\}$

The output lists $\{1,3\},\{4,5\}$ under “control measure ν” and it lists $\{6\},\{7\},\{2\}$ under “Gaussian measure G”.

Interpretation: Let $\sigma = \{\{1,3\},\{4,5\},\{6\},\{7\},\{2\}\}$. Integrating $g(x_1)\ldots g(x_7)$ over σ gives:

$$\left(\int g^2(x)\nu(dx)\right)^2 \int_{x_2 \neq x_6 \neq x_7} g(x_2)g(x_6)g(x_7)G(dx_2)G(dx_6)G(dx_7).$$

If one integrates it over “$\geq \sigma$” instead, one obtains

$$\left(\int g^2(x)\nu(dx)\right)^2 \left(\int g(x)G(dx)\right)^3.$$

Integrating a symmetric function $f(x_1, \cdots, x_7)$ over σ gives

$$\int_{x_2 \neq x_6 \neq x_7} f(x_3, x_3, x_5, x_5, x_2, x_6, x_7)\nu(dx_3)\nu(dx_5)G(dx_2)G(dx_6)G(dx_7).$$

If one integrates it over "$\geq \sigma$" instead, one obtains

$$\int f(x_3, x_3, x_5, x_5, x_2, x_6, x_7)\nu(dx_3)\nu(dx_5)G(dx_2)G(dx_6)G(dx_7).$$

A.11 Poisson Multiple Integrals

We first define some functions which are useful for the evaluation of Poisson multiple integrals. Please refer to the main text.

A.11.1 BOne

For a given set partition σ, find $B_1(\sigma)$, that is, isolate the blocks of σ that are singletons.

► Please refer to Formula (5.12.85).

Example. Find all of the singleton blocks of $\{\{1,2\},\{3,4\},\{5\}\}$.

In := BOne[{{1, 2}, {3, 4}, {5}}]
Out= {{5}}

Example. The set partition $\{\{1,2,3\},\{4,5\}\}$ has no singleton blocks.

In := BOne[{{1, 2, 3}, {4, 5}}]
Out= {}

A.11.2 BTwo

For a given set partition σ, find $B_2(\sigma)$, that is, isolate the blocks of σ that are not singletons.

► Please refer to Formula (5.12.86).

Example. Find all the blocks of $\{\{1,2\},\{3,4\},\{5\}\}$ that are not singletons.

In := BTwo[{{1, 2}, {3, 4}, {5}}]
Out= {{1, 2}, {3, 4}}

A.11.3 PBTwo

For a fixed set partition σ, find $B_2(\sigma)$; viewing $B_2(\sigma)$ as a set, find all ordered partitions of $B_2(\sigma)$ with exactly two blocks, called R_1 and R_2.

► Please refer to Formula (5.12.87).

Example. For the set partition $[\{\{1,2\},\{3,4\},\{5,6\},\{7\}\}]$, find $B_2([\{\{1,2\},\{3,4\},\{5,6\},\{7\}\}])$.

In := PBTwo$[\{\{1,2\},\{3,4\},\{5,6\},\{7\}\}]$

Out=

R1	R2
$\{\{1,2\}\}$	$\{\{3,4\},\{5,6\}\}$
$\{\{3,4\}\}$	$\{\{1,2\},\{5,6\}\}$
$\{\{5,6\}\}$	$\{\{1,2\},\{3,4\}\}$
$\{\{1,2\},\{3,4\}\}$	$\{\{5,6\}\}$
$\{\{1,2\},\{5,6\}\}$	$\{\{3,4\}\}$
$\{\{3,4\},\{5,6\}\}$	$\{\{1,2\}\}$

Example. For the set partition $\{\{1,2\},\{3\}\}$, PB$_2(\{\{1,2\},\{3\}\})$ is empty.

In := PBTwo$[\{\{1,2\},\{3\}\}]$

Out=

R1	R2

A.11.4 PoissonIntegral

Let $N(dx)$ be a Poisson measure with control measure ν and let $\widehat{N}$ be the corresponding compensated Poisson measure, that is,

$$\widehat{N}(dx) = N(dx) - \mathbb{E}N(dx) = N(dx) - \nu(dx).$$

The function "PoissonIntegral" can be used to compute multiple Poisson integrals

$$\int_\sigma f(x_1,...,x_n)\widehat{N}(dx_1)\cdots\widehat{N}(dx_n)$$

over a set partition σ, where $f(x_1,...,x_n)$ is a symmetric function, for example, $g(x_1)\cdots g(x_n)$.

► Please refer to Theorem 5.12.2, Formula (5.12.91).

This function generates the set partition data necessary to compute multiple Poisson integrals, namely

$$R_1, R_2, B_1, B_2.$$

Rules for interpreting the output. The integral of a symmetric function over σ with respect to a product of compensated Poisson measures $\widehat{N}$ is a sum of $K + 2$ terms, where K is the number of rows in the display (R_1, R_2, B_1). They are obtained as follows:

(1) The variables with indices in R_1 are identified and integrated over ν; the variables with indices in a block of R_2 are identified. These identified variables, together with the variables with indices in B_1, are integrated over $\widehat{N}$ on the off-diagonals. This is done for each row of the display (R_1, R_2, B_1).

There are two other terms:

(2) The variables with indices in blocks of B_2 are identified. These identified variables, together with the variables in B_1, are integrated over $\widehat{N}$ on the off-diagonals.

(3) The variables with indices in each block of B_2 are identified and integrated over ν; the variables with indices in B_1 are integrated over $\widehat{N}$ on the off-diagonals.

Example. Decompose a Poisson integral on $\sigma = \{\{1, 2\}, \{3, 4\}, \{5\}, \{6\}\}$.

In := PoissonIntegral[$\{\{1, 2\}, \{3, 4\}, \{5\}, \{6\}\}$]

The output is a chart with two groups of outputs. The first gives (R_1, R_2, B_1). The second gives B_2, B_1. The first group (R_1, R_2, B_1) is as follows:

$$\begin{array}{lll} \{\{1,2\}\} & \{\{3,4\}\} & \{\{5\},\{6\}\} \\ \{\{3,4\}\} & \{\{1,2\}\} & \{\{5\},\{6\}\}, \end{array}$$

the first column corresponding to R_1, the second to R_2, the last to B_1. The second group B_2, B_1 is as follows:

$$\{\{1, 2\}, \{3, 4\}\} \qquad \{\{5\}, \{6\}\}.$$

This corresponds to the following decomposition:

$$\begin{aligned}
&\int_{\sigma} g(x_1)\cdots g(x_6)\widehat{N}(dx_1)\cdots\widehat{N}(dx_6) = \\
&\quad 2\left(\int g^2(x_2)\nu(dx_2)\right)\int_{x_4\neq x_5\neq x_6} g^2(x_4)g(x_5)g(x_6)\widehat{N}(dx_4)\widehat{N}(dx_5)\widehat{N}(dx_6) \\
&\quad + \int_{x_2\neq x_4\neq x_5\neq x_6} g^2(x_2)g^2(x_4)g(x_5)g(x_6)\widehat{N}(dx_2)\widehat{N}(dx_4)\widehat{N}(dx_5)\widehat{N}(dx_6) \\
&\quad + \left(\int g^2(x_2)\nu(dx_2)\right)^2\int_{x_5\neq x_6} g(x_5)g(x_6)\widehat{N}(dx_5)\widehat{N}(dx_6).
\end{aligned}$$

Integrating a symmetric function $f(x_1,\cdots,x_7)$ over σ gives

$$\begin{aligned}
&\int_{\sigma} f(x_1,\cdots,x_6)\widehat{N}(dx_1)\cdots\widehat{N}(dx_6) = \\
&\quad 2\int_{x_4\neq x_5\neq x_6} f(x_2,x_2,x_4,x_4,x_5,x_6)\nu(dx_2)\widehat{N}(dx_4)\widehat{N}(dx_5)\widehat{N}(dx_6) \\
&\quad + \int_{x_2\neq x_4\neq x_5\neq x_6} f(x_2,x_2,x_4,x_4,x_5,x_6)\widehat{N}(dx_2)\widehat{N}(dx_4)\widehat{N}(dx_5)\widehat{N}(dx_6) \\
&\quad + \int_{x_5\neq x_6} f(x_2,x_2,x_4,x_4,x_5,x_6)\nu(dx_2)\nu(dx_4)\widehat{N}(dx_5)\widehat{N}(dx_6).
\end{aligned}$$

A.11.5 PoissonIntegralExceed

This function can be used to compute multiple Poisson integrals

$$\int_{\geq\sigma} f(x_1,...,x_n)\widehat{N}(dx_1)\cdots\widehat{N}(dx_n)$$

over all partitions at least as coarse as σ, where $f(x_1,...,x_n)$ is a symmetric function, for example, $g(x_1)\cdots g(x_n)$, and where $\widehat{N}$ is a compensated Poisson measure with control measure ν.

► Please refer to Theorem 5.12.2, Formula (5.12.90).

This function generates the set partition data necessary to compute multiple Poisson integrals, namely B_2, B_1.

Rules for interpreting the output. Equate the variables in each block of B_2 and integrate them with respect to the uncompensated Poisson measure N; the variables listed in B_1 are integrated with respect to the compensated Poisson measure $\widehat{N}$.

Example. Decompose a Poisson integral over $\geq \sigma$ where $\sigma = \{\{1,2,3\},\{4,5\},\{6\},\{7\}\}$.

In := PoissonIntegralExceed$[\{\{1,2,3\},\{4,5\},\{6\},\{7\}\}]$

The output is a chart listing B_2, and B_1 as follows:

$$\{\{1,2,3\},\{4,5\}\} \qquad \{\{6\},\{7\}\}.$$

This corresponds to the decomposition

$$\int_{\geq \sigma} g(x_1)\cdots g(x_7)\widehat{N}(dx_1)\cdots \widehat{N}(dx_7) = \int g^3(x_3)N(dx_3)\int g^2(x_5)N(dx_5)\Big(\int g(x_7)\widehat{N}(dx_7)\Big)^2.$$

Integrating a symmetric function $f(x_1,\cdots,x_7)$ over $\geq \sigma$ gives

$$\int_{\geq \sigma} f(x_1,...,x_n)\widehat{N}(dx_1)\cdots \widehat{N}(dx_n) = \int f(x_3,x_3,x_3,x_5,x_5,x_6,x_7)N(dx_3)N(dx_5)\widehat{N}(dx_6)\widehat{N}(dx_7).$$

A.12 Contraction and Symmetrization

► Please refer to Section 6.2 and Section 6.3.

A.12.1 ContractionWithSymmetrization

In the product of two symmetric functions of p and q variables, respectively, identify r variables in each function and symmetrize the result.

Input format. ContractionWithSymmetrization$[p,q,r]$, where $r \leq \min\{p,q\}$.

Example.

In := ContractionWithSymmetrization$[2,2,1]$

The input function is $f(x_1,x_2)g(x_3,x_4)$. The output is a chart corresponding to

$$(1/2)(f(x_1,x_2)g(x_1,x_3)+f(x_1,x_3)g(x_1,x_2)).$$

A.12.2 ContractionIntegration

In the product of two symmetric functions of p and q variables, respectively, identify r variables in each function, and integrate l of these r variables. The integrated variables are designated by an overtilde, e.g. $\tilde{1}$.

Input format. ContractionIntegration$[p, q, r, l]$, where $l \leq r \leq \min\{p, q\}$.

Example. The input function is $f(x_1, x_2, x_3)g(x_4, x_5)$. Identify one variable and integrate it.

In := ContractionIntegration$[3, 2, 1, 1]$
Out= $\{\{\tilde{1}, 2, 3\}, \{\tilde{1}, 4\}\}$

The output corresponds to $\int f(x_1, x_2, x_3)g(x_1, x_4)\nu(dx_1)$.

A.12.3 ContractionIntegrationWithSymmetrization

In the product of two symmetric functions of p and q variables, respectively, identify r variables in each function, integrate l of these r variables, and then symmetrize the result.

Example. The input function is $f(x_1, x_2, x_3)g(x_4, x_5)$. Identify one variable, integrate it and then symmetrize the result.

In := ContractionIntegrationWithSymmetrization$[3, 2, 1, 1]$

The output gives the number of terms (namely 3) and displays a chart corresponding to

$$(1/3)\int f(x_1, x_2, x_3)g(x_1, x_4) + f(x_1, x_2, x_4)g(x_1, x_3) + f(x_1, x_3, x_4)g(x_1, x_2)\nu(dx_1).$$

A.13 Solving Partition Equations Involving π^*

Some of the functions here are similar to earlier ones which involved an arbitrary set partition π. Here, we are interested in partitions π of the form

$$\pi^* = \{\{1, \cdots, n_1\}, \{n_1 + 1, \cdots, n_1 + n_2\}, ..., \{n_1 + \cdots + n_{k-1} + 1, \cdots, n\}\},$$

as defined in Section A.4.4. The input is a list $(n_1, ..., n_k)$ of positive integers defining π^*.

A.13.1 PiStarMeetSolve

For a finite sequence of positive integers, generate a unique set partition π^* whose block sizes are equal to these integers, and then find all set partitions σ such that $\sigma \wedge \pi^*$ is the singleton partition $\hat{0}$. The result is used in Rota and Wallstrom's Theorem.

Input format. PiStarMeetSolve[list], where 'list" is a list of positive integers that defines the set partition π^*.

Example. For the sequence $\{2, 2\}$, generate π^* and find the meets of π^* that give $\hat{0}$, that is, all partitions σ such that $\sigma \wedge \pi^* = \hat{0}$.

In := PiStarMeetSolve$[\{2, 2\}]$

Here $\pi^* = \{\{1, 2\}, \{3, 4\}\}$. The output gives the number of solutions (namely 7) and lists them:

Out= $\{\{\{1,4\},\{2,3\}\},\ \{\{1,3\},\{2,4\}\},\ \{\{1\},\{2,3\},\{4\}\},$
$\{\{1\},\{2,4\},\{3\}\},\ \{\{1,3\},\{2\},\{4\}\},\ \{\{1,4\},\{2\},\{3\}\},$
$\{\{1\},\{2\},\{3\},\{4\}\}\}$

A.13.2 PiStarJoinSolve

For a finite sequence of positive integers, generate a unique set partition π^* whose block sizes are equal to these integers, and then find all set partitions σ such that $\sigma \vee \pi^*$ is the maximal partition $\hat{1}$.

Input format. PiStarJoinSolve[list], where 'list" is a list of positive integers defining the set partition π^*.

Example. For the sequence $\{2, 2\}$, generate π^* and find the joins of π^* that give $\hat{1}$, that is, all partitions σ such that $\sigma \vee \pi^* = \hat{1}$.

In := PiStarMeetSolve$[\{2, 2\}]$

Here $\pi^* = \{\{1, 2\}, \{3, 4\}\}$. The output gives the number of solutions (namely 11) and lists them:

Out= 213]= $\{\{\{1,2,3,4\}\},\ \{\{1\},\{2,3,4\}\},\ \{\{1,3,4\},\{2\}\},\ \{\{1,2,3\},\{4\}\},$
$\{\{1,4\},\{2,3\}\},\ \{\{1,2,4\},\{3\}\},\ \{\{1,3\},\{2,4\}\},\ \{\{1\},\{2,3\},\{4\}\},$
$\{\{1\},\{2,4\},\{3\}\},\ \{\{1,3\},\{2\},\{4\}\},\ \{\{1,4\},\{2\},\{3\}\}\}$

A.13.3 PiStarMeetAndJoinSolve

For a finite sequence of positive integers, generate a unique set partition π^* whose block sizes are equal to these integers, and then find all set partitions σ such that $\sigma \wedge \pi^*$ is the singleton partition $\hat{0}$ and $\sigma \vee \pi^*$ is the maximal partition $\hat{1}$. This function is used in diagram formulae.

Example. For the sequence $\{2, 2\}$, generate π^* and find the intersection of the meets of π^* that give $\hat{0}$ and the joins of π^* that give $\hat{1}$, that is, all partitions σ such that $\sigma \wedge \pi^* = \hat{0}$ and $\sigma \vee \pi^* = \hat{1}$.

In := PiStarMeetAndJoinSolve$[\{2, 2\}]$

Here $\pi^* = \{\{1, 2\}, \{3, 4\}\}$. The output gives the number of solutions to the meet and join problem (namely 6) and lists them:

Out= $\{\{\{1,4\},\{2,3\}\},\ \{\{1,3\},\{2,4\}\},\ \{\{1\},\{2,3\},\{4\}\},$
$\{\{1\},\{2,4\},\{3\}\},\ \{\{1,3\},\{2\},\{4\}\},\ \{\{1,4\},\{2\},\{3\}\}\}$

A.14 Product of Two Gaussian Multiple Integrals

A.14.1 ProductTwoGaussianIntegrals

Computes the product $I_p^G(f) I_q^G(g)$ of two Gaussian multiple integrals, one of order $p \geq 1$ and one of order $q \geq 1$; the function f has p variables and the function g has q variables; both f and g are symmetric functions. Namely it computes

$$I_p^G(f)\, I_q^G(g) = \sum_{r=0}^{p \wedge q} r! \binom{p}{r} \binom{q}{r} I_{p+q-2r}^G (f \otimes_r g),$$

In the displayed output of the function,

$$\text{coefficient} = r! \binom{p}{r} \binom{q}{r}.$$

Observe that $I_{p+q-2r}^G (f \otimes_r g)$ is equal to $I_{p+q-2r}^G \left(\widetilde{f \otimes_r g} \right)$ where $\sim$ denotes symmetrization.

► Please refer to Proposition 6.4.1.

Input format. ProductTwoGaussianIntegrals$[p, q]$.

Example. The input functions are $f(x_1)$ and $g(x_2)$. Compute $I_1^G(f)I_1^G(g)$.

In := ProductTwoGaussianIntegrals$[1, 1]$

The output is a chart corresponding to

$$I_2^G\,(f \otimes_0 g) + I_0^G\,(f \otimes_1 g) = \\ \int_{x_1 \neq x_2} f(x_1)g(x_2)G(dx_1)G(dx_2) + \int f(x)g(x)\nu(dx).$$

A.15 Product of Two Poisson Multiple Integrals

A.15.1 ProductTwoPoissonIntegrals

Computes the product $I_p^{\widehat{N}}(f)I_q^{\widehat{N}}(g)$ of two Poisson multiple integrals, one of order $p \geq 1$ and one of order $q \geq 1$; the function f has p variables and the function g has q variables; both f and g are symmetric functions. Namely it computes

$$I_p^{\widehat{N}}(f)I_q^{\widehat{N}}(g) = \sum_{r=0}^{p\wedge q} r!\binom{p}{r}\binom{q}{r}\sum_{l=0}^{r}\binom{r}{l} I_{p+q-r-l}^{\widehat{N}}(f \star_r^l g).$$

In the displayed output of the function,

$$\text{coefficient} = r!\binom{p}{r}\binom{q}{r}\binom{r}{l}.$$

Observe that $I_{p+q-r-l}^{\widehat{N}}\left(f \star_r^l g\right)$ is equal to $I_{p+q-r-l}^{\widehat{N}}\left(\widetilde{f \star_r^l g}\right)$ where $\widetilde{\ }$ denotes symmetrization.

► Please refer to Proposition 6.5.1.

Input format. ProductTwoPoissonIntegrals$[p, q]$.

Example. The input functions are $f(x_1)$ and $g(x_2)$. Compute $I_1^{\widehat{N}}(f)I_1^{\widehat{N}}(g)$.

In := ProductTwoPoissonIntegrals[1, 1]

The output is a chart corresponding to

$$I_2^{\widehat{N}}\left(f \otimes_0^0 g\right) + I_1^{\widehat{N}}\left(f \star_1^0 g\right) + I_0^{\widehat{N}}\left(f \star_1^1 g\right) \\ = \int_{x \neq y} f(x)g(y)\widehat{N}(dx)\widehat{N}(dy) + \int f(x)g(x)\widehat{N}(dx) + \int f(x)g(x)\nu(dx) \\ = \int_{x \neq y} f(x)g(y)\widehat{N}(dx)\widehat{N}(dy) + \int f(x)g(x)N(dx).$$

A.16 MSets and MZeroSets

These sets appear in diagram formulae involving Gaussian and Poisson multiple integrals. They have similar structure. Given a list $(n_1, \cdots, n_k)$ of positive integers, let $n = n_1 + \cdots + n_k$ and let π^* be the special partition of $[n] = \{1, ..., n\}$ defined in Section A.4.4. The MSets and MZeroSets are a collection of set partitions of $[n] = \{1, ..., n\}$.

The sets $\mathcal{M}_2([n], \pi^*)$ and $\mathcal{M}_2^0([n], \pi^*)$ appear in diagram formulae when the random measure is Gaussian (see Section 7.3). The sets $\mathcal{M}_{\geq 2}([n], \pi^*)$ and $\mathcal{M}_{\geq 2}^0([n], \pi^*)$ appear when the random measure is compensated Poisson (see Section 7.4). They are respectively subsets of $\mathcal{M}([n], \pi^*)$ (involving meet and join) and $\mathcal{M}^0([n], \pi^*)$ (involving meet only).

► Please refer to Section 7.2.

Input format. FunctionName[list], where "list" is a list $(n_1, ..., n_k)$ of positive integers.

Example. Let the list be $(2, 2)$ so that $\pi^* = \{\{1, 2\}, \{3, 4\}\}$. Find the MSets and MZeroSets.

A.16.1 MSets

This function produces the same output as PiStarMeetAndJoinSolve. It provides the set partitions in $\mathcal{M}([n], \pi^*)$, namely all set partitions σ such that $\sigma \wedge \pi^* = \hat{0}$ and $\sigma \vee \pi^* = \hat{1}$.

A.16.2 MSetsEqualTwo

This function produces all solutions to PiStarMeetAndJoinSolve containing partitions only of size two. It provides the set partitions in $\mathcal{M}_2([n], \pi^*)$.

A.16.3 MSetsGreaterEqualTwo

This function produces all solutions to PiStarMeetAndJoinSolve containing no singletons. It provides the set partitions in $\mathcal{M}_{\geq 2}([n], \pi^*)$.

A.16.4 MZeroSets

This function produces the same output as PiStarMeetSolve. It provides the set partitions in $\mathcal{M}^0([n], \pi^*)$, namely all set partitions σ such that $\sigma \wedge \pi^* = \hat{0}$.

A.16.5 MZeroSetsEqualTwo

This function produces all solutions to PiStarMeetSolve containing partitions only of size two. It provides the set partitions in $\mathcal{M}_2^0([n], \pi^*)$.

A.16.6 MZeroSetsGreaterEqualTwo

This function produces all solutions to PiStarMeetSolve containing no singletons. It provides the set partitions in $\mathcal{M}_{\geq 2}^0([n], \pi^*)$.

Example. (See above.) The (full) output also indicates the number of solutions.

In := MSets[{2, 2}]
Out= {{{1, 4}, {2, 3}}, {{1, 3}, {2, 4}}, {{1}, {2, 3}, {4}},
{{1}, {2, 4}, {3}}, {{1, 3}, {2}, {4}}, {{1, 4}, {2}, {3}}}

In := MSetsEqualTwo[{2, 2}]
Out= {{{1, 4}, {2, 3}}, {{1, 3}, {2, 4}}}

In := MSetsGreaterEqualTwo[{2, 2}]
Out= {{{1, 4}, {2, 3}}, {{1, 3}, {2, 4}}}

In := MZeroSets[{2, 2}]
Out= {{{1, 4}, {2, 3}}, {{1, 3}, {2, 4}}, {{1}, {2, 3}, {4}}, {{1}, {2, 4}, {3}},
{{1, 3}, {2}, {4}}, {{1, 4}, {2}, {3}}}, {{1}, {2}, {3}, {4}}}

In := MZeroSetsEqualTwo[{2, 2}]
Out= {{{1, 4}, {2, 3}}, {{1, 3}, {2, 4}}}

In := MZeroSetsGreaterEqualTwo[{2, 2}]
Out= {{{1, 4}, {2, 3}}, {{1, 3}, {2, 4}}}

Example. A case where the cardinality of π^* is odd and, therefore, the output is empty when executing MZeroSetsEqualTwo.

In := MZeroSetsEqualTwo[{2, 2, 1}]
Out= := {}

A.17 Hermite Polynomials

A.17.1 HermiteRho

Computes Hermite polynomials with a leading coefficient of 1 and parameter $\rho > 0$. They are orthogonal with respect to the Gaussian distribution with mean zero and variance ρ. The generating function is $e^{tx-\frac{1}{2}\rho t^2}$.

These polynomials are defined by:

$$H_n(x,\rho) = (-\rho)^n e^{\frac{x^2}{2\rho}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2\rho}},\ n \geq 0,$$

and satisfy

$$H_n(x,\rho) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} (2k-1)!!(-\rho)^k x^{n-2k},\ n \geq 0.$$

► Please refer to Section 8.1.

Example. Compute the Hermite-Rho polynomial of order 5 with a leading coefficient of 1.

In := HermiteRho$[5, x, \rho]$
Out= $x^5 - 10x^3\rho + 15x\rho^2$

A.17.2 HermiteRhoGrid

Display the Hermite-Rho polynomials up to order n, using a leading coefficient of 1 for each polynomial.

Example. Make a grid of the first 11 Hermite-Rho polynomials (that is, go up to order 10).

In := HermiteRhoGrid[10]

The output is a chart:

0	1
1	x
2	$-\rho + x^2$
3	$-3\rho x + x^3$
4	$3\rho^2 - 6\rho x^2 + x^4$
5	$15\rho^2 x - 10\rho x^3 + x^5$
6	$-15\rho^3 + 45\rho^2 x^2 - 15\rho x^4 + x^6$
7	$-105\rho^3 x + 105\rho^2 x^3 - 21\rho x^5 + x^7$
8	$105\rho^4 - 420\rho^3 x^2 + 210\rho^2 x^4 - 28\rho x^6 + x^8$
9	$945\rho^4 x - 1260\rho^3 x^3 + 378\rho^2 x^5 - 36\rho x^7 + x^9$
10	$-945\rho^5 + 4725\rho^4 x^2 - 3150\rho^3 x^4 + 630\rho^2 x^6 - 45\rho x^8 + x^{10}$.

A.17.3 Hermite

Computes Hermite polynomials with a leading coefficient of 1. These polynomials are obtained by setting $\rho = 1$ and are defined as

$$H_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}}, \quad x \in \mathbb{R}.$$

Example. Compute the Hermite polynomial of order 5 with a leading coefficient of 1.

In := Hermite$[5, x]$ // TraditionalForm
Out= $x^5 - 10x^3 + 15x$

A.17.4 HermiteGrid

Display the Hermite polynomials up to order n, using a leading coefficient of 1 for each polynomial.

Example. Make a grid of the first 11 Hermite polynomials.

In := HermiteGrid[10] // TraditionalForm

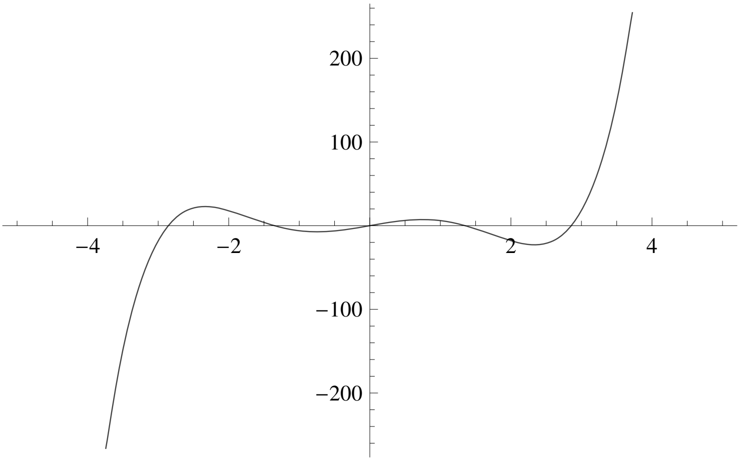

Fig. A.1. Hermite polynomial with $n = 5$

The output is a chart:

0	1
1	x
2	$x^2 - 1$
3	$x^3 - 3x$
4	$x^4 - 6x^2 + 3$
5	$x^5 - 10x^3 + 15x$
6	$x^6 - 15x^4 + 45x^2 - 15$
7	$x^7 - 21x^5 + 105x^3 - 105x$
8	$x^8 - 28x^6 + 210x^4 - 420x^2 + 105$
9	$x^9 - 36x^7 + 378x^5 - 1260x^3 + 945x$
10	$x^{10} - 45x^8 + 630x^6 - 3150x^4 + 4725x^2 - 945.$

Example. Plot $H_5(x)$ for $-5 \leq x \leq 5$. The output is Fig. A.1.

In := Plot[Hermite[5, x], {x, -5, 5}]

Example. Plot $H_k(x)$ for $k = 0, \cdots, 6$ and $-5 \leq x \leq 5$. The output is Fig. A.2.

In := Plot[Evaluate[Hermite[#, x] & /@ Range[0, 6, 1]], {x, -5, 5}, PlotStyle → {}]

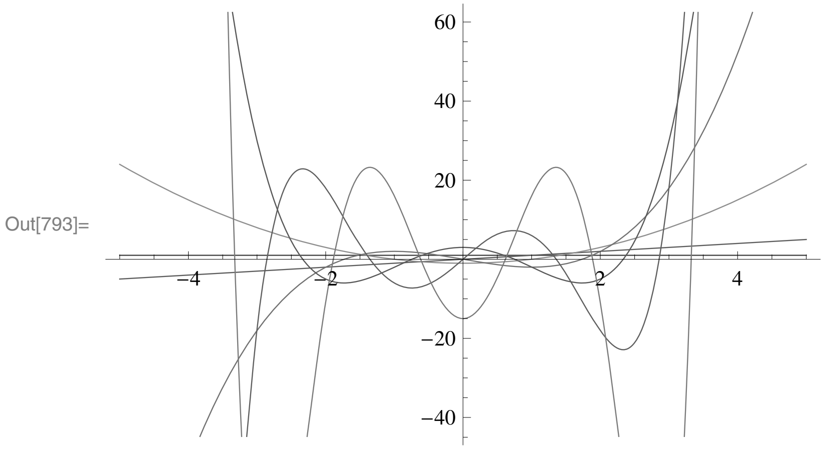

Fig. A.2. Hermite polynomials with $1 \leq n \leq 6$

A.17.5 HermiteH

The built-in *Mathematica* function HermiteH[n,x] is defined as

$$\text{HermiteH}[n, x] = (-1)^n\, e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

One has:

$$H_n(x) = 2^{-n/2}\text{HermiteH}\left[n, \frac{x}{\sqrt{2}}\right].$$

In := HermiteH$[2, x]$
Out= $-2 + 4x^2$

A.17.6 HermiteHGrid

Displays HermiteH as a chart. For example,

In := HermiteHGrid[3]

The output is a chart:

0	1
1	$2x$
2	$-2 + 4x^2$
3	$-12x + 8x^3.$

A.18 Poisson-Charlier Polynomials

A.18.1 Charlier

Computes the Poisson-Charlier Polynomials. They are orthogonal with respect to the Poisson distribution with mean a. These polynomials $c_n(x,a)$, defined for $n \geq 0$ and $a > 0$, satisfy the orthogonality relation

$$\sum_{k=0}^{\infty} c_n(k,a)c_m(k,a)e^{-a}\frac{a^k}{k!} = \frac{n!}{a^n}\delta_{nm}$$

and the recursion relation

$$\begin{aligned} c_0(x,a) &= 1 \\ c_{n+1}(x,a) &= a^{-1}xc_n(k-1,a) - c_n(x,a), \quad n \geq 1. \end{aligned}$$

► Please refer to Chapter 10.

Example. Compute the Poisson-Charlier polynomial with $n = 8$ and $a = 1$.

In := Charlier[8, x, 1] // TraditionalForm
Out= $x^8 - 36x^7 + 518x^6 - 3836x^5 + 15659x^4 - 34860x^3 + 38618x^2 - 16072x + 1$

Example. Compute the Poisson-Charlier polynomial with $n = 2$ and $a > 0$.

In := Collect[Charlier[2, x, a], x] // TraditionalForm
Out= $\frac{x^2}{a^2} + \left(-\frac{1}{a^2} - \frac{2}{a}\right)x + 1$

(“Collect” gathers together terms involving the same powers of x.)

A.18.2 CharlierGrid

Displays the Poisson-Charlier polynomials up to order n with Poisson mean $a > 0$.

Example. Make a grid of the Poisson-Charlier polynomials for $0 \leq n \leq 5$.

In := ChalierGrid[5, a] // TraditionalForm

Out=

0	1
1	$\frac{x}{a} - 1$
2	$\frac{x^2}{a^2} + \left(-\frac{1}{a^2} - \frac{2}{a}\right) x + 1$
3	$\frac{x^3}{a^3} + \left(-\frac{3}{a^3} - \frac{3}{a^2}\right) x^2 + \left(\frac{2}{a^3} + \frac{3}{a^2} + \frac{3}{a}\right) x - 1$
4	$\frac{x^4}{a^4} + \left(-\frac{6}{a^4} - \frac{4}{a^3}\right) x^3 + \left(\frac{11}{a^4} + \frac{12}{a^3} + \frac{6}{a^2}\right) x^2 + \left(-\frac{6}{a^4} - \frac{8}{a^3} - \frac{6}{a^2} - \frac{4}{a}\right) x + 1$
5	$\frac{x^5}{a^5} + \left(-\frac{10}{a^5} - \frac{5}{a^4}\right) x^4 + \left(\frac{35}{a^5} + \frac{30}{a^4} + \frac{10}{a^3}\right) x^3 + \left(-\frac{50}{a^5} - \frac{55}{a^4} - \frac{30}{a^3} - \frac{10}{a^2}\right) x^2 + \left(\frac{24}{a^5} + \frac{30}{a^4} + \frac{20}{a^3} + \frac{10}{a^2} + \frac{5}{a}\right) x - 1$

Example. Plot the Poisson-Charlier polynomial with $n = 5$ and $a = 1/2$ over the interval $-1 \leq x \leq 3$.

In := Plot[Charlier[5, x, 0.5], {x, -1, 3}, PlotRange → All]

Example. Plot the Poisson-Charlier polynomials with $a = 1$ and $1 \leq n \leq 6$ over the interval $-1 \leq x \leq 1$. The output is Fig. A.3.

In := Plot[Evaluate[Charlier[#, x, 1] & /@ Range[1, 6]], {x, -1, 1}, PlotStyle → { }]

Out[803]=

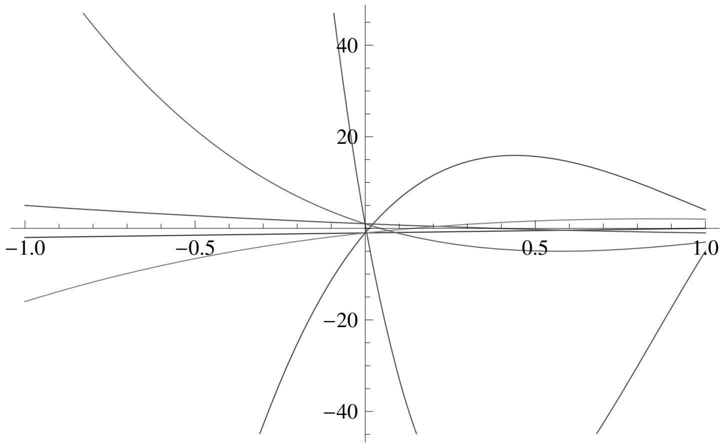

Fig. A.3. Poisson-Charlier polynomials with $1 \leq n \leq 6$

A.18.3 CharlierCentered

These polynomials are defined by

$$C_n(x, a) = a^n c_n(x + a, a).$$

They are orthogonal with respect to the centered Poisson distribution with parameter $a > 0$.

Example. Compute the centered Poisson-Charlier polynomial with $n = 3$.

In := Collect[CharlierCentered[3, x, a], x]
Out= $2a + (2 - 3a)x - 3x^2 + x^3$

Example. Compute the centered Poisson-Charlier polynomial with $n = 3$ and $a = 1$.

In := CharlierCentered[8, x, 1] // TraditionalForm
Out= $x^8 - 28x^7 + 294x^6 - 1428x^5 + 3059x^4 - 1428x^3 - 3326x^2 + 2904x - 7$

Example. Plot the centered Poisson-Charlier polynomials with $a = 1$ and $1 \le n \le 6$ over the interval $-1 \le x \le 1$. The output is Fig. A.4.

In := Plot[Evaluate[CharlierCentered[#, x, 1] & /@ Range[1, 6]], x, -1, 1, Plot Range → All]

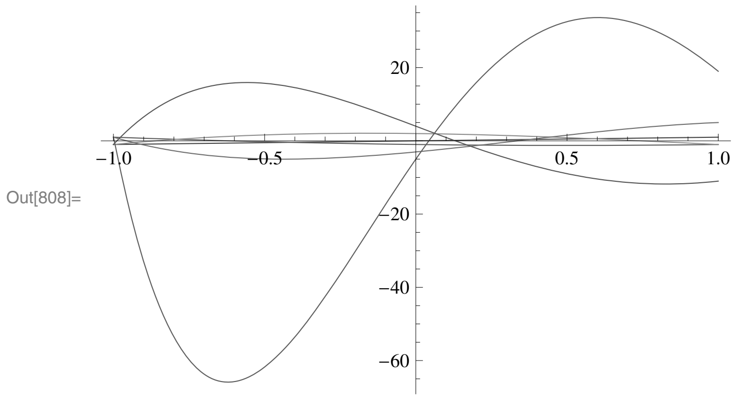

Fig. A.4. Centered Charlier polynomials with $a = 1$ and $1 \le n \le 6$

A.18.4 CharlierCenteredGrid

Displays the centered Poisson-Charlier polynomials up to order n.

Example. Make a grid of the centered Poisson-Charlier polynomials for $0 \le n \le 5$.

In := ChalierCenteredGrid[5, a]
Out=

0	1
1	x
2	$x^2 - x - a$
3	$x^3 - 3x^2 + (2 - 3a)x + 2a$
4	$x^4 - 6x^3 + (11 - 6a)x^2 + (14a - 6)x + 3a^2 - 6a$
5	$x^5 - 10x^4 + (35 - 10a)x^3 + (50a - 50)x^2 + (15a^2 - 70a + 24)x - 20a^2 + 24a$
6	$x^6 - 15x^5 + (85 - 15a)x^4 + (130a - 225)x^3 + (45a^2 - 375a + 274)x^2$ $+(-165a^2 + 404a - 120)x - 15a^3 + 130a^2 - 120a$

Appendix B

Tables of moments and cumulants

In the following tables, μ_n and κ_n denote respectively the nth moment and nth cumulant of a random variable X. If X has mean zero, then $\mu_1 = \kappa_1 = 0$ and μ_n and κ_n are said to be "centered". If, in addition, X has variance one, then $\mu_2 = \kappa_2 = 1$ as well, and μ_n and κ_n are said to be "centered and scaled".

List of tables:

G. Peccati, M.S. Taqqu: Wiener Chaos: Moments, Cumulants and Diagrams –
A survey with computer implementation.

Centered and scaled Cumulant to Moments

n	$\kappa_{\mathbf{n}} =$
1	0
2	1
3	μ_3
4	$-3 + \mu_4$
5	$-10\mu_3 + \mu_5$
6	$30 - 10\mu_3^2 - 15\mu_4 + \mu_6$
7	$210\mu_3 - 35\mu_3\mu_4 - 21\mu_5 + \mu_7$
8	$-630 + 560\mu_3^2 + 420\mu_4 - 35\mu_4^2 - 56\mu_3\mu_5 - 28\mu_6 + \mu_8$
9	$-7560\mu_3 + 560\mu_3^3 + 2520\mu_3\mu_4 + 756\mu_5 - 126\mu_4\mu_5 - 84\mu_3\mu_6 - 36\mu_7 + \mu_9$
10	$22680 - 37800\mu_3^2 - 18900\mu_4 + 4200\mu_3^2\mu_4 + 3150\mu_4^2 + 5040\mu_3\mu_5 - 126\mu_5^2 + 1260\mu_6 - 210\mu_4\mu_6 - 120\mu_3\mu_7 - 45\mu_8 + \mu_{10}$
12	$-1247400 + 3326400\mu_3^2 - 92400\mu_3^4 + 1247400\mu_4 - 831600\mu_3^2\mu_4 - 311850\mu_4^2 + 11550\mu_4^3 - 498960\mu_3\mu_5 + 55440\mu_3\mu_4\mu_5 + 16632\mu_5^2 - 83160\mu_6 + 18480\mu_3^2\mu_6 + 27720\mu_4\mu_6 - 462\mu_6^2 + 15840\mu_3\mu_7 - 792\mu_5\mu_7 + 2970\mu_8 - 495\mu_4\mu_8 - 220\mu_3\mu_9 - 66\mu_{10} + \mu_{12}$
13	$-32432400\mu_3 + 14414400\mu_3^3 + 21621600\mu_3\mu_4 - 1201200\mu_3^3\mu_4 - 2702700\mu_3\mu_4^2 + 120120\mu_3\mu_4\mu_6 + 72072\mu_5\mu_6 - 154440\mu_7 + 34320\mu_3^2\mu_7 + 51480\mu_4\mu_7 - 1716\mu_6\mu_7 + 25740\mu_3\mu_8 - 1287\mu_5\mu_8 + 4290\mu_9 - 715\mu_4\mu_9 - 286\mu_3\mu_{10} - 78\mu_{11} + \mu_{13}$
14	$97297200 - 378378000\mu_3^2 + 33633600\mu_3^4 - 113513400\mu_4 + 151351200\mu_3^2\mu_4 + 37837800\mu_4^2 - 6306300\mu_3^2\mu_4^2 - 3153150\mu_4^3 + 60540480\mu_3\mu_5 - 3363360\mu_3^3\mu_5 - 15135120\mu_3\mu_4\mu_5 - 2270268\mu_5^2 + 252252\mu_4\mu_5^2 + 7567560\mu_6 - 5045040\mu_3^2\mu_6 - 3783780\mu_4\mu_6 + 210210\mu_4^2\mu_6 + 336336\mu_3\mu_5\mu_6 + 84084\mu_6^2 - 2162160\mu_3\mu_7 + 240240\mu_3\mu_4\mu_7 + 144144\mu_5\mu_7 - 1716\mu_7^2 - 270270\mu_8 + 60060\mu_3^2\mu_8 + 90090\mu_4\mu_8 - 3003\mu_6\mu_8 + 40040\mu_3\mu_9 - 2002\mu_5\mu_9 + 6006\mu_{10} - 1001\mu_4\mu_{10} - 364\mu_3\mu_{11} - 91\mu_{12} + \mu_{14}$

Centered Cumulant to Moments

n	$\kappa_n =$
1	0
2	μ_2
3	μ_3
4	$-3\mu_2^2 + \mu_4$
5	$-10\mu_2\mu_3 + \mu_5$
6	$30\mu_2^3 - 10\mu_3^2 - 15\mu_2\mu_4 + \mu_6$
7	$210\mu_2^2\mu_3 - 35\mu_3\mu_4 - 21\mu_2\mu_5 + \mu_7$
8	$-630\mu_2^4 + 560\mu_2\mu_3^2 + 420\mu_2^2\mu_4 - 35\mu_4^2 - 56\mu_3\mu_5 - 28\mu_2\mu_6 + \mu_8$
9	$-7560\mu_2^3\mu_3 + 560\mu_3^3 + 2520\mu_2\mu_3\mu_4 + 756\mu_2^2\mu_5 - 126\mu_4\mu_5 - 84\mu_3\mu_6 - 36\mu_2\mu_7 + \mu_9$
10	$22680\mu_2^5 - 37800\mu_2^2\mu_3^2 - 18900\mu_2^3\mu_4 + 4200\mu_3^2\mu_4 + 3150\mu_2\mu_4^2 + 5040\mu_2\mu_3\mu_5 - 126\mu_5^2 + 1260\mu_2^2\mu_6 - 210\mu_4\mu_6 - 120\mu_3\mu_7 - 45\mu_2\mu_8 + \mu_{10}$
11	$415800\mu_2^4\mu_3 - 92400\mu_2\mu_3^3 - 207900\mu_2^2\mu_3\mu_4 + 11550\mu_3\mu_4^2 - 41580\mu_2^3\mu_5 + 9240\mu_3^2\mu_5 + 13860\mu_2\mu_4\mu_5 + 9240\mu_2\mu_3\mu_6 - 462\mu_5\mu_6 + 1980\mu_2^2\mu_7 - 330\mu_4\mu_7 - 165\mu_3\mu_8 - 55\mu_2\mu_9 + \mu_{11}$
12	$-1247400\mu_2^6 + 3326400\mu_2^3\mu_3^2 - 92400\mu_3^4 + 1247400\mu_2^4\mu_4 - 831600\mu_2\mu_3^2\mu_4 - 311850\mu_2^2\mu_4^2 + 11550\mu_4^3 - 498960\mu_2^2\mu_3\mu_5 + 55440\mu_3\mu_4\mu_5 + 16632\mu_2\mu_5^2 - 83160\mu_2^3\mu_6 + 18480\mu_3^2\mu_6 + 27720\mu_2\mu_4\mu_6 - 462\mu_6^2 + 15840\mu_2\mu_3\mu_7 - 792\mu_5\mu_7 + 2970\mu_2^2\mu_8 - 495\mu_4\mu_8 - 220\mu_3\mu_9 - 66\mu_2\mu_{10} + \mu_{12}$
13	$-32432400\mu_2^5\mu_3 + 14414400\mu_2^2\mu_3^3 + 21621600\mu_2^3\mu_3\mu_4 - 1201200\mu_3^3\mu_4 - 2702700\mu_2\mu_3\mu_4^2 + 3243240\mu_2^4\mu_5 - 2162160\mu_2\mu_3^2\mu_5 - 1621620\mu_2^2\mu_4\mu_5 + 90090\mu_4^2\mu_5 + 72072\mu_3\mu_5^2 - 1081080\mu_2^2\mu_3\mu_6 + 120120\mu_3\mu_4\mu_6 + 72072\mu_2\mu_5\mu_6 - 154440\mu_2^3\mu_7 + 34320\mu_3^2\mu_7 + 51480\mu_2\mu_4\mu_7 - 1716\mu_6\mu_7 + 25740\mu_2\mu_3\mu_8 - 1287\mu_5\mu_8 + 4290\mu_2^2\mu_9 - 715\mu_4\mu_9 - 286\mu_3\mu_{10} - 78\mu_2\mu_{11} + \mu_{13}$

Cumulant to Moments

$\mathbf{n}$	$\kappa_{\mathbf{n}} =$
1	μ_1
2	$-\mu_1^2 + \mu_2$
3	$2\mu_1^3 - 3\mu_1\mu_2 + \mu_3$
4	$-6\mu_1^4 + 12\mu_1^2\mu_2 - 3\mu_2^2 - 4\mu_1\mu_3 + \mu_4$
5	$24\mu_1^5 - 60\mu_1^3\mu_2 + 30\mu_1\mu_2^2 + 20\mu_1^2\mu_3 - 10\mu_2\mu_3 - 5\mu_1\mu_4 + \mu_5$
6	$-120\mu_1^6 + 360\mu_1^4\mu_2 - 270\mu_1^2\mu_2^2 + 30\mu_2^3 - 120\mu_1^3\mu_3 + 120\mu_1\mu_2\mu_3 - 10\mu_3^2 + 30\mu_1^2\mu_4 - 15\mu_2\mu_4 - 6\mu_1\mu_5 + \mu_6$
7	$720\mu_1^7 - 2520\mu_1^5\mu_2 + 2520\mu_1^3\mu_2^2 - 630\mu_1\mu_2^3 + 840\mu_1^4\mu_3 - 1260\mu_1^2\mu_2\mu_3 + 210\mu_2^2\mu_3 + 140\mu_1\mu_3^2 - 210\mu_1^3\mu_4 + 210\mu_1\mu_2\mu_4 - 35\mu_3\mu_4 + 42\mu_1^2\mu_5 - 21\mu_2\mu_5 - 7\mu_1\mu_6 + \mu_7$
8	$-5040\mu_1^8 + 20160\mu_1^6\mu_2 - 25200\mu_1^4\mu_2^2 + 10080\mu_1^2\mu_2^3 - 630\mu_2^4 - 6720\mu_1^5\mu_3 + 13440\mu_1^3\mu_2\mu_3 - 5040\mu_1\mu_2^2\mu_3 - 1680\mu_1^2\mu_3^2 + 560\mu_2\mu_3^2 + 1680\mu_1^4\mu_4 - 2520\mu_1^2\mu_2\mu_4 + 420\mu_2^2\mu_4 + 560\mu_1\mu_3\mu_4 - 35\mu_4^2 - 336\mu_1^3\mu_5 + 336\mu_1\mu_2\mu_5 - 56\mu_3\mu_5 + 56\mu_1^2\mu_6 - 28\mu_2\mu_6 - 8\mu_1\mu_7 + \mu_8$
9	$40320\mu_1^9 - 181440\mu_1^7\mu_2 + 272160\mu_1^5\mu_2^2 - 151200\mu_1^3\mu_2^3 + 22680\mu_1\mu_2^4 + 60480\mu_1^6\mu_3 - 151200\mu_1^4\mu_2\mu_3 + 90720\mu_1^2\mu_2^2\mu_3 - 7560\mu_2^3\mu_3 + 20160\mu_1^3\mu_3^2 - 15120\mu_1\mu_2\mu_3^2 + 560\mu_3^3 - 15120\mu_1^5\mu_4 + 30240\mu_1^3\mu_2\mu_4 - 11340\mu_1\mu_2^2\mu_4 - 7560\mu_1^2\mu_3\mu_4 + 2520\mu_2\mu_3\mu_4 + 630\mu_1\mu_4^2 + 3024\mu_1^4\mu_5 - 4536\mu_1^2\mu_2\mu_5 + 756\mu_2^2\mu_5 + 1008\mu_1\mu_3\mu_5 - 126\mu_4\mu_5 - 504\mu_1^3\mu_6 + 504\mu_1\mu_2\mu_6 - 84\mu_3\mu_6 + 72\mu_1^2\mu_7 - 36\mu_2\mu_7 - 9\mu_1\mu_8 + \mu_9$
10	$-362880\mu_1^{10} + 1814400\mu_1^8\mu_2 - 3175200\mu_1^6\mu_2^2 + 2268000\mu_1^4\mu_2^3 - 567000\mu_1^2\mu_2^4 + 22680\mu_2^5 - 604800\mu_1^7\mu_3 + 1814400\mu_1^5\mu_2\mu_3 - 1512000\mu_1^3\mu_2^2\mu_3 + 302400\mu_1\mu_2^3\mu_3 - 252000\mu_1^4\mu_3^2 + 302400\mu_1^2\mu_2\mu_3^2 - 37800\mu_2^2\mu_3^2 - 16800\mu_1\mu_3^3 + 151200\mu_1^6\mu_4 - 378000\mu_1^4\mu_2\mu_4 + 226800\mu_1^2\mu_2^2\mu_4 - 18900\mu_2^3\mu_4 + 100800\mu_1^3\mu_3\mu_4 - 75600\mu_1\mu_2\mu_3\mu_4 + 4200\mu_3^2\mu_4 - 9450\mu_1^2\mu_4^2 + 3150\mu_2\mu_4^2 - 30240\mu_1^5\mu_5 + 60480\mu_1^3\mu_2\mu_5 - 22680\mu_1\mu_2^2\mu_5 - 15120\mu_1^2\mu_3\mu_5 + 5040\mu_2\mu_3\mu_5 + 2520\mu_1\mu_4\mu_5 - 126\mu_5^2 + 5040\mu_1^4\mu_6 - 7560\mu_1^2\mu_2\mu_6 + 1260\mu_2^2\mu_6 + 1680\mu_1\mu_3\mu_6 - 210\mu_4\mu_6 - 720\mu_1^3\mu_7 + 720\mu_1\mu_2\mu_7 - 120\mu_3\mu_7 + 90\mu_1^2\mu_8 - 45\mu_2\mu_8 - 10\mu_1\mu_9 + \mu_{10}$

Centered and scaled Moment to Cumulants

$\mathbf{n}$	$\mu_{\mathbf{n}} =$
1	0
2	1
3	κ_3
4	$3 + \kappa_4$
5	$10\kappa_3 + \kappa_5$
6	$15 + 10\kappa_3^2 + 15\kappa_4 + \kappa_6$
7	$105\kappa_3 + 35\kappa_3\kappa_4 + 21\kappa_5 + \kappa_7$
8	$105 + 280\kappa_3^2 + 210\kappa_4 + 35\kappa_4^2 + 56\kappa_3\kappa_5 + 28\kappa_6 + \kappa_8$
9	$1260\kappa_3 + 280\kappa_3^3 + 1260\kappa_3\kappa_4 + 378\kappa_5 + 126\kappa_4\kappa_5 + 84\kappa_3\kappa_6 + 36\kappa_7 + \kappa_9$
10	$945 + 6300\kappa_3^2 + 3150\kappa_4 + 2100\kappa_3^2\kappa_4 + 1575\kappa_4^2 + 2520\kappa_3\kappa_5 + 126\kappa_5^2 + 630\kappa_6 + 210\kappa_4\kappa_6 + 120\kappa_3\kappa_7 + 45\kappa_8 + \kappa_{10}$
11	$17325\kappa_3 + 15400\kappa_3^3 + 34650\kappa_3\kappa_4 + 5775\kappa_3\kappa_4^2 + 6930\kappa_5 + 4620\kappa_3^2\kappa_5 + 6930\kappa_4\kappa_5 + 4620\kappa_3\kappa_6 + 462\kappa_5\kappa_6 + 990\kappa_7 + 330\kappa_4\kappa_7 + 165\kappa_3\kappa_8 + 55\kappa_9 + \kappa_{11}$
12	$10395 + 138600\kappa_3^2 + 15400\kappa_3^4 + 51975\kappa_4 + 138600\kappa_3^2\kappa_4 + 51975\kappa_4^2 + 5775\kappa_4^3 + 83160\kappa_3\kappa_5 + 27720\kappa_3\kappa_4\kappa_5 + 8316\kappa_5^2 + 13860\kappa_6 + 9240\kappa_3^2\kappa_6 + 13860\kappa_4\kappa_6 + 462\kappa_6^2 + 7920\kappa_3\kappa_7 + 792\kappa_5\kappa_7 + 1485\kappa_8 + 495\kappa_4\kappa_8 + 220\kappa_3\kappa_9 + 66\kappa_{10} + \kappa_{12}$
13	$270270\kappa_3 + 600600\kappa_3^3 + 900900\kappa_3\kappa_4 + 200200\kappa_3^3\kappa_4 + 450450\kappa_3\kappa_4^2 + 135135\kappa_5 + 360360\kappa_3^2\kappa_5 + 270270\kappa_4\kappa_5 + 45045\kappa_4^2\kappa_5 + 36036\kappa_3\kappa_5^2 + 180180\kappa_3\kappa_6 + 60060\kappa_3\kappa_4\kappa_6 + 36036\kappa_5\kappa_6 + 25740\kappa_7 + 17160\kappa_3^2\kappa_7 + 25740\kappa_4\kappa_7 + 1716\kappa_6\kappa_7 + 12870\kappa_3\kappa_8 + 1287\kappa_5\kappa_8 + 2145\kappa_9 + 715\kappa_4\kappa_9 + 286\kappa_3\kappa_{10} + 78\kappa_{11} + \kappa_{13}$
14	$135135 + 3153150\kappa_3^2 + 1401400\kappa_3^4 + 945945\kappa_4 + 6306300\kappa_3^2\kappa_4 + 1576575\kappa_4^2 + 1051050\kappa_3^2\kappa_4^2 + 525525\kappa_4^3 + 2522520\kappa_3\kappa_5 + 560560\kappa_3^3\kappa_5 + 2522520\kappa_3\kappa_4\kappa_5 + 378378\kappa_5^2 + 126126\kappa_4\kappa_5^2 + 315315\kappa_6 + 840840\kappa_3^2\kappa_6 + 630630\kappa_4\kappa_6 + 105105\kappa_4^2\kappa_6 + 168168\kappa_3\kappa_5\kappa_6 + 42042\kappa_6^2 + 360360\kappa_3\kappa_7 + 120120\kappa_3\kappa_4\kappa_7 + 72072\kappa_5\kappa_7 + 1716\kappa_7^2 + 45045\kappa_8 + 30030\kappa_3^2\kappa_8 + 45045\kappa_4\kappa_8 + 3003\kappa_6\kappa_8 + 20020\kappa_3\kappa_9 + 2002\kappa_5\kappa_9 + 3003\kappa_{10} + 1001\kappa_4\kappa_{10} + 364\kappa_3\kappa_{11} + 91\kappa_{12} + \kappa_{14}$

Centered Moment to Cumulants

n	$\mu_n =$
1	0
2	κ_2
3	κ_3
4	$3\kappa_2^2 + \kappa_4$
5	$10\kappa_2\kappa_3 + \kappa_5$
6	$15\kappa_2^3 + 10\kappa_3^2 + 15\kappa_2\kappa_4 + \kappa_6$
7	$105\kappa_2^2\kappa_3 + 35\kappa_3\kappa_4 + 21\kappa_2\kappa_5 + \kappa_7$
8	$105\kappa_2^4 + 280\kappa_2\kappa_3^2 + 210\kappa_2^2\kappa_4 + 35\kappa_4^2 + 56\kappa_3\kappa_5 + 28\kappa_2\kappa_6 + \kappa_8$
9	$1260\kappa_2^3\kappa_3 + 280\kappa_3^3 + 1260\kappa_2\kappa_3\kappa_4 + 378\kappa_2^2\kappa_5 + 126\kappa_4\kappa_5 + 84\kappa_3\kappa_6 + 36\kappa_2\kappa_7 + \kappa_9$
10	$945\kappa_2^5 + 6300\kappa_2^2\kappa_3^2 + 3150\kappa_2^3\kappa_4 + 2100\kappa_3^2\kappa_4 + 1575\kappa_2\kappa_4^2 + 2520\kappa_2\kappa_3\kappa_5 + 126\kappa_5^2 + 630\kappa_2^2\kappa_6 + 210\kappa_4\kappa_6 + 120\kappa_3\kappa_7 + 45\kappa_2\kappa_8 + \kappa_{10}$
11	$17325\kappa_2^4\kappa_3 + 15400\kappa_2\kappa_3^3 + 34650\kappa_2^2\kappa_3\kappa_4 + 5775\kappa_3\kappa_4^2 + 6930\kappa_2^3\kappa_5 + 4620\kappa_3^2\kappa_5 + 6930\kappa_2\kappa_4\kappa_5 + 4620\kappa_2\kappa_3\kappa_6 + 462\kappa_5\kappa_6 + 990\kappa_2^2\kappa_7 + 330\kappa_4\kappa_7 + 165\kappa_3\kappa_8 + 55\kappa_2\kappa_9 + \kappa_{11}$
12	$10395\kappa_2^6 + 138600\kappa_2^3\kappa_3^2 + 15400\kappa_3^4 + 51975\kappa_2^4\kappa_4 + 138600\kappa_2\kappa_3^2\kappa_4 + 51975\kappa_2^2\kappa_4^2 + 5775\kappa_4^3 + 83160\kappa_2^2\kappa_3\kappa_5 + 27720\kappa_3\kappa_4\kappa_5 + 8316\kappa_2\kappa_5^2 + 13860\kappa_2^3\kappa_6 + 9240\kappa_3^2\kappa_6 + 13860\kappa_2\kappa_4\kappa_6 + 462\kappa_6^2 + 7920\kappa_2\kappa_3\kappa_7 + 792\kappa_5\kappa_7 + 1485\kappa_2^2\kappa_8 + 495\kappa_4\kappa_8 + 220\kappa_3\kappa_9 + 66\kappa_2\kappa_{10} + \kappa_{12}$
13	$270270\kappa_2^5\kappa_3 + 600600\kappa_2^2\kappa_3^3 + 900900\kappa_2^3\kappa_3\kappa_4 + 200200\kappa_3^3\kappa_4 + 450450\kappa_2\kappa_3\kappa_4^2 + 135135\kappa_2^4\kappa_5 + 360360\kappa_2\kappa_3^2\kappa_5 + 270270\kappa_2^2\kappa_4\kappa_5 + 45045\kappa_4^2\kappa_5 + 36036\kappa_3\kappa_5^2 + 180180\kappa_2^2\kappa_3\kappa_6 + 60060\kappa_3\kappa_4\kappa_6 + 36036\kappa_2\kappa_5\kappa_6 + 25740\kappa_2^3\kappa_7 + 17160\kappa_3^2\kappa_7 + 25740\kappa_2\kappa_4\kappa_7 + 1716\kappa_6\kappa_7 + 12870\kappa_2\kappa_3\kappa_8 + 1287\kappa_5\kappa_8 + 2145\kappa_2^2\kappa_9 + 715\kappa_4\kappa_9 + 286\kappa_3\kappa_{10} + 78\kappa_2\kappa_{11} + \kappa_{13}$
14	$135135\kappa_2^7 + 3153150\kappa_2^4\kappa_3^2 + 1401400\kappa_2\kappa_3^4 + 945945\kappa_2^5\kappa_4 + 6306300\kappa_2^2\kappa_3^2\kappa_4 + 1576575\kappa_2^3\kappa_4^2 + 1051050\kappa_3^2\kappa_4^2 + 525525\kappa_2\kappa_4^3 + 2522520\kappa_2^3\kappa_3\kappa_5 + 560560\kappa_3^3\kappa_5 + 2522520\kappa_2\kappa_3\kappa_4\kappa_5 + 378378\kappa_2^2\kappa_5^2 + 126126\kappa_4\kappa_5^2 + 315315\kappa_2^4\kappa_6 + 840840\kappa_2\kappa_3^2\kappa_6 + 630630\kappa_2^2\kappa_4\kappa_6 + 105105\kappa_4^2\kappa_6 + 168168\kappa_3\kappa_5\kappa_6 + 42042\kappa_2\kappa_6^2 + 360360\kappa_2^2\kappa_3\kappa_7 + 120120\kappa_3\kappa_4\kappa_7 + 72072\kappa_2\kappa_5\kappa_7 + 1716\kappa_7^2 + 45045\kappa_2^3\kappa_8 + 30030\kappa_3^2\kappa_8 + 45045\kappa_2\kappa_4\kappa_8 + 3003\kappa_6\kappa_8 + 20020\kappa_2\kappa_3\kappa_9 + 2002\kappa_5\kappa_9 + 3003\kappa_2^2\kappa_{10} + 1001\kappa_4\kappa_{10} + 364\kappa_3\kappa_{11} + 91\kappa_2\kappa_{12} + \kappa_{14}$

Moment to Cumulants

n	$\mu_{\mathbf{n}} =$
1	κ_1
2	$\kappa_1^2 + \kappa_2$
3	$\kappa_1^3 + 3\kappa_1\kappa_2 + \kappa_3$
4	$\kappa_1^4 + 6\kappa_1^2\kappa_2 + 3\kappa_2^2 + 4\kappa_1\kappa_3 + \kappa_4$
5	$\kappa_1^5 + 10\kappa_1^3\kappa_2 + 15\kappa_1\kappa_2^2 + 10\kappa_1^2\kappa_3 + 10\kappa_2\kappa_3 + 5\kappa_1\kappa_4 + \kappa_5$
6	$\kappa_1^6 + 15\kappa_1^4\kappa_2 + 45\kappa_1^2\kappa_2^2 + 15\kappa_2^3 + 20\kappa_1^3\kappa_3 + 60\kappa_1\kappa_2\kappa_3 + 10\kappa_3^2 + 15\kappa_1^2\kappa_4 + 15\kappa_2\kappa_4 + 6\kappa_1\kappa_5 + \kappa_6$
7	$\kappa_1^7 + 21\kappa_1^5\kappa_2 + 105\kappa_1^3\kappa_2^2 + 105\kappa_1\kappa_2^3 + 35\kappa_1^4\kappa_3 + 210\kappa_1^2\kappa_2\kappa_3 + 105\kappa_2^2\kappa_3 + 70\kappa_1\kappa_3^2 + 35\kappa_1^3\kappa_4 + 105\kappa_1\kappa_2\kappa_4 + 35\kappa_3\kappa_4 + 21\kappa_1^2\kappa_5 + 21\kappa_2\kappa_5 + 7\kappa_1\kappa_6 + \kappa_7$
8	$\kappa_1^8 + 28\kappa_1^6\kappa_2 + 210\kappa_1^4\kappa_2^2 + 420\kappa_1^2\kappa_2^3 + 105\kappa_2^4 + 56\kappa_1^5\kappa_3 + 560\kappa_1^3\kappa_2\kappa_3 + 840\kappa_1\kappa_2^2\kappa_3 + 280\kappa_1^2\kappa_3^2 + 280\kappa_2\kappa_3^2 + 70\kappa_1^4\kappa_4 + 420\kappa_1^2\kappa_2\kappa_4 + 210\kappa_2^2\kappa_4 + 280\kappa_1\kappa_3\kappa_4 + 35\kappa_4^2 + 56\kappa_1^3\kappa_5 + 168\kappa_1\kappa_2\kappa_5 + 56\kappa_3\kappa_5 + 28\kappa_1^2\kappa_6 + 28\kappa_2\kappa_6 + 8\kappa_1\kappa_7 + \kappa_8$
9	$\kappa_1^9+36\kappa_1^7\kappa_2+378\kappa_1^5\kappa_2^2+1260\kappa_1^3\kappa_2^3+945\kappa_1\kappa_2^4+84\kappa_1^6\kappa_3+1260\kappa_1^4\kappa_2\kappa_3+3780\kappa_1^2\kappa_2^2\kappa_3+1260\kappa_2^3\kappa_3+840\kappa_1^3\kappa_3^2+2520\kappa_1\kappa_2\kappa_3^2+280\kappa_3^3+126\kappa_1^5\kappa_4+1260\kappa_1^3\kappa_2\kappa_4+1890\kappa_1\kappa_2^2\kappa_4+1260\kappa_1^2\kappa_3\kappa_4 + 1260\kappa_2\kappa_3\kappa_4 + 315\kappa_1\kappa_4^2 + 126\kappa_1^4\kappa_5 + 756\kappa_1^2\kappa_2\kappa_5 + 378\kappa_2^2\kappa_5 + 504\kappa_1\kappa_3\kappa_5+126\kappa_4\kappa_5+84\kappa_1^3\kappa_6+252\kappa_1\kappa_2\kappa_6+84\kappa_3\kappa_6+36\kappa_1^2\kappa_7+36\kappa_2\kappa_7+9\kappa_1\kappa_8+\kappa_9$
10	$\kappa_1^{10} + 45\kappa_1^8\kappa_2 + 630\kappa_1^6\kappa_2^2 + 3150\kappa_1^4\kappa_2^3 + 4725\kappa_1^2\kappa_2^4 + 945\kappa_2^5 + 120\kappa_1^7\kappa_3 + 2520\kappa_1^5\kappa_2\kappa_3 + 12600\kappa_1^3\kappa_2^2\kappa_3 + 12600\kappa_1\kappa_2^3\kappa_3 + 2100\kappa_1^4\kappa_3^2 + 12600\kappa_1^2\kappa_2\kappa_3^2 + 6300\kappa_2^2\kappa_3^2 + 2800\kappa_1\kappa_3^3 + 210\kappa_1^6\kappa_4 + 3150\kappa_1^4\kappa_2\kappa_4 + 9450\kappa_1^2\kappa_2^2\kappa_4 + 3150\kappa_2^3\kappa_4 + 4200\kappa_1^3\kappa_3\kappa_4 + 12600\kappa_1\kappa_2\kappa_3\kappa_4 + 2100\kappa_3^2\kappa_4 + 1575\kappa_1^2\kappa_4^2 + 1575\kappa_2\kappa_4^2 + 252\kappa_1^5\kappa_5 + 2520\kappa_1^3\kappa_2\kappa_5 + 3780\kappa_1\kappa_2^2\kappa_5 + 2520\kappa_1^2\kappa_3\kappa_5 + 2520\kappa_2\kappa_3\kappa_5 + 1260\kappa_1\kappa_4\kappa_5 + 126\kappa_5^2 + 210\kappa_1^4\kappa_6 + 1260\kappa_1^2\kappa_2\kappa_6 + 630\kappa_2^2\kappa_6 + 840\kappa_1\kappa_3\kappa_6 + 210\kappa_4\kappa_6 + 120\kappa_1^3\kappa_7 + 360\kappa_1\kappa_2\kappa_7 + 120\kappa_3\kappa_7 + 45\kappa_1^2\kappa_8 + 45\kappa_2\kappa_8 + 10\kappa_1\kappa_9 + \kappa_{10}$
11	$\kappa_1^{11} + 55\kappa_1^9\kappa_2 + 990\kappa_1^7\kappa_2^2 + 6930\kappa_1^5\kappa_2^3 + 17325\kappa_1^3\kappa_2^4 + 10395\kappa_1\kappa_2^5 + 165\kappa_1^8\kappa_3 + 4620\kappa_1^6\kappa_2\kappa_3 + 34650\kappa_1^4\kappa_2^2\kappa_3 + 69300\kappa_1^2\kappa_2^3\kappa_3 + 17325\kappa_2^4\kappa_3 + 4620\kappa_1^5\kappa_3^2 + 46200\kappa_1^3\kappa_2\kappa_3^2 + 69300\kappa_1\kappa_2^2\kappa_3^2 + 15400\kappa_1^2\kappa_3^3 + 15400\kappa_2\kappa_3^3 + 330\kappa_1^7\kappa_4 + 6930\kappa_1^5\kappa_2\kappa_4 + 34650\kappa_1^3\kappa_2^2\kappa_4 + 34650\kappa_1\kappa_2^3\kappa_4 + 11550\kappa_1^4\kappa_3\kappa_4 + 69300\kappa_1^2\kappa_2\kappa_3\kappa_4 + 34650\kappa_2^2\kappa_3\kappa_4 + 23100\kappa_1\kappa_3^2\kappa_4 + 5775\kappa_1^3\kappa_4^2 + 17325\kappa_1\kappa_2\kappa_4^2 + 5775\kappa_3\kappa_4^2 + 462\kappa_1^6\kappa_5 + 6930\kappa_1^4\kappa_2\kappa_5 + 20790\kappa_1^2\kappa_2^2\kappa_5 + 6930\kappa_2^3\kappa_5 + 9240\kappa_1^3\kappa_3\kappa_5+ 27720\kappa_1\kappa_2\kappa_3\kappa_5 + 4620\kappa_3^2\kappa_5 + 6930\kappa_1^2\kappa_4\kappa_5 + 6930\kappa_2\kappa_4\kappa_5 + 1386\kappa_1\kappa_5^2 + 462\kappa_1^5\kappa_6 + 4620\kappa_1^3\kappa_2\kappa_6 + 6930\kappa_1\kappa_2^2\kappa_6 + 4620\kappa_1^2\kappa_3\kappa_6 + 4620\kappa_2\kappa_3\kappa_6 + 2310\kappa_1\kappa_4\kappa_6 + 462\kappa_5\kappa_6 + 330\kappa_1^4\kappa_7 + 1980\kappa_1^2\kappa_2\kappa_7 + 990\kappa_2^2\kappa_7 + 1320\kappa_1\kappa_3\kappa_7 + 330\kappa_4\kappa_7 + 165\kappa_1^3\kappa_8 + 495\kappa_1\kappa_2\kappa_8 + 165\kappa_3\kappa_8 + 55\kappa_1^2\kappa_9 + 55\kappa_2\kappa_9 + 11\kappa_1\kappa_{10} + \kappa_{11}$

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