Chapter 9

The functional equation for the Riemann zeta function

We will eventually deduce a functional equation, relating $\zeta(s)$ to $\zeta(1-s)$. There are various methods to derive this functional equation, see E.C. Titchmarsh, The theory of the Riemann zeta function. We give a proof based on a functional equation for the Jacobi theta function $\theta(z) = \sum_{m=-\infty}^{\infty} e^{-\pi m^2 z}$. We start with some preparations.

9.1 Poisson's summation formula

We start with a simple result from Fourier analysis. Given a function $f : [0,1] \to \mathbb{C}$, we define the Fourier coefficients of f by

$$c_n(f) := \int_0^1 f(t)e^{-2\pi int}dt \quad \text{for } n \in \mathbb{Z}.$$

Theorem 9.1. *Let f be a complex analytic function, defined on an open subset of $\mathbb{C}$ containing the real interval $[0,1]$. Then*

$$\lim_{N\to\infty} \sum_{n=-N}^{N} c_n(f)e^{2\pi inx} = \begin{cases} \frac{1}{2}\big(f(0)+f(1)\big) & \text{if } x=0 \text{ or } x=1, \\ f(x) \quad \text{if } 0<x<1. \end{cases}$$

Remarks 1. The condition that f be analytic on an open subset containing $[0,1]$ is much too strong, but it has been inserted first since it is sufficient for our purposes, and second since it considerably simplifies the proof. Dirichlet proved the above theorem for functions $f : [0,1] \to \mathbb{C}$ that are differentiable and whose derivative is piecewise continuous.

2. It may be that a doubly infinite series $\sum_{n=-\infty}^{\infty} a_n = \lim_{M,N\to\infty} \sum_{n=-M}^{N} a_n$ diverges, while $\lim_{N\to\infty} \sum_{n=-N}^{N} a_n$ converges. For instance, if $a_{-n} = -a_n$ for $n \in \mathbb{Z} \setminus \{0\}$, then $\lim_{N\to\infty} \sum_{n=-N}^{N} a_n = a_0$, while $\sum_{n=-\infty}^{\infty} a_n$ may be horribly divergent.

Proof. We first assume that either $0 < x < 1$, or that $x \in \{0,1\}$ and $f(0) = f(1)$.

We use the so-called *Dirichlet kernel*

$$\begin{aligned} D_N(x) &= \sum_{n=-N}^{N} e^{2\pi inx} = e^{-2\pi iNx} \sum_{n=0}^{2N} e^{2\pi inx} \\ &= e^{-2\pi iNx} \cdot \frac{e^{2\pi i(2N+1)x} - 1}{e^{2\pi ix} - 1} \\ &= \frac{e^{\pi i(2N+1)x} - e^{-\pi i(2N+1)x}}{e^{\pi ix} - e^{-\pi ix}} = \frac{\sin(2N+1)\pi x}{\sin \pi x}. \end{aligned}$$

Further, we use

$$\int_0^1 e^{2\pi int} dt = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases}$$

Using these facts, we obtain

$$f(x) - \sum_{n=-N}^{N} c_n(f) e^{2\pi inx} = f(x) - \sum_{n=-N}^{N} \left(\int_0^1 f(t) e^{-2\pi int} dt \right) e^{2\pi inx}$$

$$= \sum_{n=-N}^{N} \left(\int_0^1 f(x) e^{-2\pi int} dt \right) e^{2\pi inx} - \sum_{n=-N}^{N} \left(\int_0^1 f(t) e^{-2\pi int} dt \right) e^{2\pi inx}$$

(the first integral is $f(x)$ if $n = 0$ and 0 if $n \neq 0$)

$$= \sum_{n=-N}^{N} \int_0^1 \big(f(x) - f(t)\big) \cdot e^{-2\pi in(t-x)} dt$$

$$= \int_0^1 \big(f(x) - f(t)\big) \left(\sum_{n=-N}^{N} e^{-2\pi in(t-x)} \right) dt$$

$$= \int_0^1 \big(f(x) - f(t)\big) \cdot \frac{\sin\big((2N+1)\pi(t-x)\big)}{\sin \pi(t-x)} \cdot dt.$$

Fix x and define

$$g(z) := \frac{f(x) - f(z)}{\sin \pi(z - x)}.$$

We show that g is analytic on an open set containing $[0, 1]$. First, suppose that $0 < x < 1$. By assumption, f is analytic on an open set $U \subset \mathbb{C}$ containing $[0, 1]$. By shrinking U if needed, we may assume that U contains $[0, 1]$ but not $x + n$ for any non-zero integer n. Then $\sin \pi(z - x)$ has a simple zero at $z = x$ but is otherwise non-zero on U. This shows that $g(z)$ is analytic on $U \setminus \{x\}$. But $g(z)$ is also analytic at $z = x$, since the simple zero of $\sin \pi(z-x)$ is cancelled by the zero of $f(x) - f(z)$. In case that $x \in \{0, 1\}$ and $f(0) = f(1)$ one proceeds in the same manner.

Using integration by parts, we obtain

$$f(x) - \sum_{n=-N}^{N} c_n(f) e^{2\pi i n x} = \int_0^1 g(t) \sin\{(2N + 1)\pi(t - x)\} dt$$

$$= \frac{-1}{(2N + 1)\pi} \int_0^1 g(t) d \cos\{(2N + 1)\pi(t - x)\}$$

$$= \frac{-1}{(2N + 1)\pi} \Big\{ g(1) \cos\{(2N + 1)\pi(1 - x)\} - g(0) \cos\{(2N + 1)\pi x\} +$$

$$+ \int_0^1 g'(t) \cos\{(2N + 1)\pi(t - x)\} dt \Big\}.$$

The functions $g(t)$, $g'(t)$ are continuous, hence bounded on $[0, 1]$ since g is analytic, and also the cosine terms are bounded on $[0, 1]$. It follows that the above expression converges to 0 as $N \to \infty$.

We are left with the case $x \in \{0, 1\}$ and $f(0) \neq f(1)$. Let

$$\widetilde{f}(z) := f(z) + (f(0) - f(1))z.$$

Then $\widetilde{f}$ is analytic on U and $\widetilde{f}(0) = \widetilde{f}(1) = f(0)$. It is easy to check that the function $id : z \mapsto z$ has Fourier coefficients $c_0(id) = \frac{1}{2}$, $c_n(id) = -1/2\pi i n$ for $n \neq 0$. In particular, $c_{-n}(id) = -c_n(id)$ for $n \neq 0$. Consequently,

$$\begin{aligned} \lim_{N\to\infty} \sum_{n=-N}^{N} c_n(f) &= \lim_{N\to\infty} \left(\sum_{n=-N}^{N} c_n(\widetilde{f}) + \big(f(1) - f(0)\big) \sum_{n=-N}^{N} c_n(id) \right) \\ &= f(0) + \tfrac{1}{2}\big(f(1) - f(0)\big) = \tfrac{1}{2}\big(f(0) + f(1)\big). \end{aligned}$$

This completes our proof. ∎

Theorem 9.2 (Poisson's summation formula for finite sums)**.** *Let a, b be integers with $a < b$ and let f be a complex analytic function, defined on an open set containing the interval $[a, b]$. Then*

$$\begin{aligned}\sum_{m=a}^{b} f(m) &= \tfrac{1}{2}\big(f(a)+f(b)\big) + \lim_{N\to\infty} \sum_{n=-N}^{N} \int_a^b f(t)e^{-2\pi i n t}dt \\ &= \tfrac{1}{2}\big(f(a)+f(b)\big)\big) + \int_a^b f(t)dt + 2\sum_{n=1}^{\infty} \int_a^b f(t)\cos 2\pi n t \cdot dt.\end{aligned}$$

Proof. Pick $m \in \{a, \dots, b-1\}$. Then by Theorem 9.1,

$$\begin{aligned}\tfrac{1}{2}\big(f(m)+f(m+1)\big) &= \lim_{N\to\infty} \sum_{n=-N}^{N} \int_m^{m+1} f(t)e^{-2\pi i n t}dt \\ &= \int_m^{m+1} f(t)dt + \lim_{N\to\infty} \sum_{n=1}^{N} \int_m^{m+1} f(t)\left(e^{2\pi i n t} + e^{-2\pi i n t}\right)dt \\ &= \int_m^{m+1} f(t)dt + 2\sum_{n=1}^{\infty} \int_m^{m+1} f(t)\cos 2\pi n t \cdot dt.\end{aligned}$$

Now take the sum over $m = a, a+1, \dots, b-1$. ∎

We need a variation on Theorem 9.2, dealing with infinite sums $\sum_{m=-\infty}^{\infty} f(m)$.

Theorem 9.3. *Let f be a complex function such that:*
(i) $f(z)$ is analytic on $U(\delta) := \{z \in \mathbb{C} : |\mathrm{Im}\, z| < \delta\}$ for some $\delta > 0$;
(ii) there are $C > 0, \varepsilon > 0$ such that

$$|f(z)| \leqslant C \cdot (|z|+1)^{-1-\varepsilon} \quad \text{for } z \in U(\delta).$$

Then

$$\sum_{n=-\infty}^{\infty} f(n) = \lim_{N\to\infty} \sum_{n=-N}^{N} \int_{-\infty}^{\infty} f(t)e^{-2\pi i n t}dt.$$

The idea is to apply Theorem 9.1 to the function $F(z) := \sum_{m=-\infty}^{\infty} f(z+m)$. We first prove some properties of this function.

Lemma 9.4. *(i)* $F(0) = F(1) = \sum_{m=-\infty}^{\infty} f(m)$.
(ii) The function $F(z)$ *is analytic on an open set containing* $[0,1]$.
(iii) For every $n \in \mathbb{Z}$ *we have* $\int_0^1 F(t)e^{-2\pi int}dt = \int_{-\infty}^{\infty} f(t)e^{-2\pi int}dt$.

Proof. (i) Obvious.

(ii) Let $U := \{z \in \mathbb{C} : -\delta < \operatorname{Re} z < 1+\delta, |\operatorname{Im} z| < \delta\}$. Assuming that δ is sufficiently small, we have $|f(z+m)| \leqslant C(|m|-\delta)^{-1-\varepsilon} =: A_m$ for $z \in U$, $m \in \mathbb{Z} \setminus \{0\}$. All summands $f(z+m)$ are analytic on U', and the series $\sum_{m\neq 0} A_m$ converges. So by Corollary 2.26, the function $F(z)$ is analytic on U.

(iii) Since $|f(t+m)e^{-2\pi int}| \leqslant A_m$ for $t \in [0,1]$, $m \in \mathbb{Z} \setminus \{0\}$, and $\sum_{m\neq 0} A_m$ converges, the series $\sum_{m=\infty}^{\infty} f(t+m)e^{-2\pi int}$ converges uniformly on $[0,1]$. Therefore, we may interchange the integral and the infinite sum, and obtain

$$\begin{aligned}
\int_0^1 F(t)e^{-2\pi int}dt &= \int_0^1 \Big(\sum_{m=-\infty}^{\infty} f(t+m)\Big)e^{-2\pi int}dt = \sum_{m=-\infty}^{\infty} \int_0^1 f(t+m)e^{-2\pi int}dt \\
&= \sum_{m=-\infty}^{\infty} f(t+m)e^{-2\pi in(t+m)}dt = \sum_{m=-\infty}^{\infty} \int_m^{m+1} f(t)e^{-2\pi int}dt \\
&= \int_{-\infty}^{\infty} f(t)e^{-2\pi int}dt.
\end{aligned}$$

In the last step we have used that the integral $\int_{-\infty}^{\infty} f(t)e^{-2\pi int}dt$ converges, due to our assumption $|f(z)| \leqslant C(|z|+1)^{-1-\varepsilon}$ for $z \in U(\delta)$. □

Proof of Theorem 9.3. By combining Theorem 9.1 with Lemma 9.4 we obtain

$$\begin{aligned}
\sum_{m=-\infty}^{\infty} f(m) &= \tfrac{1}{2}\big(F(0)+F(1)\big) = \lim_{N\to\infty} \sum_{n=-N}^{N} \int_0^1 F(t)e^{-2\pi int}dt \\
&= \lim_{N\to\infty} \sum_{n=-N}^{N} \int_{-\infty}^{\infty} f(t)e^{-2\pi int}dt.
\end{aligned}$$

□

9.2 A functional equation for the theta function

The Jacobi theta function is given by

$$\theta(z) := \sum_{m=-\infty}^{\infty} e^{-\pi m^2 z} \quad (z \in \mathbb{C},\ \mathrm{Re}\, z > 0).$$

Verify yourself that $\theta(z)$ converges and is analytic on $\{z \in \mathbb{C} : \mathrm{Re}\, z > 0\}$.

Theorem 9.5. *$\theta(z^{-1}) = \sqrt{z} \cdot \theta(z)$ for $z \in \mathbb{C}$, $\mathrm{Re}\, z > 0$, where $\sqrt{z}$ is chosen such that $|\arg \sqrt{z}| < \frac{\pi}{4}$.*

Remark. Let $A := \{z \in \mathbb{C} : \mathrm{Re}\, z > 0\}$. We may choose the argument of $z \in A$ such that $|\arg z| < \frac{\pi}{2}$. Then indeed, we may choose $\sqrt{z}$ such that $|\arg \sqrt{z}| < \frac{\pi}{4}$.

Proof. Both $\theta(z^{-1})$ and $\sqrt{z}\theta(z)$ are analytic on A. Hence it suffices to prove the identity in Theorem 9.5 on a subset of A having a limit point in A. For this subset we take $\mathbb{R}_{>0}$. Thus, it suffices to prove that

$$\sum_{m=-\infty}^{\infty} e^{-\pi m^2/x} = \sqrt{x} \cdot \sum_{m=-\infty}^{\infty} e^{-\pi m^2 x} \quad \text{for } x > 0.$$

We apply Theorem 9.3 to $f(z) := e^{-\pi z^2/x}$ with $x > 0$ fixed. Verify that f satisfies all conditions of that Theorem. Thus, for any $x > 0$,

$$\sum_{m=-\infty}^{\infty} e^{-\pi m^2/x} = \lim_{N\to\infty} \sum_{n=-N}^{N} \int_{-\infty}^{\infty} e^{-(\pi t^2/x) - 2\pi i n t} dt.$$

We compute the integrals by substituting $u = t\sqrt{x}$. Thus,

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-(\pi t^2/x)-2\pi i n t} dt &= \sqrt{x} \cdot \int_{-\infty}^{\infty} e^{-\pi u^2 - 2\pi i n\sqrt{x}\cdot u} du \\ &= \sqrt{x} \cdot \int_{-\infty}^{\infty} e^{-\pi(u+in\sqrt{x})^2 - \pi n^2 x} du \\ &= \sqrt{x} e^{-\pi n^2 x} \int_{-\infty}^{\infty} e^{-\pi(u+in\sqrt{x})^2} du. \end{aligned}$$

In the lemma below we prove that the last integral is equal to 1. Then it follows that

$$\sum_{m=-\infty}^{\infty} e^{-\pi m^2/x} = \lim_{N\to\infty} \sum_{n=-N}^{N} \sqrt{x} e^{-\pi n^2 x} = \sqrt{x} \sum_{n=-\infty}^{\infty} e^{-\pi n^2 x},$$

since the last series converges. This proves our Theorem. □

Lemma 9.6. *Let* $z \in \mathbb{C}$*. Then* $\int_{-\infty}^{\infty} e^{-\pi(u+z)^2} du = 1$*.*

Proof. The following proof was suggested to me by Michiel Kosters. Let

$$F(z) := \int_{\infty}^{\infty} e^{-\pi(u+z)^2} du.$$

We show that this defines an analytic function on $\mathbb{C}$. To this end, we prove that F is analytic on $D(0, R) := \{z \in \mathbb{C} : |z| < R\}$ for every $R > 0$. We apply Theorem 2.29. First, $(u, z) \mapsto e^{-\pi(u+z)^2}$ is continuous, hence measurable, on $\mathbb{R} \times D(0, R)$. Second, for every fixed $u \in \mathbb{R}$, $z \mapsto e^{-\pi(u+z)^2}$ is analytic on $D(0, R)$. Third,

$$\begin{aligned} |e^{-\pi(u+z)^2}| &= e^{-\mathrm{Re}\,\pi(u+z)^2} = e^{-(\pi u^2 + 2\pi u \mathrm{Re}\, z + \pi \mathrm{Re}\, z^2)} \\ &\leqslant e^{-\pi u^2 + 2\pi R u + \pi R^2} = e^{-\pi(u-R)^2 + 2\pi R^2}, \end{aligned}$$

and $\int_{-\infty}^{\infty} e^{-\pi(u-R)^2 + 2\pi R^2} du$ converges. So by Theorem 2.29, F is analytic on $D(0, R)$.

Knowing that F is analytic on $\mathbb{C}$, in order to prove that $F(z) = 1$ for $z \in \mathbb{C}$ it is sufficient to prove, for any set $S \subset \mathbb{C}$ with a limit point in $\mathbb{C}$, that $F(z) = 1$ for $z \in S$. For the set S we take $\mathbb{R}$. For $z \in \mathbb{R}$ we obtain, by substituting $v = u + z$,

$$F(z) = \int_{-\infty}^{\infty} e^{-\pi(u+z)^2} du = \int_{-\infty}^{\infty} e^{-\pi v^2} dv = 2 \int_{0}^{\infty} e^{-\pi v^2} dv.$$

Now a second substitution $t = \pi v^2$ yields

$$F(z) = \pi^{-1/2} \int_0^{\infty} e^{-t} t^{-1/2} dt = \pi^{-1/2} \Gamma(\tfrac{1}{2}) = 1.$$

□

9.3 The functional equation for the Riemann zeta function

Put

$$\xi(s) := \tfrac{1}{2}s(s-1)\pi^{-s/2}\Gamma(\tfrac{1}{2}s)\zeta(s) \;=\; (s-1)\pi^{-s/2}\Gamma(\tfrac{1}{2}s+1)\zeta(s),$$

where we have used the identity $\frac{1}{2}s\Gamma(\frac{1}{2}s) = \Gamma(\frac{1}{2}s+1)$.

Theorem 9.7. *The function ξ has an analytic continuation to $\mathbb{C}$.*
For this continuation we have $\xi(1-s) = \xi(s)$ for $s \in \mathbb{C}$.

Before proving this, we deduce some consequences.

Corollary 9.8. *The function ζ has an analytic continuation to $\mathbb{C}\setminus\{1\}$ with a simple pole with residue 1 at $s=1$.*
For this continuation we have

$$\zeta(1-s) = 2^{1-s}\pi^{-s}\cos(\tfrac{1}{2}\pi s)\Gamma(s)\cdot\zeta(s) \quad \textit{for } s \in \mathbb{C}\setminus\{0,1\}.$$

Proof. We define the analytic continuation of ζ by

$$\zeta(s) = \frac{\xi(s)\pi^{s/2}\cdot 1/\Gamma(\frac{1}{2}s+1)}{s-1}.$$

By Corollary 8.5, $1/\Gamma$ is analytic on $\mathbb{C}$, and the other functions in the numerator are also analytic on $\mathbb{C}$. Hence ζ is analytic on $\mathbb{C}\setminus\{1\}$. The analytic continuation of ζ defined here coincides with the one defined in Theorem 5.2 on $\{s \in \mathbb{C} : \operatorname{Re} s > 0\}\setminus\{1\}$ since analytic continuations to connected sets are uniquely determined. Hence $\zeta(s)$ has a simple pole with residue 1 at $s=1$.

We derive the functional equation. By Theorem 9.7 we have, for $s \in \mathbb{C}\setminus\{0,1\}$,

$$\begin{aligned}
\zeta(1-s) &= \frac{\xi(1-s)}{\frac{1}{2}(1-s)(-s)\pi^{-(1-s)/2}\Gamma(\frac{1}{2}(1-s))} = \frac{\xi(s)}{\frac{1}{2}s(s-1)\pi^{-(1-s)/2}\Gamma(\frac{1}{2}(1-s))} \\
&= \frac{\frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(\frac{1}{2}s)}{\frac{1}{2}s(s-1)\pi^{-(1-s)/2}\Gamma(\frac{1}{2}(1-s))}\cdot\zeta(s) = F(s)\zeta(s),
\end{aligned}$$

say. Now we have

$$\begin{aligned} F(s) &= \pi^{(1/2)-s} \cdot \frac{\Gamma(\frac{1}{2}s)\Gamma(\frac{1}{2}s+\frac{1}{2})}{\Gamma(\frac{1}{2}-\frac{1}{2}s)\Gamma(\frac{1}{2}+\frac{1}{2}s)} \\ &= \pi^{(1/2)-s} \frac{2^{1-s}\sqrt{\pi}\Gamma(s)}{\pi/\sin(\pi(\frac{1}{2}-\frac{1}{2}s))} \quad \text{(by Corollary 8.12, Theorem 8.3)} \\ &= \pi^{-s}2^{1-s}\cos(\tfrac{1}{2}\pi s)\Gamma(s). \end{aligned}$$

This implies Corollary 9.8. □

Corollary 9.9. *ζ has simple zeros at $s = -2, -4, -6, \ldots$.*
ζ has no other zeros outside the critical strip $\{s \in \mathbb{C} : 0 < \operatorname{Re} s < 1\}$.

Proof. We first show that $\xi(s) \neq 0$ if $\operatorname{Re} s \geqslant 1$ or $\operatorname{Re} s \leqslant 0$. We use the second expression for $\xi(s)$. By Corollary 5.4 and Theorem 4.5, we know that $\zeta(s) \neq 0$ for $s \in \mathbb{C}$ with $\operatorname{Re} s \geqslant 1$, $s \neq 1$. Further, $\lim_{s \to 1}(s-1)\zeta(s) = 1$, hence $(s-1)\zeta(s) \neq 0$ if $\operatorname{Re} s \geqslant 1$. By Corollary 8.5, we know that $\Gamma(\frac{1}{2}s+1) \neq 0$ if $\operatorname{Re} s \geqslant 1$. hence $\xi(s) \neq 0$ if $\operatorname{Re} s \geqslant 1$. But then by Theorem 9.7, $\xi(s) \neq 0$ if $\operatorname{Re} s \leqslant 0$.

We consider $\zeta(s)$ for $\operatorname{Re} s \leqslant 0$. For $s \neq -2, -4, -6, \ldots$, the function $\Gamma(\frac{1}{2}s+1)$ is analytic. Further, for these values of s, we have $\xi(s) \neq 0$, hence $\zeta(s)$ must be $\neq 0$ as well. The function $\Gamma(\frac{1}{2}s)$ has simple poles at $s = -2, -4, -6, \ldots$. To make $\xi(s)$ analytic and non-zero for these values of s, the function ζ must have simple zeros at $s = -2, -4, -6, \ldots$. □

Proof of Theorem 9.7 (Riemann). Let for the moment, $s \in \mathbb{C}$, $\operatorname{Re} s > 1$. Recall that

$$\Gamma(\tfrac{1}{2}s) = \int_0^\infty e^{-t} t^{(s/2)-1} dt.$$

Substituting $t = \pi n^2 u$ gives

$$\Gamma(\tfrac{1}{2}s) = \int_0^\infty e^{-\pi n^2 u}(\pi n^2 u)^{(s/2)-1} d(\pi n^2 u) = \pi^{s/2} n^s \int_0^\infty e^{-\pi n^2 u} u^{(s/2)-1} du.$$

Hence

$$\pi^{-s/2}\Gamma(\tfrac{1}{2}s)n^{-s} = \int_0^\infty e^{-\pi n^2 u} u^{(s/2)-1} du,$$

and so, by summing over n,

$$\pi^{-s/2}\Gamma(\tfrac{1}{2}s)\zeta(s) = \sum_{n=1}^\infty \int_0^\infty e^{-\pi n^2 u} \cdot u^{(s/2)-1} du.$$

We justify that the infinite integral and infinite sum can be interchanged. We use the following special case of the Fubini-Tonelli theorem: if $\{f_n : (0,\infty) \to \mathbb{C}\}_{n=1}^{\infty}$ is a sequence of measurable functions such that $\sum_{n=1}^{\infty} \int_0^{\infty} |f_n(u)|du$ converges, then all integrals $\int_0^{\infty} f_n(u)du$ $(n \geqslant 1)$ converge, the series $\sum_{n=1}^{\infty} f_n(u)$ converges almost everywhere on $(0,\infty)$ and moreover,

$$\sum_{n=1}^{\infty} \int_0^{\infty} f_n(u)du, \quad \int_0^{\infty} \left(\sum_{n=1}^{\infty} f_n(u) \right) du$$

converge and are equal. In our situation we have that indeed (putting $\sigma := \operatorname{Re} s$)

$$\begin{aligned} \sum_{n=1}^{\infty} \int_0^{\infty} |e^{-\pi n^2 u} \cdot u^{(s/2)-1}|du &= \sum_{n=1}^{\infty} \int_0^{\infty} e^{-\pi n^2 u} u^{(s/2)-1} du \\ &= \sum_{n=1}^{\infty} \pi^{-\sigma/2} \Gamma(\tfrac{1}{2}\sigma) n^{-\sigma} \quad \text{(reversing the above argument)} \\ &= \pi^{-\sigma/2} \Gamma(\tfrac{1}{2}\sigma) \zeta(\sigma) \end{aligned}$$

converges. Thus, we conclude that for $s \in \mathbb{C}$ with $\operatorname{Re} s > 1$,

$$\pi^{-s/2} \Gamma(\tfrac{1}{2}s) \zeta(s) = \int_0^{\infty} \omega(u) \cdot u^{(s/2)-1} du, \quad \text{where } \omega(u) = \sum_{n=1}^{\infty} e^{-\pi n^2 u}. \tag{9.1}$$

Recall that $\theta(u) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 u} = 1 + 2\omega(u)$.

We want to replace the right-hand side of (9.1) by something that converges for every $s \in \mathbb{C}$. Obviously, for $s \in \mathbb{C}$ with $\operatorname{Re} s < 0$ there are problems if $u \downarrow 0$. To overcome these, we split the integral $\int_0^{\infty}$ into $\int_1^{\infty} + \int_0^1$ and then transform $\int_0^1$ into an integral $\int_1^{\infty}$ by means of a substitution $v = u^{-1}$. After this substitution, the integral contains a term $\omega(v^{-1})$. By Theorem 9.5, we have

$$\begin{aligned} \omega(v^{-1}) &= \tfrac{1}{2}(\theta(v^{-1}) - 1) = \tfrac{1}{2} v^{1/2} \theta(v) - \tfrac{1}{2} \\ &= \tfrac{1}{2} v^{1/2} \big(2\omega(v) + 1\big) - \tfrac{1}{2} = v^{1/2} \omega(v) + \tfrac{1}{2} v^{1/2} - \tfrac{1}{2}. \end{aligned}$$

We work out in detail the approach sketched above. We keep for the moment our assumption $\operatorname{Re} s > 1$. Thus,

$$\begin{aligned}\pi^{-\frac{1}{2}s}\Gamma(\tfrac{1}{2}s)\zeta(s) &= \int_1^\infty \omega(u)u^{(s/2)-1}du - \int_1^\infty \omega(v^{-1})v^{1-s/2}dv^{-1} \\ &= \int_1^\infty \omega(u)u^{(s/2)-1}du + \int_1^\infty \big(v^{1/2}\omega(v) + \tfrac{1}{2}v^{1/2} - \tfrac{1}{2}\big)v^{1-s/2}v^{-2}dv \\ &= \int_1^\infty \tfrac{1}{2}\big(v^{-(s+1)/2} - v^{-(s/2)-1}\big)dv + \int_1^\infty \omega(v)\big(v^{(s/2)-1} + v^{-(s+1)/2}\big)dv\end{aligned}$$

where we have combined the terms without ω into one integral, and the terms involving ω into another integral. Since we are still assuming $\operatorname{Re} s > 1$, the first integral is equal to

$$\tfrac{1}{2}\left[-\frac{2}{s-1}v^{-(s-1)/2} + \frac{2}{s}v^{-s/2}\right]_1^\infty = \frac{1}{s-1} - \frac{1}{s} = \frac{1}{s(s-1)}.$$

Hence

$$\pi^{-s/2}\Gamma(\tfrac{1}{2}s)\zeta(s) = \frac{1}{s(s-1)} + \int_1^\infty \omega(v)\big(v^{(s/2)-1} + v^{-(s+1)/2}\big)dv.$$

For our function $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(\frac{1}{2}s)\zeta(s)$ this gives

$$\xi(s) = \tfrac{1}{2} + \tfrac{1}{2}s(s-1)\int_1^\infty \omega(v)\big(v^{(s/2)-1} + v^{-(s+1)/2}\big)dv \quad \text{if } \operatorname{Re} s > 1. \tag{9.2}$$

Assume for the moment that $F(s) := \int_1^\infty \omega(v)\big(v^{(s/2)-1} + v^{-(s+1)/2}\big)dv$ defines an analytic function on $\mathbb{C}$. Then we can use the right-hand side of (9.2) to define the analytic continuation of $\xi(s)$ to $\mathbb{C}$. By substituting $1-s$ for s in the right-hand side, we see that $\xi(1-s) = \xi(s)$.

It remains to prove that $F(s)$ defines an analytic function on $\mathbb{C}$. To this end, it suffices to prove that $F(s)$ is analytic on $U_A := \{s \in \mathbb{C} : |\operatorname{Re} s| < A\}$ for every $A > 0$.

We apply as usual Theorem 2.29. We check that $f(v,s) = \omega(v)\big(v^{(s/2)-1} + v^{-(s+1)/2}\big)$ satisfies the conditions of that theorem.

a) $f(v,s)$ is measurable on $(1,\infty)\times U_A$. For $\omega(v) = \sum_{n=1}^\infty e^{-\pi n^2 v}$ is measurable, being a pointwise convergent series of continuous, hence measurable functions, and also $v^{(s/2)-1} + v^{-(s+1)/2}$ is measurable, since it is continuous.

b) $s \mapsto \omega(v)\big(v^{(s/2)-1} + v^{-(s+1)/2}\big)$ is analytic on U_A for every fixed v. This is obvious.

c) There is a measurable function $M(v)$ on $(1,\infty)$ such that $|f(v,s)| \leqslant M(v)$ for $s \in U_A$ and $\int_1^\infty M(v)dv < \infty$. Indeed, we first have for $v \in (1,\infty)$

$$0 \leqslant \omega(v) \leqslant e^{-\pi v}\big(1 + e^{-3\pi v} + e^{-8\pi v}\big) \leqslant 2e^{-\pi v}$$

and second, for $v \in (1,\infty)$, $s \in U_A$

$$|v^{(s/2)-1} + v^{-(s+1)/2}| \leqslant v^{(A/2)-1} + v^{(-(A+1)/2} \leqslant 2v^{(A/2)-1}.$$

Hence

$$|f(v,s)| \leqslant 4e^{-\pi v}v^{(A/2)-1} =: M(v).$$

Further,

$$\int_1^\infty M(v)dv \leqslant 4\int_0^\infty e^{-v}v^{(A/2)-1)}dv \leqslant 4\cdot\Gamma(\tfrac{1}{2}A) < \infty.$$

So $f(v,s)$ satisfies all conditions of Theorem 2.29, and it follows that $F(s) = \int_1^\infty f(v,s)dv$ is analytic on U_A. □

9.4 The functional equations for L-functions

Let q be an integer $\geqslant 2$ and χ a Dirichlet character modulo q with $\chi \neq \chi_0^{(q)}$. We give, without proof, a functional equation for $L(s,\chi)$ in the case that χ is primitive, i.e., that it is not induced by a character modulo d for any proper divisor d of q.

Notice that for any character χ modulo q we have $\chi(-1)^2 = \chi(1) = 1$, hence $\chi(-1) \in \{-1,1\}$. A character χ is called *even* if $\chi(-1) = 1$, and *odd* if $\chi(-1) = -1$. There will be different functional equations for even and odd characters.

In Chapter 4 we defined the Gauss sum related to a character χ mod q by

$$\tau(1,\chi) = \sum_{a=0}^{q-1} \chi(a)e^{2\pi i a/q}.$$

According to Theorem 4.17, if χ is primitive then $|\tau(1,\chi)| = \sqrt{q}$.

By $\overline{\chi}$ we denote the complex conjugate of a character χ.

Theorem 9.10. *Let q be an integer with $q \geqslant 2$, and χ a primitive character mod q. Put*

$$\xi(s,\chi) := \left(\frac{q}{\pi}\right)^{s/2}\Gamma(\tfrac12 s)L(s,\chi), \qquad c(\chi) := \frac{\sqrt{q}}{\tau(1,\chi)} \quad \textit{if } \chi \textit{ is even,}$$
$$\xi(s,\chi) := \left(\frac{q}{\pi}\right)^{(s+1)/2}\Gamma\big(\tfrac12(s+1)\big)L(s,\chi), \quad c(\chi) := \frac{i\sqrt{q}}{\tau(1,\chi)} \quad \textit{if } \chi \textit{ is odd.}$$

Then $\xi(s,\chi)$ has an analytic continuation to $\mathbb{C}$, and

$$\xi(1-s,\overline{\chi}) = c(\chi)\xi(s,\chi) \quad \textit{for } s \in \mathbb{C}.$$

Remark. We know that $|c(\chi)| = 1$. In general, it is a difficult problem to compute $c(\chi)$.

The proof of Theorem 9.10 is similar to that of that of the functional equation for $L(s,\chi)$, but with some additional technicalities, see H. Davenport, Multiplicative Number Theory, Chapter 9.

We deduce some consequences.

Corollary 9.11. *Let q be an integer $\geqslant 2$ and χ a character mod q with $\chi \neq \chi_0^{(q)}$. Then $L(s,\chi)$ has an analytic continuation to $\mathbb{C}$.*

Proof. First assume that χ is primitive and χ is even. Then

$$L(s,\chi) = \xi(s,\chi)(\pi/q)^{s/2}/\Gamma(\tfrac12 s).$$

The functions $\xi(s,\chi)$ and $(\pi/q)^{s/2}$ are both analytic on $\mathbb{C}$, and according to Corollary 7.5, $1/\Gamma(\frac12 s)$ is analytic on $\mathbb{C}$ as well. Hence $L(s,\chi)$ is analytic on $\mathbb{C}$.

In a completely similar manner one shows that $L(s,\chi)$ is analytic on $\mathbb{C}$ if χ is primitive and odd.

Now suppose that χ is not primitive. Let q' be the conductor of χ. By Corollary 4.13, χ is induced by a character χ' mod q'. Verify yourself that χ' is primitive. We have $q' > 1$, since otherwise, χ would be equal to $\chi_0^{(q)}$.

For $s \in \mathbb{C}$ with $\operatorname{Re} s > 1$ we have, noting that $\chi(p) = \chi'(p)$ if p is a prime not dividing q and $\chi(p) = 0$ if p is a prime dividing q,

$$\begin{aligned} L(s,\chi) &= \prod_{p} \frac{1}{1-\chi(p)p^{-s}} = \prod_{p\nmid q} \frac{1}{1-\chi'(p)p^{-s}} \\ &= \prod_{p} \frac{1}{1-\chi'(p)p^{-s}} \cdot \prod_{p\mid q} \big(1-\chi'(p)p^{-s}\big) = L(s,\chi') \cdot \prod_{p\mid q} \big(1-\chi'(p)p^{-s}\big). \end{aligned}$$

Now clearly, we can extend $L(s,\chi)$ analytically to $\mathbb{C}$ by defining

$$L(s,\chi) = L(s,\chi') \cdot \prod_{p\mid q} \big(1-\chi'(p)p^{-s}\big) \quad \text{for } s \in \mathbb{C}. \tag{9.3}$$

□

We consider the zeros of L-functions. Notice that (9.3) implies that if χ is induced by a primitive character χ', then $L(s,\chi)$ has the same set of zeros as $L(s,\chi)$, except for possible zeros of $\prod_{p\mid q}\big(1-\chi'(p)p^{-s}\big)$, which all lie on the line $\mathrm{Re}\, s = 0$.

We consider henceforth only the zeros of $L(s,\chi)$ for primitive characters χ. We have proved in Chapter 5 that $L(s,\chi) \neq 0$ if $\mathrm{Re}\, s \geqslant 1$. The next corollary considers the zeros s with $\mathrm{Re}\, s \leqslant 0$.

Corollary 9.12. *Let q be an integer $\geqslant 2$ and χ a primitive character mod q.*

(i) If χ is even, then $L(s,\chi)$ has simple zeros at $s = 0, -2, -4, \ldots$ and $L(s,\chi) \neq 0$ if $\mathrm{Re}\, s \leqslant 0$, $s \notin \{0, -2, -4, \ldots\}$.

(ii) If χ is odd, then $L(s,\chi)$ has simple zeros at $s = -1, -3, -5, \ldots$ and $L(s,\chi) \neq 0$ if $\mathrm{Re}\, s \leqslant 0$, $s \notin \{-1, -3, -5, \ldots\}$.

Proof. Exercise. □