Ph.D. Qualifying Examination

Complex Analysis

ANSWER ALL 6 QUESTIONS.

Convention: *Throughout this paper,* $\mathbf{R}$ *denotes the set of real numbers, and* $\mathbf{C}$ *denotes the set of complex numbers. For* $z \in \mathbf{C}$, $\operatorname{Re} z$ *denotes the real part of* z, $\operatorname{Im} z$ *denotes the imaginary part of* z*. In addition, for a complex variable* $z = x + iy$, x *denotes the real part of* z*, and* y *denotes the imaginary part of* z*, unless otherwise stated.*

1. Determine whether the following statements are true or false. Justify your answers.

   (i) If f is an analytic function on the annulus $\{z \in \mathbf{C} \mid 1 < |z| < 2\}$, then f extends to a meromorphic function on the disc $\{z \in \mathbf{C} \mid |z| < 2\}$.

   (ii) If g is a non-constant entire function such that $g(0) = 1$, then there exists a number $r > 0$ such that $g(z) \neq 1$ for all z satisfying $0 < |z| < r$.

2. Find a conformal map from the domain
$$D_1 := \{z \in \mathbf{C} \mid -3 < \operatorname{Re} z < 3\}$$
onto the domain $D_2 := \{z \in \mathbf{C} \mid |z + i| < 2\}$.

3. Use the Cauchy residue theorem to evaluate the improper integral
$$\int_0^\infty \frac{x^2 \cos 5x}{x^4 + 16}\, dx.$$
Justify your steps.

4. Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be n given complex numbers. Prove that there exists a complex number β such that
$$|\beta| = 1 \quad \text{and} \quad \prod_{i=1}^{n} |\beta - \alpha_i| \geq 1.$$
[Suggestion: Consider the polynomial $(z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_n)$.]

5. Consider the punctured coordinate plane $\mathbf{D} := \{(x, y) \in \mathbf{R}^2 \mid (x, y) \neq (0, 0)\}$, and let $h : \mathbf{D} \to \mathbf{R}$ be a harmonic function such that
$$h(x, y) > -1 \quad \text{for all} \quad (x, y) \in \mathbf{D}.$$
Is it true that h must be a constant function on $\mathbf{D}$? Justify your answer.

6. Let z_o be a given complex number. It is given that a function $f : \mathbf{C} \setminus \{z_o\} \to \mathbf{C}$ has an isolated singular point at z_o. Consider the function $g : \mathbf{C} \setminus \{z_o\} \to \mathbf{C}$ given by
$$g(z) := \cos(f(z)), \quad z \in \mathbf{C} \setminus \{z_o\}.$$
Can g have a pole at z_o? Justify your answer.