On the Betti numbers of birationally isomorphic projective varieties with trivial canonical bundles

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Abstract

Let X and Y be two birationally isomorphic smooth projective n-dimensional algebraic varieties X and Y over $\mathbb{C}$ having trivial canonical line bundles. Using methods of the p-adic analysis on algebraic varieties over local number fields, we prove that in the above situation the Betti numbers of X and Y must be the same.

1 Introduction

The purpose of this note is to show that the elementary theory of the p-adic integrals on algebraic varieties help to prove some cohomological properties of birationally isomorphic algebraic varieties over $\mathbb{C}$. We prove the following theorem which has been used by Beauville in his recent explanation of a Yau-Zaslow formula for the number of rational curves on a $K3$-surface [1] (see also [3, 10]):

Theorem 1.1 *Let X and Y be two irreducible birationally isomorphic smooth n-dimensional projective algebraic varieties over $\mathbb{C}$. Assume that the canonical line bundles Ω^n_X and Ω^n_Y are trivial. Then X and Y must have the same Betti numbers, i.e.,*

$$H^i(X, \mathbb{C}) \cong H^i(Y, \mathbb{C}) \quad \forall i \geq 0.$$

We remark that Theorem 1.1 is obvious for $n = 1$. In the case $n = 2$, Theorem 1.1 follows from the uniqueness of minimal models of surfaces of nonnegative Kodaira dimension, i.e. from the property that any birational isomorphism between two such minimal models extends to a biregular one [4]. The uniqueness of minimal models of n-dimensional algebraic varieties of nonnegative Kodaira dimension fails for $n \geq 3$ in general. However, Theorem 1.1 for $n = 3$ can be proved using a result of

Kawamata ([5], §6), who has shown that any two birationally isomorphic minimal models of 3-folds are conected by a sequence of flops (see also [6]). By simple topological arguments, one can prove that if two 3-dimensional projective algebraic varieties over $\mathbb{C}$ with at worst $\mathbb{Q}$-factorial terminal singularities are birationally isomorphic via a flop, then their singular Betti numbers are the same. Since one still knows very little about flops in dimension $n \geq 4$, it seems unlikely to expect that a consideration of flops could help to prove 1.1 in arbitary dimension $n \geq 4$. Moreover, for projective algebraic varieties with at worst $\mathbb{Q}$-factorial Gorenstein terminal singularities of dimension $n \geq 4$ Theorem 1.1 is not true in general. For this reason the condition of *smoothness* for X and Y in 1.1 becomes very important in the case $n \geq 4$.

2 Gauge-forms and p-adic measures

Let F be a local number field, i.e., a finite extension of $\mathbb{Q}_p$ for some prime $p \in \mathbb{Z}$. Let $R \subset F$ be the maximal compact subring, $\mathfrak{q} \subset R$ the maximal ideal, $F_{\mathfrak{q}} = R/\mathfrak{q}$ the residue field with $|F_{\mathfrak{q}}| = q = p^r$. We denote by $\| \cdot \| \,:\, F \to \mathbb{R}_{\geq 0}$ the multiplicative p-adic norm:

$$a \mapsto \|a\| = p^{-Ord(N_{F/\mathbb{Q}_p}(a))},$$

where

$$N_{F/\mathbb{Q}_p} \;:\; F \to \mathbb{Q}_p$$

is the standard norm mapping.

Definition 2.1 Let $\mathfrak{X}$ be an arbitrary reduced algebraic S-scheme, where $S = Spec\, R$. We denote by $\mathfrak{X}(R)$ the set of S-morphisms $S \to \mathfrak{X}$ (or sections of $\mathfrak{X} \to S$). We call $\mathfrak{X}(R)$ the set of R**-integral points** in $\mathfrak{X}$. The set of sections of the morphism $\mathfrak{X} \times_S Spec\, F \to Spec\, F$ we denote by $\mathfrak{X}(F)$ and call the set of F**-rational points** in $\mathfrak{X}$.

Remark 2.2 (i) If $\mathfrak{X}$ is an affine S-scheme, then one can identify $\mathfrak{X}(R)$ with the subset in $\mathfrak{X}(F)$ consisting of all points $x \in \mathfrak{X}(F)$ such that $f(x) \in R$ for all $f \in \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$.

(ii) If $\mathfrak{X}$ is a projective (or proper) S-scheme, then $\mathfrak{X}(R) = \mathfrak{X}(F)$.

Now let X be a smooth n-dimensional algebraic variety over F. Denote by Ω^n_X the canonical line bundle over X. We assume that X admits an extension $\mathfrak{X}$ to a regular S-scheme.

Recall the following definition introduced by A. Weil in [9]:

Definition 2.3 A global section $\omega \in \Gamma(\mathfrak{X}, \Omega^n_{\mathfrak{X}/S})$ is called a **gauge-form** if the n-form ω has no zeros in $\mathfrak{X}$. By definition, a gauge-form ω defines an isomorphism $\mathcal{O}_{\mathfrak{X}} \cong \Omega^n_{\mathfrak{X}/S}$ which sends 1 to ω, i.e., it exists if and only if $\Omega^n_{\mathfrak{X}/S}$ is a trivial line bundle.

It was observed by A. Weil that a gauge form ω determines a canonical p-adic measure $d\mu_\omega$ on the locally compact p-adic topological space $\mathfrak{X}(F)$ of F-rational points in $\mathfrak{X}$. The p-adic measure $d\mu_\omega$ is defined as follows:

Let $x \in \mathfrak{X}(F)$ be an F-point, $t_1, \ldots, t_n$ local p-adic analytic parameters at x. Then $t_1, \ldots, t_n$ define a p-adic homeomorphism $\theta : U \to \mathbb{A}^n(F)$ of an open subset $U \subset \mathcal{X}(F)$ containing x with an open subset $\theta(U) \subset \mathbb{A}^n(F)$. One should stress that both subsets $U \subset \mathfrak{X}(F)$ and $\theta(U) \subset \mathbb{A}^n(F)$ are considered to be open in p-adic topology, but not in Zariski topology. We write

$$\omega = \theta^* \left(g dt_1 \wedge \cdots \wedge dt_n \right),$$

where $g = g(t)$ is a p-adic analytic function on $\theta(U)$ having no zeros. Then a p-adic measure $d\mu_\omega$ on U is defined to be the pull-back with respect to θ of the p-adic measure $\|g(t)\|\mathbf{dt}$ on $\theta(U)$, where $\mathbf{dt}$ is a standard p-adic Haar measure on $\mathbb{A}^n(F)$ with the normalizing condition

$$\int_{\mathbb{A}^n(R)} \mathbf{dt} = 1.$$

It is a standard excercise with the Jacobian to check that two p-adic measures $d\mu'_\omega, d\mu''_\omega$ constructed by the above method on any two open subsets $U', U'' \subset \mathfrak{X}(F)$ coincide on the intersection $U' \cap U''$.

Definition 2.4 The measure $d\mu_\omega$ on $\mathfrak{X}(F)$ constructed as above we call a **p-adic measure of Weil** associated with a gauge-form ω.

Theorem 2.5 ([9], Th. 2.2.5) *Assume that $\mathfrak{X}$ is a regular S-scheme as above, ω is a gauge-form on $\mathfrak{X}$, and $d\mu_\omega$ the corresponding p-adic measure of Weil on $\mathfrak{X}(F)$. Then*

$$\int_{\mathfrak{X}(R)} d\mu_\omega = \frac{|\mathfrak{X}(F_\mathfrak{q})|}{q^n},$$

where $\mathfrak{X}(F_\mathfrak{q})$ is the set of closed points of $\mathfrak{X}$ over the finite residue field $F_\mathfrak{q}$.

Proof. Let

$$\phi \; : \; \mathfrak{X}(R) \to \mathfrak{X}(F_\mathfrak{q}), \; x \mapsto \overline{x} \in \mathfrak{X}(F_\mathfrak{q})$$

be the natural surjecive mapping. The idea of proof of the theorem is based on the fact that if $\overline{x} \in \mathfrak{X}(F_\mathfrak{q})$ is a closed $F_\mathfrak{q}$-point of $\mathfrak{X}$ and $g_1, \ldots, g_n$ are generators of the maximal ideal of $\overline{x}$ in $\mathcal{O}_{\mathfrak{X},\overline{x}}$ modulo the ideal $\mathfrak{q}$, then the elements $g_1, \ldots, g_n$ define a p-adic analytic homeomorphism

$$\gamma \; : \; \phi^{-1}(\overline{x}) \to \mathbb{A}^n(\mathfrak{q}),$$

where $\phi^{-1}(\overline{x})$ is the fiber of ϕ over $\overline{x}$ and $\mathbb{A}^n(\mathfrak{q})$ is the set of all R-integral points of $\mathbb{A}^n$ whose coordinates belong to the ideal $\mathfrak{q} \subset R$. Moreover, the p-adic norm of the

Jacobian of γ is identically equal to 1 on the whole fiber $\phi^{-1}(\overline{x})$. The latter follows from the fact that if n formal power series

$$g_1(t), \ldots, g_n(t) \in R[[t_1, \ldots, t_n]]$$

are generators of the prime ideal $(t_1, \ldots, t_n)$, then the series $g_1(t), \ldots, g_n(t)$ converge absolutely in p-adic norm on the compact $\mathbb{A}^n(\mathfrak{q})$ and the Jacobian of the corresponding mapping

$$\mathbb{A}^n(\mathfrak{q}) \to \mathbb{A}^n(\mathfrak{q}), \ (t_1, \ldots, t_n) \mapsto (g_1(t), \ldots, g_n(t))$$

is equal to a nonzero element of $F_{\mathfrak{q}}$ modulo $\mathfrak{q}$ on the whole subset $\mathbb{A}^n(\mathfrak{q}) \subset \mathbb{A}^n(R)$. So, using the p-adic analytic homeomorphism γ, one obtains

$$\int_{\phi^{-1}(\overline{x})} d\mu_\omega = \int_{\mathbb{A}^n(\mathfrak{q})} \mathbf{dt} = \frac{1}{q^n}$$

for each $\overline{x} \in \mathcal{X}(F_{\mathfrak{q}})$. □

Now we consider a slightly more general situation. Let us only assume that $\mathfrak{X}$ is a regular scheme over S, but do not assume the existence of a gauge-form on $\mathfrak{X}$ (i.e. of an isomorphism $\mathcal{O}_{\mathfrak{X}} \cong \Omega^n_{\mathfrak{X}/S}$). Nevertheless under these weaker assumptions we can define a unique natural p-adic measure $d\mu$ at least on the compact $\mathfrak{X}(R) \subset \mathfrak{X}(F)$ (but may be not on the whole p-adic topological space $\mathfrak{X}(F)$!):

Let $\mathcal{U}_1, \ldots, \mathcal{U}_k$ be a finite covering of $\mathfrak{X}$ by Zariski open S-subschemes such that the restriction of $\Omega^n_{\mathfrak{X}/S}$ on each $\mathcal{U}_i$ is isomorphic to $\mathcal{O}_{\mathcal{U}_i}$. Then each $\mathcal{U}_i$ admits a gauge-form ω_i and we define a p-adic measure $d\mu_i$ on each compact $\mathcal{U}_i(R)$ as the restriction of the p-adic measure of Weil $d\mu_{\omega_i}$ associated with ω_i on $\mathcal{U}_i(F)$. We note that the gauge-forms ω_i are defined uniquely up to elements $s_i \in \Gamma(\mathcal{U}_i, \mathcal{O}^*_{\mathfrak{X}})$. On the other hand, the p-adic norm $\|s_i(x)\|$ equals 1 for any element $s_i \in \Gamma(\mathcal{U}_i, \mathcal{O}^*_{\mathfrak{X}})$ and any R-rational point $x \in \mathcal{U}_i(R)$. Therefore, the constructed p-adic measure on $\mathcal{U}_i(R)$ does not depend on the choice of a gauge-form ω_i. Moreover, the p-adic measures $d\mu_i$ on $\mathcal{U}_i(R)$ glue together to a p-adic measure $d\mu$ on the whole compact $\mathfrak{X}(R)$, since one has

$$\mathcal{U}_i(R) \cap \mathcal{U}_j(R) = (\mathcal{U}_i \cap \mathcal{U}_j)(R) \ \ \forall i, j \in \{1, \ldots, k\}$$

and

$$\mathcal{U}_1(R) \cup \cdots \cup \mathcal{U}_k(R) = (\mathcal{U}_1 \cup \cdots \cup \mathcal{U}_k)(R) = \mathcal{X}(R).$$

Definition 2.6 The constructed above p-adic measure defined on the set $\mathfrak{X}(R)$ of R-integral points of a S-scheme $\mathfrak{X}$ will be called the **canonical p-adic measure**.

For the canonical p-adic measure $d\mu$, we obtain the same property as for the p-adic measure of Weil $d\mu_\omega$:

Theorem 2.7

$$\int_{\mathfrak{X}(R)} d\mu = \frac{|\mathfrak{X}(F_{\mathfrak{q}})|}{q^n}.$$

Proof. Using a covering of $\mathfrak{X}$ by some Zariski open subsets $\mathcal{U}_1, \ldots, \mathcal{U}_k$, we obtain

$$\int_{\mathfrak{X}(R)} d\mu = \sum_{i_1} \int_{\mathcal{U}_{i_1}(R)} d\mu - \sum_{i_1 < i_2} \int_{(\mathcal{U}_{i_1} \cap \mathcal{U}_{i_2})(R)} d\mu + \cdots + (-1)^k \int_{(\mathcal{U}_1 \cap \cdots \cap \mathcal{U}_k)(R)} d\mu$$

and

$$|\mathfrak{X}(F_{\mathfrak{q}})| = \sum_{i_1} |\mathcal{U}_{i_1}(F_{\mathfrak{q}})| - \sum_{i_1 < i_2} |(\mathcal{U}_{i_1} \cap \mathcal{U}_{i_2})(F_{\mathfrak{q}})| + \cdots + (-1)^k |(\mathcal{U}_1 \cap \cdots \cap \mathcal{U}_k)(F_{\mathfrak{q}})|.$$

It remains to apply 2.5 to every intersection $\mathcal{U}_{i_1} \cap \cdots \cap \mathcal{U}_{i_s}$. □

Theorem 2.8 *Let $\mathfrak{X}$ be a regular integral S-scheme and $\mathcal{Z} \subset \mathfrak{X}$ is a closed reduced subscheme of codimension 1. Then the subset $\mathcal{Z}(R) \subset \mathfrak{X}(R)$ has zero measure with respect to the canonical p-adic measure $d\mu$ on $\mathfrak{X}(R)$.*

Proof. Using a covering of $\mathfrak{X}$ by some Zariski open affine subsets $\mathcal{U}_1, \ldots, \mathcal{U}_k$, one can always reduce the situation to the case when $\mathfrak{X}$ is an affine regular integral S-scheme and $\mathcal{Z} \subset \mathfrak{X}$ is an irreducible principal divisor defined by an equation $f = 0$, where f is a prime element of $A = \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$.

Let us consider a special case $\mathfrak{X} = \mathbb{A}^n_S = Spec\, R[X_1, \ldots, X_n]$ and $\mathcal{Z} = \mathbb{A}^{n-1}_S = Spec\, R[X_2, \ldots, X_n]$, i.e., $f = X_1$. For every positive integer m, we denote by $\mathcal{Z}_m(R)$ the subset in $\mathbb{A}^n(R)$ consisting of all points $x = (x_1, \ldots, x_d) \in R^n$ such that the x_1 belongs to the m-th power of $\mathfrak{q}$. One computes straightforward the p-adic integral

$$\int_{\mathcal{Z}_m(R)} \mathbf{dx} = \int_{\mathbb{A}^1(\mathfrak{q}^m)} dx_1 \prod_{i=2}^{n} \left(\int_{\mathbb{A}^1(R)} dx_i \right) = \frac{1}{q^m}.$$

On the other hand, we have

$$\mathcal{Z}(R) = \bigcap_{m=1}^{\infty} \mathcal{Z}_m(R).$$

Hence

$$\int_{\mathcal{Z}(R)} \mathbf{dx} = \lim_{m \to \infty} \int_{\mathcal{Z}_m(R)} \mathbf{dx} = 0,$$

and in this case the statement is proved. Using the Noether normalization theorem, one reduces the more general case to the above special one. □

3 The Betti numbers

Proposition 3.1 *Let X and Y be two birationally isomorphic smooth projective n-dimensional algebraic varieties over $\mathbb{C}$ having trivial canonical line bundles. Then there exist two Zariski open dense subsets $U \subset X$ and $V \subset Y$ such that U is biregularly isomorphic to V and $codim_X(X \setminus U)$, $codim_Y(Y \setminus V) \geq 2$.*

Proof. Consider a birational rational map $\varphi : X - - > Y$. Since X is smooth, there exists a Zariski open dense subset $U_0 \subset X$ with $codim_X(X \setminus U_0) \geq 2$ such that φ extends to a regular morphism $\varphi_0 : U_0 \to Y$. Define $Z \subset U$ to be the Zariski closed subset consisting of all points $x \in X$ such that $\varphi_0^{-1}(\varphi_0(x)) \neq x$. Since both line bundles Ω_X^n and Ω_Y^n are trivial, Z can not be a divisor in U: otherwise Z would be the set of zeros of the φ_0-pullback of nowhere vanishing holomorphic differential n-form $\omega \in H^0(Y, \Omega_Y^n)$. If we set $U_1 = U_0 \setminus Z$, then the restriction of φ_0 on U_1 is a regular injective birational morphism $\varphi_1 : U_1 \to Y$. Again we have $codim_X(X \setminus U_1) \geq 2$. Let $\psi := \varphi^{-1} : Y - - > X$ be the inverse birational rational map. By the same arguments as above, there exists a a Zariski open dense subset $V_1 \subset Y$ with $codim_Y(Y \setminus V_1) \geq 2$ such that ψ extends to a regular injective birational morphism $\psi_1 : V_1 \to X$. Now we define $U := \varphi_1^{-1}(V_1)$ and $V := \psi_1^{-1}(U)$. By the construction, both $U \subset X$ and $V \subset Y$ are Zariski open subsets whose complements have codimensions at least 2. Moreover, the restriction Φ of φ_1 on U induces a biregular isomorphism between U and V. □

Proof of Theorem 1.1. Let X and Y be two smooth projective birationally isomorphic varieties of dimension n over $\mathbb{C}$ with the trivial canonical bundles. By 3.1, there exist two Zariski open dense subsets $U \subset X$ and $V \subset Y$ with $codim_X(X \setminus U) \geq 2$ and $codim_Y(Y \setminus V) \geq 2$ and a biregular isomorphism $\varphi : U \to V$.

By standard arguments, one can choose a finitely generated $\mathbb{Z}$-subalgebra $\mathcal{R} \subset \mathbb{C}$ such that the projective varieties X and Y and the Zariski open subsets $U \subset X$ and $V \subset Y$ can be obtained by the base change $* \times_{\mathcal{S}} Spec\, \mathbb{C}$ from some regular projective $\mathcal{S}$-schemes $\mathcal{X}$ and $\mathcal{Y}$ together with Zariski open $\mathcal{S}$-subschemes $\mathcal{U} \subset \mathcal{X}$ and $\mathcal{V} \subset \mathcal{Y}$, where $\mathcal{S} := Spec\, \mathcal{R}$. Moreover, one can choose $\mathcal{R}$ in such a way that both relative canonical line bundles $\Omega^n_{\mathcal{X}/\mathcal{S}}$ and $\Omega^n_{\mathcal{Y}/\mathcal{S}}$ are trivial, both codimensiona $codim_{\mathcal{X}}(\mathcal{X} \setminus \mathcal{U})$ and $codim_{\mathcal{Y}}(\mathcal{Y} \setminus \mathcal{V})$ are at least 2, and the biregular isomorphism $\varphi : U \to V$ is obtained by the base change from a biregular $\mathcal{S}$-isomorphism $\Phi : \mathcal{U} \to \mathcal{V}$.

For almost all prime numbers $p \in \mathbb{N}$, there exist a regular R-integral point $\pi \in \mathcal{S} \times_{Spec\, \mathbb{Z}} Spec\, \mathbb{Z}_p$, where R is the maximal compact subring with a maximal ideal $\mathfrak{q}$ in some local p-adic field F. By an appropriate choice of $\pi \in \mathcal{S} \times_{Spec\, \mathbb{Z}} Spec\, \mathbb{Z}_p$, we can get that both $\mathcal{X}$ and $\mathcal{Y}$ have good reduction modulo $\mathfrak{q}$. Moreover, we can assume that the maximal ideal $I(\overline{\pi})$ of the unique closed point in

$$S := Spec\, R \stackrel{\pi}{\hookrightarrow} \mathcal{S} \times_{Spec\, \mathbb{Z}} Spec\, \mathbb{Z}_p$$

is obtained by the base change from some maximal ideal $J(\overline{\pi}) \subset \mathcal{R}$ over the prime ideal $(p) \subset \mathbb{Z}$.

Let $\omega_{\mathcal{X}}$ and $\omega_{\mathcal{Y}}$ be gauge-forms on $\mathcal{X}$ and $\mathcal{Y}$ respectively. We denote by $\omega_{\mathcal{U}}$ (resp. by $\omega_{\mathcal{V}}$) the restriction of $\omega_{\mathcal{X}}$ to $\mathcal{U}$ (resp. of $\omega_{\mathcal{Y}}$ to $\mathcal{V}$). Since Φ^* is a biregular $\mathcal{S}$-morphism, $\Phi^*\omega_{\mathcal{Y}}$ is another gauge-form on $\mathcal{U}$. Hence there exists a nowhere vanishing regular function $h \in \Gamma(\mathcal{U}, \mathcal{O}^*_{\mathcal{X}})$ such that

$$\Phi^*\omega_{\mathcal{V}} = h\omega_{\mathcal{U}}.$$

The property $codim_{\mathcal{X}}(\mathcal{X} \setminus \mathcal{U}) \geq 2$ implies that h is an element of $\Gamma(\mathcal{X}, \mathcal{O}^*_{\mathcal{X}}) = \mathcal{R}^*$. Hence, one has $\|h(x)\| = 1$ for all $x \in \mathcal{X}(F)$, i.e., the p-adic measures of Weil on $\mathcal{U}(F)$ associated with $\Phi^*\omega_{\mathcal{V}}$ and $\omega_{\mathcal{U}}$ are the same. The latter implies the following equality of the p-adic integrals

$$\int_{\mathcal{U}(F)} d\mu_{\mathcal{X}} = \int_{\mathcal{V}(F)} d\mu_{\mathcal{Y}}.$$

By 2.8 and 2.2(ii), we obtain

$$\int_{\mathcal{U}(F)} d\mu_{\mathcal{X}} = \int_{\mathcal{X}(F)} d\mu_{\mathcal{X}} = \int_{\mathcal{X}(R)} d\mu_{\mathcal{X}}$$

and

$$\int_{\mathcal{V}(\mathcal{F})} d\mu_{\mathcal{Y}} = \int_{\mathcal{Y}(\mathcal{F})} d\mu_{\mathcal{Y}} = \int_{\mathcal{Y}(R)} d\mu_{\mathcal{Y}}.$$

Now, applying the formula in 2.7, we come to the equality

$$\frac{|\mathcal{X}(F_{\mathfrak{q}})|}{q^n} = \frac{|\mathcal{Y}(F_{\mathfrak{q}})|}{q^n}.$$

This shows that the numbers of $F_{\mathfrak{q}}$-rational points in $\mathcal{X}$ and $\mathcal{Y}$ modulo the ideal $J(\overline{\pi}) \subset \mathcal{R}$ are the same. By the consideration of a cyclotomic extension $\mathcal{R}^{(r)} \subset \mathbb{C}$ containing all complex $(q^r - 1)$-th roots of unity, we can repeat the same arguments and obtain that both projective schemes $\mathcal{X}$ and $\mathcal{Y}$ have the same number of $F_{\mathfrak{q}}^{(r)}$-rational points, where $F_{\mathfrak{q}}^{(r)}$ is the degree-r extension of the finite field $F_{\mathfrak{q}}$. In particular, we obtain that the zeta-functions of Weil

$$Z(\mathcal{X}, p, t) = \exp\left(\sum_{r=1}^{\infty} |\mathcal{X}(F_{\mathfrak{q}}^{(r)})| \frac{t^r}{r}\right)$$

and

$$Z(\mathcal{Y}, p, t) = \exp\left(\sum_{r=1}^{\infty} |\mathcal{Y}(F_{\mathfrak{q}}^{(r)})| \frac{t^r}{r}\right)$$

are the same. Using the Weil's conjectures proved by Deligne [8] and the comparison theorem between the étale and singular cohomology, we obtain

$$Z(\mathcal{X}, p, t) = \frac{P_1(t)P_3(t)\cdots P_{2n-1}(t)}{P_0(t)P_2(t)\cdots P_{2n}(t)}$$

and

$$Z(\mathcal{Y}, p, t) = \frac{Q_1(t)Q_3(t)\cdots Q_{2n-1}(t)}{Q_0(t)Q_2(t)\cdots Q_{2n}(t)},$$

where $P_i(t)$ and $Q_i(t)$ are polynomials with integer coefficients having the properties

$$deg\, P_i(t) = dim\, H^i(X, \mathbb{C}), \quad deg\, Q_i(t) = dim\, H^i(Y, \mathbb{C}) \quad \forall i \geq 0.$$

Since the standart archimedian absolute value of each root of polynomials $P_i(t)$ and $Q_i(t)$ must be $q^{-i/2}$ and $P_i(0) = Q_i(0) = 1 \quad \forall i \geq 0$, the equality $Z(\mathcal{X}, p, t) = Z(\mathcal{Y}, p, t)$ implies $P_i(t) = Q_i(t) \quad \forall i \geq 0$. Therefore, we have $dim\, H^i(X, \mathbb{C}) = dim\, H^i(Y, \mathbb{C}) \quad \forall i \geq 0$. □

4 Remarks

As we have seen from the proof of Theorem 3.1, the zeta-fuctions of Weil $Z(\mathcal{X}, p, t)$ and $Z(\mathcal{Y}, p, t)$ are the same for almost all primes $p \in Spec\, \mathbb{Z}$. This fact being expressed in terms of the associated L-functions indicates that the establisched isomorphism $H^i(X, \mathbb{C}) \cong H^i(Y, \mathbb{C})$ for all $i \geq 0$ must have some more deep motivic nature. Recently Kontsevich suggested an idea of a motivic integration [7], which has been developed by Denef and Loeser [2]. In particular, this technique allows to prove that not only the Betti numbers, but also the Hodge numbers of X and Y in 1.1 must be the same.

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