

Introduction

In this lecture we introduce and explain the three functions that are typically used to characterize the growth of algorithms. $O(f(n))$, or big Oh, is a ceiling saying that the algorithm grows no faster than $f(n)$. $\Omega(g(n))$ is a floor saying that the algorithm grows no slower than $g(n)$. $\Theta(f(n))$ is the defining function for an algorithm. When we can find a ceiling and a floor that employ the same function $f(n)$, we say that the algorithm is $\Theta(f(n))$.

Big-O Notation

DEFINITION 1

Let T and f be functions from the set of integers or the set of real numbers to the set of real numbers. We say that $T(n)$ is $O(f(n))$ if there are positive constants n_0 and C such that

$$|T(n)| \leq C|f(n)|$$

whenever

$$n > n_0$$

[This is read as “ $T(n)$ is big-oh of $f(n)$.”]

- The constants C and n_0 in the definition of big- O notation are called **witnesses** to the relationship $T(n)$ is $O(f(n))$.
- There are *infinitely many* witnesses to the relationship $T(n)$ is $O(f(n))$.

Finding C , n_0 , and $f(n)$ for $T(n)$

Steps: Assume $T(n) = \frac{3}{2}n^2 + \frac{5}{2}n + 10$

- Find $f(n)$. Let $f(n)$ be the fastest growing term in $T(n)$ with its coefficient removed.

$$f(n) = n^2$$

- Find C .

2.1. $C = C_{min} + \Delta$, where $\Delta = 1$ (in many cases).

$$2.2. C_{min} = \lim_{n \rightarrow \infty} \frac{T(n)}{f(n)} = \lim_{n \rightarrow \infty} \frac{\frac{3}{2}n^2 + \frac{5}{2}n + 10}{n^2} = \frac{3}{2}$$

2.3. In practice, C_{min} is the coefficient of the fastest growing term in $T(n)$.

$$3. C = \Delta + \frac{3}{2} = 1 + \frac{3}{2} = \frac{5}{2}$$

- Find n_0 .

4.1. Solve $|T(n_0)| \leq C|f(n_0)|$

$$\begin{aligned} \frac{3}{2}n_0^2 + \frac{5}{2}n_0 + 10 &\leq \frac{5}{2}n_0^2 \\ n_0^2 - \frac{5}{2}n_0 - 10 &\geq 0 \\ n_0 &= \left\lceil \frac{\frac{5}{2} \mp \sqrt{\left(\frac{5}{2}\right)^2 + 4 \cdot 1 \cdot 10}}{2} \right\rceil, n_0 > 0 \\ n_0 &= [4.65], n_0 > 0 \\ n_0 &= 5 \end{aligned}$$

4.2. Choose an integer value for n_0 . Let $n_0 = 5$.

We have shown that $T(n) = \frac{3}{2}n^2 + \frac{5}{2}n + 10$ is $O(n^2)$ because we have found witnesses $C = \frac{5}{2}$ and $n_0 = 5$.

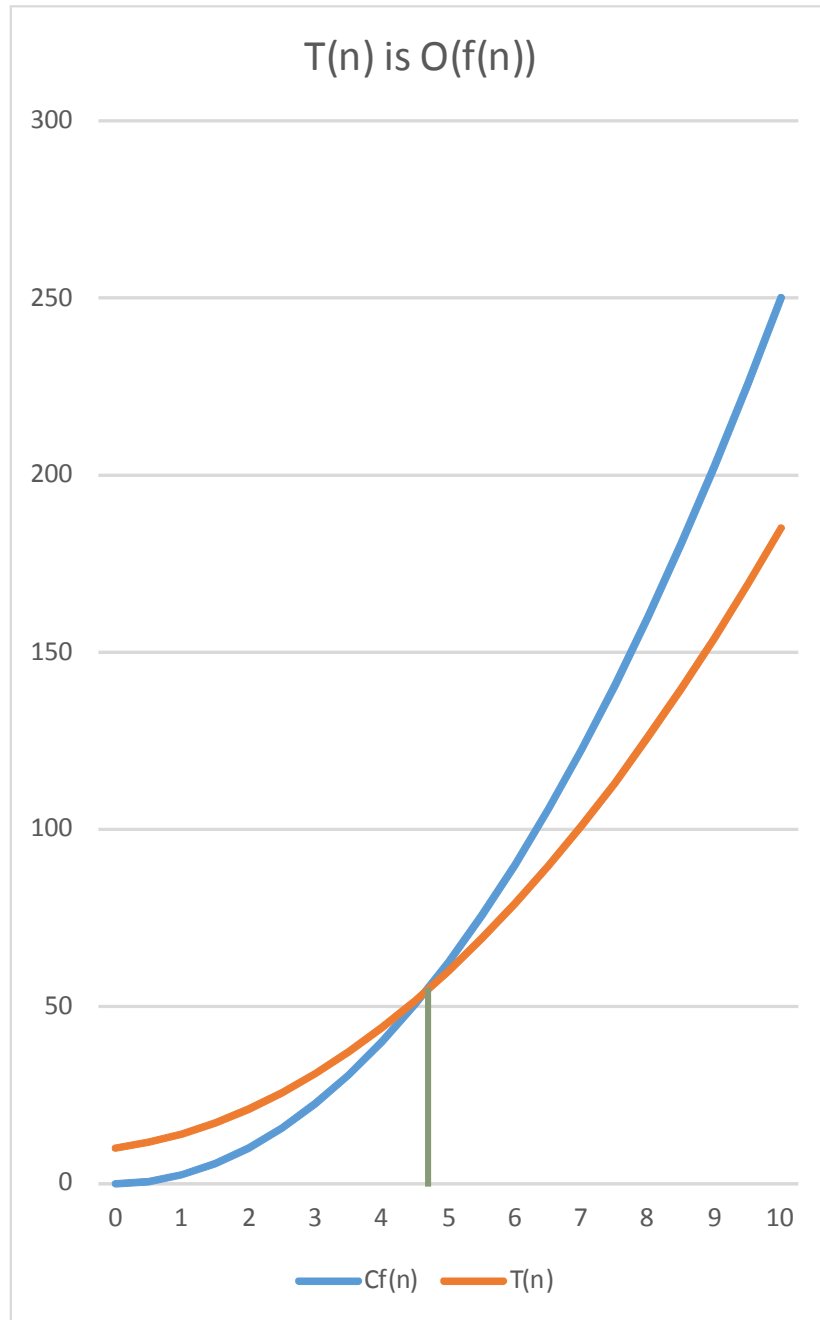


Figure 1. $T(n)$ is $O(f(n))$



REMARK

As an estimate of the growth of $T(n)$, $O(f(n))$ is a conservative estimate because it places a **ceiling** on the growth $T(n)$. When $n > n_0$, $O(f(n)) \geq T(n)$. Let us call $O(f(n))$, big-O of (n) , the Eeyore estimate because the actual running time of a function, given by $T(n)$ will be **less** than the estimate $O(f(n))$ because Eeyore is a pessimist always promising **less** than he actually delivers.

Seek to emulate Eeyore.

Big-Ω Notation

DEFINITION 2

Let T and f be functions from the set of integers or the set of real numbers to the set of real numbers. We say that $T(n)$ is $\Omega(g(n))$ if there are positive constants n_0 and C such that

$$|T(n)| \geq C|g(n)|$$

whenever

$$n > n_0$$

[This is read as “ $T(n)$ is big-Omega of $g(n)$.”]

Finding C , n_0 , and $f(n)$ for $T(n)$

Steps: Assume $T(n) = \frac{3}{2}n^2 + \frac{5}{2}n - 10$

5. Find $g(n)$. Let $f(n)$ be the fastest growing term in $T(n)$ without its coefficient.

$$f(n) = n^2$$

6. Find C .

6.1. $C = C_{min} - \Delta$, where $\Delta = 1$ (in many cases).

$$6.2. C_{min} = \lim_{n \rightarrow \infty} \frac{T(n)}{g(n)} = \lim_{n \rightarrow \infty} \frac{\frac{3}{2}n^2 + \frac{5}{2}n - 10}{n^2} = \frac{3}{2}$$

6.3. In practice, C_{min} is the coefficient of the fastest growing term in $T(n)$.

$$6.4. C = \frac{3}{2} - \Delta = \frac{3}{2} - 1 = \frac{1}{2}$$

7. Find n_0 .

7.1. Solve $|T(n_0)| \geq C|g(n_0)|$

$$\begin{aligned} \frac{3}{2}n_0^2 + \frac{5}{2}n_0 - 10 &\leq \frac{1}{2}n_0^2 \\ n_0^2 + \frac{5}{2}n_0 - 10 &\leq 0 \\ n_0 &= \left\lceil \frac{-\frac{5}{2} \mp \sqrt{\left(\frac{5}{2}\right)^2 + 4 \cdot 1 \cdot 10}}{2} \right\rceil, n_0 > 0 \\ n_0 &= \lfloor 2.15 \rfloor, n_0 > 0 \\ n_0 &\geq 2 \end{aligned}$$

7.2. For $n > 0$, $T(n) \geq Cg(n)$. Select $n_0 = 0$.

We have shown that $T(n) = \frac{3}{2}n^2 + \frac{5}{2}n - 10$ is $\Omega(n^2)$ because we have found witnesses $C = \frac{1}{2}$ and $n_0 = 2$.

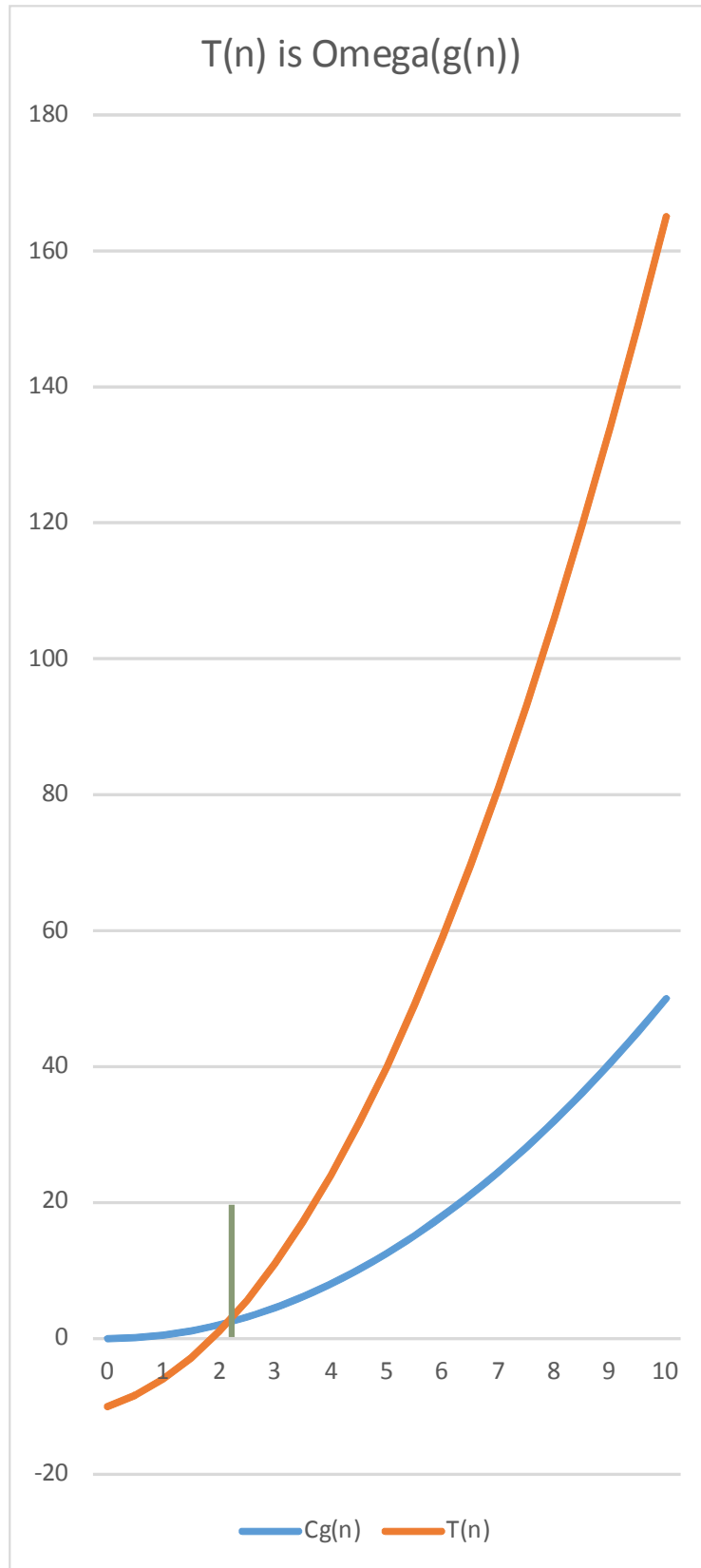


Figure 2. $T(n)$ is $\Omega(g(n))$



REMARK

As an estimate of the growth of $T(n)$, $\Omega(g(n))$ is an optimistic estimate because it places a **floor** on the growth of $T(n)$. When $n > n_0$, $T(n) \geq \Omega(g(n))$. Let us call $\Omega(g(n))$, big-omega of (n) , the Tigger estimate because the actual running time of a function, given by $T(n)$ will be **greater** than the estimate $Cg(n)$ and Tigger is an optimist always promising **more** than he can deliver.

Big- Θ Notation

DEFINITION 3

Let T and f be functions from the set of integers or the set of real numbers to the set of real numbers. We say that $T(n)$ is $\Theta(f(n))$ if $T(n)$ is $O(f(n))$ and if $T(n)$ is $\Omega(f(n))$. When $T(n)$ is $\Theta(f(n))$, we say that " T is big-Theta of $f(n)$ and we also say that $T(n)$ is of *order* $f(n)$ ".

Let us find a function, $T(n)$, that is both $O(f(n))$ and $\Omega(f(n))$.

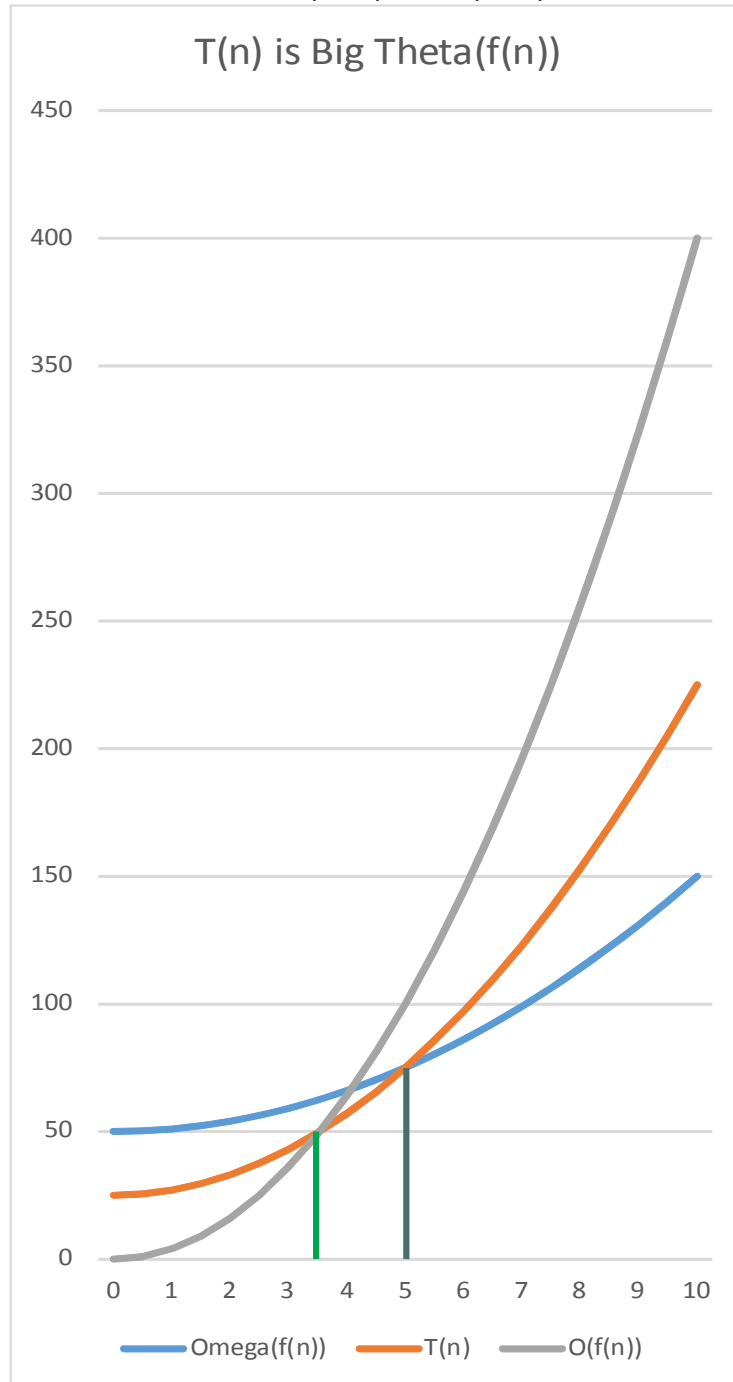


Figure 3. $T(n)$ is $\Theta(n)$