A CLASS OF STATIONARY SEQUENCES

Florentin Smarandache, Ph D
Department of Math & Sciences
University of New Mexico
200 College Road
Gallup, NM 87301, USA
E-mail:smarand@unm.edu

§1. We define a class of sequences $\{a_n\}$ by $a_1 = a$ and $a_{n+1} = P(a_n)$, where P is a polynomial with real coefficients. For which a values, and for which polynomials P will these sequences be constant after a certain rank? Then we generalize it from polynomials P to real functions f.

In this note, the author answers this question using as reference F. Lazebnik & Y. Pilipenko's E 3036 problem from A. M. M., Vol. 91, No. 2/1984, p. 140.

An interesting property of functions admitting fixed points is obtained.

§2. Because $\{a_n\}$ is constant after a certain rank, it results that $\{a_n\}$ converges. Hence, $(\exists)e \in R:\ e = P(e)$, that is the equation $P(x) - x = 0$ admits real solutions. Or P admits fixed points $((\exists)x \in R:\ P(x) = x)$.

Let $e_1, ..., e_m$ be all real solutions of this equation. We construct the recurrent set E as follows:

1) $e_1, ..., e_m \in E$;

2) if $b \in E$ then all real solutions of the equation $P(x) = b$ belong to E;

3) no other element belongs to E, except those elements obtained from the rules 1) and/or 2), applied for a finite number of times.

We prove that this set E, and the set A of the "a" values for which $\{a_n\}$ becomes constant after a certain rank, are indistinct.

Let's show that "$E \subseteq A$":

1) If $a = e_i,\ 1 \le i \le m$, then $(\forall)n \in \mathbb{N}^*$ $a_n = e_i = \text{constant}$.

2) If for $a = b$ the sequence $a_1 = b,\ a_2 = P(b)$ becomes constant after a certain rank, let x_0 be a real solution of the equation $P(x) - b = 0$, the new formed sequence: $a_1^{'} = x_0,\ a_2^{'} = P(x_0) = b,\ a_3^{'} = P(b), ...$ is indistinct after a certain rank with the first one, hence it becomes constant too, having the same limit.

3) Beginning from a certain rank, all these sequences converge towards the same limit e (that is: they have the same e value from a certain rank) are indistinct, equal to e.

Let's show that "$A \subseteq E$":

Let "a" be a value such that: $\{a_n\}$ becomes constant (after a certain rank) equal to e. Of course $e \in \{e_1, ..., e_m\}$ because $e_1, ..., e_m$ are the only values towards these sequences can tend.

If $a \in \{e_1, ..., e_m\}$, then $a \in E$.

Let $a \notin \{e_1, ..., e_m\}$, then $(\exists) n_0 \in \mathbb{N}: a_{n_0+1} = P(a_{n_0}) = e$, hence we obtain by applying the rules 1) or 2) a finite number of times. Therefore, because $e \in \{e_1, ..., e_m\}$ and the equation $P(x) = e$ admits real solutions we find a_{n_0} among the real solutions of this equation: knowing a_{n_0} we find a_{n_0-1} because the equation $P(a_{n_0-1}) = a_{n_0}$ admits real solutions (because $a_{n_0} \in E$ and our method goes on until we find $a_1 = a$ hence $a \in E$.

Remark. For $P(x) = x^2 - 2$ we obtain the E 3036 Problem (A. M. M.).

Here, the set E becomes equal to

$$\{\pm 1, 0, \pm 2\} \cup \left\{ \underbrace{\pm\sqrt{2 \pm \sqrt{2 \pm ... \sqrt{2}}}}_{n_0 \text{ times}},\ n \in \mathbb{N}^* \right\} \cup \left\{ \underbrace{\pm\sqrt{2 \pm \sqrt{2 \pm ... \sqrt{2 \pm \sqrt{3}}}}}_{n_0 \text{ times}},\ n \in \mathbb{N} \right\}$$

Hence, for all $a \in E$ the sequence $a_1 = a,\ a_{n+1} = a_n^2 - 2$ becomes constant after a certain rank, and it converges (of course) towards –1 or 2:

$$(\exists) n_0 \in \mathbb{N}^* : (\forall) n \geq n_0 \qquad a_n = -1$$

or

$$(\exists) n_0 \in \mathbb{N}^* : (\forall) n \geq n_0 \qquad a_n = 2.$$

Generalization.

This can be generalized to defining a class of sequences $\{a_n\}$ by $a_1 = a$ and $a_{n+1} = f(a_n)$, where f: R → R is a real function. For which a values, and for which functions f will these sequences be constant after a certain rank?

In a similar way, because $\{a_n\}$ is constant after a certain rank, it results that $\{a_n\}$ converges. Hence, $(\exists) e \in R:\ e = f(e)$, that is the equation $f(x) - x = 0$ admits real solutions. Or f admits fixed points $((\exists) x \in R:\ \mathrm{f}(x) = x)$.

Let $e_1, ..., e_m$ be all real solutions of this equation. We construct the recurrent set E as follows:

1) $e_1, ..., e_m \in E$;

2) if $b \in E$ then all real solutions of the equation $f(x) = b$ belong to E;

3) no other element belongs to E, except those elements obtained from the rules 1) and/or 2), applied for a finite number of times.

Analogously, this set E, and the set A of the "a" values for which $\{a_n\}$ becomes constant after a certain rank, are indistinct.

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