AARMS TROPICAL GEOMETRY - LECTURE 10

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In this lecture we will consider a large class of examples whose tropical varieties can be completely described. These are the $X \subset T^n$ whose equations are all linear.

Example 1. Let $X = V(3x_1 + 5x_2 - 7x_3) \subset T^3_K$ where $K = \mathbb{C}\{\{t\}\}$. Then $\mathrm{trop}(X) = \{(w_1, w_1, w_2) \in \mathbb{R}^3 : w_1 = w_2 \leq w_3 \text{ or } w_1 = w_3 \leq 3_2 \text{ or } w_2 = w_3 \leq w_1\}$ This is the union of the three cones $\mathrm{span}((1,1,1)) + \mathrm{pos}((1,0,0))$, $\mathrm{span}((1,1,1)) + \mathrm{pos}((0,1,0))$, and $\mathrm{span}((1,1,1)) + \mathrm{pos}((0,0,1))$. Here $\mathrm{span}((1,1,1)) + \mathrm{pos}((1,0,0))$ is the set $\{a(1,1,1) + b(1,0,0) : a, b \in \mathbb{R}, b \geq 0\}$. Note $\mathrm{trop}(X)$ is a fan, rather than just a polyhedral complex. This will be the case whenever the coefficients of the polynomials defining the ideal of X live in a subfield of K (such as $\mathbb{C} \subset \mathbb{C}\{\{t\}\}$) where all elements have valuation zero. Note also that $(1,1,1)$ lies in the lineality space of all cones in $\mathrm{trop}(X)$. This will be the case whenever the equations defining the ideal of X are all homogeneous.

We assume here that there is an inclusion $i\colon \Bbbk \to K$ of the residue field $\Bbbk = R/\mathfrak{m}$ into our field K. This is true when $K = \mathbb{C}\{\{t\}\}$. Let $S = \Bbbk[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, and let $I = \langle f_1, \dots, f_r \rangle \subset S$ be an ideal in S minimally generated by linear forms $f_1, \dots, f_r$. We now describe the tropical variety of $X = V(I) \subset T$.

Example 2. Let $I = \langle x_1+x_2+x_3+x_4+x_5, x_1+2x_2+3x_3 \rangle \subset \Bbbk[x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}, x_4^{\pm 1}, x_5^{\pm 1}]$. Then $V(I)$ is a two-dimensional subvariety of $T \cong (\Bbbk^*)^5$.

Let $f_i = a_{i1}x_1 + \dots a_{in}x_n$ for $1 \leq i \leq r$. Let A be the $r \times n$ matrix with entries $a_{ij} \in \Bbbk$, and let B be a $(n-r) \times n$ matrix whose rows are a basis for $\ker(A)$. Thus $V(I)$ is equal to the intersection of the row space of B with the torus T. Let $\mathcal{B} = \{\mathbf{b}_0, \dots, \mathbf{b}_n\} \subset \Bbbk^{n-r}$ be the columns of the matrix B. While $\mathcal{B}$ depends on the choice of the matrix B, it is determined up to the action of $\mathrm{GL}(n-r, \Bbbk)$.

The lattice of flats $\mathcal{L}(B)$ of the linear space $\mathrm{row}(B)$ has elements the subspaces (flats) of $\Bbbk^{n-r}$ spanned by subsets of $\mathcal{B}$. We make $\mathcal{L}(B)$ into a poset (partially ordered set) by setting $S_1 \preceq S_2$ if $S_1 \subseteq S_2$ for two subspaces S_1, S_2 of K^{n-r} spanned by subsets of $\mathcal{B}$. The poset $\mathcal{L}(B)$ is actually a lattice of rank $n-r$. This means that every maximal chain in $\mathcal{L}(B)$ has length $n-r$. See, for example, [Sta97, Chapter 3] for more on lattices.

Example 3. We continue Example 2. In this case the matrices A and B are

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -2 & 1 & 0 & 1 & 0 \\ -2 & 1 & 0 & 0 & 1 \\ -1 & -1 & 1 & 1 & 0 \end{pmatrix}.$$

We thus have $\mathbf{b}_1 = (-2,-2,-1)$, $\mathbf{b}_2 = (1,1,-1)$, $\mathbf{b}_3 = (0,0,1)$, $\mathbf{b}_4 = (1,0,1)$, and $\mathbf{b}_5 = (0,1,0)$. There are fifteen subspaces of $\Bbbk^3$ spanned by subsets of $\mathcal{B} =$

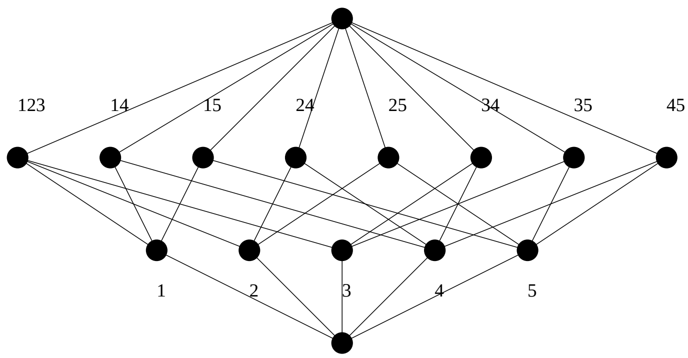

FIGURE 1. The lattice of flats for the linear space of Example 2

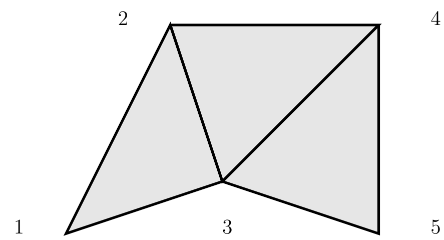

FIGURE 2.

$\{(-2,-2,-1),(1,1,-1),(0,0,1),(1,0,1),(0,1,0)\}$. These are

$$\begin{aligned}&\{0\}\cup\{\text{span}(\mathbf{b}_i) : 1 \leq i < j \leq 5\}\\ \cup&\{\text{span}(\mathbf{b}_1,\mathbf{b}_2,\mathbf{b}_3\}, \text{span}(\mathbf{b}_1,\mathbf{b}_4), \text{span}(\mathbf{b}_1,\mathbf{b}_5), \text{span}(\mathbf{b}_2,\mathbf{b}_4),\\ &\text{span}(\mathbf{b}_2,\mathbf{b}_5), \text{span}(\mathbf{b}_3,\mathbf{b}_4), \text{span}(\mathbf{b}_3,\mathbf{b}_5), \text{span}(\mathbf{b}_4,\mathbf{b}_5)\}\end{aligned}$$

This gives the lattice shown in Figure 1.

A *simplicial complex* on a set S is a collection of subsets of S ("simplices") that is closed under taking subsets. For example, if $S = \{1,2,3,4,5\}$, then one simpicial complex is

$$\Delta = \{\{1,2,3\},\{2,3,4\},\{3,4,5\},\{1,2\},\{1,3\},\{2,3\},\{2,4\},\{3,4\},\{3,5\},\{4,5\},$$
$$\{1\},\{2\},\{3\},\{4\},\{5\},\{\emptyset\}\}.$$

We draw simplical complexes as collections of triangles (or higher-dimensional simplices) in $\mathbb{R}^n$. For example, the above simplicial complex is illustrated in Figure 2

There is a simplicial complex, called the *order complex*, associated to any poset. A *chain* in a poset is a set $x_0 \prec x_1 \prec \cdots \prec x_s$. We say that this chain has length s. Thh has vertices the elements of the poset, and simplices all proper chains, which are totally ordered subsets of the poset not using the bottom or top elements (0 or $\mathbb{R}^n$ in our case). The order complex of $\mathcal{L}(B)$ is pure of dimension $n-r-1$. In the case of our poset there is a nice geometric realization of this simplicial complex, which we now describe.

Definition 4. Let $\mathbf{e}_i$ be the ith standard basis vector on $\mathbb{R}^n$. Given a subset $\sigma \subset \{1, \dots, n\}$ we set $\mathbf{e}_\sigma = \sum_{i \in \sigma} \mathbf{e}_i$. If V is a subspace of $\mathbb{R}^{n-r}$ spanned by some of the $\mathbf{b}_i$, set $\sigma(V)$ to be $\{i : \mathbf{b}_i \in V\}$. Let $\Delta(\mathcal{B})$ be the fan whose cones are

$$\operatorname{pos}(\mathbf{e}_{\sigma(V_i)} : 1 \leq i \leq s) + \operatorname{span}(\mathbf{1}),$$

where $V_1 \prec V_2 \prec \cdots \prec V_s$ is a chain in $\mathcal{L}(\mathcal{B})$, and $\mathbf{1}$ is the all-ones vector in $\mathbb{R}^n$.

Example 5. We continue Example 2. The fan $\Delta(\mathcal{B}) \subset \mathbb{R}^5$ has lineality space the span of $\mathbf{1} = (1, 1, 1, 1, 1)$, meaning that every cone contains the span of this ray, so we describe the quotient fan in $\mathbb{R}^5/\mathbf{1}$. This has 13 rays, corresponding to the five rays spanned by the $\mathbf{b}_i$ and the eight planes spanned by them. There is a two-dimensional cone for every inclusion of a ray into a plane, of which there are 17 in total.

The point of this construction is that the tropical variety $\operatorname{trop}(V(I))$ is equal to $\Delta(\mathcal{B})$.

Theorem 6. *Let I be a linear ideal in S. The tropical variety of $X = V(I) \subset T$ is equal to $\Delta(\mathcal{B})$.*

Proof. We first show that $\operatorname{trop}(X) \subseteq \Delta(\mathcal{B})$. Suppose $w \notin \Delta(\mathcal{B})$. Let $W^j = \{\mathbf{b}_i : w_i \geq j\}$. Note that there are only finitely many subspaces $\operatorname{span}(W^j)$ as j varies, and these form a chain in the lattice of flats $\mathcal{L}(\mathcal{B})$. Let $l = \min\{j :$ there exists $\mathbf{b}_i \in \operatorname{span}(W^j) \setminus W^j\}$. If no such l existed, then w would live in the cone of $\Delta(\mathcal{B})$ defined by the chain $\{\operatorname{span}(W^j)\} \subset \mathcal{L}(\mathcal{B})$. Let $F = \operatorname{span}(W^l l)$. Pick $\mathbf{b}_k \in F \setminus W^l$. Then $w_k < l$ by the definition of W^l. Since $\{\mathbf{b}_i : i \in W^l\}$ spans F, we can write $\mathbf{b}_k = \sum_{i : \mathbf{b}_i \in W^l} \lambda_i \mathbf{b}_i$ for $\lambda_i \in \mathbb{k}$. This means that $\mathbf{e}_k - \sum \lambda_i \mathbf{e}_i \in \ker(B) = \operatorname{row}(A)$. Thus $f = x_k - \sum_{i \in W^l} \lambda_i x_i \in I$. Now $\operatorname{in}_w(f) = x_k$, so $\operatorname{in}_w(I) = \langle 1 \rangle$, and so $w \notin \operatorname{trop}(X)$.

We next show that $\Delta(\mathcal{B}) \subseteq \operatorname{trop}(X)$ by exhibiting for each $w \in \Delta(\mathcal{B})$ an element $y \in X$ with $\operatorname{val}(y) = w$. Given $w \in \Delta(\mathcal{B})$, let $V_1 \subset V_2 \subset \cdots \subset V_{n-r} = \mathbb{k}^{n-r}$ be the chain of flats labelling a maximal cone of $\Delta(\mathcal{B})$ containing w, so $\dim(V_i) = i$. Pick $\mathbf{b}_{i_1} \in V_1$, and $\mathbf{b}_{i_j} \in V_j \setminus V_{j-1}$ for $2 \leq j \leq n - r$. Note that

$$w_{i_j} \geq w_{i_{j+1}}. \tag{1}$$

After renumbering if necessary we may assume that $i_j = j$, and that the matrix B has been chosen so that the first $(n - r) \times (n - r)$ square submatrix is the identity, which is possible as the $\mathbf{b}_{i_j}$ are linearly independent by construction. This implies (exercise!) that the last $r \times r$ submatrix of A must be invertible, so we may assume that it is the identity matrix (since performing row operations on A corresponds to choosing a different generating set for I). We then have

$$A = \left(\begin{array}{c|c} A' & I_r \end{array} \right), \quad B = \left(\begin{array}{c|c} I_{n-r} & A'^T \end{array} \right).$$

Set $y = (t^{w_1}, \dots, t^{w_{n-r}})B$. Explicitly, $y_i = t^{w_i}$ for $1 \leq i \leq n-r$. For $n-r+1 \leq i \leq n$, $y_i = \sum_{j=1}^{n-r} -a_{j(i-n+r)} t^{w_j}$. Then $Ay = 0$ by construction, so $y \in X$. The valuation $\operatorname{val}(y_i) = w_i$ for $1 \leq i \leq n - r$ by construction. For $n - r + 1 \leq i \leq n$ we have $\operatorname{val}(y_i) = \min\{w_j : a_{(i-n+r)j} \neq 0, 1 \leq j \leq n - r\} = w_s$ for $s = \max\{j : a_{(i-n+r)j} \neq 0, 1 \leq j \leq n - r\}$, by Equation 1. Now if $\operatorname{val}(y_i) = w_j$, then $\mathbf{b}_i \in V_j \setminus V_{j-1}$, so by the choice of w we have $w_i = w_j$, and thus $\operatorname{val}(y_i) = w_i$. So y is the desired element of X with $\operatorname{val}(y) = w$, and so $\Delta(\mathcal{B}) \subseteq \operatorname{trop}(X)$. $\square$

Remark 7. To visualize these tropical varieties we can do two tricks. Firstly, since we are assuming that the ideals are homogeneous, we can quotient out by the lineality space span $\mathbf{1}$ when drawing pictures. Dehomogenizing by setting one variable equal to one has the same effect. Secondly, since the tropical varieties are fans, we can intersect the fan with the sphere $S^{n-1} \subset \mathbb{R}^n$ to get an (abstract) polyhedral complex of dimension $\dim(X) - 2$.

References

[Sta97] Richard P. Stanley, *Enumerative combinatorics. Vol. 1*, Cambridge Studies in Advanced Mathematics, vol. 49, Cambridge University Press, Cambridge, 1997. With a foreword by Gian-Carlo Rota; Corrected reprint of the 1986 original. MR **1442260 (98a:**05001)