

SYSTEM IDENTIFICATION

IEEE Press
445 Hoes Lane, P.O. Box 1331
Piscataway, NJ 08855-1331

IEEE Press Editorial Board

Stamatios V. Kartalopoulos, *Editor in Chief*

M. Akay	M. Eden	M. S. Newman
J. B. Anderson	M. E. El-Hawary	M. Padgett
R. J. Baker	R. J. Herrick	W. D. Reeve
J. E. Brewer	R. F. Hoyt	G. Zobrist
	D. Kirk	

Kenneth Moore, *Director of IEEE Press*
Catherine Faduska, *Senior Acquisitions Editor*
John Griffin, *Acquisitions Editor*
Robert H. Bedford, *Assistant Acquisitions Editor*
Anthony VenGraitis, *Project Editor*

Cover design: William T. Donnelly, *WT Design*

Technical Reviewers

Keith Godfrey, *University of Warwick, Coventry, UK*
Brett Ninness, *University of Newcastle, Australia*

Books of Related Interest from the IEEE Press

PERSPECTIVES IN CONTROL ENGINEERING: Technologies, Applications, and New Directions

Edited by Tariq Samad

2001 Hardcover 536 pp IEEE Order No. PC5798 ISBN 0-7803-5356-0

ROBUST VISION FOR VISION-BASED CONTROL OF MOTION

Edited by Markus Vincze and Gregory D. Hager

2000 Hardcover 272 pp IEEE Order No. PC5403 ISBN 0-7803-5378-1

FUZZY CONTROL AND MODELING: Analytical Foundations and Applications

Hao Ying

2000 Hardcover 344 pp IEEE Order No. PC5729 ISBN 0-7803-3497-3

PHYSIOLOGICAL CONTROL SYSTEMS

Michael C. K. Khoo

2000 Hardcover 344 pp IEEE Order No. PC5680 ISBN 0-7803-3408-6

SYSTEM IDENTIFICATION

A Frequency Domain Approach

Rik Pintelon

Department of Electrical Engineering

Vrije Universiteit Brussel

BELGIUM

Johan Schoukens

Department of Electrical Engineering

Vrije Universiteit Brussel

BELGIUM



**IEEE
PRESS**

The Institute of Electrical and Electronics Engineers, Inc. New York

This book and other books may be purchased at a discount from the publisher when ordered in bulk quantities. Contact:

IEEE Press Marketing
Attn: Special Sales
445 Hoes Lane
P.O. Box 1331
Piscataway, NJ 08855-1331
Fax: +1 732 981 9334

For more information about IEEE Press products, visit the
IEEE Online Catalog & Store: <http://www.ieee.org/store>.

© 2001 by the Institute of Electrical and Electronics Engineers, Inc.
3 Park Avenue, 17th Floor, New York, NY 10016-5997.

*All rights reserved. No part of this book may be reproduced in any form,
nor may it be stored in a retrieval system or transmitted in any form,
without written permission from the publisher.*

Printed in the United States of America.

10 9 8 7 6 5 4 3 2 1

ISBN 0-7803-6000-1
IEEE Order No. PC5832

Library of Congress Cataloging-in-Publication Data

Pintelon, R. (Rik)

System identification: a frequency domain approach / Rik Pintelon, Johan Schoukens.
p. cm.

ISBN 0-7803-6000-1

1. System identification. I. Schoukens, J. (Johan) II. Title.

QA402.P56 2001
003'.85—dc21

2001016782

To all people of goodwill

Rik

To Annick, Ine, Maarten, and Sanne

Johan

Contents

Preface xxiii

Acknowledgments xxvii

List of Operators and Notational Conventions xxix

List of Symbols xxxiii

List of Abbreviations xxxvii

CHAPTER 1 An Introduction to Identification 1

- 1.1 What Is Identification? 1
- 1.2 Identification: A Simple Example 2
 - 1.2.1 Estimation of the Value of a Resistor 2
 - 1.2.2 Simplified Analysis of the Estimators 7
 - 1.2.3 Interpretation of the Estimators: A Cost Function–Based Approach 11
- 1.3 Description of the Stochastic Behavior of Estimators 13
 - 1.3.1 Location Properties: Unbiased and Consistent Estimates 13
 - 1.3.2 Dispersion Properties: Efficient Estimators 14
- 1.4 Basic Steps in the Identification Process 17
 - 1.4.1 Collect Information about the System 17
 - 1.4.2 Select a Model Structure to Represent the System 17
 - 1.4.3 Match the Selected Model Structure to the Measurements 19
 - 1.4.4 Validate the Selected Model 19
 - 1.4.5 Conclusion 19
- 1.5 A Statistical Approach to the Estimation Problem 20
 - 1.5.1 Least Squares Estimation 20
 - 1.5.2 Weighted Least Squares Estimation 22
 - 1.5.3 The Maximum Likelihood Estimator 23

- 1.5.4 The Bayes Estimator 25
- 1.5.5 Instrumental Variables 27
- 1.6 Exercises 29

CHAPTER 2 Measurements of Frequency Response Functions 33

- 2.1 Introduction 33
- 2.2 An Introduction to the Discrete Fourier Transform 34
 - 2.2.1 The Sampling Process 34
 - 2.2.2 The Discrete Fourier Transform (DFT-FFT) 35
 - 2.2.3 DFT Properties of Periodic Signals 40
 - 2.2.4 DFT of Burst Signals 42
 - 2.2.5 Conclusion 43
- 2.3 Spectral Representations of Periodic Signals 43
- 2.4 Analysis of FRF Measurements Using Periodic Excitations 44
 - 2.4.1 Measurement Setup 44
 - 2.4.2 Error Analysis 45
- 2.5 Reducing FRF Measurement Errors for Periodic Excitations 48
 - 2.5.1 Basic Principles 48
 - 2.5.2 Processing Repeated Measurements 50
 - 2.5.3 Improved Averaging Methods for Nonsynchronized Measurements 50
 - 2.5.4 Coherence 51
- 2.6 FRF Measurements Using Random Excitations 52
 - 2.6.1 Basic Principles 53
 - 2.6.2 Reducing the Noise Influence 53
 - 2.6.3 Leakage Errors 57
 - 2.6.4 Improved FRF Measurements for Random Excitations 59
- 2.7 FRF Measurements of Multiple Input, Multiple Output Systems 61
- 2.8 Guidelines for FRF Measurements 61
 - 2.8.1 Guideline 1: Use Periodic Excitations 62
 - 2.8.2 Guideline 2: Select the Best FRF Estimator 62
 - 2.8.3 Guideline 3: Pretreatment of Data 62
- 2.9 Conclusion 63
- 2.10 Exercises 63
- 2.11 Appendixes 64
 - Appendix 2.A: Asymptotic Behavior of Averaging Techniques 64
 - Appendix 2.B: Proof of Theorem 2.6 (On Decaying Leakage Errors) 66

CHAPTER 3 Frequency Response Function Measurements in the Presence of Nonlinear Distortions 69

- 3.1 Introduction 69
- 3.2 Intuitive Understanding of the Behavior of Nonlinear Systems 70
- 3.3 A Formal Framework to Describe Nonlinear Distortions 71
 - 3.3.1 Class of Excitation Signals 72
 - 3.3.2 Selection of a Model Structure for the Nonlinear System 73

3.4	Study of the Properties of FRF Measurements in the Presence of Nonlinear Distortions	74
3.4.1	Study of the Expected Value of the FRF for a Constant Number of Harmonics	76
3.4.2	Asymptotic Behavior of the FRF if the Number of Harmonics Tends to Infinity	77
3.4.3	Further Comments on the Related Linear Dynamic System	79
3.4.4	Further Comments on the Stochastic Nonlinear Contributions	82
3.4.5	Extension to Discrete-Time Modeling	83
3.4.6	Experimental Illustration	84
3.5	Detection of Nonlinear Distortions	86
3.5.1	Detection of Nonlinear Distortions Using Periodic Excitations	86
3.5.2	Illustration on the Electrical Simulator	88
3.5.3	A Short Overview of Other Methods to Detect Nonlinear Distortions	88
3.6	Minimizing the Impact of Nonlinear Distortions on FRF Measurements	89
3.6.1	Guidelines	89
3.6.2	Discussions and Illustrations	90
3.7	Conclusion	93
3.8	Exercises	93
3.9	Appendixes	94
	Appendix 3.A: Bias and Stochastic Contributions of the Nonlinear Distortions	94
	Appendix 3.B: Study of the Moments of the Stochastic Nonlinear Contributions	95
	Appendix 3.C: Mixing Property of the Stochastic Nonlinear Contributions	99
	Appendix 3.D: Structure of the Indecomposable Sets	102
	Appendix 3.E: Distribution of the Stochastic Nonlinearities	104
	Appendix 3.F: Extension to Random Amplitudes and Nonuniform Phases	107
	Appendix 3.G: Response of a Nonlinear System to a Gaussian Excitation	107
	Appendix 3.H: Proof of Theorem 3.12	109
	Appendix 3.I: Proof of Theorem 3.15	111
	Appendix 3.J: Proof of Theorem 3.16	111
	Appendix 3.K: Proof of Theorem 3.8	112

CHAPTER 4 Design of Excitation Signals 115

4.1	Introduction	115
4.2	General Remarks on Excitation Signals for Nonparametric Frequency Response Measurements	116
4.2.1	Quantifying the Quality of an Excitation Signal	117
4.2.2	Stepped Sine versus Broadband Excitations	118
4.3	Study of Broadband Excitation Signals	119
4.3.1	General Purpose Signals	119
4.3.2	Optimized Test Signals	126
4.3.3	Advanced Test Signals	129

- 4.4 Optimization of Excitation Signals for Parametric Measurements 131
 - 4.4.1 Introduction 131
 - 4.4.2 Optimization of the Power Spectrum of a Signal 132
- 4.5 Appendix 137
 - Appendix 4.A: Minimizing the Crest Factor of a Multisine 137

CHAPTER 5 Models of Linear Time-Invariant Systems 139

- 5.1 Introduction 139
- 5.2 Plant Models 144
- 5.3 Relation between the Input-Output DFT Spectra 146
 - 5.3.1 Models for Periodic Signals 147
 - 5.3.2 Models for Arbitrary Signals 147
 - 5.3.3 Models for Records with Missing Data 150
- 5.4 Models for Damped (Complex) Exponentials 151
- 5.5 Identifiability 152
 - 5.5.1 Models for Periodic Signals 152
 - 5.5.2 Models for Arbitrary Signals 153
 - 5.5.3 Models for Records with Missing Data 154
- 5.6 Multivariable Systems 154
- 5.7 Noise Models 155
 - 5.7.1 Introduction 155
 - 5.7.2 Nonparametric Noise Model 156
 - 5.7.3 Parametric Noise Model 156
- 5.8 Nonlinear Systems 158
- 5.9 Exercises 159
- 5.10 Appendixes 161
 - Appendix 5.A: Stability and Minimum Phase Regions 161
 - Appendix 5.B: Relation between DFT Spectra and Transfer Function for Arbitrary Signals 162
 - Appendix 5.C: Parameterizations of the Extended Transfer Function Model 164
 - Appendix 5.D: Convergence Rate of the Equivalent Initial Conditions 165
 - Appendix 5.E: Some Integral Expressions 165
 - Appendix 5.F: Convergence Rate of the Residual Alias Errors 167
 - Appendix 5.G: Relation between DFT Spectra and Transfer Function for Arbitrary Signals with Missing Data 172
 - Appendix 5.H: Free Decay Response of a Finite-Dimensional System 172
 - Appendix 5.I: Relation between the Free Decay Parameters and the Partial Fraction Expansion 172
 - Appendix 5.J: Some Properties of Polynomials 173
 - Appendix 5.K: Proof of the Identifiability of Transfer Function Model (5-32) (Theorem 5.9) 174
 - Appendix 5.L: Proof of the Identifiability of Transfer Function Models (5-35) and (5-36) 174

CHAPTER 6 An Intuitive Introduction to Frequency Domain Identification 177

- 6.1 Intuitive Approach 177
- 6.2 The Errors-In-Variables Formulation 178
- 6.3 Generating Starting Values 180
- 6.4 Differences from and Similarities to the “Classical” Time Domain Identification Framework 181
- 6.5 Extensions of the Model: Dealing with Unknown Delays and Transients 182

CHAPTER 7 Estimation with Known Noise Model 183

- 7.1 Introduction 183
- 7.2 Frequency Domain Data 184
- 7.3 Plant Model 186
- 7.4 Estimation Algorithms 187
- 7.5 Quick Tools to Analyze Estimators 189
- 7.6 Assumptions 191
 - 7.6.1 Stochastic Convergence 191
 - 7.6.2 Stochastic Convergence Rate 193
 - 7.6.3 Systematic and Stochastic Errors 193
 - 7.6.4 Asymptotic Normality 193
 - 7.6.5 Deterministic Convergence 194
 - 7.6.6 Consistency 194
 - 7.6.7 Asymptotic Bias 195
 - 7.6.8 Asymptotic Efficiency 195
- 7.7 Asymptotic Properties 196
- 7.8 Linear Least Squares 199
 - 7.8.1 Introduction 199
 - 7.8.2 Linear Least Squares 199
 - 7.8.3 Iterative Weighted Linear Least Squares 201
 - 7.8.4 A Simple Example 202
- 7.9 Nonlinear Least Squares 203
 - 7.9.1 Output Error 203
 - 7.9.2 Logarithmic Least Squares 206
 - 7.9.3 A Simple Example—Continued 207
- 7.10 Total Least Squares 208
 - 7.10.1 Introduction 208
 - 7.10.2 Total Least Squares 210
 - 7.10.3 Generalized Total Least Squares 210
- 7.11 Maximum Likelihood 212
 - 7.11.1 The Maximum Likelihood Solution 212
 - 7.11.2 Discussion 214
 - 7.11.3 Asymptotic Properties 215
 - 7.11.4 Calculation of Uncertainty Bounds 216
- 7.12 Approximate Maximum Likelihood 217
 - 7.12.1 Introduction 217
 - 7.12.2 Iterative Quadratic Maximum Likelihood 217
 - 7.12.3 Bootstrapped Total Least Squares 218
 - 7.12.4 Weighted (Total) Least Squares 219
- 7.13 Instrumental Variables 221

7.14	Subspace Algorithms	221
7.14.1	Model Equations	221
7.14.2	Subspace Identification Algorithms	223
7.14.3	Stochastic Properties	226
7.15	Illustration and Overview of the Properties	227
7.15.1	Simulation Example 1	228
7.15.2	Simulation Example 2	230
7.15.3	Real Measurement Examples	231
7.15.4	Overview of the Properties	233
7.16	High-Order Systems	237
7.16.1	Scalar Orthogonal Polynomials	239
7.16.2	Vector Orthogonal Polynomials	240
7.16.3	Application to the Estimators	241
7.16.4	Notes	241
7.17	Systems with Time Delay	242
7.18	Identification in Feedback	242
7.19	Modeling in the Presence of Nonlinear Distortions	244
7.20	Missing Data	244
7.21	Multivariable Systems	245
7.22	Transfer Function Models with Complex Coefficients	246
7.23	Exercises	247
7.24	Appendixes	248
	Appendix 7.A: A Second-Order Simulation Example	248
	Appendix 7.B: Signal-to-Noise Ratio of DFT Spectra Measurements of Random Excitations	249
	Appendix 7.C: Signal-to-Noise Ratio of DFT Spectra Measurements of Periodic Excitations	249
	Appendix 7.D: Asymptotic Behavior Cost Function for a Time Domain Experiment	250
	Appendix 7.E: Asymptotic Properties of Frequency Domain Estimators with Deterministic Weighting (Theorem 7.21)	251
	Appendix 7.F: Asymptotic Properties of Frequency Domain Estimators with Stochastic Weighting (Corollary 7.22)	255
	Appendix 7.G: Expected Value of an Analytic Function	255
	Appendix 7.H: Total Least Squares Solution—Equivalences (Lemma 7.23)	257
	Appendix 7.I: Expected Value Total Least Squares Cost Function	258
	Appendix 7.J: Explicit Form of the Total Least Squares Cost Function	258
	Appendix 7.K: Rank of the Column Covariance Matrix	259
	Appendix 7.L: Calculation of the Gaussian Maximum Likelihood Estimate	260
	Appendix 7.M: Number of Free Parameters in an Errors-in- Variables Problem	264
	Appendix 7.N: Uncertainty of the BTLS Estimator in the Absence of Model Errors	264
	Appendix 7.O: Asymptotic Properties of the Instrumental Variables Method	264

Appendix 7.P:	Equivalences between Range Spaces	264
Appendix 7.Q:	Estimation of the Range Space	265
Appendix 7.R:	Subspace Algorithm for Discrete-Time Systems (Algorithm 7.24)	267
Appendix 7.S:	Subspace Algorithm for Continuous-Time Systems (Algorithm 7.25)	270
Appendix 7.T:	Sensitivity Estimates to Noise Model Errors	274
Appendix 7.U:	IWLS Solution in Case of Vector Orthogonal Polynomials	276
Appendix 7.V:	Asymptotic Properties in the Presence of Nonlinear Distortions	277
Appendix 7.W:	Consistency of the Missing Data Problem	277
Appendix 7.X:	Normal Equation for Complex Parameters and Analytic Residuals	279
Appendix 7.Y:	Total Least Squares for Complex Parameters	279

CHAPTER 8 Estimation with Unknown Noise Model 281

8.1	Introduction	281
8.2	Discussion of the Disturbing Noise Assumptions	283
8.2.1	Assuming Independent Normally Distributed Noise for Time Domain Experiments	283
8.2.2	Considering Successive Periods as Independent Realizations	284
8.3	Properties of the ML Estimator Using a Sample Covariance Matrix	284
8.3.1	The Sample Maximum Likelihood Estimator: Definition of the Cost Function	284
8.3.2	Properties of the Sample Maximum Likelihood Estimator	285
8.3.3	Discussion	286
8.3.4	Estimation of Covariance Matrix of the Model Parameters	287
8.3.5	Properties of the Cost Function in Its Global Minimum	287
8.4	Properties of the GTLS Estimator Using a Sample Covariance Matrix	288
8.5	Properties of the BTLS Estimator Using a Sample Covariance Matrix	290
8.6	Properties of the SUB Estimator Using a Sample Covariance Matrix	292
8.7	Identification in the Presence of Nonlinear Distortions	294
8.8	Illustration and Overview of the Properties	296
8.8.1	Real Measurement Example	296
8.8.2	Overview of the Properties	298
8.9	Identification of Parametric Noise Models	299
8.10	Identification in Feedback	301
8.11	Appendixes	303
Appendix 8.A:	Expected Value and Variance of the Inverse of Chi-Square Random Variable	303
Appendix 8.B:	First and Second Moments of the Ratio of the True and the Sample Variance of the Equation Error	303

Appendix 8.C:	Calculation of Some First- and Second-Order Moments	304
Appendix 8.D:	Proof of Theorem 8.3	306
Appendix 8.E:	Approximation of the Derivative of the Cost Function	307
Appendix 8.F:	Loss in Efficiency of the Sample Estimator	307
Appendix 8.G:	Mean and Variance of the Sample Cost in Its Global Minimum	309
Appendix 8.H:	Asymptotic Properties of the SGTLS Estimator (Theorem 8.6)	310
Appendix 8.I:	Relationship between the GTLS and the SGTLS Estimates (Theorem 8.7)	311
Appendix 8.J:	Asymptotic Properties of SBTLS Estimator (Theorem 8.8)	311
Appendix 8.K:	Relationship between the BTLS and the SBTLS Estimates (Theorem 8.9)	312
Appendix 8.L:	Asymptotic Properties of SSUB Algorithms (Theorem 8.10)	312
Appendix 8.M:	Related Linear Dynamic System of a Cascade of Nonlinear Systems	314
Appendix 8.N:	Asymptotic Properties of the Prediction Error Estimate (Theorem 8.15)	314

CHAPTER 9 Model Selection and Validation 321

9.1	Introduction	321
9.2	Assessing the Model Quality: Quantifying the Stochastic Errors	322
9.2.1	Uncertainty Bounds on the Calculated Transfer Functions	323
9.2.2	Uncertainty Bounds on the Residuals	323
9.2.3	Uncertainty Bounds on the Poles/Zeros	326
9.3	Avoiding Overmodeling	327
9.3.1	Introduction: Impact of an Increasing Number of Parameters on the Uncertainty	327
9.3.2	Balancing the Model Complexity versus the Model Variability	329
9.4	Detection of Undermodeling	331
9.4.1	Undermodeling: A Good Idea?	332
9.4.2	Detecting Model Errors	332
9.4.3	Qualifying and Quantifying the Model Errors	334
9.4.4	Illustration on a Mechanical System	337
9.5	Model Selection	338
9.5.1	Model Structure Selection Based on Preliminary Data Processing: Initial Guess	339
9.5.2	"Postidentification" Model Structure Updating	340
9.6	Guidelines for the User	341
9.7	Exercises	342
9.8	Appendixes	343
Appendix 9.A:	Properties of the Global Minimum of the Maximum Likelihood Cost Function (Theorem 9.2)	343

Appendix 9.B:	Calculation of Improved Uncertainty Bounds for the Estimated Poles and Zeros	343
Appendix 9.C:	Proof of Theorem 9.5	345
Appendix 9.D:	Calculation of the Sample Correlation at Lag Zero (Proof of Theorem 9.5)	347
Appendix 9.E:	Study of the Sample Correlation at Lag One (Proof of Theorem 9.7)	347
Appendix 9.F:	Calculation of the Variance of the Sample Correlation	348

CHAPTER 10 Basic Choices in System Identification 351

10.1	Introduction	351
10.2	Intersample Assumptions: Facts	352
10.2.1	Formal Description of the ZOH and BL Assumptions	352
10.2.2	Relation between the Intersample Behavior and the Model	354
10.2.3	Mixing the Intersample Behavior and the Model	356
10.3	The Intersample Assumption: Appreciation of the Facts	357
10.3.1	Intended Use of the Model	358
10.3.2	Impact of the Intersample Assumption on the Setup	360
10.3.3	Impact of the Intersample Behavior Assumption on the Identification Methods	361
10.4	Periodic Excitations: Facts	361
10.5	Periodic Excitations: Detailed Discussion and Appreciation of the Facts	362
10.5.1	Data Reduction Linked to an Improved Signal-to-Noise Ratio of the Raw Data	362
10.5.2	Separation of the Signal from the Noise Disturbances	363
10.5.3	Elimination of Nonexcited Frequencies	363
10.5.4	Independent Estimation of Nonparametric Noise Models	364
10.5.5	Improved Frequency Response Measurements	364
10.5.6	Detection, Qualification, and Quantification of Nonlinear Distortions	366
10.5.7	Improved Model Validation	366
10.5.8	Detection of Trends	366
10.6	Periodic versus Random Excitations: User Aspects	367
10.6.1	Design Aspects: Required User Interaction	367
10.7	Time and Frequency Domain Identification	368
10.8	Time and Frequency Domain Identification: Equivalences	369
10.8.1	Initial Conditions: Transient versus Leakage Errors	369
10.8.2	Windowing in the Frequency Domain, (Noncausal) Filtering in the Time Domain	369
10.8.3	Cost Function Interpretation	370
10.9	Time and Frequency Domain Identification: Differences	371
10.9.1	Choice of the Model	371
10.9.2	Unstable Plants	371
10.9.3	Noise Models: Parametric or Nonparametric Noise Models	372
10.9.4	Extended Frequency Range: Combination of Different Experiments	372
10.9.5	The Errors-in-Variables Problem	373
10.10	Conclusions	373

- 10.11 Exercises 374
- 10.12 Appendix 375
 - Appendix 10.A: Frequency Domain Maximum Likelihood
Solution of the Prediction Error Problem 375

CHAPTER 11 Guidelines for the User 377

- 11.1 Introduction 377
- 11.2 Selection of an Identification Scheme 377
 - 11.2.1 Questions 378
 - 11.2.2 Advice 378
 - 11.2.3 Special Case 381
- 11.3 Identification Step-by-Step 381
 - 11.3.1 Check and Selection of the Experimental Setup 382
 - 11.3.2 Experiment Design 383
 - 11.3.3 Preprocessing 384
 - 11.3.4 Identification 386
- 11.4 Validation 388
- 11.5 Appendixes 389
 - Appendix 11.A: Independent Experiments 389
 - Appendix 11.B: Relationship between Averaged DFT Spectra
and Transfer Function for Arbitrary
Excitations 390
 - Appendix 11.C: Relationship between DFT Spectra of
Concatenated Data Sets and Transfer
Function 390

CHAPTER 12 Applications 393

- 12.1 Introduction 393
- 12.2 Compact Disc Player 394
 - 12.2.1 Measurement Setup 394
 - 12.2.2 Introduction 394
 - 12.2.3 Experiment Design, Preliminary Measurement 395
 - 12.2.4 Quantifying the Nonlinear Distortions 396
 - 12.2.5 Identification 396
- 12.3 Extraction of a Simulation Model 398
 - 12.3.1 Experimental Setup and Measurements 398
 - 12.3.2 Processing 398
- 12.4 Identification of the Best Linear Approximation in the Presence
of Nonlinear Distortions 400
- 12.5 Identification of a Parametric Noise Model 401
 - 12.5.1 Measurement Setup 401
 - 12.5.2 Identification 402
 - 12.5.3 Cross-Validation 403
- 12.6 Synchronous Machine 404
 - 12.6.1 Measurement Setup 404
 - 12.6.2 Measurement Results 404
 - 12.6.3 Identification Results 406
- 12.7 Electrochemical Processes 408
- 12.8 Microwave Device 409
 - 12.8.1 Measurement Setup 409
 - 12.8.2 Measurement Results 410

CHAPTER 13 Some Linear Algebra Fundamentals 413

- 13.1 Notations and Definitions 413
- 13.2 Operators and Functions 414
- 13.3 Norms 415
- 13.4 Decompositions 416
 - 13.4.1 Singular Value Decomposition 416
 - 13.4.2 Generalized Singular Value Decomposition 417
 - 13.4.3 The QR Factorization 418
 - 13.4.4 Square Root of a Positive (Semi-)Definite Matrix 418
- 13.5 Moore-Penrose Pseudoinverse 418
- 13.6 Idempotent Matrices 419
- 13.7 Kronecker Algebra 420
- 13.8 Isomorphism between Complex and Real Matrices 421
- 13.9 Derivatives 422
 - 13.9.1 Derivatives of Functions and Vectors w.r.t. a Vector 422
 - 13.9.2 Derivative of a Function w.r.t. a Matrix 423
- 13.10 Inner Product 424
- 13.11 Gram-Schmidt Orthogonalization 426
- 13.12 Calculating the Roots of Polynomials 428
 - 13.12.1 Scalar Orthogonal Polynomials 428
 - 13.12.2 Vector Orthogonal Polynomials 429
- 13.13 Sensitivity of the Least Squares Solution 430
- 13.14 Exercises 431
- 13.15 Appendix 433
 - Appendix 13.A: Calculation of the Roots of a Polynomial 433

CHAPTER 14 Some Probability and Stochastic Convergence Fundamentals 435

- 14.1 Notations and Definitions 435
- 14.2 The Covariance Matrix of a Function of a Random Variable 439
- 14.3 Sample Variables 440
- 14.4 Mixing Random Variables 441
 - 14.4.1 Definition 441
 - 14.4.2 Properties 442
- 14.5 Preliminary Example 444
- 14.6 Definitions of Stochastic Limits 446
- 14.7 Interrelations between Stochastic Limits 447
- 14.8 Properties of Stochastic Limits 450
- 14.9 Laws of Large Numbers 451
- 14.10 Central Limit Theorems 453
- 14.11 Properties of Estimators 454
- 14.12 Cramér-Rao Lower Bound 456
- 14.13 How to Prove Asymptotic Properties of Estimators? 459
 - 14.13.1 Convergence—Consistency 459
 - 14.13.2 Convergence Rate 460
 - 14.13.3 Asymptotic Bias 462
 - 14.13.4 Asymptotic Normality 463
 - 14.13.5 Asymptotic Efficiency 463

14.14	Pitfalls	463
14.15	Preliminary Example—Continued	464
14.15.1	Consistency	465
14.15.2	Convergence Rate	466
14.15.3	Asymptotic Normality	466
14.15.4	Asymptotic Efficiency	467
14.15.5	Asymptotic Bias	468
14.15.6	Robustness	468
14.16	Properties of the Noise after a Discrete Fourier Transform	469
14.17	Exercises	473
14.18	Appendixes	475
	Appendix 14.A: Indecomposable Sets	475
	Appendix 14.B: Proof of Lemma 14.5	476
	Appendix 14.C: Proof of Lemma 14.8	477
	Appendix 14.D: Almost Sure Convergence Implies Convergence in Probability	479
	Appendix 14.E: Convergence in Mean Square Implies Convergence in Probability	479
	Appendix 14.F: The Borel-Cantelli Lemma	479
	Appendix 14.G: Proof of the (Strong) Law of Large Numbers for Mixing Sequences	480
	Appendix 14.H: Proof of the Central Limit Theorem for Mixing Sequences	482
	Appendix 14.I: Generalized Cauchy-Schwarz Inequality for Random Vectors	483
	Appendix 14.J: Proof of the Generalized Cramér-Rao Inequality (Theorem 14.18)	483
	Appendix 14.K: Proof of Lemma 14.23	484
	Appendix 14.L: Proof of Lemma 14.24	484
	Appendix 14.M: Proof of Lemma 14.26	485
	Appendix 14.N: Proof of Lemma 14.27	487
	Appendix 14.O: Proof of Theorem 14.28	487
	Appendix 14.P: Proof of Theorem 14.29	490
	Appendix 14.Q: Proof of Corollary 14.30	491
	Appendix 14.R: Proof of Lemma 14.31	491
	Appendix 14.S: Proof of Theorem 14.32	492
	Appendix 14.T: Proof of Theorem 14.33	493

CHAPTER 15 Properties of Least Squares Estimators with Deterministic Weighting 495

15.1	Introduction	495
15.2	Strong Convergence	496
15.2.1	Strong Convergence of the Cost Function	497
15.2.2	Strong Convergence of the Minimizer	498
15.3	Strong Consistency	499
15.4	Convergence Rate	500
15.4.1	Convergence of the Derivatives of the Cost Function	501
15.4.2	Convergence Rate of $\hat{\theta}(z)$ to $\hat{\theta}(z_0)$	501
15.4.3	Convergence Rate of $\hat{\theta}(z_0)$ to θ_*	502

- 15.5 Asymptotic Bias 502
- 15.6 Asymptotic Normality 504
- 15.7 Asymptotic Efficiency 505
- 15.8 Overview of the Asymptotic Properties 505
- 15.9 Exercises 506
- 15.10 Appendixes 507
 - Appendix 15.A: Proof of the Strong Convergence of the Cost Function (Lemma 15.3) 507
 - Appendix 15.B: Proof of the Strong Convergence of the Minimizer (Theorem 15.6) 508
 - Appendix 15.C: Lemmas 509
 - Appendix 15.D: Proof of the Convergence Rate of the Minimizer (Theorem 15.19) 514
 - Appendix 15.E: Proof of the Improved Convergence Rate of the Minimizer (Theorem 15.21) 515
 - Appendix 15.F: Equivalence between the Truncated and the Original Minimizer (Lemma 15.27) 516
 - Appendix 15.G: Proof of the Asymptotic Bias on the Truncated Minimizer (Theorem 15.28) 517
 - Appendix 15.H: Cumulants of the Partial Sum of a Mixing Sequence 517
 - Appendix 15.I: Proof of the Asymptotic Distribution of the Minimizer (Theorem 15.29) 518
 - Appendix 15.J: Proof of the Existence and the Convergence of the Covariance Matrix of the Truncated Minimizer (Theorem 15.30) 518

CHAPTER 16 Properties of Least Squares Estimators with Stochastic Weighting 521

- 16.1 Introduction—Notational Conventions 521
- 16.2 Strong Convergence 522
 - 16.2.1 Strong Convergence of the Cost Function 523
 - 16.2.2 Strong Convergence of the Minimizer 523
- 16.3 Strong Consistency 524
- 16.4 Convergence Rate 524
 - 16.4.1 Convergence of the Derivatives of the Cost Function 525
 - 16.4.2 Convergence Rate of $\hat{\theta}(z)$ to $\tilde{\theta}(z_0)$ 526
- 16.5 Asymptotic Bias 527
- 16.6 Asymptotic Normality 528
- 16.7 Overview of the Asymptotic Properties 529
- 16.8 Exercises 530
- 16.9 Appendixes 531
 - Appendix 16.A: Proof of the Strong Convergence of the Cost Function (Lemma 16.4) 531
 - Appendix 16.B: Proof of the Convergence Rate of the Minimizer (Theorem 16.16) 531
 - Appendix 16.C: Proof of the Asymptotic Bias of the Truncated Minimizer (Theorem 16.22) 532
 - Appendix 16.D: Proof of the Asymptotic Normality of the Minimizer (Theorem 16.25) 533

CHAPTER 17 Identification of Semilinear Models 535

- 17.1 The Semilinear Model 535
 - 17.1.1 Signal Model 536
 - 17.1.2 Transfer Function Model 536
- 17.2 The Markov Estimator 537
 - 17.2.1 Real Case 537
 - 17.2.2 Complex Case 538
- 17.3 Cramér-Rao Lower Bound 538
 - 17.3.1 Real Case 538
 - 17.3.2 Complex Case 540
- 17.4 Properties Markov Estimator 540
 - 17.4.1 Consistency 541
 - 17.4.2 Strong Convergence 542
 - 17.4.3 Convergence Rate 542
 - 17.4.4 Asymptotic Normality 543
 - 17.4.5 Asymptotic Efficiency 544
 - 17.4.6 Robustness 544
 - 17.4.7 Practical Calculation of Uncertainty Bounds 544
- 17.5 Residuals of the Model Equation 545
 - 17.5.1 Real Case 545
 - 17.5.2 Complex Case 548
- 17.6 Mean and Variance of the Cost Function 548
 - 17.6.1 Real Case 548
 - 17.6.2 Complex Case 550
- 17.7 Model Selection and Model Validation 550
 - 17.7.1 Real Case 550
 - 17.7.2 Complex Case 551
- 17.8 Exercises 552
- 17.9 Appendixes 554
 - Appendix 17.A: Constrained Minimization (17-6) 554
 - Appendix 17.B: Proof of the Cramér-Rao Lower Bound for Semilinear Models 555
 - Appendix 17.C: Markov Estimates of the Observations for Signal Models and Transfer Function Models with Known Input 557
 - Appendix 17.D: Proof of the Convergence Rate of the Markov Estimates for Large Signal-to-Noise Ratios and Small Model Errors (Theorem 17.2) 557
 - Appendix 17.E: Proof of the Asymptotic Distribution of the Markov Estimates without Model Errors 560
 - Appendix 17.F: Proof of the Asymptotic Efficiency of the Markov Estimates (Theorem 17.4) 560
 - Appendix 17.G: Proof of the Convergence Rate of the Residuals (Lemma 17.8) 560
 - Appendix 17.H: Properties of the Projection Matrix in Lemma 17.9 561
 - Appendix 17.I: Proof of the Improved Convergence Rate of the Residuals (Lemma 17.9) 561
 - Appendix 17.J: Proof of the Properties of the Sample Correlation of the Residuals (Theorem 17.10) 562

Appendix 17.K: Proof of the Convergence Rate of the Minimum of the Cost Function (Lemma 17.11)	563
Appendix 17.L: Proof of the Properties of the Cost Function (Theorem 17.12)	566
Appendix 17.M: Model Selection Criteria	567

CHAPTER 18 Identification of Invariants of (Over)Parameterized Models 569

18.1	Introduction	569
18.2	(Over)Parameterized Models and Their Invariants	570
18.3	Cramér-Rao Lower Bound for Invariants of (Over)Parameterized Models	572
18.4	Estimates of Invariants of (Over)Parameterized Models—Finite Sample Results	573
18.4.1	The Estimators	573
18.4.2	Main Result	575
18.5	A Simple Numerical Example	576
18.6	Exercises	578
18.7	Appendixes	578
	Appendix 18.A: Proof of Theorem 18.8 [Cramér-Rao Bound of (Over)Parameterized Models]	578
	Appendix 18.B: Proof of Theorem 18.15 [Jacobian Matrix of (Over)Parameterized Models]	579
	Appendix 18.C: Proof of Theorem 18.16	579

REFERENCES 581

SUBJECT INDEX 593

REFERENCE INDEX 601

ABOUT THE AUTHORS 605

Preface

Identification is a powerful technique for building accurate models of complex systems from noisy data. It consists of three basic steps, which are interrelated: (1) the design of an experiment; (2) the construction of a model, black box or from physical laws; and (3) the estimation of the model parameters from the measurements. The art of modeling lies in proper use of the skills and specialized knowledge of experts in the field of study, who decide what approximations can be made, suggest how to manipulate the system, reveal the important aspects, and so on. Consequently, modeling should preferably be executed by these experts themselves. Naturally, they require relevant tools for extracting information of interest. However, most experts will not be familiar with identification theory and will struggle in each new situation with the same difficulties while developing their own identification techniques, losing time over problems already solved in the literature of identification.

This book presents a thorough description of methods to model linear dynamic time-invariant systems by their transfer function. The relations between the transfer function and the physical parameters of the system are very dependent upon the specific problem. Because transfer function models are generally valid, we have restricted the scope of the book to these alone, so as to develop and study general purpose identification techniques. This should not be unnecessarily restricting for readers who are more interested in the physical parameters of a system: the transfer function still contains all the information that is available in the measurements, and it can be considered to be an intermediate model between the measurements and the physical parameters. Also, the transfer function model is very suitable for those readers looking for a black box description of the input-output relations of a system. And, of course, the model is directly applicable to predict the output of the system.

In this book, we use mainly frequency domain representations of the data. In combination with periodic excitations, this opens many possibilities to identify continuous-time (Laplace-domain) or discrete-time (z -domain) models, if necessary extended with an arbitrary and unknown delay. Although we strongly advocate using periodic excitations, we also extend the methods and models to deal with arbitrary excitations. The “classical” time-domain identification methods that are specifically directed toward these signals are briefly covered and encapsulated in the identification framework that we offer to the reader.

This book provides answers to questions at different levels, such as: What is identification and why do I need it? How to measure the frequency response function of a linear dynamic system? How to identify a dynamic system? All these are very basic questions, directly focused on the interests of the practitioner. Especially for these readers, we have added guidelines to many chapters for the user, giving explicit and clear advice on what are good choices in order to attain a sound solution. Another important part of the material is intended for readers who want to study identification techniques at a more profound level. Questions on how to analyze and prove the properties of an identification scheme are addressed in this part. This study is not restricted to the identification of linear dynamic systems; it is valid for a very wide class of weighted, nonlinear least squares estimators. As such, this book provides a great deal of information for readers who want to set up their own identification scheme to solve their specific problem.

The structure of the book can be split into four parts: (1) collection of raw data or non-parametric identification; (2) parametric identification; (3) comparison with existing frameworks, guidelines, and illustrations; (4) profound development of theoretical tools.

In the first part, after the introductory chapter on identification, we discuss the collection of the raw data: How to measure a frequency response function of a system. What is the impact of nonlinear distortions? How to recognize, qualify, and quantify nonlinear distortions. How to select the excitation signals in order to get the best measurements. This non-parametric approach to identification is discussed in detail in Chapters 2, 3, and 4.

In the second part, we focus on the identification of parametric models. Signal and system models are presented, using a frequency and a time domain representation. The equivalence and impact of leakage effects and initial conditions are shown. Nonparametric and parametric noise models are introduced. The estimation of the parameters in these models is studied in detail. Weighted (nonlinear) least squares methods, maximum likelihood, and subspace methods are discussed and analyzed. First, we assume that the disturbing noise model is known; next, the methods are extended to the more realistic situation of unknown noise models that have to be extracted from the data, together with the system model. Special attention is paid to the numerical conditioning of the sets of equations to be solved. Taking some precautions, very high order systems, with 100 poles and zeros or even more, can be identified. Finally, validation tools to verify the quality of the models are explained. The presence of unmodeled dynamics or nonlinear distortions is detected, and simple rules to guide even the inexperienced user to a good solution are given. This material is presented in Chapters 5 to 9.

The third part begins with an extensive comparison of what is classically called *time and frequency domain identification*. It is shown that, basically, both approaches are equivalent, but some questions are more naturally answered in one domain instead of the other. The most important question is periodic excitations versus nonperiodic or arbitrary excitations. Next, we provide the practitioner with detailed guidelines to help avoid pitfalls from the very beginning of the process (collecting the raw data), over the selection of appropriate identification methods until the model validation. Finally, we illustrate many of the developed ideas in a wide variety of examples from different fields. This part covers Chapters 10, 11, and 12.

The last part of the book is intended for readers who want to acquire a thorough understanding of the material or those who want to develop their own identification scheme. Not only do we give an introduction to the stochastic concepts we use, but we also show, in a structured approach, how to prove the properties of an estimator. This avoids the need for each freshman in this field to find out, time and again, the basic steps to solve such a problem. Starting from this background, a general but detailed framework is set up to analyze the properties of nonlinear least squares estimators with deterministic and stochastic weighting.

For the special and quite important class of semilinear models, it is possible to make this analysis in much more detail. This material is covered in Chapters 13 to 18.

It is possible to extract a number of undergraduate courses from this book. In most of the chapters that can be used in these courses, we added exercises that introduce the students to the typical problems that appear when applying the methods to solve practical problems.

A first, quite general undergraduate course subject is the measurement of frequency response functions of dynamic systems, as discussed in Chapters 2 to 4.

A second possibility is a first introduction to the identification of linear dynamic systems. Such an undergraduate course should include Chapter 1 and some selected parts of Chapters 5, 6, 7, 8, and 9.

A last course, at the graduate level, is an advanced course on identification based on the methods that are explained in Chapters 15, 16, 17, and 18. This gives an excellent introduction for students who want to develop their own algorithms.

A complete MATLAB[®] toolbox, which includes the techniques developed in this book, is available. It can be used with a graphical user interface, avoiding most problems and nasty questions for the inexperienced user. At the basic level, this toolbox produces almost autonomously a good model. At the intermediate or advanced level, the user obtains access to some of the parameters in order to optimize the operation of the toolbox to solve dedicated modeling problem. Finally, for those who want to use it as a research tool, there is also a command level that gives full access to all the parameters that can be set to optimize and influence the behavior of the algorithms. More information on this package can be obtained by sending an E-mail to one of the authors: rik.pintelon@vub.ac.be or johan.schoukens@vub.ac.be

Rik Pintelon
Department of Electrical Engineering
Vrije Universiteit Brussel
BELGIUM

Johan Schoukens
Department of Electrical Engineering
Vrije Universiteit Brussel
BELGIUM

Acknowledgments

This book is the culmination of 20 years of research in the identification group of the Department of Fundamental Electricity and Instrumentation (ELEC) of the Vrije Universiteit Brussel (VUB). It is the result of close and harmonious cooperation between many of the workers in the department, and we would like to thank all of them for exchanges of ideas and important discussions that have taken place over the gestation period of this, the end result. We are greatly indebted to ELEC, which is directed by Professor Alain Barel, to the R&D Department of the VUB, to the FWO-Vlaanderen (the Fund for Scientific Research in Flanders), and to the Flemish and the Federal Government (GOA and IUAP research programs). Without their sustained support, this work would never have seen the light of day.

We are grateful to many colleagues and coworkers in the field who imparted new ideas and methods to us, took time for discussions, and were prepared to listen to some of our less conventional ideas that did not fit in the mainstream of the “classical” identification. Their critical remarks and constructive suggestions contributed significantly to our view on the field.

Last, but not least, we want to thank Yves Rolain for contributing to our work as a friend, a colleague, and a coauthor of many of our papers. Without his sustained support, we would never have been able to access the advanced measurement equipment we used for the many experiments that are reported in this book.

Rik Pintelon

Johan Schoukens

List of Operators and Notational Conventions

\mathbf{A}	outline uppercase font denotes a set, for example, \mathbf{N} , \mathbf{Z} , \mathbf{Q} , \mathbf{R} and \mathbf{C} are, respectively, the natural, the integer, the rational, the real, and the complex numbers
\otimes	the Kronecker matrix product
$*$	convolution operator
$\text{Re}(\)$	real part of
$\text{Im}(\)$	imaginary part of
$\arg \min_x f(x)$	the minimizing argument of $f(x)$
$O(x)$	an arbitrary function with the property $\lim_{x \rightarrow 0} O(x)/x < \infty$
$o(x)$	an arbitrary function with the property $\lim_{x \rightarrow 0} o(x)/x = 0$
$\hat{\theta}$	estimated value of θ
\bar{x}	complex conjugate of x
subscript 0	true value
subscript Re	$A_{\text{Re}} = \begin{bmatrix} \text{Re}(A) - \text{Im}(A) \\ \text{Im}(A) \quad \text{Re}(A) \end{bmatrix}$
subscript re	$A_{\text{re}} = \begin{bmatrix} \text{Re}(A) \\ \text{Im}(A) \end{bmatrix}$
subscript u	with respect to the input of the system
subscript y	with respect to the output of the system
subscript $*$	limiting estimate
superscript T	matrix transpose

superscript $-T$ superscript H superscript $-H$ superscript $+$ superscript \perp $\angle x$ $X_{[ij]}(s)$ $A_{[i, j]}(s)$ $A_{[:, j]}$ $A_{[i, :]}$ $X^{[k]}(s)$ $\lambda(A)$ $\sigma(A)$ $\kappa(A) = (\max_i \sigma_i(A)) / (\min_i \sigma_i(A))$ $|x| = \sqrt{(\operatorname{Re}(x))^2 + (\operatorname{Im}(x))^2}$ $\|A\|_1 = \max_{1 \leq j \leq m} \sum_{i=1}^n |A_{[i, j]}|$ $\|A\|_2 = \max_{1 \leq i \leq m} \sigma_i(A)$ $\|X\|_2 = \sqrt{X^H X}$ $\|A\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^m |A_{[i, j]}|$ $\|A\|_F = \sqrt{\operatorname{tr}(A^H A)}$ $\operatorname{diag}(A_1, A_2, \dots, A_K)$ $\operatorname{herm}(A) = (A + A^H)/2$ $\operatorname{null}(A)$ $\operatorname{range}(A)$ $\operatorname{rank}(A)$

transpose of the inverse matrix

Hermitian transpose: complex conjugate
transpose of a matrix

Hermitian transpose of the inverse matrix

Moore-Penrose pseudoinverse

orthogonal complement of a subspace or
a matrixphase (argument) of the complex number
 x i th entry of the vector function $X(s)$ i, j th entry of the matrix function $A(s)$ j th column of A i th row of A k th realization of a random process $X(s)$ eigenvalue of a square matrix A singular value of an $n \times m$ matrix A condition number of an $n \times m$ matrix A magnitude of a complex number x 1-norm of an $n \times m$ matrix A 2-norm of an $n \times m$ ($n \geq m$) matrix A 2-norm of the column vector X ∞ -norm of an $n \times m$ matrix A Frobenius norm of an $n \times m$ matrix A block diagonal matrix with blocks A_k ,
 $k = 1, 2, \dots, K$ Hermitian symmetric part of an $n \times m$
matrix A null space of the $n \times m$ matrix A , linear
subspace of \mathbb{C}^m defined by $Ax = 0$ range of the $n \times m$ matrix A , linear
subspace of \mathbb{C}^n that is reachable by
making linear combinations of the
columns of A ($\operatorname{range}(A) = (\operatorname{null}(A^T))^\perp$)rank of the $n \times m$ matrix A , maximum
number of linear independent rows
(columns) of A

$\text{span}\{a_1, a_2, \dots, a_m\}$

$\text{tr}(A) = \sum_{i=1}^n A_{[i,i]}$

$\text{vec}(A)$

a.s.lim

l.i.m.

plim

Lim

$\mathcal{E}\{ \}$

$\text{Prob}()$

$b_X = X - \mathcal{E}\{X\}$

$\text{Cov}(X, Y) = \mathcal{E}\{(X - \mathcal{E}\{X\})(Y - \mathcal{E}\{Y\})^H\}$

$\text{covar}(x, y) = \mathcal{E}\{(x - \mathcal{E}\{x\})(y - \mathcal{E}\{y\})^H\}$

$\text{cum}()$

$\text{var}(x) = \mathcal{E}\{|x - \mathcal{E}\{x\}|^2\}$

$C_X = \text{Cov}(X) = \text{Cov}(X, X)$

$\hat{C}_X = \frac{1}{M-1} \sum_{m=1}^M (X^{[m]} - \hat{X})(X^{[m]} - \hat{X})^H$

$C_{XY} = \text{Cov}(X, Y)$

$\hat{C}_{XY} = \frac{1}{M-1} \sum_{m=1}^M (X^{[m]} - \hat{X})(Y^{[m]} - \hat{Y})^H$

$CR(X)$

$\text{DFT}(x(t))$

$Fi(X)$

I_m

q

$\text{MSE}(X) = \mathcal{E}\{(X - X_0)(X - X_0)^H\}$

$R_{xx}(\tau) = \mathcal{E}\{x(t)x^H(t - \tau)\}$

$R_{xy}(\tau) = \mathcal{E}\{x(t)y^H(t - \tau)\}$

$S_{XX}(j\omega)$

$S_{XY}(j\omega)$

the span of the vectors a_1, a_2, \dots, a_m is the linear subspace obtained by making all possible linear combinations of a_1, a_2, \dots, a_m

trace of an $n \times n$ matrix A

a column vector formed by stacking the columns of the matrix A on top of each other

almost sure limit, limit with probability one

limit in mean square

limit in probability

limit in distribution

mathematical expectation

probability

bias of the estimate X

cross-covariance matrix of X and Y

covariance of x and y

cumulant

variance of x

covariance matrix of X

sample covariance matrix of M realizations of X

cross-covariance matrix of X and Y

sample cross-covariance matrix of M realizations of X and Y

Cramér-Rao lower bound on X

discrete Fourier transform of the samples $x(t)$, $t = 0, 1, \dots, N-1$

Fisher information matrix with respect to the parameters X

$m \times m$ identity matrix

backward shift operator:

$$qu(kT_s) = u((k-1)T_s)$$

mean square error of the estimate X

autocorrelation of $x(t)$

cross-correlation of $x(t)$ and $y(t)$

Fourier transform of $R_{xx}(\tau)$ (autopower spectrum of $x(t)$)

Fourier transform of $R_{xy}(\tau)$ (cross-power spectrum of $x(t)$ and $y(t)$)

$$\hat{X} = \frac{1}{M} \sum_{m=1}^M X^{[m]}$$

$$\mu_x = \mathcal{E}\{x\}$$

$$\sigma_x^2 = \text{var}(x)$$

$$\hat{\sigma}_x^2 = \frac{1}{M-1} \sum_{m=1}^M |x^{[m]} - \hat{x}|^2$$

$$\sigma_{xy}^2 = \text{covar}(x, y)$$

$$\hat{\sigma}_{xy}^2 = \frac{1}{M-1} \sum_{m=1}^M (x^{[m]} - \hat{x})(\overline{y^{[m]} - \hat{y}})$$

sample mean of M realizations
(experiments) of X

mean value of x

variance of the x

sample variance of M realizations of x

covariance of x and y

sample covariance of M realizations of x
and y

List of Symbols

$$A(\Omega, \theta) = \sum_{r=0}^{n_p} a_r p_r(\Omega)$$

$$A(\Omega, \theta) = \sum_{r=0}^{n_a} a_r \Omega^r$$

$$B(\Omega, \theta) = \sum_{r=0}^{n_q} b_r q_r(\Omega)$$

$$B(\Omega, \theta) = \sum_{r=0}^{n_b} b_r \Omega^r$$

$$C(z^{-1}, \theta) = \sum_{r=0}^{n_c} c_r z^{-r}$$

$$D(z^{-1}, \theta) = \sum_{k=0}^{n_d} d_k z^{-k}$$

$e(t)$

$E(k)$

f

F

f_s

$G(j\omega)$

$G_R(j\omega)$

$$G(\Omega, \theta) = B(\Omega, \theta)/A(\Omega, \theta)$$

$$H(z^{-1}, \theta) = C(z^{-1}, \theta)/D(z^{-1}, \theta)$$

$$I(\Omega, \theta) = \sum_{r=0}^{n_i} i_r \Omega^r$$

j

denominator polynomial plant model
expanded in the polynomial basis
 $p_r(\Omega)$

denominator polynomial plant model

numerator plant model expanded in the
polynomial basis $q_r(\Omega)$

numerator polynomial plant model

numerator polynomial noise model

denominator polynomial noise model

white noise at time t

discrete Fourier transform of the samples
 $e(tT_s)$, $t = 0, 1, \dots, N-1$

frequency

number of frequency domain data
samples

sampling frequency

frequency response function

best linear approximation of a nonlinear
plant

parametric plant model

parametric noise model

polynomial of the initial and the final
conditions of the plant model
 $B(\Omega, \theta)/A(\Omega, \theta)$

$j^2 = -1$

$$J(z^{-1}, \theta) = \sum_{r=0}^{n_d} j_r z^{-r}$$

M

N

n_a, n_b, n_c, n_d, n_i and n_j

n_θ

$n_u(t), n_y(t)$

$N_U(k), N_Y(k)$

s

s_k

$$T(\Omega, \theta) = I(\Omega, \theta)/A(\Omega, \theta)$$

t

T_s

$$U(e^{j\omega T_s}), Y(e^{j\omega T_s})$$

$$U(k), Y(k)$$

$$U_k, Y_k$$

$$U(j\omega), Y(j\omega)$$

$$U(s), Y(s)$$

$$u(t), y(t)$$

$$U(z), Y(z)$$

$$V_*(\theta)$$

$$V_F(\theta, z)$$

$$V_F'(\theta, z)$$

$$V_F''(\theta, z)$$

$$Z(k) = [Y(k)U(k)]^T$$

$$Z = [Z^T(1)Z^T(2)\dots Z^T(F)]^T$$

polynomial of the initial and the final conditions of the noise model $C(z^{-1}, \theta)/D(z^{-1}, \theta)$

number of (repeated) experiments

number of time domain data samples

order of the polynomials $A(\Omega, \theta)$, $B(\Omega, \theta)$, $C(z^{-1}, \theta)$, $D(z^{-1}, \theta)$, $I(\Omega, \theta)$, and $J(z^{-1}, \theta)$

dimension of the parameter vector θ

disturbing time domain noise on the input $u(t)$ and output $y(t)$ signals, respectively

discrete Fourier transform of the samples $n_u(tT_s)$ and $n_y(tT_s)$, $k = 0, 1, \dots, N-1$, respectively

Laplace transform variable

Laplace transform variable evaluated along the imaginary axis at DFT frequency k : $s_k = j\omega_k$

parametric transient model of the plant $B(\Omega, \theta)/A(\Omega, \theta)$

continuous or discrete time variable

sampling period

Fourier transform of $u(tT_s)$ and $y(tT_s)$

discrete Fourier transform of the samples $u(tT_s)$ and $y(tT_s)$, $t = 0, 1, \dots, N-1$

Fourier coefficients of the periodic signals $u(t)$, $y(t)$

Fourier transform of $u(t)$ and $y(t)$

one-sided Laplace transform of $u(t)$ and $y(t)$

input and output time signals

one-sided Z-transform of $u(tT_s)$, $y(tT_s)$

asymptotic ($F \rightarrow \infty$) cost function

cost function based on F measurements

derivative cost function w.r.t. θ (dimension $1 \times n_\theta$)

second-order derivative (Hessian) cost function w.r.t. θ (dimension $n_\theta \times n_\theta$)

data vector containing the measured input and output (DFT) spectra at (DFT) frequency k

data vector containing the measured input and output DFT spectra (dimension $2F$)

z	Z-transform variable
z_k	Z-transform variable evaluated along the unit circle at DFT frequency k : $z_k = e^{j\omega_k T_s} = e^{j2\pi k/N}$
$\varepsilon(\theta, Z)$	column vector of the (weighted) model residuals (dimension F)
θ	column vector of the model parameters
$\tilde{\theta}(Z_0)$	minimizing argument of the cost function $V_F(\theta)$
$\hat{\theta}(Z)$	estimated model parameters, minimizing argument of the cost function $V_F(\theta, Z)$
$\hat{\theta}(Z)$	truncated estimator
$\sigma_{\hat{Y}}^2(k) = \text{var}(U(k))$	variance of the measured input DFT spectrum
$\sigma_Y^2(k) = \text{var}(Y(k))$	variance of the measured output DFT spectrum
$\sigma_{YU}^2(k) = \text{covar}(Y(k), U(k))$	covariance of the measured output and input DFT spectra
τ	time delay (normalized with the sampling period for discrete time systems)
$J(\theta, Z) = \partial \varepsilon(\theta, Z) / \partial \theta$	gradient of residuals $\varepsilon(\theta, Z)$ w.r.t. the parameters θ (dimension $F \times n_\theta$)
$\omega = 2\pi f$	angular frequency
Ω	generalized transform variable: Laplace domain $\Omega = s$, Z-domain $\Omega = z^{-1}$, Richardson domain $\Omega = \tanh(\tau_R s)$, and diffusion phenomena $\Omega = \sqrt{s}$
Ω_k	generalized transform variable evaluated at DFT frequency k : Laplace domain $\Omega_k = j\omega_k$, Z-domain $\Omega_k = e^{-j\omega_k T_s}$, Richardson domain $\Omega_k = \tanh(\tau_R j\omega_k)$, and diffusion phenomena $\Omega_k = \sqrt{j\omega_k}$, with $\omega_k = 2\pi k/N$

List of Abbreviations

ARMA	AutoRegressive Moving Average
ARMAX	AutoRegressive Moving Average with eXternal input
ARX	AutoRegressive with eXternal input
BJ	Box-Jenkins (model structure)
BTLS	Bootstrapped Total Least Squares
CRB	Cramér-Rao bound for biased estimators
DFT	Discrete Fourier Transform
DUT	Device Under Test
EV	Errors-in-Variables
FFT	Fast Fourier Transform
FRF	Frequency Response Function
GSVD	Generalized Singular Value Decomposition
GTLS	Generalized Total Least Squares
iid	independent identically distributed
IV	Instrumental Variables
IWLS	Iterative weighted linear least squares
IQML	Iterative Quadratic Maximum Likelihood
LS	Least Squares
ML	Maximum Likelihood
NLS	Nonlinear Least Squares
NLS-FRF	Nonlinear Least Squares based on FRF measurements
NLS-IO	Nonlinear Least Squares based on Input- Output measurements

OE	Output Error (model structure)
pdf	probability density function
PE	Prediction Error
rms	root mean square value
SBTLS	sample BTLS
SGTLS	sample GTLS
SISO	Single Input, Single Output
SML	sample ML
SNR	Signal-to-Noise Ratio
SSUB	sample SUB
SUB	subspace
SVD	Singular Value Decomposition
TLS	Total Least Squares
UCRB	Cramér-Rao Bound for Unbiased estimators
w.p.1	with probability one
WGTLS	Weighted Generalized Total Least Squares
WLS	Weighted Least Squares