COMPLEX ANALYSIS–LECTURES 3 AND 4

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Contents

1. Curves in the complex plane $\mathbb{C}$

In this note, we review the notion of curves in $\mathbb{C}$, and many special types of curves. We begin by recalling what a curve in $\mathbb{C}$ looks like.

Definition 1.1. A curve $\mathcal{C}$ in the complex plane $\mathbb{C}$ is a collection of points of the form $z(t) = x(t)+iy(t)$ with $a \le t \le b$ in $\mathbb{C}$, where a, b are real numbers, and $x(t), y(t)$ are real-valued functions in t. In notation, we write

$$\mathcal{C} : z(t) = x(t) + iy(t),\ a \le t \le b$$

to denote the curve $\mathcal{C}$.

Example 1.2. $\mathcal{C} : z(t) = 2t + i(1 + t^2),\ 0 \le t \le 1$ is a curve in $\mathbb{C}$. The points on $\mathcal{C}$ can be obtained by replacing t by a real number between 0 and 1. For example, $z(0) = 2 \cdot 0 + i(1 + 0^2) = i$, is a point on $\mathcal{C}$.

Definition 1.3. (simple curves)
Let $\mathcal{C} : z(t) = x(t)+iy(t),\ a \le t \le b$ be a curve in $\mathcal{C}$. The curve $\mathcal{C}$ is said to be **simple** if $z(t_1) \ne z(t_2)$ for any $a \le t_1 \ne t_2 \le b$, except that the $z(a) = z(b)$ is allowed.

Example 1.4. Let $\mathcal{C} : z(t) = 1 + t + i(2t),\ 0 \le t \le 1$ be a curve in $\mathbb{C}$. We contend that $\mathcal{C}$ is simple. Indeed, take any $t_1 \ne t_2$ with $0 \le t_1, t_1 \le 1$. We see that $1 + t_1 \ne 1 + t_2$, and thus

$$z(t_1) = 1 + t_1 + i(2t_1) \ne 1 + t_2 + i(2t_2) = z(t_2).$$

Thus $\mathcal{C}$ is simple.

Example 1.5. Let $\mathcal{C} : z(t) = t^2 + 1 + i(9t^4 + t^2 + 2017),\ -2 \le t \le 2$ be a curve in $\mathbb{C}$. We claim that $\mathcal{C}$ is a **non-simple** curve. Indeed, we see that

$$z(-1) = (-1)^2 + 1 + i(9(-1)^2 + (-1)^2 + 2017) = 2 + 2027i = z(1),$$

which proves that $\mathcal{C}$ is non-simple.

Definition 1.6. (closed curves)
A curve $\mathcal{C} : z(t) = x(t) + iy(t),\ a \le t \le b$ is said to be **closed** if $z(a) = z(b)$.

Example 1.7. Note that the circle of radius R with center at 0, denoted by C_R, can be realized as a curve in $\mathbb{C}$ by representing each point on the circle by $z(t) = Re^{it}$ for some $0 \le t \le 2\pi$. In this way, one can write

$$C_R : z(t) = Re^{it} = R(\cos(t) + i\sin(t)) = R\cos(t) + i\sin(t),\ 0 \le t \le 2\pi.$$

The circle C_R is an example of **closed** curves. Indeed since $z(0) = Re^{i\cdot 0} = R = Re^{i(2\pi)} = z(2\pi)$, C_R is closed.

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Definition 1.8. (contours)

Let $\mathcal{C} : z(t) = x(t)+iy(t),\ a \leq t \leq b$ be a curve in $\mathbb{C}$. We say that $\mathcal{C}$ is a **contour** if $z(t)$ is continuous on $[a, b]$, and $z'(t)$ is piecewise continuous on $[a, b]$, i.e., there exist real numbers $a \leq a_1 \leq a_2 \leq \cdots \leq a_n \leq b$ such that $z'(t)$ is continuous on each of the intervals $[a, a_1], [a_1, a_2], \ldots, [a_n, b]$.

2. Integrals of complex-valued functions over curves

In this lecture, we are mainly interested in **simple, closed contours** because of its connection to the notion of integrals of functions on curves in $\mathbb{C}$.

We say that a function $f(z)$ is **continuous on a curve** $\mathcal{C} : z(t) = x(t) + iy(t),\ a \leq t \leq b$ **in** $\mathbb{C}$ if the function $f(z(t))$ (viewed as a complex-valued function on the interval $[a, b]$) is continuous on $[a, b]$.

Let $\mathcal{C} : z(t) = x(t) + iy(t),\ a \leq t \leq b$ be a contour in the complex plane $\mathbb{C}$, and let $f(z)$ be a function which is continuous on $\mathcal{C}$. The **integral of** f **over** $\mathcal{C}$ is defined by

$$\int_{\mathcal{C}} f(z)dz := \int_a^b f(z(t))z'(t)dt.$$

Example 2.1. Let $C_R : z(t) = Re^{it},\ 0 \leq t \leq 2\pi$ be the circle of radius $R > 0$ with center at 0, and let $f(z) = 1 + 2\bar{z}^2$. **Compute the integral** $\int_{C_R} f(z)dz$**.**

By definition, we see that

$$\begin{aligned}
\int_{\mathcal{C}} f(z)dz = \int_0^{2\pi} f(z(t))z'(t)dt &= \int_0^{2\pi} (1 + 2\overline{z(t)}^2)z'(t)dt \\
&= \int_0^{2\pi} (1 + 2(e^{-it})^2)(Rie^{it})dt \ \ (\text{since } \overline{z(t)} = e^{-it} \text{ and } z'(t) = iRe^{it}) \\
&= \int_0^{2\pi} (1 + 2(e^{-i2t})(Rie^{it})dt \\
&= Ri\int_0^{2\pi} e^{it}dt + 2Ri\int_0^{2\pi} (e^{-it})dt \\
&= R(e^{it})|_0^{2\pi} - 2R(e^{-it})|_0^{2\pi} \\
&= R(e^{i(2\pi)} - e^{i\cdot 0}) - 2R(e^{-i(2\pi)} - e^{-i\cdot 0}) \\
&= 0.
\end{aligned}$$

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